Manipulation Estimation and Control Problem Set 1

Luka Eerens

1. a) Grounded with four 1DOF joints so: **4 DOF**

   b) Using Gruebler's Equation: DOF = 3(n-1) -2l = 3(4-1) – 2(4) = **1 DOF**
   (Where n = number of links, l = number of 1DOF joints)

   c) Mechanism is not grounded and closed loop, so can translate in x, y and z AND rotate about 3 axis, so: **6 DOF**.

   d) Has 2 parallel prismatic joints that are unable to twist so translates in 1-dimension: **1DOF**

   e) One can say that there are **infinite DOF**, however the trunk of an elephant is not composed of an infinite amount of atoms let alone muscle fibres and muscle bellies. We can simplify our lives by stating infinity because it is a very large number but to be mathematically correct one must put a range of values where this must lie. According to realclearscience.com, an elephant trunk contains 40,000 muscles, a finite number of contracting muscles that act together at the systems level to produce many more degrees of freedom. However having infinite degrees of freedom on a series link of joints would require an infinite amount of joints, and infinite amount of links. Though parallel mechanisms may emulate the motion of certain parts better than series ones (because of multiple muscle inserts which are grounded at attach points) there still aren't enough links or joints in an elephant trunk to reach infinite DOF. Even if the answer was a gigantic number such as 10^{100}, this number would still pale in comparison to infinity. I thus propose a DOF domain lying between much greater than 3 (from the stem of the trunk) and much lower than infinity: $\mathbf{3} \ll \boldsymbol{DOF} \ll \infty$

2. Transforming from point 1 to point 0 using a homogenous transformation matrix H_1^0 can be expressed in the equation:

   $$P^0 = H_1^0 P^1 \qquad (1)$$

   To add, the orientation and displacement can be used to describe this transformation using:

   $$P^0 = d_1^0 + R_1^0 P^1 \qquad (2)$$

   In order to find H_0^1 one must isolate the P^1 in (2) and this is done by:

   $$P^0 - d_1^0 = R_1^0 P^1$$

   $$[R_1^0]^{-1}(P^0 - d_1^0) = [R_1^0]^{-1} R_1^0 P^1$$

   $$[R_1^0]^{-1} R_1^0 = I \ \therefore [R_1^0]^{-1}(P^0 - d_1^0) = P^1$$

   Now apply the distributive property:

   $$[R_1^0]^{-1} P^0 - [R_1^0]^{-1} d_1^0 = P^1 \qquad (3)$$

   Now if we decide to isolate P^1 from equation (1) then we get:

   $$[H_1^0]^{-1} P^0 = [H_1^0]^{-1} H_1^0 P^1$$
   $$\therefore H_0^1 P^0 = P^1$$

   Substitute $H_0^1 P^0$ for P^1 in equation (3) and get:

   $$[R_1^0]^{-1} P^0 - [R_1^0]^{-1} d_1^0 = H_0^1 P^0$$

This can be represented in matrix form as:

$$\begin{bmatrix} [R_1^0]^{-1} & -[R_1^0]^{-1}d_1^0 \\ 0\,0\,0 & 1 \end{bmatrix} * \begin{bmatrix} P^0 \\ 1 \end{bmatrix} = H_0^1 P^0$$

$$\therefore H_0^1 = \begin{bmatrix} [R_1^0]^{-1} & -[R_1^0]^{-1}d_1^0 \\ 0\,0\,0 & 1 \end{bmatrix}$$

3. The transformation from frame 0 to 1 will be broken down into 3 sequential homogenous transformation matrices. These are:
   H_A^0 = A translation from point 0 to 1 (fixed frame)
   H_B^A = A rotation at point 1 about the original y_0 axis (fixed frame) to orient x_0 as x_1
   H_1^B = A rotation at point 1 about the new x_0 axis (relative frame) to orient z_0 as z_1

   Given the fixed/relative frame nature of this problem, transformations are thus ordered as:
   $$H_1^0 = H_A^0 * H_B^A * H_1^B$$

   Given that the interior angle of the hexagon is 120 degrees and that the length of each side is $2\sqrt{3}$, the z-axis height of the polygon is found to have a length of 6 as seen by the cosine rule:
   $$h^2 = \left(2\sqrt{3}\right)^2 + \left(2\sqrt{3}\right)^2 - 2\left(2\sqrt{3}\right)\left(2\sqrt{3}\right)cos120$$
   $$h^2 = 36$$
   $$\therefore h = 6$$
   Formally representing these homogenous transformation matrices yields:

   $$H_A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   $$H_B^A = \begin{bmatrix} \cos(60) & 0 & \sin(60) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(60) & 0 & \cos(60) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   $$H_1^B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(90) & -\sin(90) & 0 \\ 0 & \sin(90) & \cos(90) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
   Combining the homogenous transformation matrices together in format $H_1^0 = H_B^A * H_A^0 * H_1^B$:

   $$H_1^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} \cos(60) & 0 & \sin(60) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(60) & 0 & \cos(60) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(90) & -\sin(90) & 0 \\ 0 & \sin(90) & \cos(90) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   Multiplying the first 2 transformation operations together (H_A^0 and H_B^A) yields:

   $$= \begin{bmatrix} \cos(60)+0+0+0 & 0+0+0+0 & 0+0+\sin(60)+0 & 0+0+0+0 \\ 0+0+0+0 & 0+1+0+0 & 0+0+0+0 & 0+0+0+5 \\ -\sin(60)+0+0+0 & 0+0+0+0 & 0+0+\cos(60)+0 & 0+0+0+6 \\ 0+0+0+0 & 0+0+0+0 & 0+0+0+0 & 0+0+0+1 \end{bmatrix}$$

$$\therefore H_B^0 = \begin{bmatrix} \cos(60) & 0 & \sin(60) & 0 \\ 0 & 1 & 0 & 5 \\ -\sin(60) & 0 & \cos(60) & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Multiplying the H_B^0 matrix with H_1^B to get H_1^0 yields:

$$H_1^0 = \begin{bmatrix} \cos(60) & \sin(60)\sin(90) & \sin(60)\cos(90) & 0 \\ 0 & cos90 & -\sin(90) & 5 \\ -\sin(60) & \sin(90)\cos(60) & \cos(90)\cos(60) & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

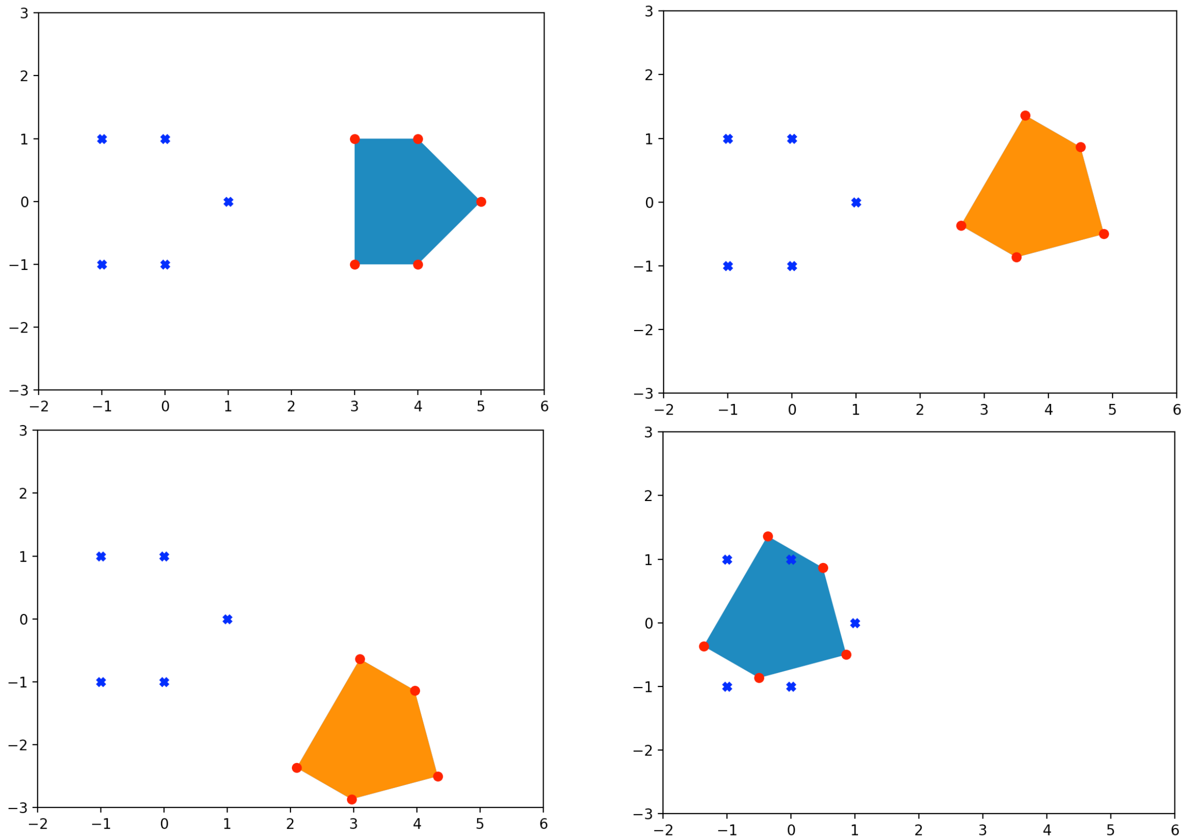
If the values are actually calculated with identical significant figures we obtain:

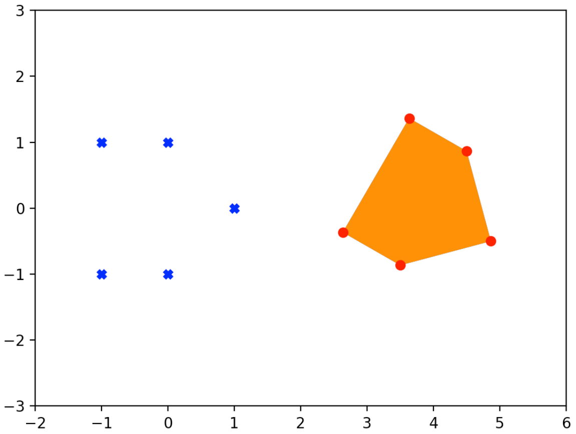
$$H_1^0 = \begin{bmatrix} 0.500 & 0.866 & 0.000 & 0.000 \\ 0.000 & 0.000 & -1.000 & 5.000 \\ -0.866 & 0.500 & 0.000 & 6.000 \\ 0.000 & 0.000 & 0.000 & 1.000 \end{bmatrix}$$

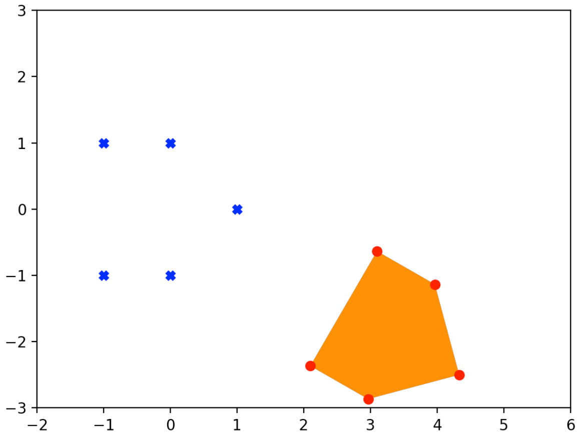
4. If the 5 points on the rigid body are in a list called “points” then the order of homogenous transformation matrix operations (from matrix “a” and “b”) are:
   a) Transformation = a*points
   b) Transformation = a*b*points
   c) Transformation = b*a*points
   d) Transformation = b*points
   e) Transformation = a*b*points
   f) Transformation = b*a*points

   Note: The code provided is in python, and imports the numpy library as well as matplotlib

   Here are some screenshots of the transformations (They are ordered from a -> f with the a) in the top, b) in the top right, c) in the middle left, d) in the middle right being, e) in the bottom left, f) in the bottom right. (Coloured-in shape is AFTER transform from original position which itself is marked by the b point crosses):







5. The objective of the robot is to perfectly align and center the trip of the screwdriver into the slot of the screw. This means that they must have coinciding frame origins, and maintain identical orientation of frames relative to each other.

   It thus follows that the homogeneous transformation matrix from the screw to the base frame will be identical to the homogenous transformation matrix from the tip of the screwdriver to the base frame. Thus:

$$H_S^b = H_T^b$$

   Breaking these up individually reveals homogeneous transformation matrices that are already available to us:

$$H_S^b = H_S^c * H_c^b$$

   and

$$H_T^b = H_t^W * H_W^b$$

   Substituting for H_S^b gives:

$$H_t^W * H_W^b = H_S^c * H_c^b$$

   So

$$[H_t^W]^{-1} * H_t^W * H_W^b = [H_t^W]^{-1} * H_S^c * H_c^b$$

$$\therefore H_W^b = [H_t^W]^{-1} * H_S^c * H_c^b$$

6. The process has been broken down into 5 steps. There are:
   - Translate to Centre of the Earth from Pittsburgh (translation in the Z-axis by +6000 units) $= H_c^p$
   - Next, the tilted north that pointed to the magnetic North, will be realigned with true North by rotating this frame about its y-axis by 40.5 degrees$= H_{y40.5}^c$
   - Since Greenwhich lies on the prime meridian the frame can then be rotated 80 degrees about the x-axis which is now collinear with the True North $= H_{x80}^{y40.5}$
   - Greenwhich's inclination can then be faced "for lack of a better word" by rotating the current frame about its y-axis by -51 degrees (according to the RHR). $= H_{-y51}^{x80}$
   - Relative to it's z-axis which now points directly opposite the direction of Greenwhich, the frame can be translated by -6000 units back onto the surface. $= H_G^{-y51}$

Each of these homogenous transformation matrices can be defined as:

$$H_c^p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 6000 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_{y40.5}^{c} = \begin{bmatrix} \cos(40.5) & 0 & -sin(40.5) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(40.5) & 0 & \cos(40.5) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_{x80}^{y40.5} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(80) & -sin(80) & 0 \\ 0 & \sin(80) & \cos(80) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_{-y51}^{x80} = \begin{bmatrix} \cos(-51) & 0 & -sin(-51) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(-51) & 0 & \cos(-51) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_{G}^{-y51} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -6000 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Again, the order that these matrices are going to be multiplied in is very simple, every transformation is relative to a the current frame and so all homogeneous transformation matrices will be multiplied to the right such as:

$$H_G^P = H_c^p * H_{y40.5}^{c} * H_{x80}^{y40.5} * H_{-y51}^{x80} * H_G^{-y51}$$

$$H_G^P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 6000 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} \cos(40.5) & 0 & -\sin(40.5) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(40.5) & 0 & \cos(40.5) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(80) & -\sin(80) & 0 \\ 0 & \sin(80) & \cos(80) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$* \begin{bmatrix} \cos(-51) & 0 & -\sin(-51) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(-51) & 0 & \cos(-51) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -6000 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

As this involves tedious amounts of calculations, the overall Pittsburgh to

$$H_G^P = \begin{bmatrix} 0.566 & 0.6396 & -0.52 & 3119.8 \\ -0.765 & -0.174 & -0.6198 & 3718.6 \\ -0.306 & -0.749 & 0.5878 & 2473.1 \\ 0.000 & 0.000 & 0.000 & 1 \end{bmatrix}$$

7. A rotational transformation from one point to another can be represented as:

$$P^1 = R_0^1 * P^0$$

In quaternion form, it is represented as:

$$P^1 = Q * P^0 * Q^{-1}$$

Because both variables share a P^1 substituting for P^1 in one of the expressions leads to:

$$R_0^1 * P^0 = Q * P^0 * Q^{-1}$$

Q, P^0 and Q^{-1} can be represented as:

$$Q = q_0 + q_1 i + q_2 j + q_3 k \qquad (1)$$
$$P^0 = p_1 i + p_2 j + p_3 k \qquad (2)$$

$$Q^{-1} = q_0 - q_1 i - q_2 j - q_3 k \qquad (3)$$

Multiplying the $Q * P^0 * Q^{-1}$ gives:

$$Q * P^0 * Q^{-1} = [\,(-\,q_1p_1 - q_2p_2 - \ q_3p_3) + \mathrm{i}\,(\,q_0p_1 + \ q_2p_3 - \ q_3p_2) + \mathrm{j}\,(q_0p_2 \ - \ q_1p_3 \ + \ q_3p_1) + \mathrm{k}\,(q_0p_3 \ + \ q_1p_2 \ - \ q_2p_1\,)] \ * \ (q_0 - q_1 i - q_2 j - q_3 k)$$

Applying the distributive property in order to multiply the $Q * P^0$ with the Q^{-1} leads to:

$$\begin{aligned} Q * P^0 * Q^{-1} = {} & [(-q_1p_1 - q_2p_2 - \ q_3p_3)q_0 - \ (\,q_0p_1 \ + \ q_2p_3 - \ q_3p_2)(-q_1) \\ & - (q_0p_2 - q_1p_3 \ + \ q_3p_1)(-q_2) - \ (q_0p_3 \ + \ q_1p_2 - q_2p_1\,)(-q_3)] \\ & + \big[\mathbf{i}\big((q_0p_1 + q_2p_3 - \ q_3p_2)q_0 + (-q_1p_1 - q_2p_2 - q_3p_3)(-q_1\,) \\ & + (q_0p_2 - q_1p_3 + q_3p_1)(-q_3) - (q_0p_3 + q_1p_2 - q_2p_1\,)(-q_2)\big)\big] \\ & + \big[\mathbf{j}\big((-\,q_1p_1 - q_2p_2 - q_3p_3)(-q_2) - (\,q_0p_1 + q_2p_3 - q_3p_2)(-q_3) \\ & + (q_0p_2 - q_1p_3 + q_3p_1)q_0 + (q_0p_3 + q_1p_2 - q_2p_1)(-q_1)\big)\big] \\ & + [\mathbf{k}\,\big((-\,q_1p_1 - q_2p_2 - q_3p_3)(-q_3) + \ (\,q_0p_1 + q_2p_3 - q_3p_2)(-q_2) \\ & - \ (q_0p_2 - q_1p_3 + q_3p_1)(-q_1) + (q_0p_3 + q_1p_2 - q_2p_1\,)q_0\big)] \end{aligned}$$

Again, applying the distributive property in order to expand out everything on the right hand side, yields a really long expression, however since we care about only rotation, the only values that we care about are **i**, **j**, **k** :
So multiplying out the **i** values and partitioning these values into p_0, p_1, p_2 yields:

$$\big[\mathbf{i}\big((q_0q_0 + q_1q_1 + -q_3q_3 - q_2q_2)(p_1) + (q_2q_1 + q_1q_2 - \ q_3q_0 - q_3q_0)(p_2) + (q_2q_0 + q_3q_1 + q_1q_3 + q_2q_0)(p_3)\big)\big]$$

So multiplying out the j values and partitioning these values into p_0, p_1, p_2 yields:

$$\big[\mathbf{j}\big((q_2\,q_1 + q_3\,q_0 + q_0q_3 + q_1q_2)(p_1) + (q_0q_0 - q_3q_3 - q_1q_1 + q_2q_2)(p_2) + (-q_1q_0 + q_2q_3 + q_3q_2 - q_0q_1)(p3)\big)\big]$$

So multiplying out the k values and partitioning these values into p_0, p_1, p_2 yields:

$$[\mathbf{k}\,\big((+q_3\,q_1 - q_2q_0 - q_0q_2 + q_1q_3)(p_1) + \ (q_2q_3 + q_0q_1 + q_1q_0 + q_3q_2)(p_2) + (q_0q_0 - q_2q_2 - q_1q_1 + +q_3q_3)(p_3)\big)]$$

Placing them into a table (rows = **i**, **j**, **k**, columns = p_0, p_1, p_2) yields:

$q_0q_0 + q_1q_1 - q_3q_3 - q_2q_2$	$q_2q_1 + q_1q_2 - \ q_3q_0 - q_3q_0$	$q_2q_0 + q_3q_1 + q_1q_3 + q_2q_0$
$q_2\,q_1 + q_3\,q_0 + q_0q_3 + q_1q_2$	$q_0q_0 - q_3q_3 - q_1q_1 + q_2q_2$	$-q_1q_0 + q_2q_3 + q_3q_2 - q_0q_1$
$q_3\,q_1 - q_2q_0 - q_0q_2 + q_1q_3$	$q_2q_3 + q_0q_1 + q_1q_0 + q_3q_2$	$q_0q_0 - q_2q_2 - q_1q_1 + q_3q_3$

Simplifying each element of the table yields:

$q_0q_0 + q_1q_1 - q_3q_3 - q_2q_2$	$2(q_1q_2 - \ q_3q_0)$	$2(q_2q_0 + q_3q_1)$
$2(q_2\,q_1 + q_3\,q_0)$	$q_0q_0 - q_3q_3 - q_1q_1 + q_2q_2$	$2(q_2q_3 - q_0q_1)$
$2(q_3\,q_1 - q_2q_0)$	$2(q_2q_3 + q_0q_1)$	$q_0q_0 - q_2q_2 - q_1q_1 + q_3q_3$

This is the rotation matrix R_0^1 that corresponds to the rotation represented by Q

8. Each revolute joint rotates about a single axis but because of how they are connected, they rotate relative to each other. Therefore starting from the current frame (frame 1: end effector) all rotation matrices will sequentially be multiplied rightwards. Because Frame 1 is directly fixed on a revolute joint and is coaxial in 1 dimension (z) with this joint, $R_{\theta 2}^{1} = \ R_{\theta 2}^{\theta 3}$

$$\therefore R_0^1 = \ R_{\theta 2}^{\theta 3} * R_{\theta 1}^{\theta 2} * R_0^{\theta 1}$$

WARNING! *I assume that the z-axis points upwards along the page, and that y-axis points perpendicular AND out of the page. I also assume that in the diagram the curved arrows for* θ_1 *and* θ_3 *indicated a counter clockwise rotation. All calculations from now will be set based on these assumptions.*

$R_{\theta 2}^{\theta 3}$ rotates about the z-axis and using the RHR θ_3 is positive and so is expressed as:

$$R_{\theta 2}^{\theta 3} = \begin{bmatrix} cos\theta_3 & -sin\theta_3 & 0 \\ sin\theta_3 & cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_{0}^{\theta 1}$ rotates about the z-axis and using the RHR θ_1 is positive and so is expressed as:

$$R_{0}^{\theta 1} = \begin{bmatrix} cos\theta_1 & -sin\theta_1 & 0 \\ sin\theta_1 & cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_{\theta 1}^{\theta 2}$ rotates about the y-axis and using the RHR θ_2 is positive and so is expressed as:

$$R_{\theta 1}^{\theta 2} = \begin{bmatrix} cos\theta_2 & 0 & sin\theta_2 \\ 0 & 1 & 0 \\ -sin\theta_2 & 0 & cos\theta_2 \end{bmatrix}$$

$$\therefore R_0^1 = \begin{bmatrix} cos\theta_3 & -sin\theta_3 & 0 \\ sin\theta_3 & cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} cos\theta_2 & 0 & sin\theta_2 \\ 0 & 1 & 0 \\ -sin\theta_2 & 0 & cos\theta_2 \end{bmatrix} * \begin{bmatrix} cos\theta_1 & -sin\theta_1 & 0 \\ sin\theta_1 & cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Starting off with $R_{\theta 2}^{\theta 3} * R_{\theta 1}^{\theta 2}$:

$$R_{\theta 1}^{1} = \begin{bmatrix} cos\theta_3 cos\theta_2 & -sin\theta_3 & cos\theta_3 sin\theta_2 \\ sin\theta_3 cos\theta_2 & cos\theta_3 & sin\theta_2 sin\theta_3 \\ -sin\theta_2 & 0 & cos\theta_2 \end{bmatrix}$$

And now $R_1^{\theta 1} * R_0^{\theta 1}$:

$$R_0^1 = \begin{bmatrix} cos\theta_3 cos\theta_2 cos\theta_1 - sin\theta_3 sin\theta_1 & -cos\theta_3 cos\theta_2 sin\theta_1 - sin\theta_3 cos\theta_1 & cos\theta_3 sin\theta_2 \\ sin\theta_3 cos\theta_2 cos\theta_1 + sin\theta_1 cos\theta_3 & -sin\theta_3 cos\theta_2 sin\theta_1 + cos\theta_3 cos\theta_1 & sin\theta_2 sin\theta_3 \\ -sin\theta_2 cos\theta_1 & sin\theta_2 sin\theta_1 & cos\theta_2 \end{bmatrix}$$

This problem looks familiar in the sense that it relates to Euler Angles and finding an equivalent rotation matrix from 3 separate ones, by multiplying successive rotation matrices together on the right.