Star transposition Gray codes for multiset permutations

Torsten Mütze
(University of Warwick + Charles University Prague)

joint work with Arturo Merino (TU Berlin) and
Petr Gregor (Charles University Prague)

Permutations

- one of the most fundamental classes of combinatorial objects

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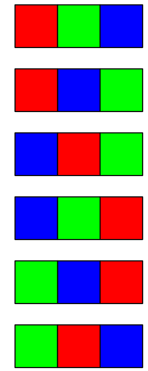
312
321
231
213



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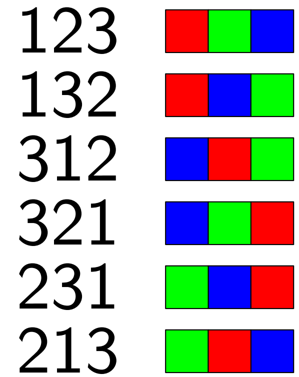

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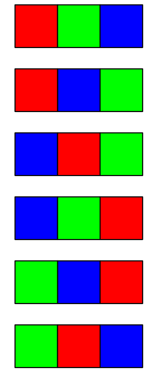

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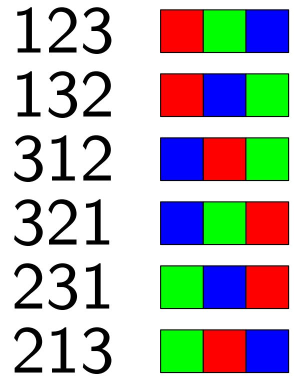

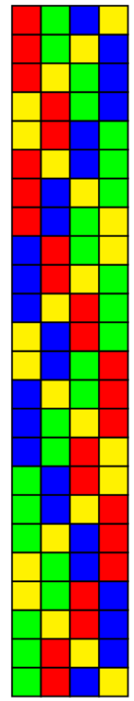

SJT

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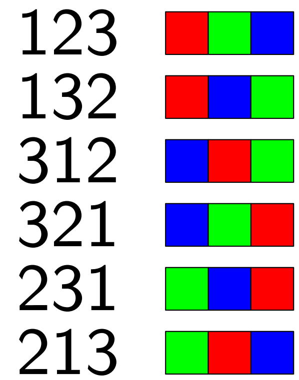

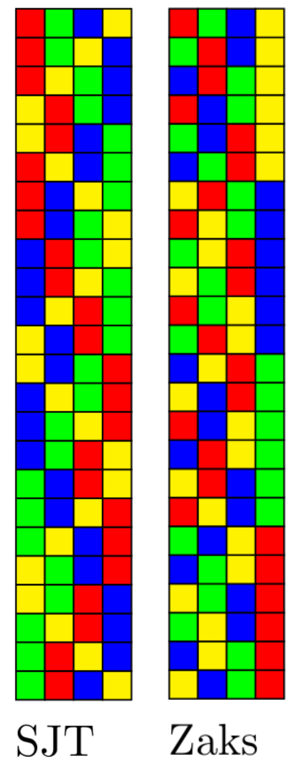

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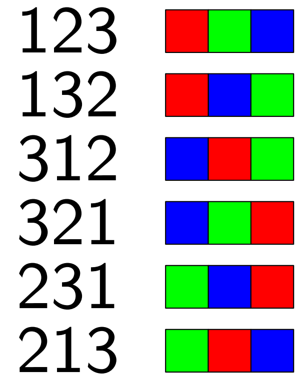

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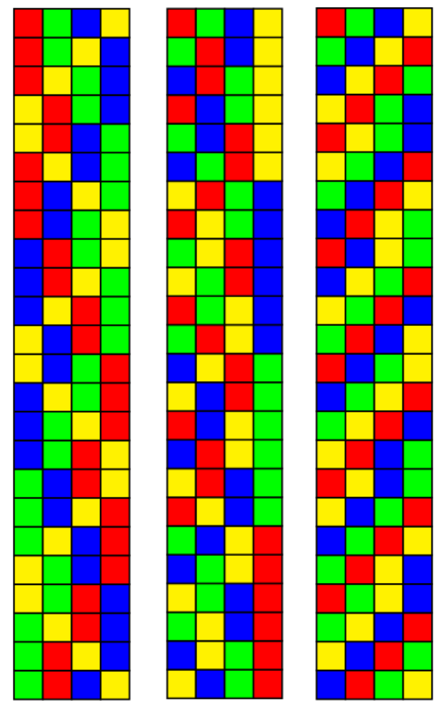

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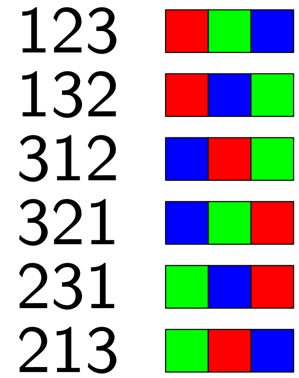

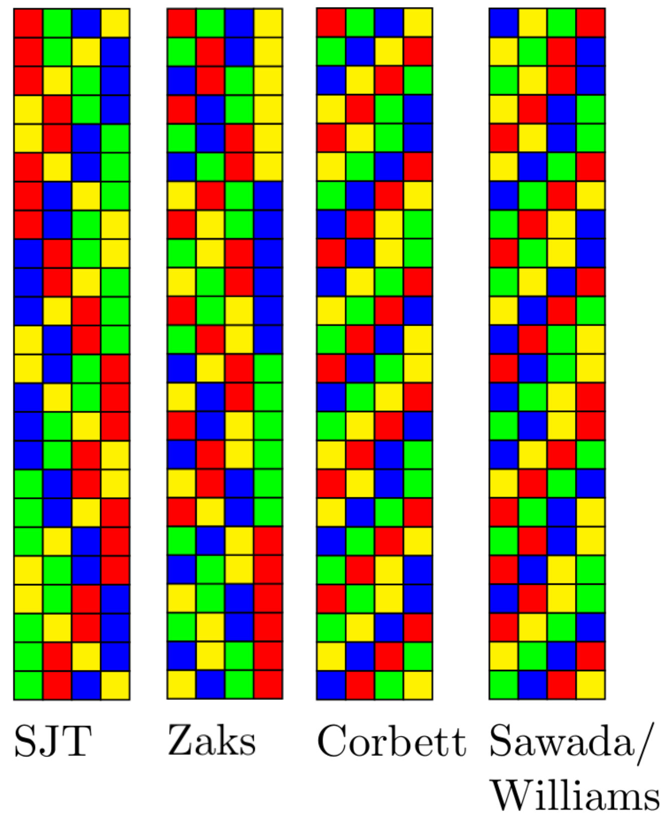

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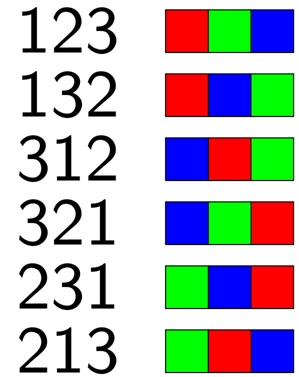

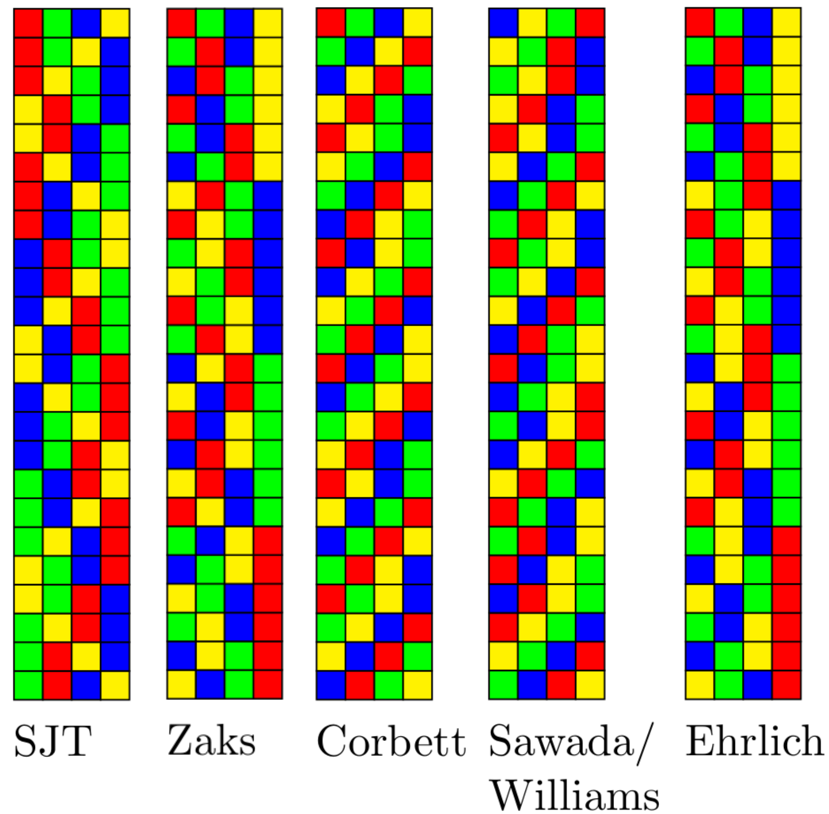

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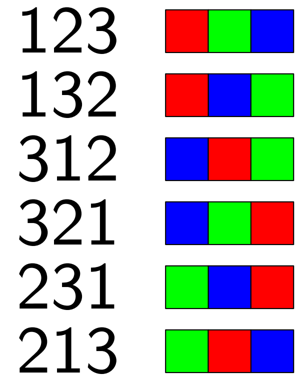

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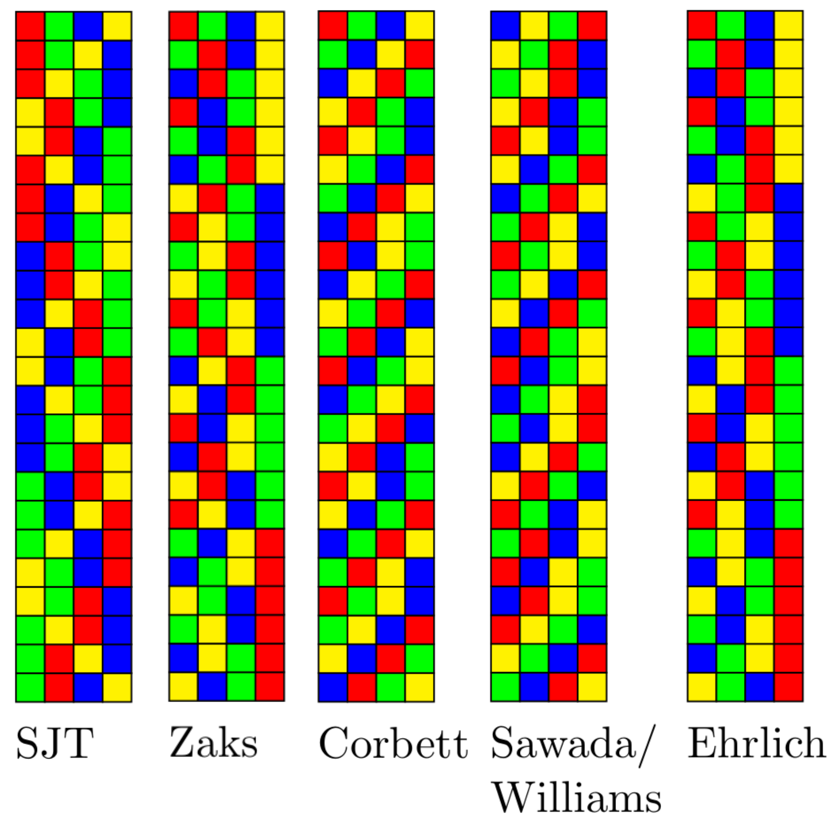

- Hamilton cycles in Cayley graphs of the symmetric group for different generators $\rightarrow$ Lovász' conjecture [Lovász 70]

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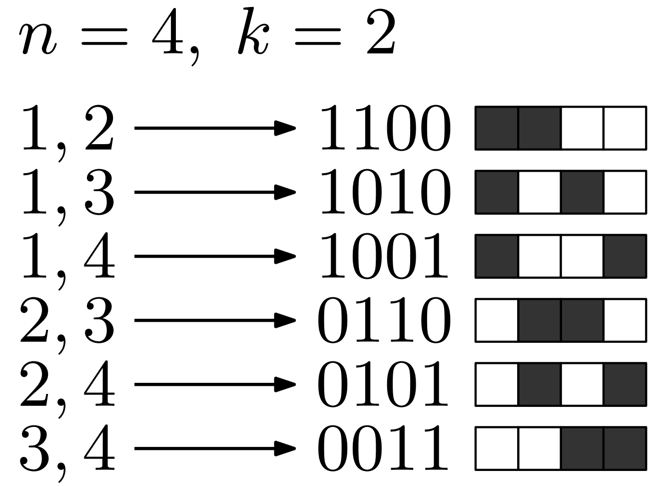

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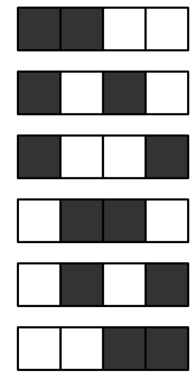

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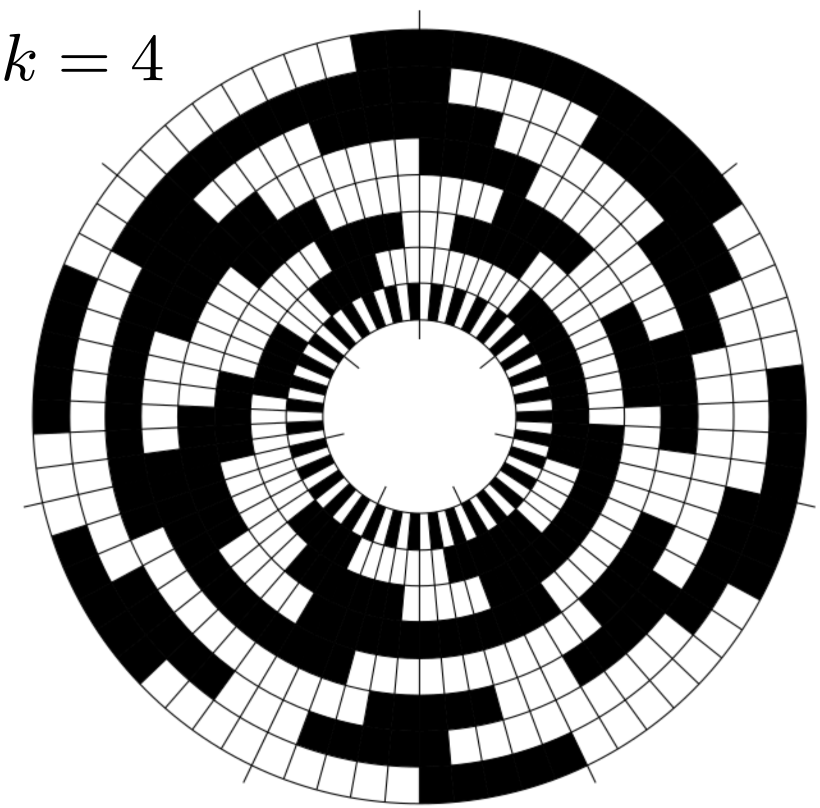

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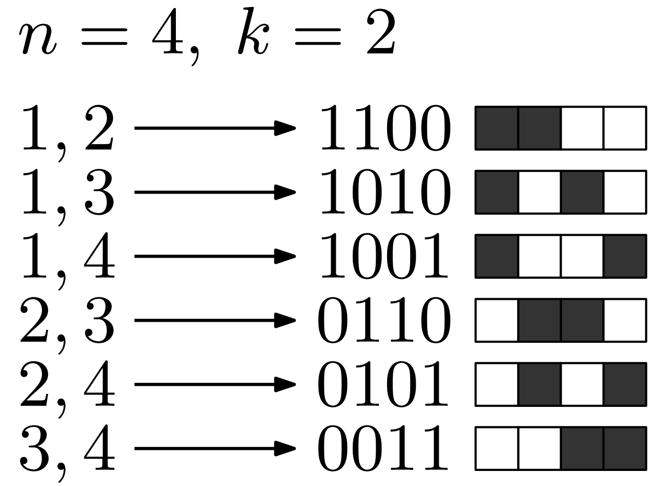

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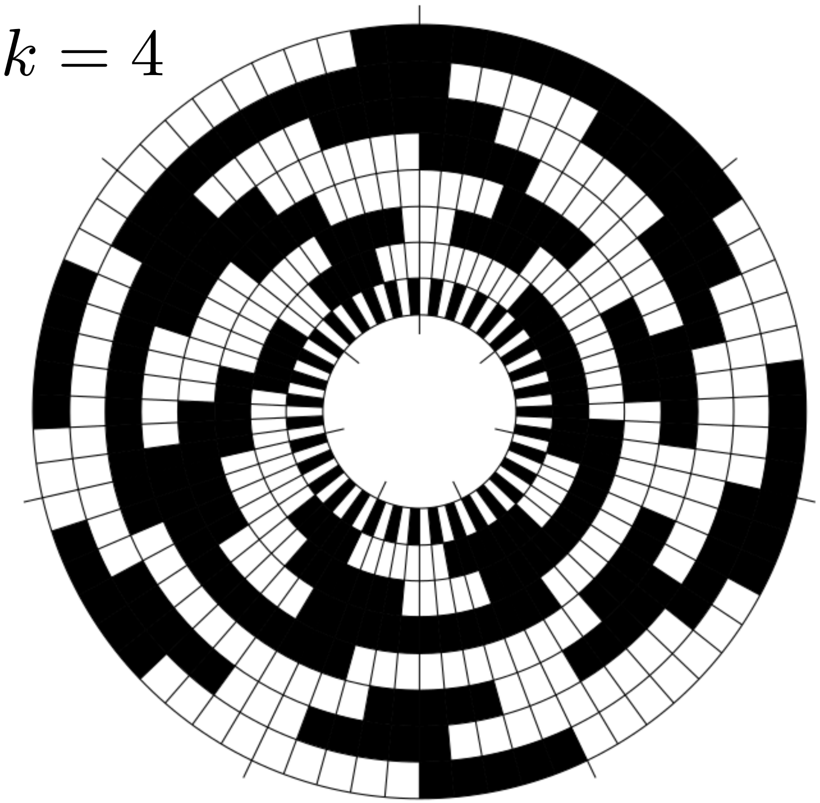

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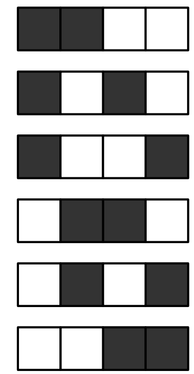

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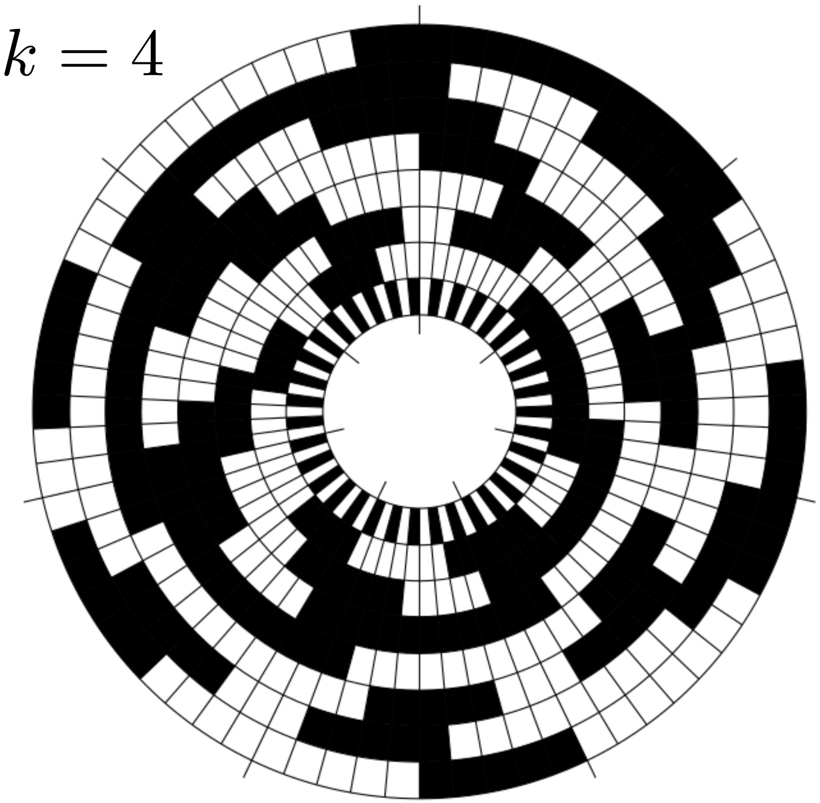

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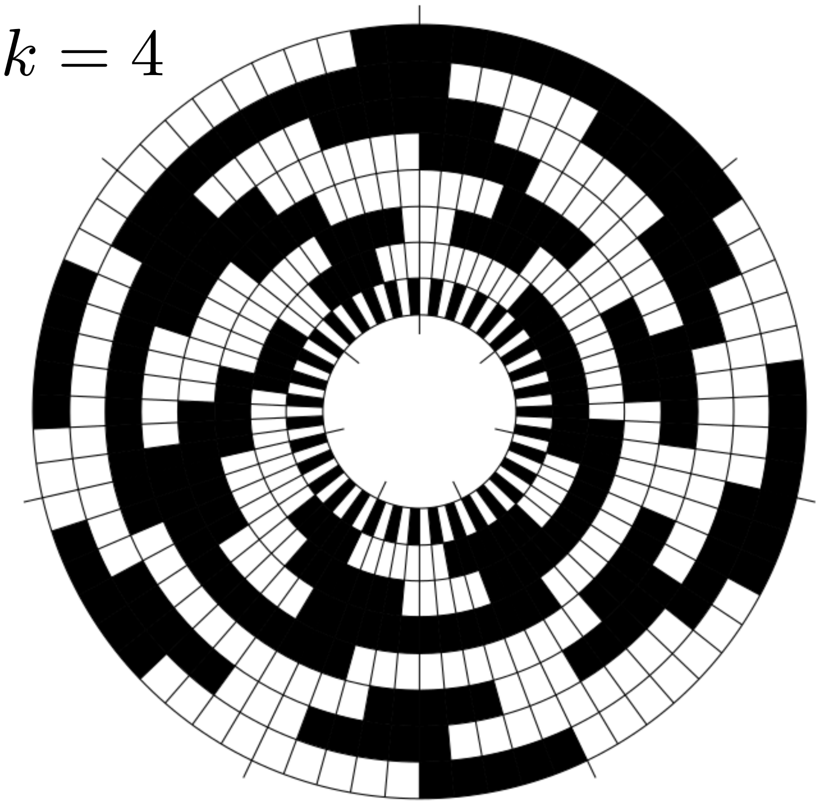

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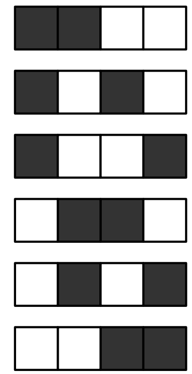

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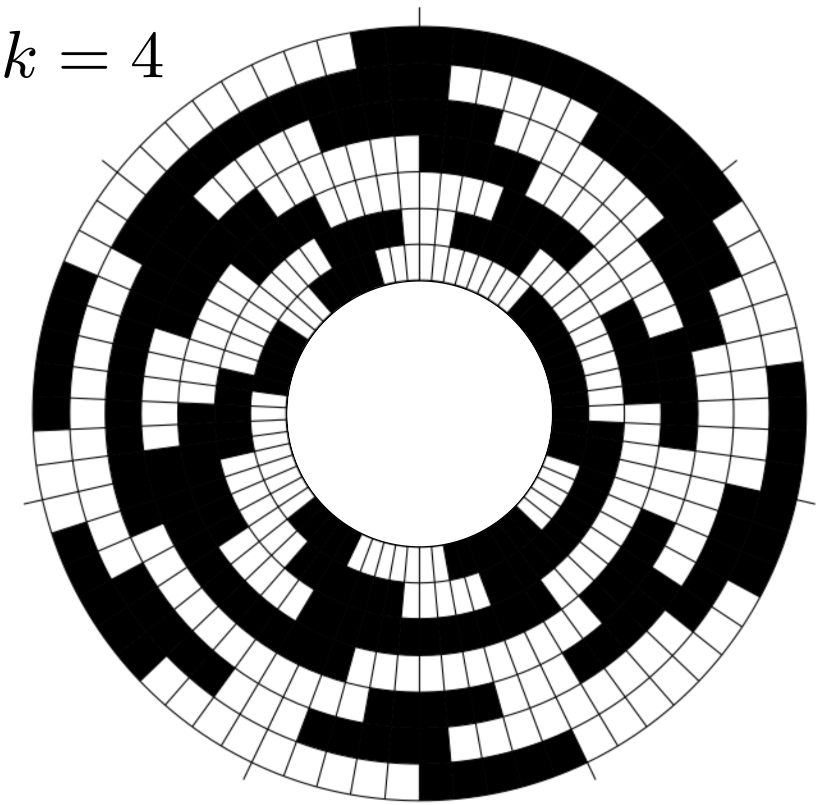

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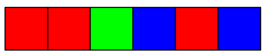
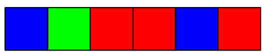
  112313 

  321131 

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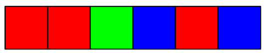
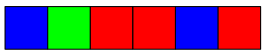
  112313 

  321131 

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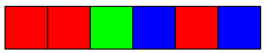
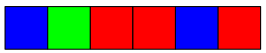
  112313 

  321131 

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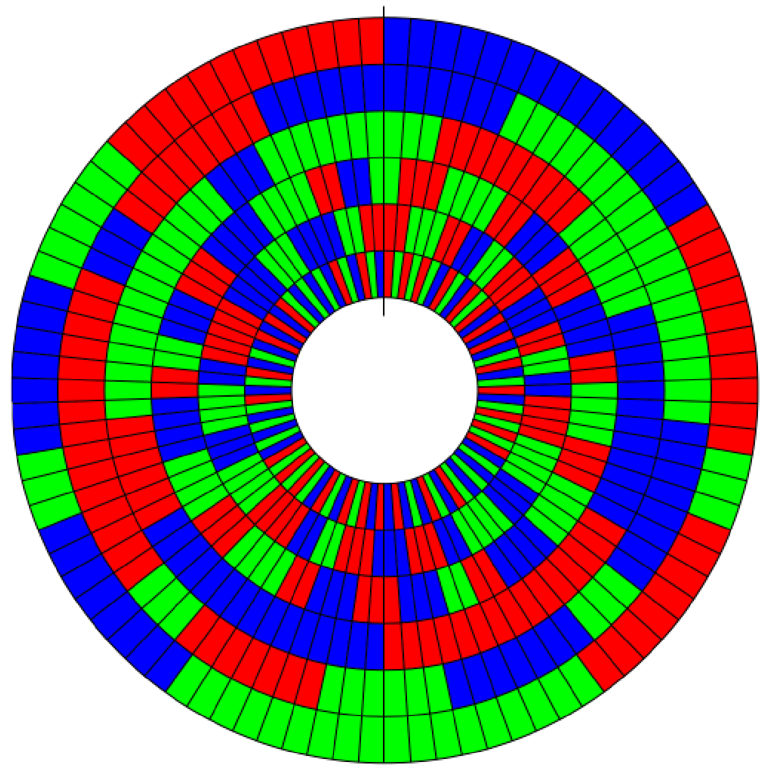

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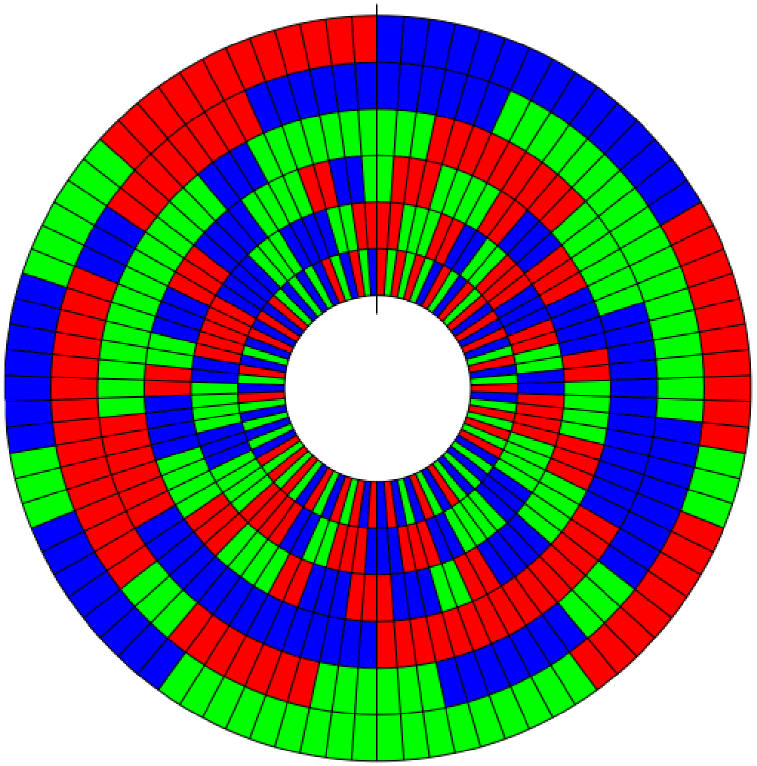

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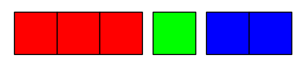

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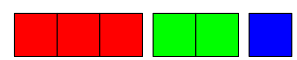

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- think of $\mathbf{a}$ as an integer partition

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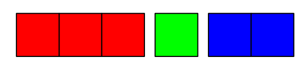

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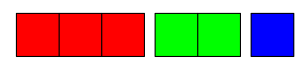

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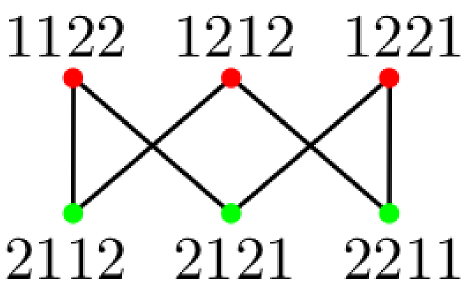

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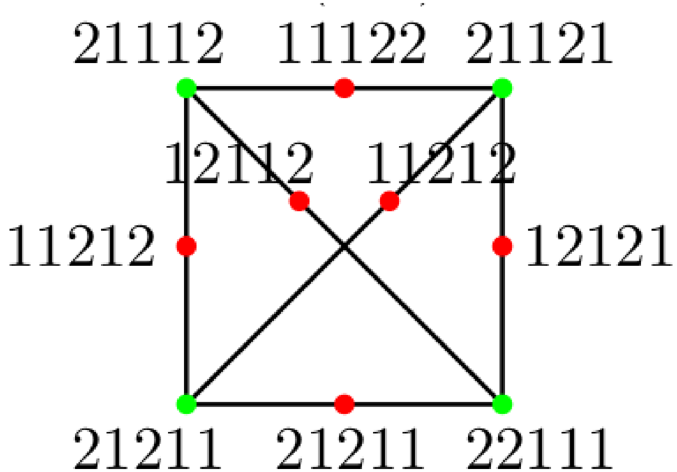
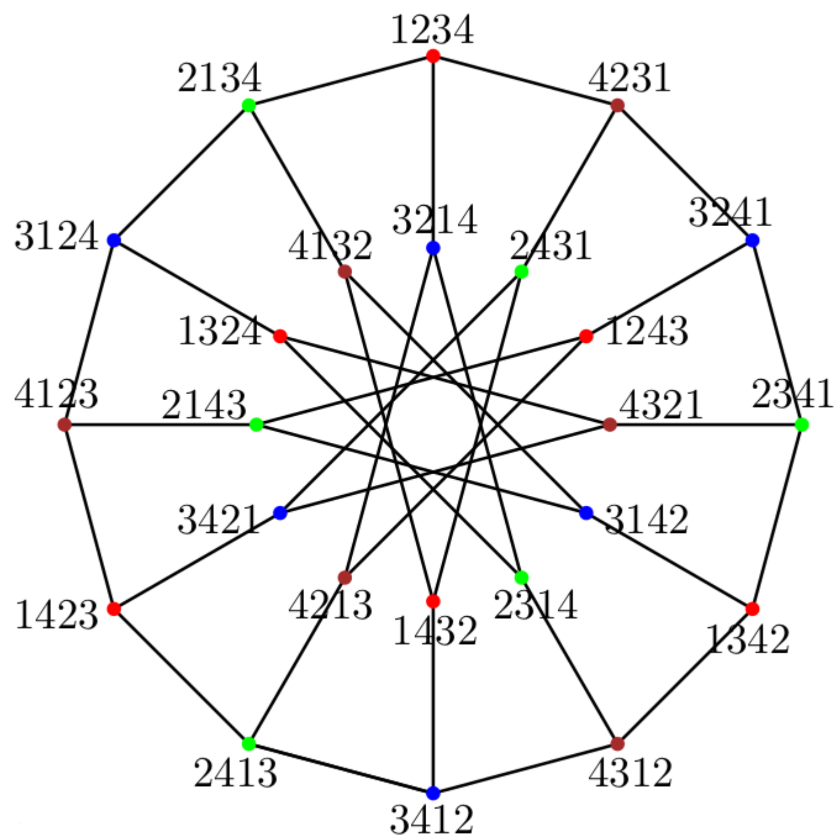


$\mathbf{a} = (1, 1, 1, 1)$



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Flip graph $G(\mathbf{a})$

Conjecture **[Shen, Williams 21]:** $G(\alpha^k)$ has a HC for any $\alpha \geq 1$ and $k \geq 2$.

Thm **[Ehrlich 87]:** $G(1^k)$ has a HC for any $k \geq 2$.

Thm **[Mütze 16]:** $G(\alpha, \alpha)$ has a HC for any $\alpha \geq 1$.

- **Question:** For which $\mathbf{a}$ does $G(\mathbf{a})$ admit a Hamilton cycle?

Obstacles for Hamiltonicity

- $G(\mathbf{a})$ is k-partite, with partition classes given by first symbol

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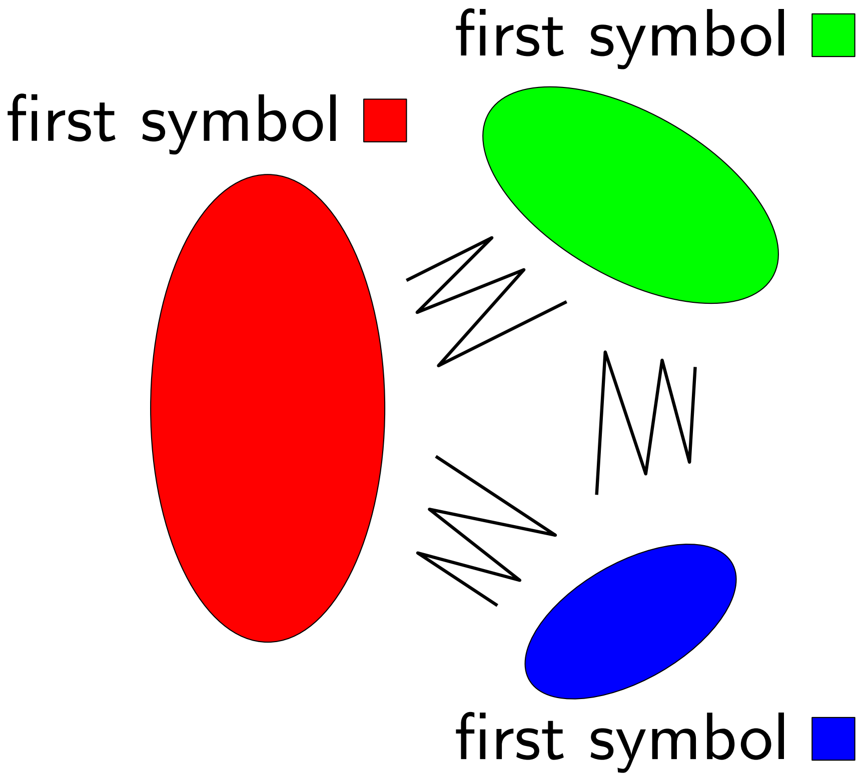
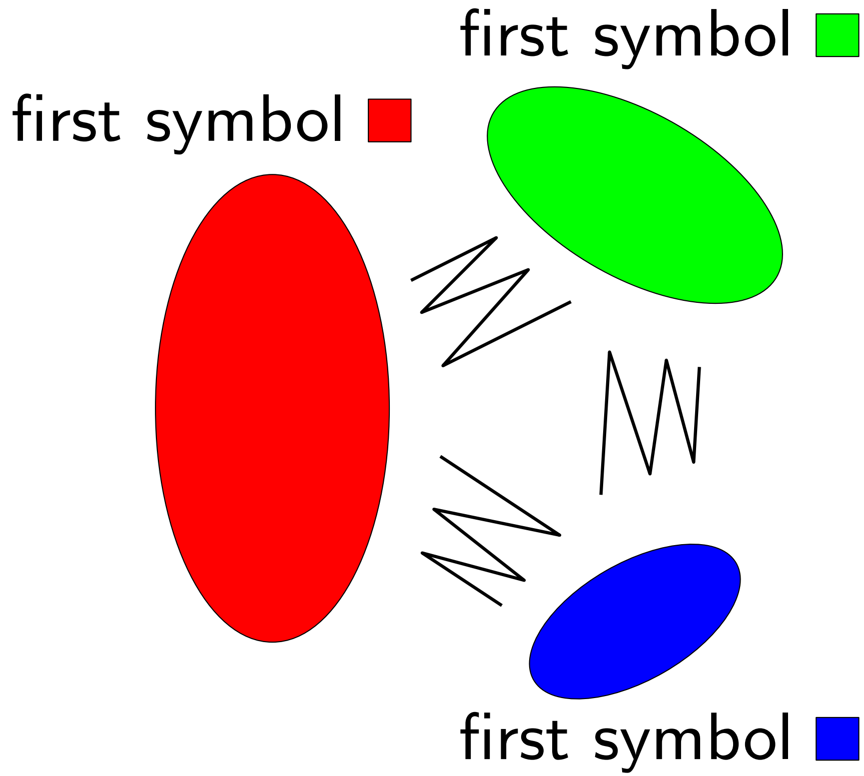


Obstacles for Hamiltonicity

- $G(\mathbf{a})$ is k-partite, with partition classes given by first symbol
- one partition class bigger than all others combined $\rightarrow$ no HC



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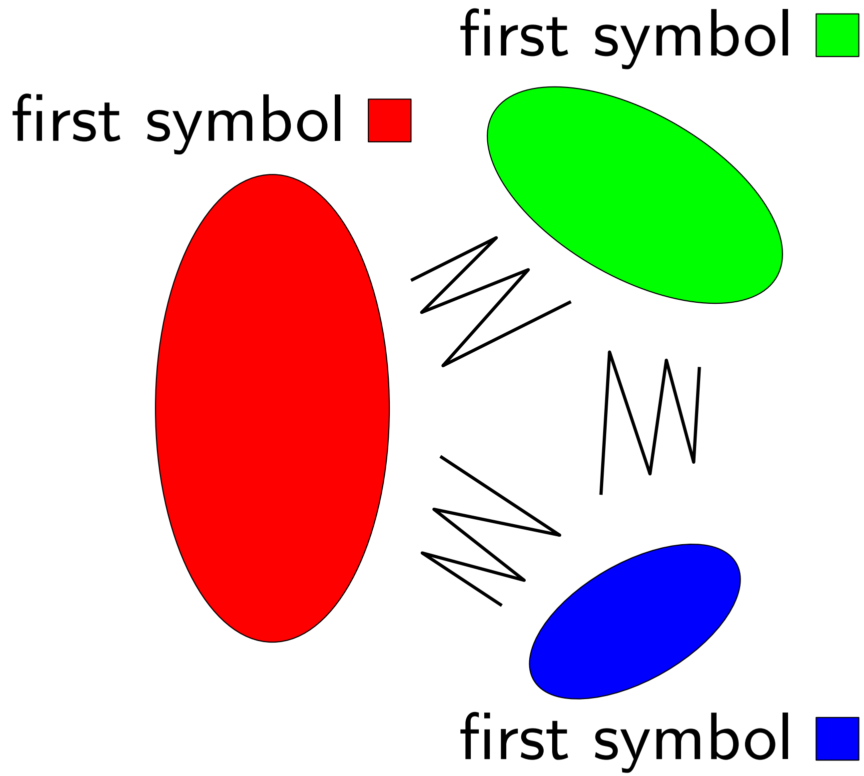
$$\iff 2a_1 > \sum_{i=1}^{k} a_i$$



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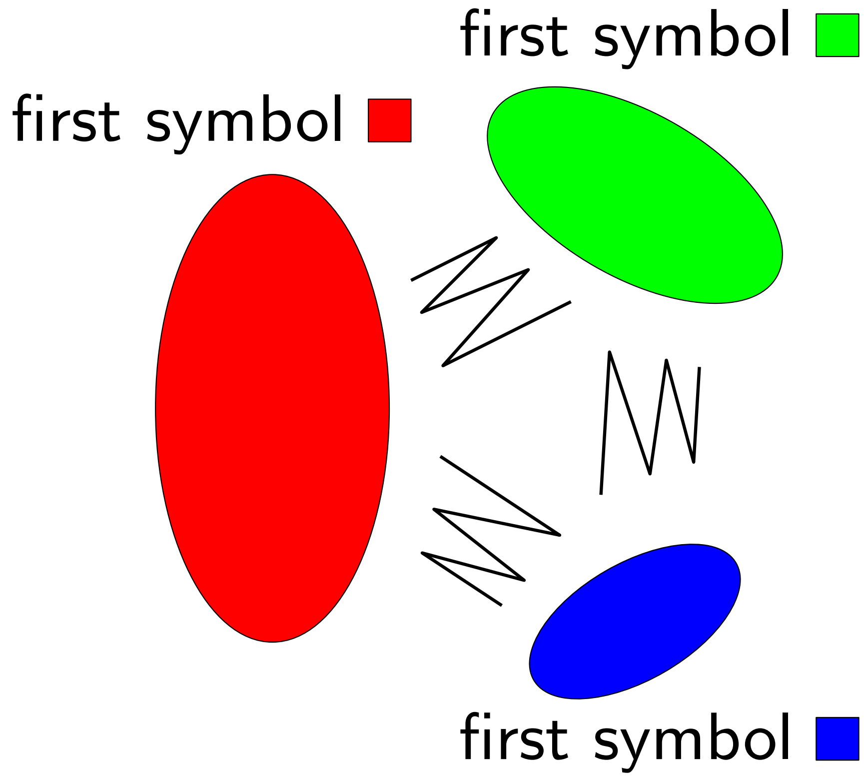
$$\iff 2a_1 > \sum_{i=1}^{k} a_i$$

$$\iff \left(\sum_{i=1}^{k} a_i\right) - 2a_1 < 0$$



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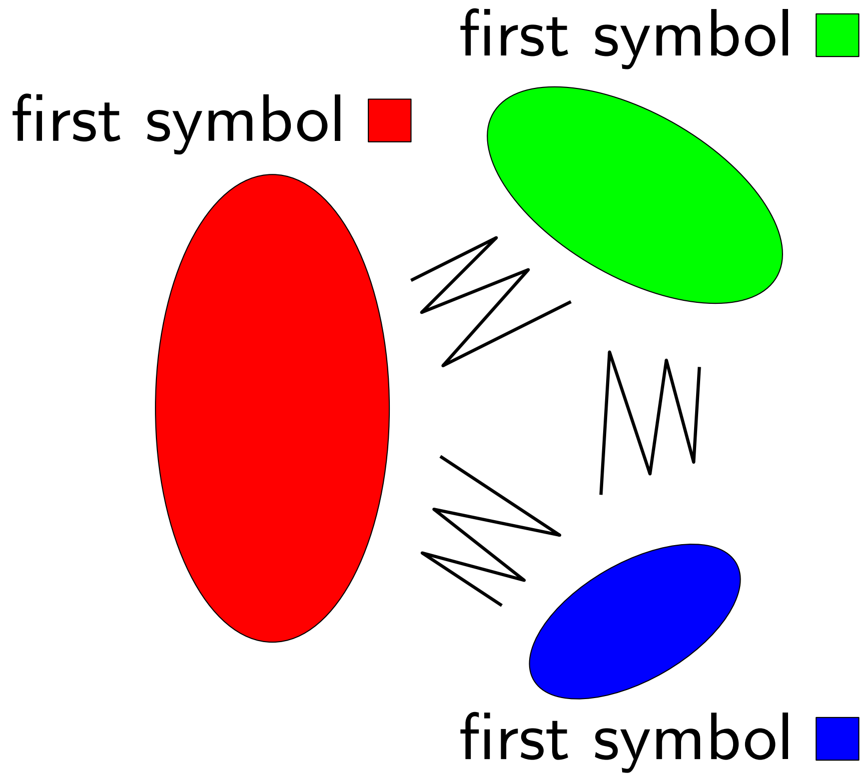
$$\iff 2a_1 > \sum_{i=1}^{k} a_i$$

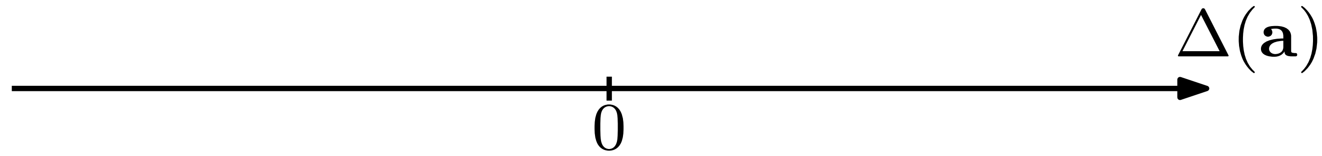
$$\iff \underbrace{\left(\sum_{i=1}^{k} a_i\right) - 2a_1}_{\Delta(\mathbf{a}) :=} < 0$$



Our results

Thm: If $\Delta(\mathbf{a}) < 0$, then $G(\mathbf{a})$ has no HC.

Our results



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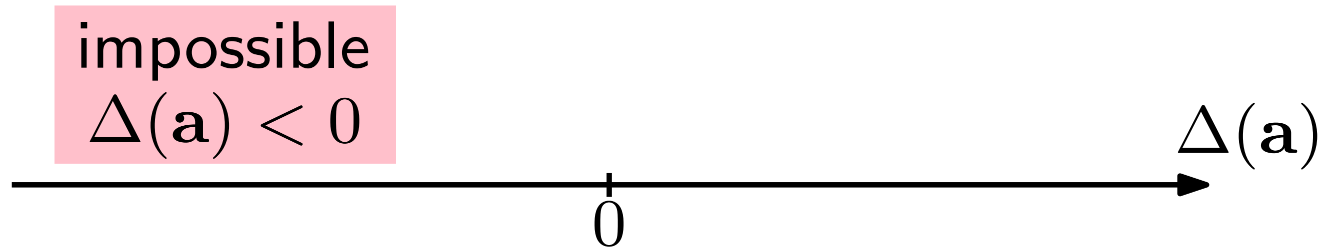

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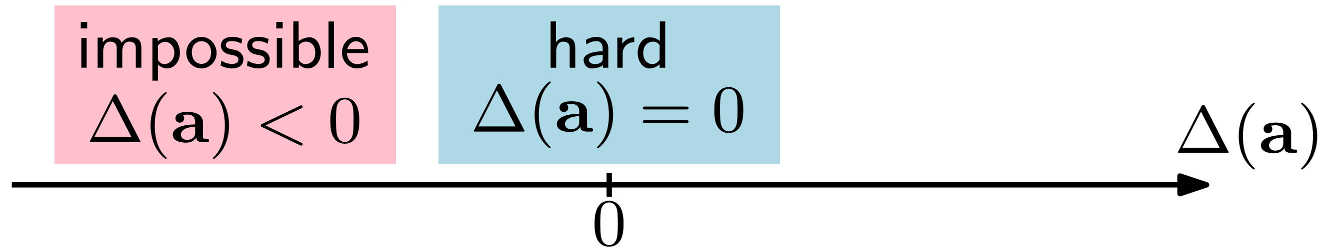

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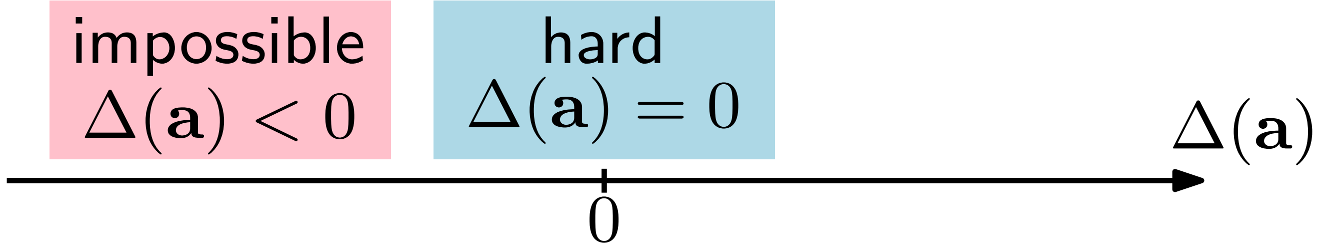

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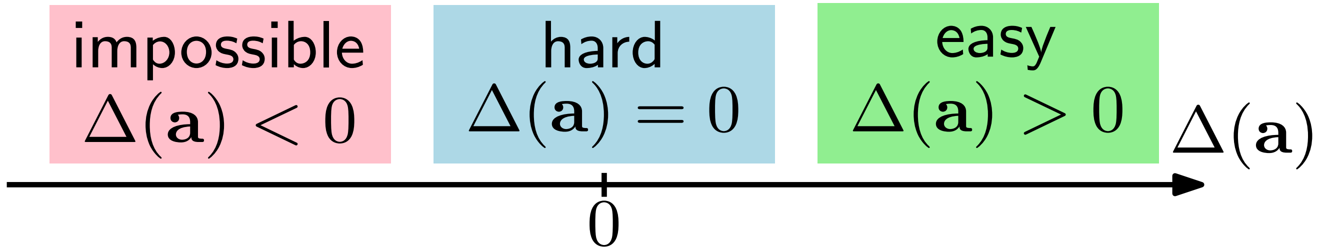
Generalized middle levels problem

Our results



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Thm: If conjecture holds, then $G(\mathbf{a})$ with $\Delta(\mathbf{a}) > 0$ has a HC.

Our results

impossible	hard	easy
$\Delta(\mathbf{a}) < 0$	$\Delta(\mathbf{a}) = 0$	$\Delta(\mathbf{a}) > 0$

0 $\longrightarrow$ $\Delta(\mathbf{a})$

Thm: If $\Delta(\mathbf{a}) < 0$, then $G(\mathbf{a})$ has no HC.

Conjecture: If $\Delta(\mathbf{a}) = 0$, then $G(\mathbf{a})$ is Hamilton 1-laceable [...]

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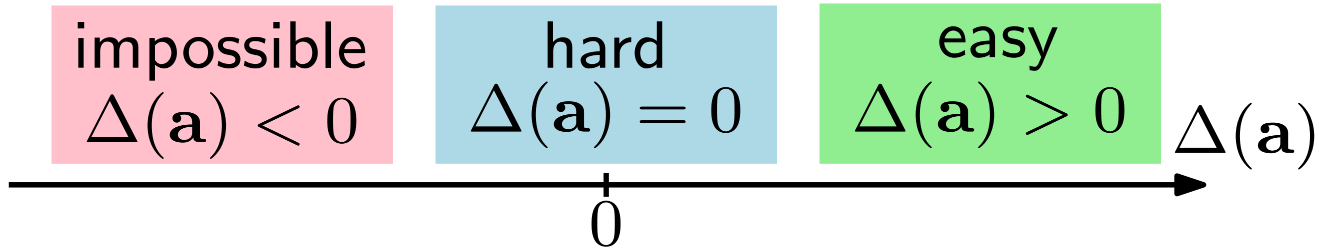

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Hamilton 1-laceable: Hamilton path between any vertex in partition class 1 and any other vertex

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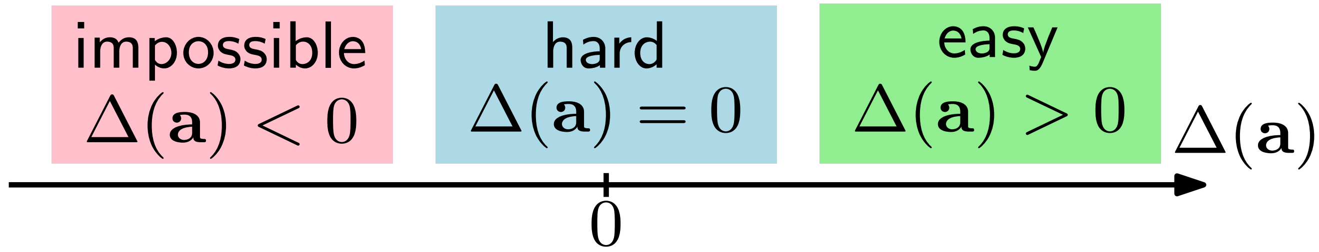

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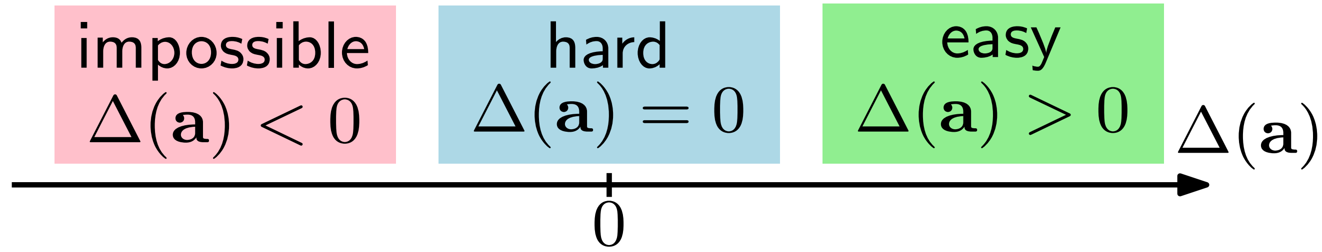

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Hamilton 1-laceable: Hamilton path between any vertex in partition class 1 and any other vertex

Hamilton-connected: Hamilton path between any two vertices

Our results

impossible $\Delta(\mathbf{a}) < 0$

hard $\Delta(\mathbf{a}) = 0$

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$\Delta(\mathbf{a})$

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Evidence:

- small cases checked by computer

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answers Shen/Williams' conjecture $G(\alpha^k)$ for $\alpha \in \{2, 3, 4\}$, $k \geq 2$

Proof ideas for $\Delta(\mathbf{a}) > 0$

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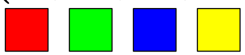

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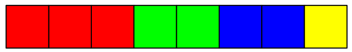

y

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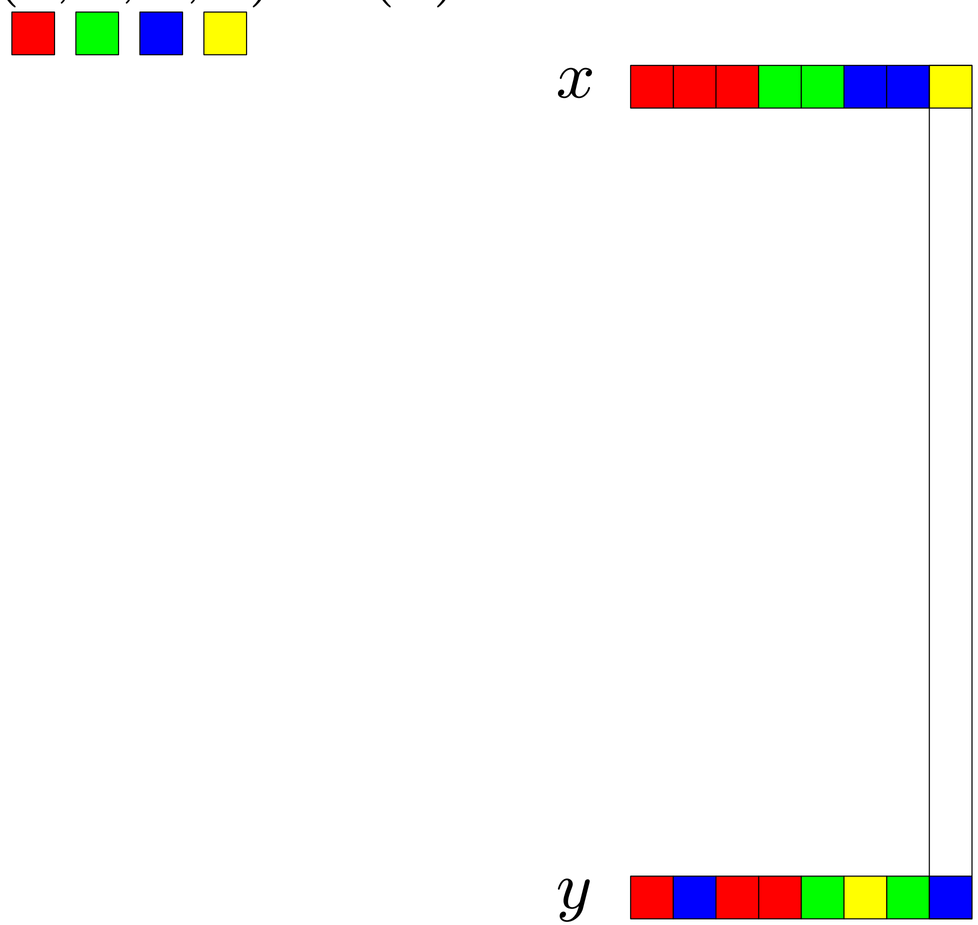

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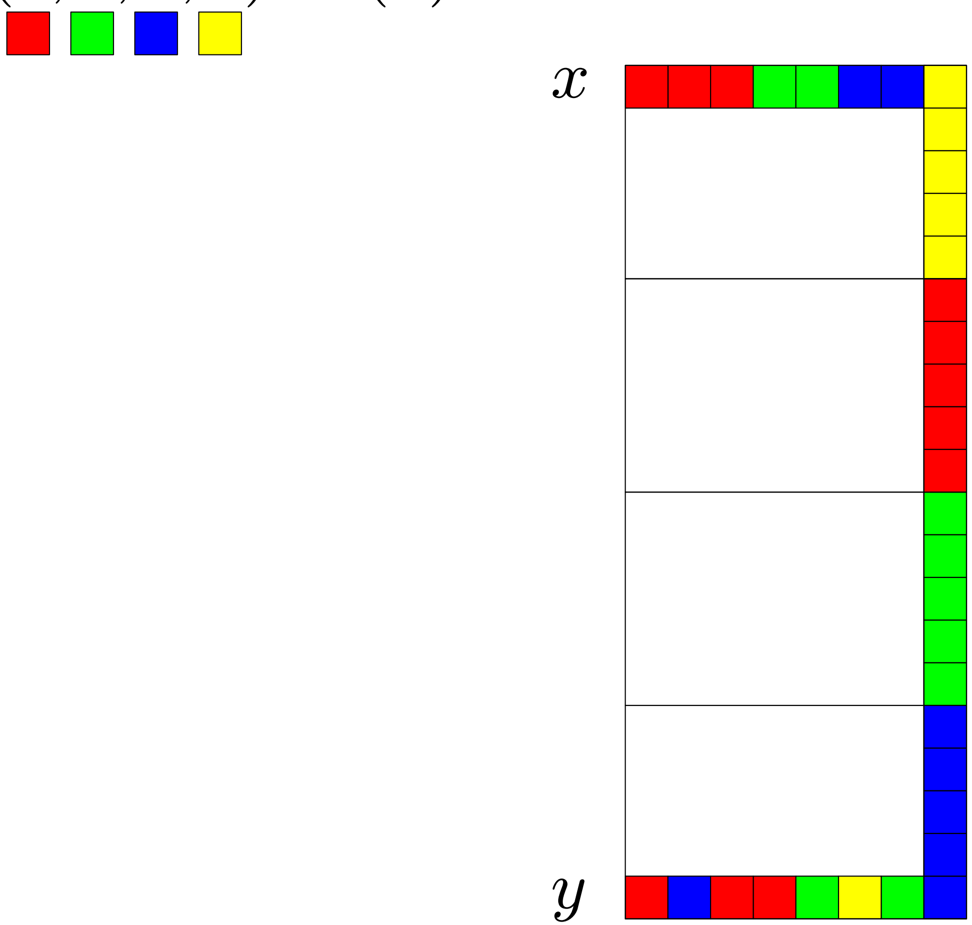

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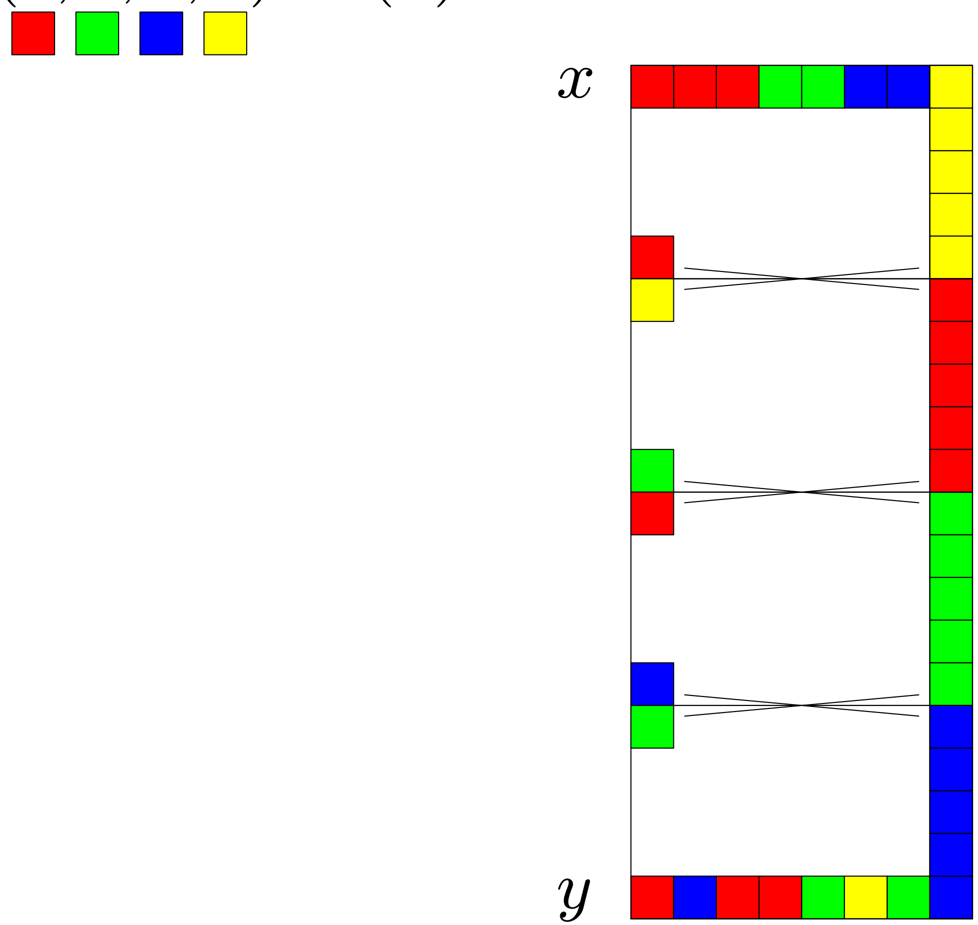

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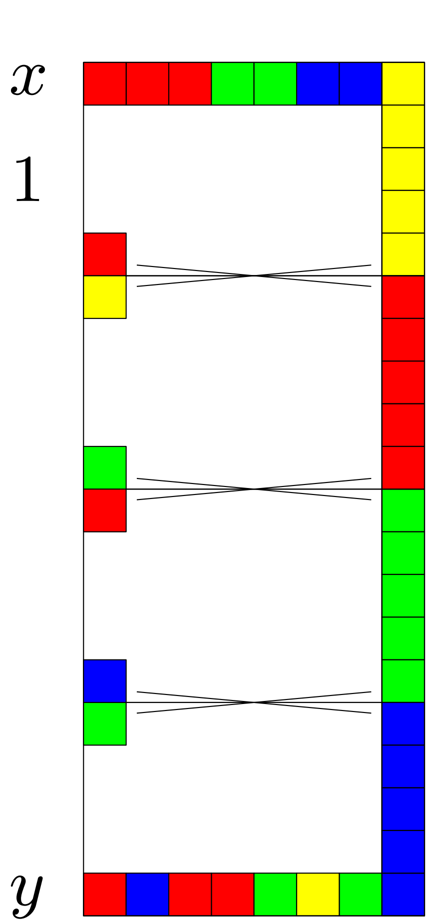

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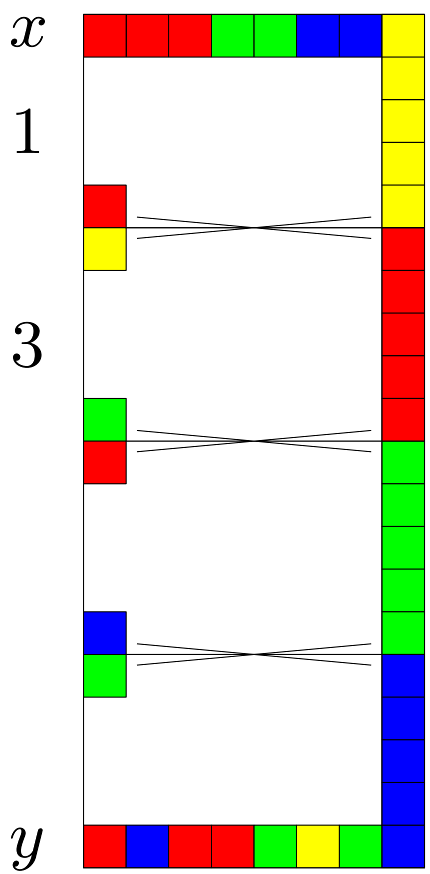

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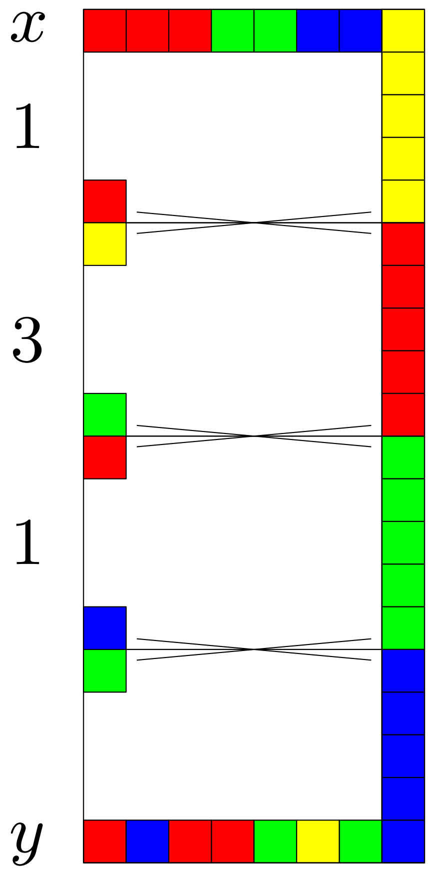

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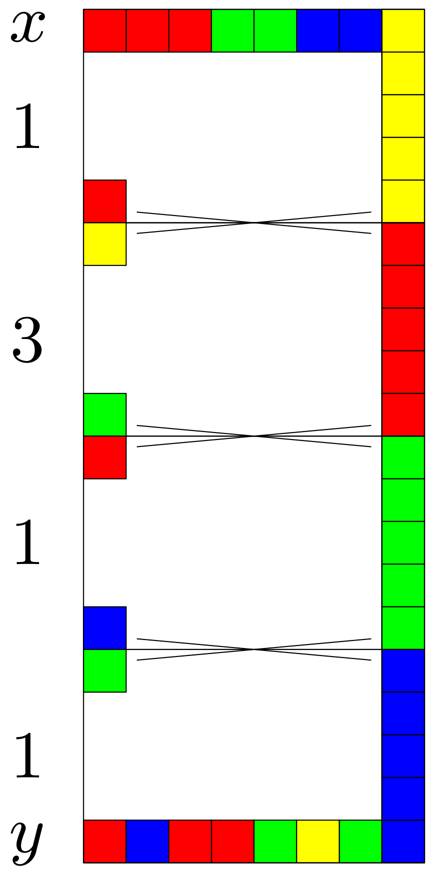

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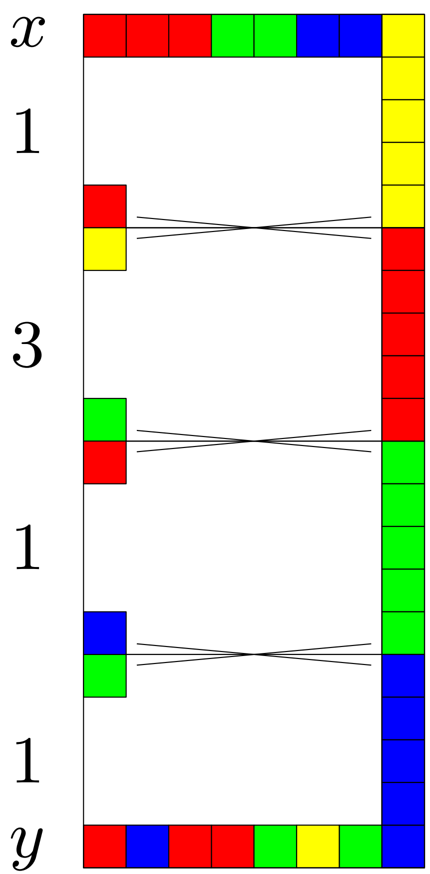

if $\Delta(\mathbf{b}) > 0$ then Ham.-connected by induction

Proof ideas for $\Delta(\mathbf{a}) > 0$

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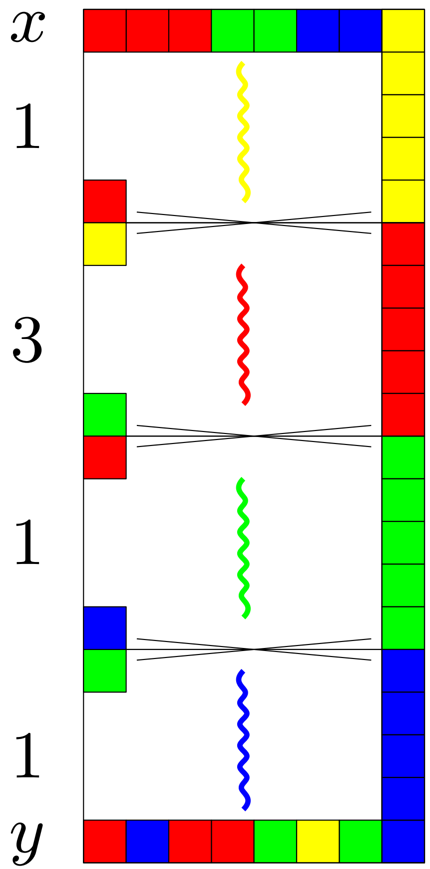

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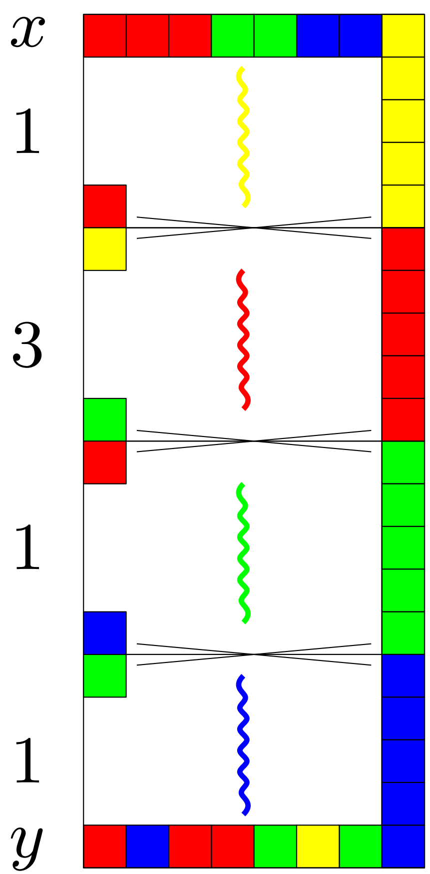

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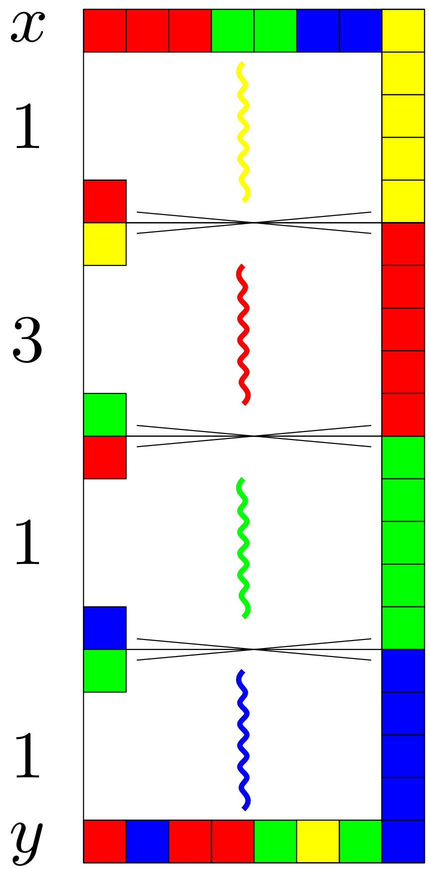

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stitch HPs together

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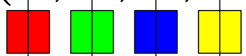

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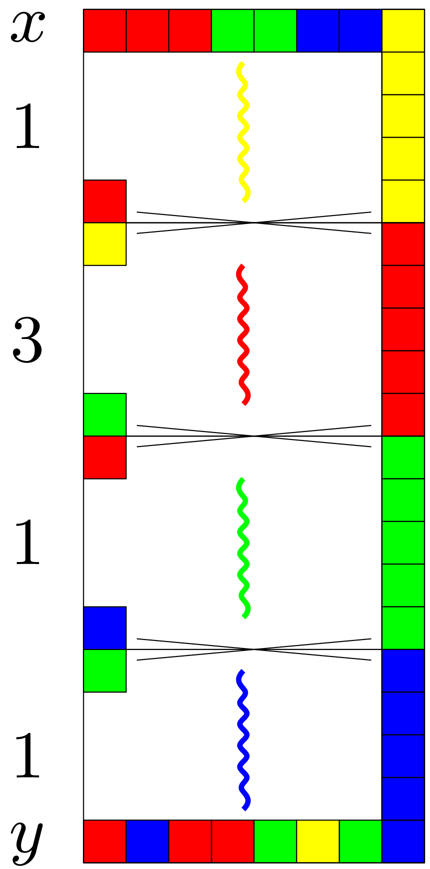

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stronger assumptions (Ham.-connected and Hamilton-1-laceable) make the proof easier

Proof ideas for $\Delta(\mathbf{a}) = 0$

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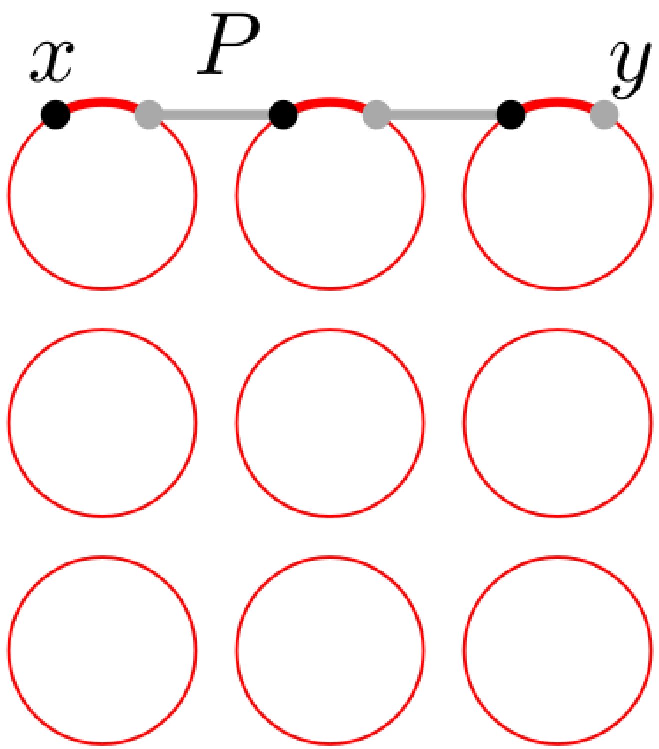

1. build cycle factor

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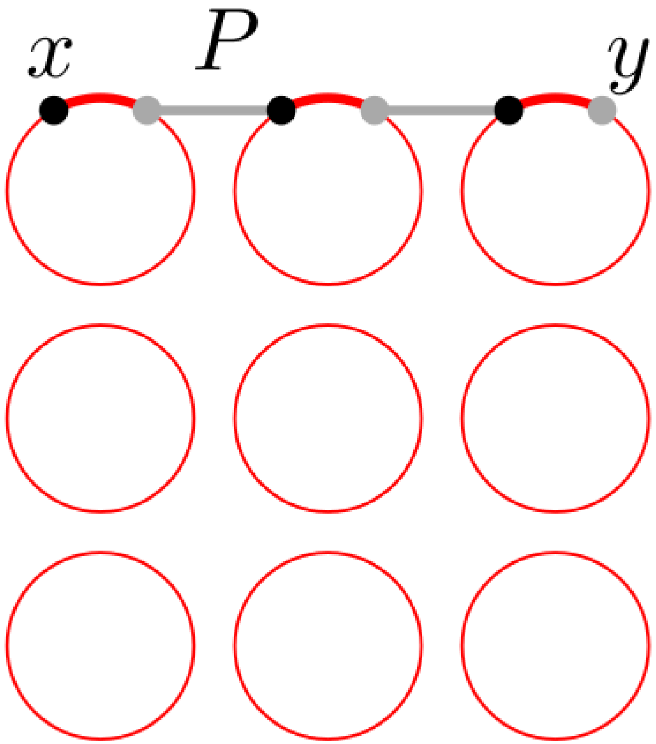

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2. join x and y by alternating path P

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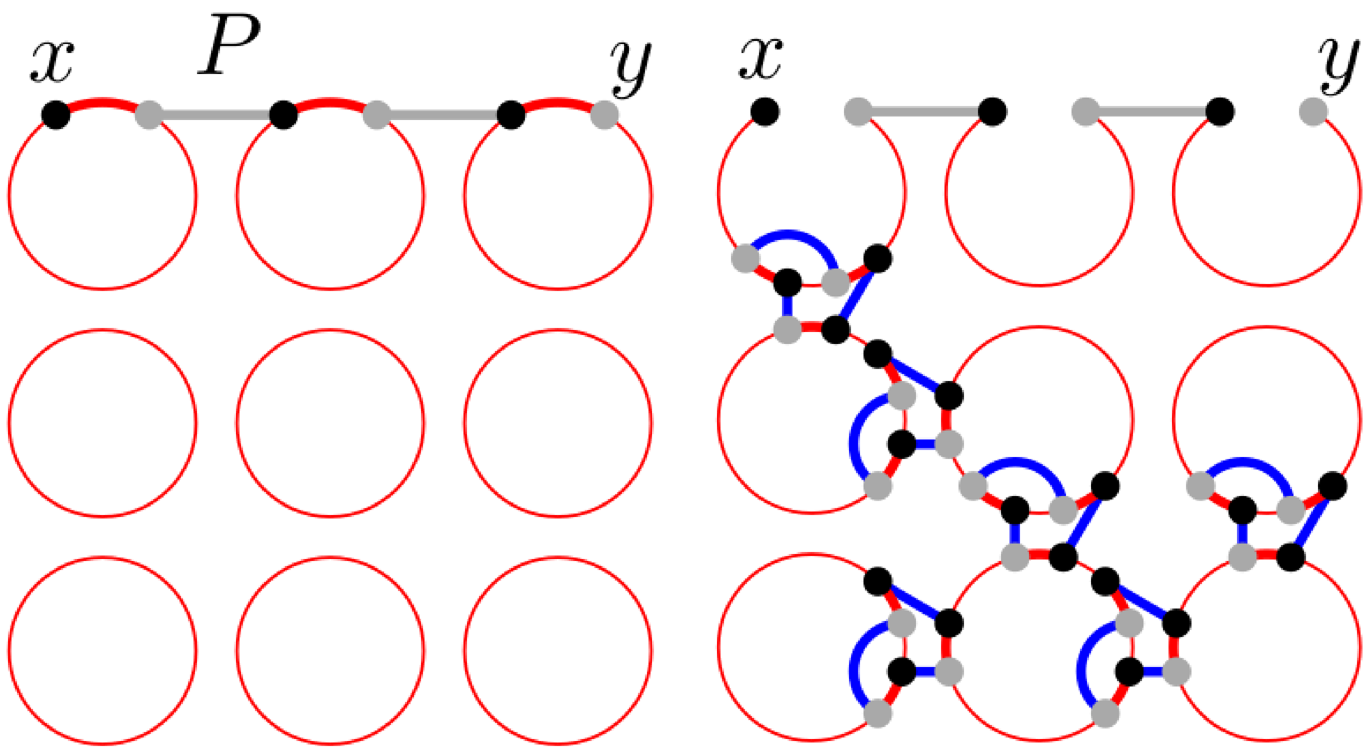

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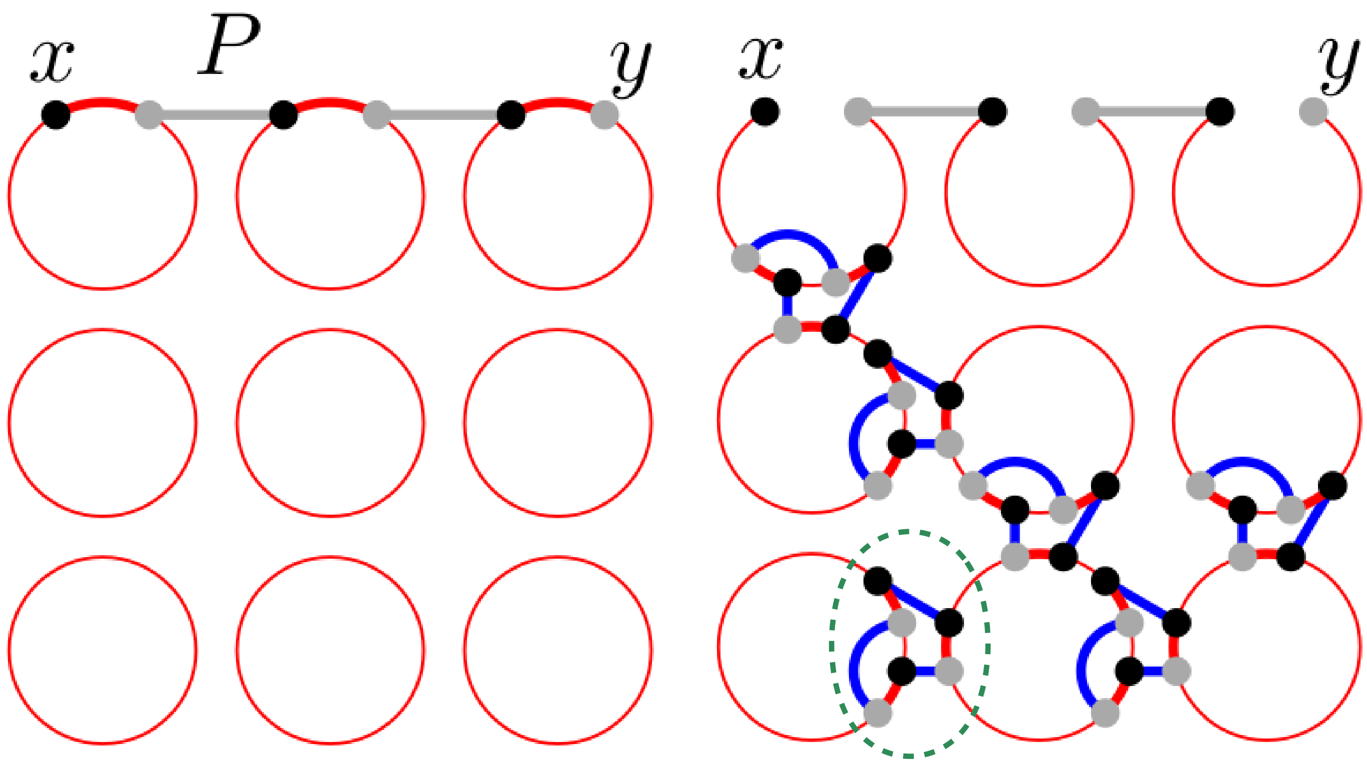

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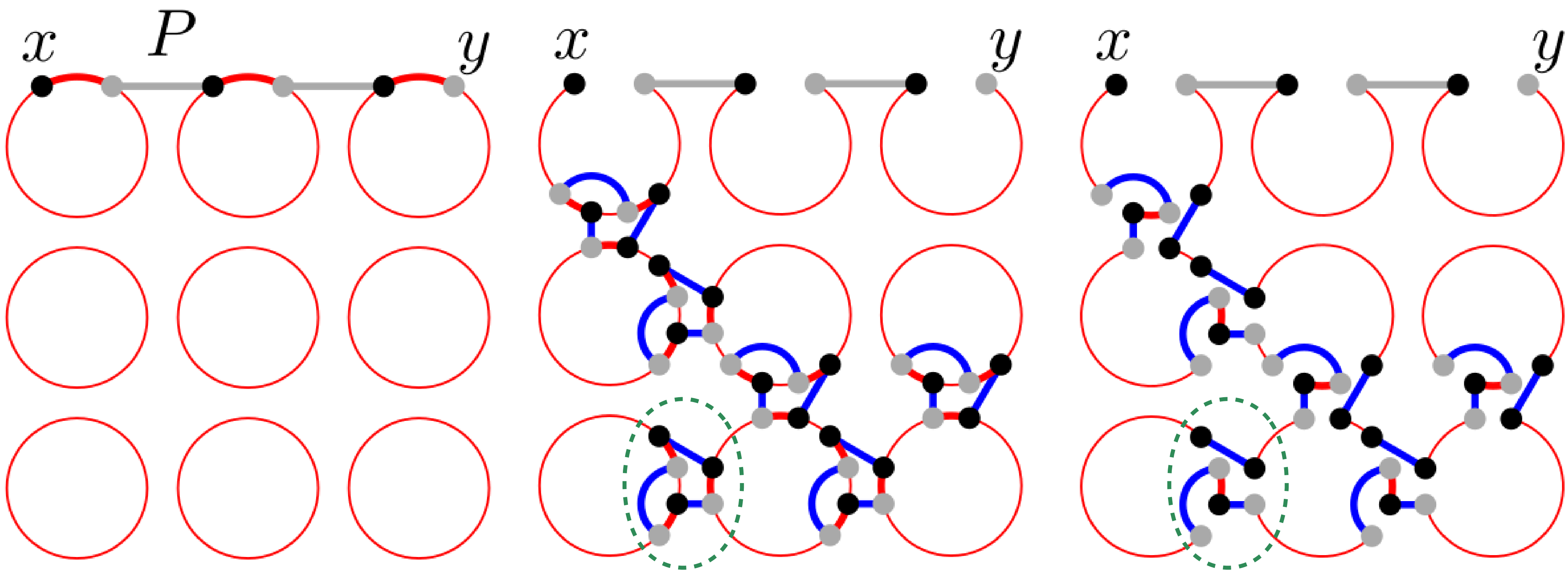
3. join remaining cycles via gluing 6-cyles

Proof ideas for $\Delta(\mathbf{a}) = 0$

Thm: $G(\alpha, \alpha)$ is Hamilton-(1-)laceable for any $\alpha \geq 3$.



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- simpler constructions / successor rules?
- other generators (adjacent transp. are known [Stachowiak 92])?