1. For something to be stable (globally, assymptotically) all its eigen values need to have negative real parts.

So let's look at our linear equation now...

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 5 & 7 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{B} u(t)$$

First thing we wanna do is find eigen vals of A:

$$\det(\lambda \underline{I} - A) = \det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 5 & 7 \end{bmatrix} \right)$$

$$= \det\left(\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & -5 & \lambda-7 \end{bmatrix} \right)$$

$$= \lambda\left[(\lambda)(\lambda-7) - (-1)(-5)\right] - (-1)\left[0 - (-1)(-1)\right] + 0\left[\dots\right]$$

$$= \lambda\left[\lambda(\lambda-7) + 6\right] + \left[2\right] = 0$$

Now solve to find eigen values...

$$\lambda\left[\lambda^2 - 7\lambda + 6\right] + 2 = 0$$

$$\lambda^3 - 7\lambda^2 + 6\lambda + 2 = 0$$

The zeros of this escpression were found through wolfram alpha. Here they are:

① $\lambda = \frac{7}{3} - \frac{31(1+i\sqrt{3})}{6\sqrt[3]{127+3i\sqrt{1518}}} - \frac{1}{6}(1-i\sqrt{3})\sqrt[3]{127+3i\sqrt{1518}}$

This is suppose to be cube root...

To save space, let

$C = \frac{31}{\sqrt[3]{127+3i\sqrt{1518}}}$ and $D = \sqrt[3]{127+3i\sqrt{1518}}$

The other zeros are:

② $\lambda = \frac{7}{3} - \frac{C(1-i\sqrt{3})}{6} - \frac{1}{6}(1+i\sqrt{3})D$

③ $\lambda = \frac{7}{3} + \frac{1}{3}(C + D)$ ← Obviously both contain imaginary

Now... going back to the beginning comment, we can see that A itself is unstable as non of its real values, are negative, and all need to be negative.

Since $y(t)$ is dependent ~~the complecs~~ on $x(t)$, we can it ignore it, since the stable as a whole.

Since the "vector" attached to $U(t)$ isn't a square matrix, we can't find its determinant, so leave it...

b) For the system to be controllable, ~~the~~ matrix Q where $Q = [B \quad AB \quad A^2B \ldots \quad A^{n-1}B]$ needs to have a rank of n where n is the rank of matrix A.

In this case rank is 3, therefore $n=3$.

So $Q_c = [B \quad AB \quad A^2B]$

This is a ~~...~~ equal to:

$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 5 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$A^2B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 5 & 7 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 5 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 7 \end{bmatrix}$

$\therefore Q_C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 7 \end{bmatrix}$

The rank of this matrix is 3 as well, so Q has a full rank and therefore the system is controllable

c) For the system to system to be observable, the following matrix needs to have full rank:

$$Q_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

, since we have a matrix with rank 3, we only have to deal with

$$Q_o = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$$

C in our case is $[0 \quad 1 \quad 3]$

so $CA = [0 \quad 1 \quad 3] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 5 & 7 \end{bmatrix}$

$\therefore CA = [3 \quad 15 \quad 22]$

and $CA^2 = [22 \quad 113 \quad 169]$

so $Q_o = \begin{bmatrix} 0 & 1 & 3 \\ 3 & 15 & 22 \\ 22 & 113 & 169 \end{bmatrix}$

This matrix also has a rank of 3 and so has full rank.

Therefore the system is observable as well.

d) Will be done on the computer

e) Will be done on the computer

f) Will be done on the computer

I'm sorry! x_c should be x_1 $\quad$ $\dot{x}_c$ should be x_3

ϕ should be x_2 $\quad$ $\dot{\phi}$ should be x_4

2. $\gamma \ddot{x}_c - \beta \ddot{\phi} \cos\phi + \beta \dot{\phi}^2 \sin\phi + \mu \dot{x}_c = F \quad (1)$

$\alpha \ddot{\phi} - \beta \ddot{x}_c \cos\phi - D \sin\phi = 0 \quad (2)$

a) Now to find the state vector:

$$x = \begin{bmatrix} x_c \\ \phi \\ \dot{x}_c \\ \dot{\phi} \end{bmatrix} \quad \text{so } \dot{x} = \begin{bmatrix} \dot{x}_c \\ \dot{\phi} \\ \ddot{x}_c \\ \ddot{\phi} \end{bmatrix}$$

These 2 are already known from x

These 2 are unknown

Now let's look at matrix M and see how it can be deployed to find $\ddot{x}_c$ and $\ddot{\phi}$...

$$\underbrace{\begin{bmatrix} \gamma & -\beta\cos\phi \\ -\beta\cos\phi & \alpha \end{bmatrix}}_{M} \begin{bmatrix} \ddot{x}_c \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} F - \mu\dot{x}_c - \beta\dot{\phi}^2\sin\phi \\ D\sin\phi \end{bmatrix}$$

These values are obtained from equations (1) and (2).

Now, let's isolate the $\begin{bmatrix} \ddot{x}_c \\ \ddot{\phi} \end{bmatrix}$ so that we can ~~solf~~ solve for its terms.

$$\begin{bmatrix} \ddot{x}_c \\ \ddot{\phi} \end{bmatrix} = [M]^{-1} \begin{bmatrix} F - \mu\dot{x}_c - \beta\dot{\phi}^2\sin\phi \\ D\sin\phi \end{bmatrix}$$

$$\begin{bmatrix} \ddot{x}_c \\ \ddot{\phi} \end{bmatrix} = \frac{1}{\alpha\gamma - \beta^2\cos^2\phi} \begin{bmatrix} \alpha & \beta\cos\phi \\ \beta\cos\phi & \gamma \end{bmatrix} \begin{bmatrix} F - \mu\dot{x}_c - \beta\dot{\phi}^2\sin\phi \\ D\sin\phi \end{bmatrix}$$

so $$\ddot{x}_c = \frac{1}{\alpha\gamma - \beta^2\cos^2\phi}\left[\alpha F - \mu\alpha\dot{x}_c - \alpha\beta\dot{\phi}^2\sin\phi + BD\sin\phi\cos\phi\right]$$

and $$\ddot{\phi} = \frac{1}{\alpha\gamma - \beta^2\cos^2\phi}\left[\beta\cos(\phi)F - \mu\dot{x}_c\beta\cos\phi - \beta^2\dot{\phi}^2\sin\phi\cos\phi + \gamma D\sin\phi\right]$$

∴ The state vector looks like:

$$\dot{x} = \begin{bmatrix} \dot{x}_c \\ \dot{\phi} \\ \ddot{x}_c \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} \dot{x}_c \\ \dot{\phi} \\ \frac{1}{\alpha\gamma - \beta^2\cos^2\phi}\left[\alpha F - \mu\alpha\dot{x}_c - \alpha\beta\dot{\phi}^2\sin\phi + BD\sin\phi\cos\phi\right] \\ \frac{1}{\alpha\gamma - \beta^2\cos^2\phi}\left[\beta F\cos\phi - \mu\dot{x}_c\beta\cos\phi - \beta^2\dot{\phi}^2\sin\phi\cos\phi + \gamma D\sin\phi\right] \end{bmatrix}$$

This should be:

$$= \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{matrix} \dot{x}_1 \\ \dot{x}_2 \\ \frac{1}{\alpha\gamma - \beta^2\cos^2 x_2}\left[\alpha F - \mu\alpha\dot{x}_1 - \alpha\beta\dot{x}_2^2\sin x_2 + BD\sin x_2\cos x_2\right] \\ \frac{1}{\quad}\left[\cancel{\alpha F}\ \beta F\cos x_2 - \mu\dot{x}_1\beta\cos x_2 - \beta^2\dot{x}_2^2\sin x_2\cos x_2 + \gamma D\sin x_2\right] \end{matrix}$$

Again, $\dot{x}_c$ should be $\dot{x}_1$, $\dot{\phi}$ should be $\dot{x}_2$, $\ddot{x}_c$ should be $\dot{x}_3$ and $\ddot{\phi}$ should be $\dot{x}_4$

b) The set of equilibrium points in the system are those for which $\dot{x} = 0$

$\therefore \dot{x}_c = 0$

$\dot{\phi} = 0$

$\ddot{x}_c = 0$

$\ddot{\phi} = 0$

Now looking at the equations for $\ddot{x}_c$ and $\ddot{\phi}$ from previous page, we get:

$$\cancel{\alpha F} - \cancel{\mu\alpha\dot{x}_c} - \cancel{\alpha\beta\dot{\phi}^2}\sin\phi + BD\sin\phi\cos\phi = 0$$

$F=0$ because of equilibrium

so we are left with either $\boxed{\sin\phi = 0 \text{ or } \cos\phi = 0}$

$$\text{and } \cancel{\beta F\cos\phi} - \cancel{\mu\dot{x}_c\beta\cos\phi} - \cancel{\beta^2\dot{\phi}^2}\sin\phi\cos\phi + \gamma D\sin\phi = 0$$

$F=0$ because of equilibrium

So we are left with ~~either $\gamma D = 0$~~ $\sin\phi = 0$

and therefore $\phi = \sin^{-1}(0)$ $\therefore \phi = n\pi$

where $n \in [1, \infty]$ integer if π in radians

or

$n \in [0, 180K]$ where K is integer $\in [1, \infty]$ if π in degrees.

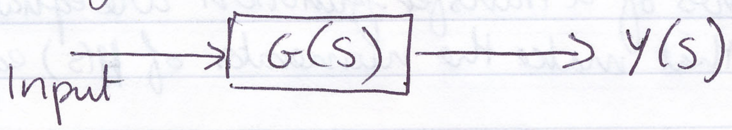
Therefore because in this system ϕ is angle from vertical dy-axis for the pendulum, states of equilibrium are when pendulum is completely vertical either by pointing up, or by pointing down

c) Done on the computer

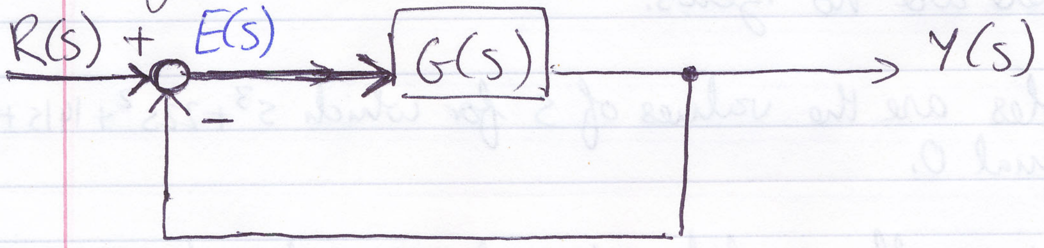
d) Done on the computer.

3. a) $G(s) = \dfrac{200}{s^3 + 22s^2 + 141s + 2}$

So since it is an open loop transfer function, the block diagram looks like:



Now with unity negative feedback, the closed loop block diagram looks like:



The closed loop transfer function is espressed as:

$$T(s) = \frac{G(s)}{1 + G(s)} \quad \text{which is} \quad \frac{\dfrac{200}{s^3 + 22s^2 + 141s + 2}}{1 + \dfrac{200}{s^3 + 22s^2 + 141s + 2}}$$

~~which is~~: The denominator is equal to

$$\frac{s^3 + 22s^2 + 141s + 202}{s^3 + 22s^2 + 141s + 2}$$

So $T(s) = \dfrac{200}{s^3 + 22s^2 + 141s + 202}$

b) The poles of a transfer function are equal to the values that make the denominator of $H(s)$ equal to 0.

The zeros of a transfer function are equal to the values that make the numerator of $H(s)$ equal to 0.

$\therefore$ Since $H(s) = \dfrac{200}{s^3 + 22s^2 + 141s + 202}$

There are no zeros.

Poles are the values of s for which $s^3 + 22s^2 + 141s + 202$ equal 0.

Using wolfram alpha, the poles were found to be:

$s = -10 - i$, $s = -10 + i$, $s = -2$

c) Will be submitted by PC.

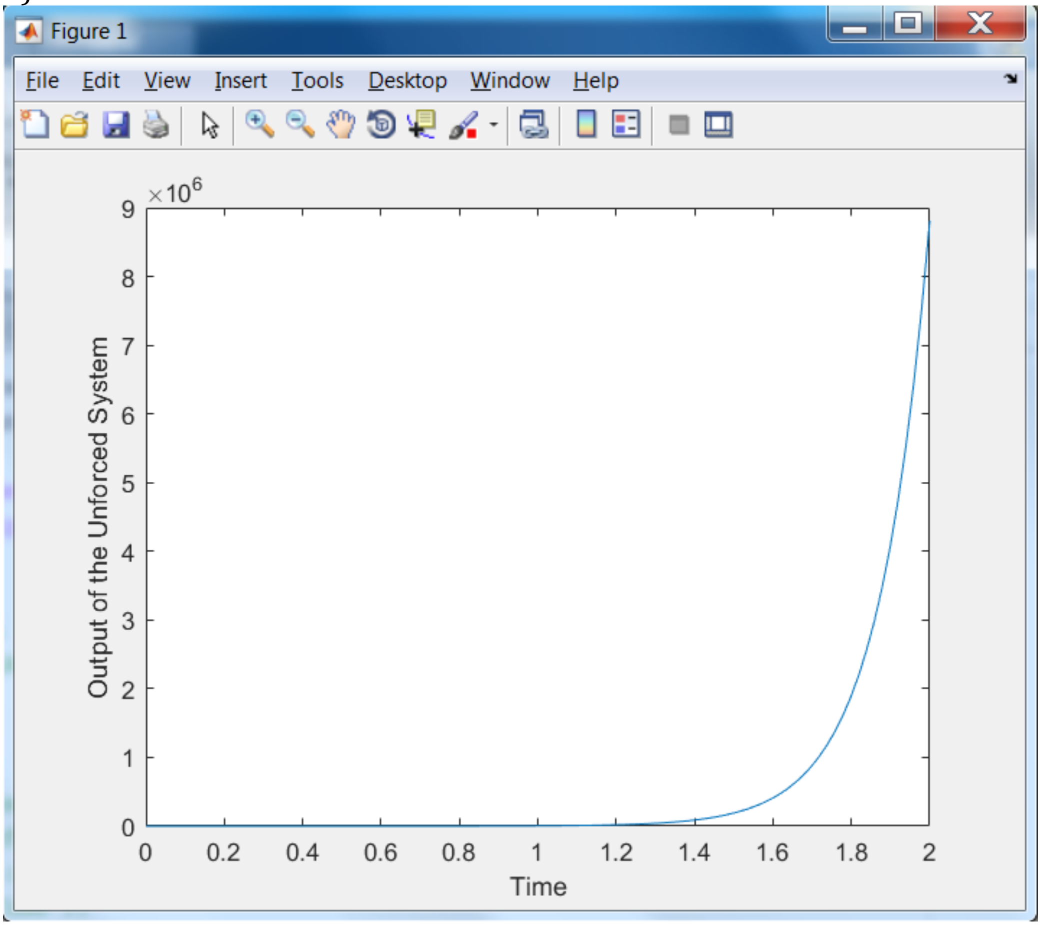
d) According to final value theorem, the steady state value can be found from:

$\lim_{s \to 0} T(s)$ so as $s \to 0$ $H(s) = \dfrac{200}{0^3 + 22 \cdot 0^2 + 141 \cdot 0 + 202}$

so $= \dfrac{200}{202} = 0.99$

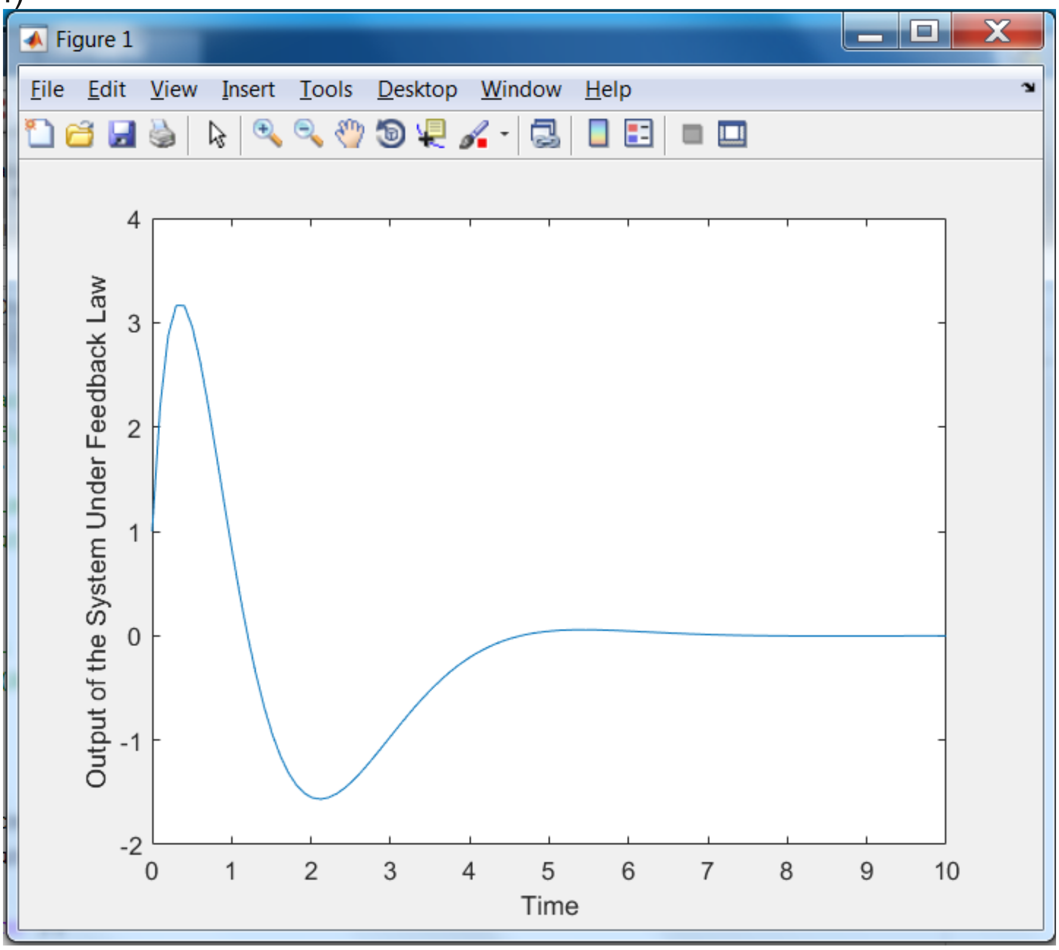
Question1:

d)



e) K = [11.0000 60.0000 88.0000]

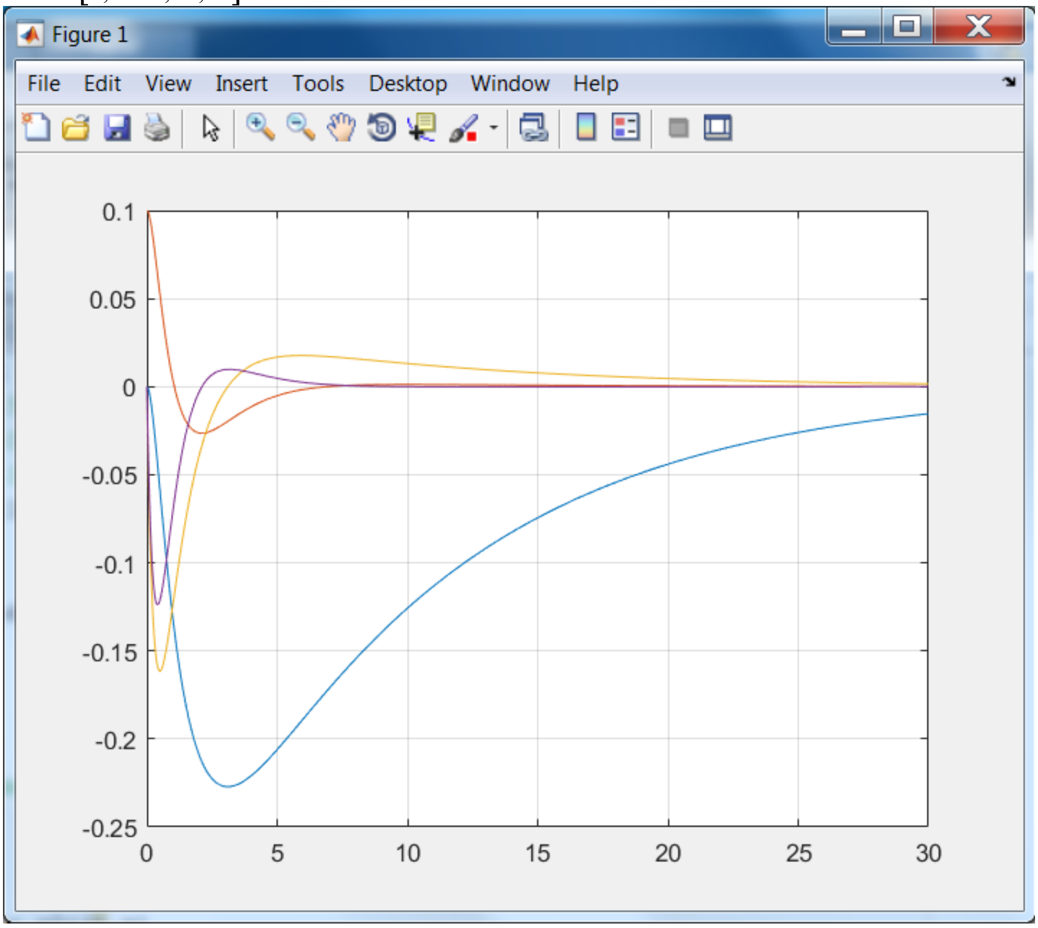
f)



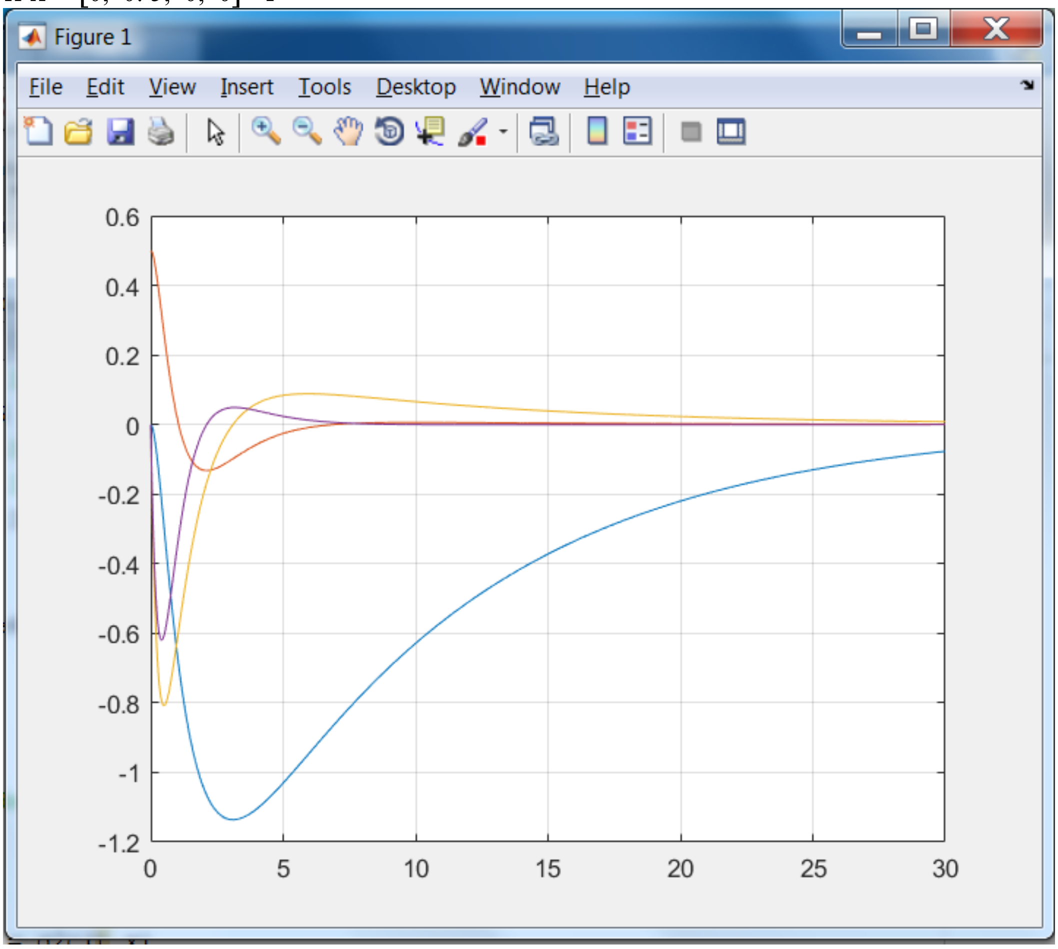
Question 2:

c) *Note: For all graphs below, X axis is time, Y-axis is State of System*

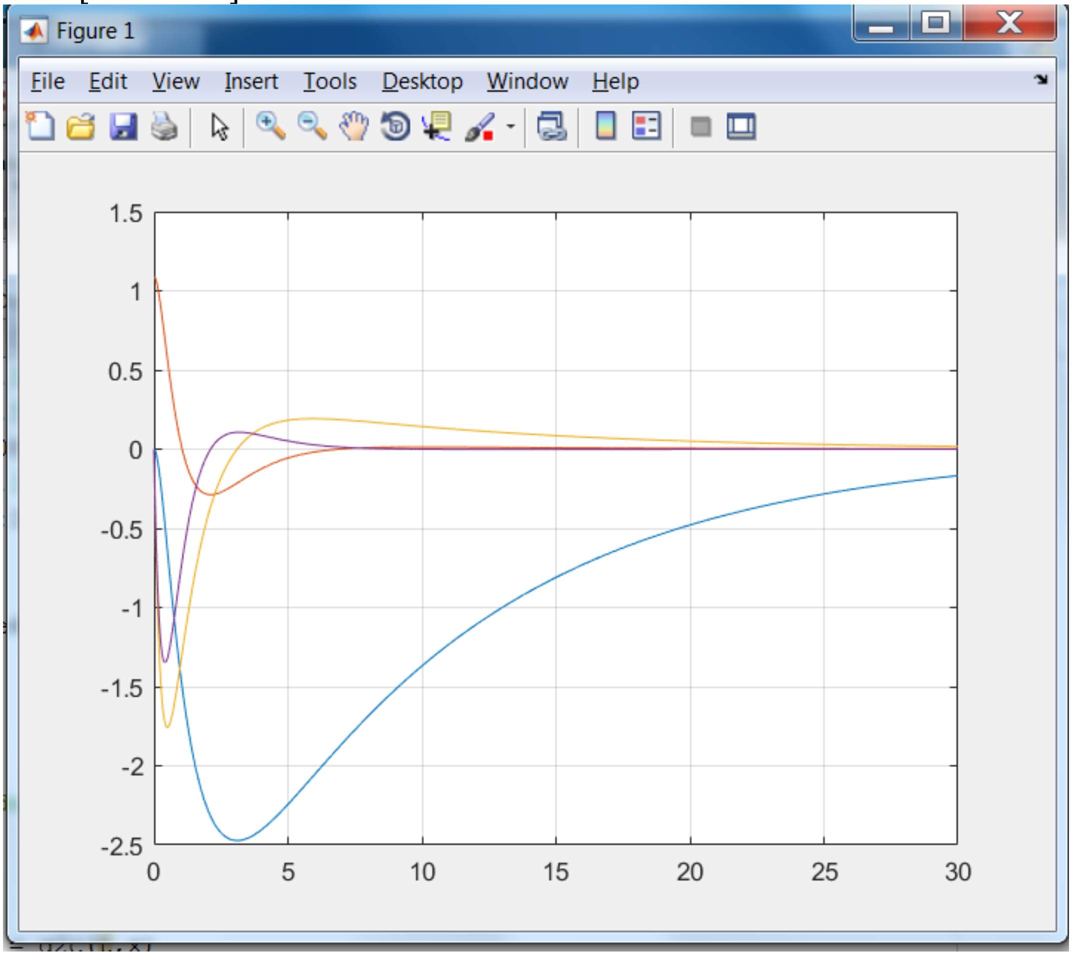
If x = [0; 0: 1; 0; 0]^T



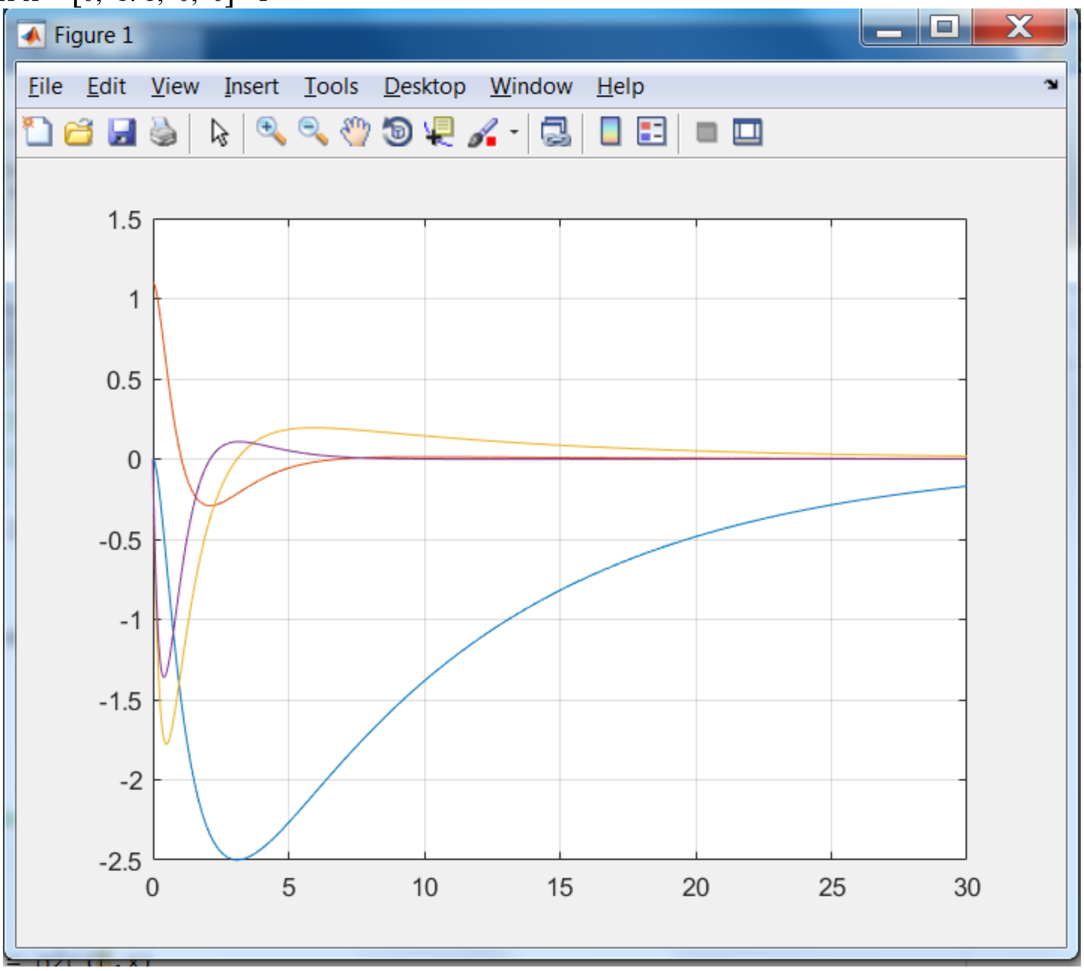
If x = [0; 0: 5; 0; 0]^T



If x = [0 1: 0886 0 0]^T

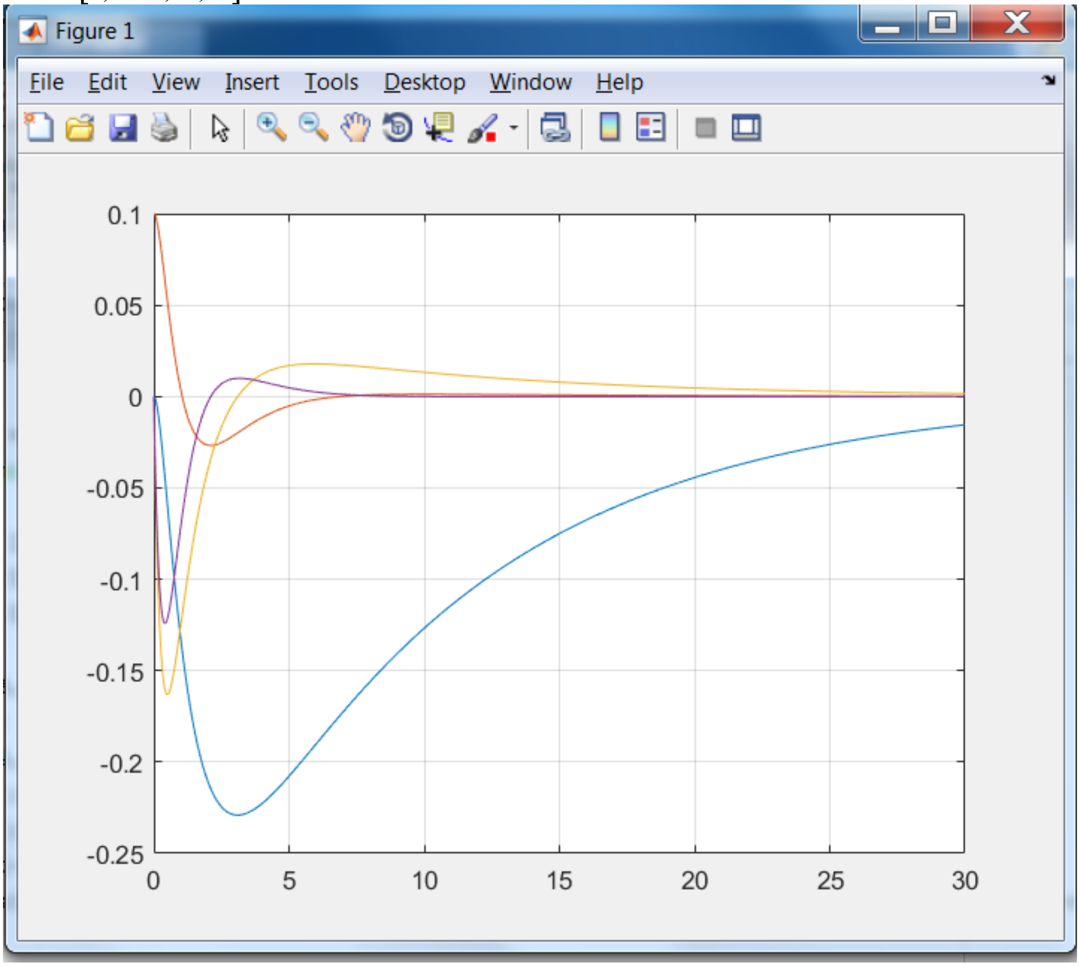


If x = [0; 1: 1; 0; 0]^T

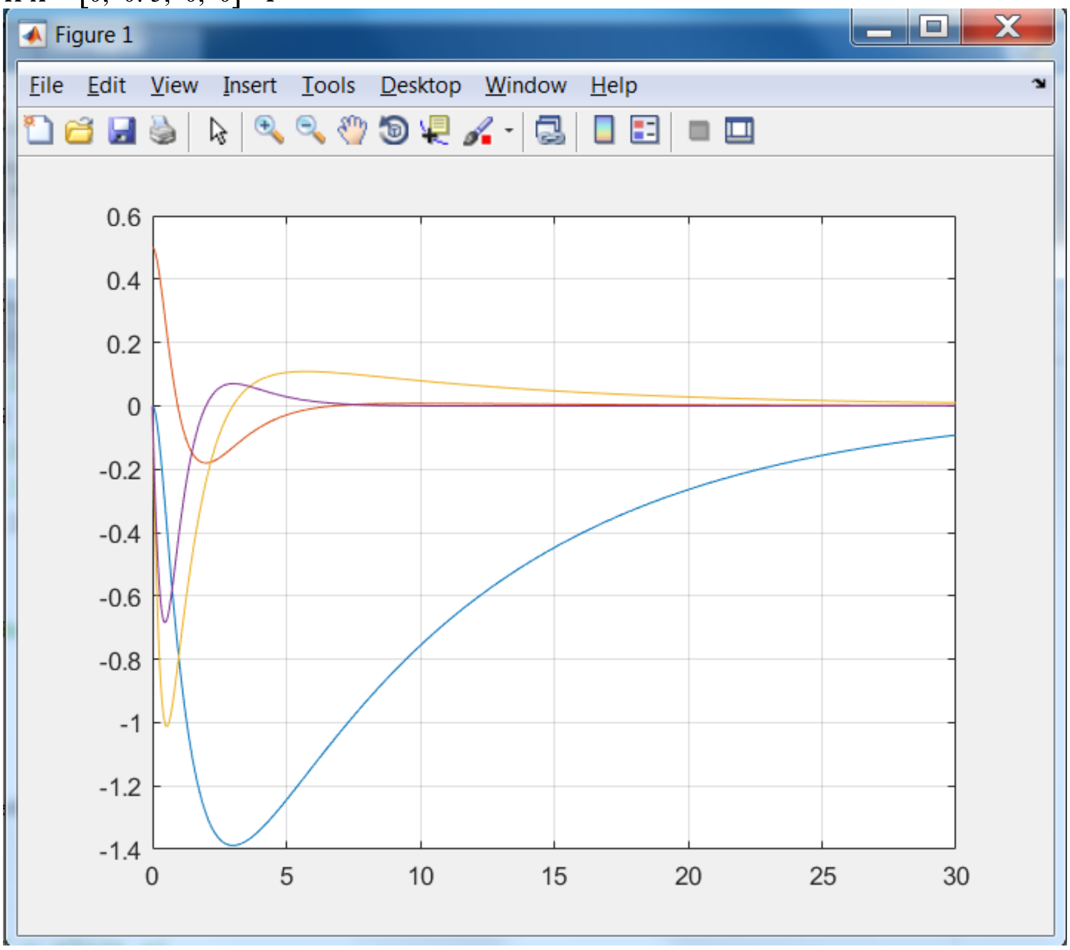


d)

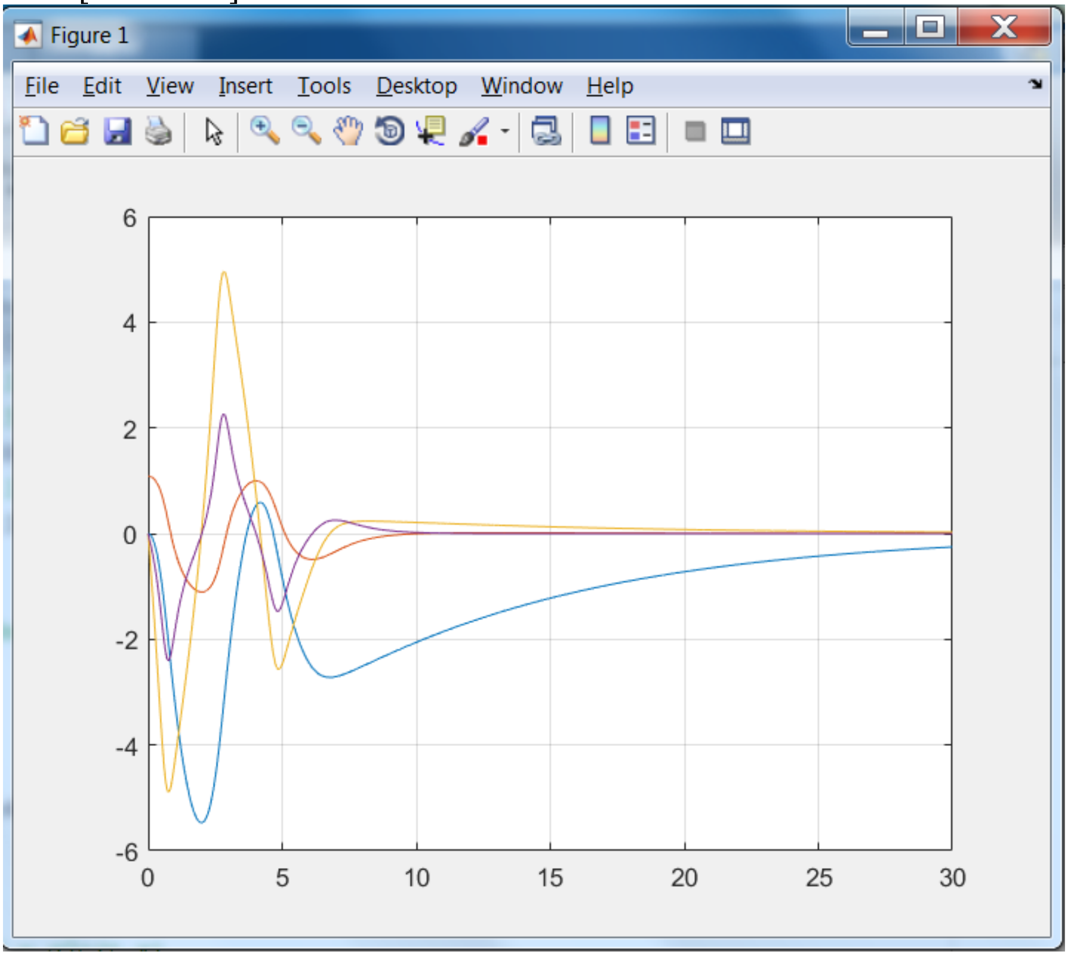
If x = [0; 0: 1; 0; 0]^T



If x = [0; 0: 5; 0; 0]^T



If x = [0 1: 0886 0 0]^T



If x = [0; 1: 1; 0; 0]^T
It takes forever for matlab to run this... There are 2 possible reasons for this:

Reason1: Taking the small angle approximation of it doesn't work very well for high order systems where a whole host of non-linearities ensue.

Reason2: The equilibrium point seems to be very far away, and so it takes it a very long time for it to get pulled and for there to be convergence.

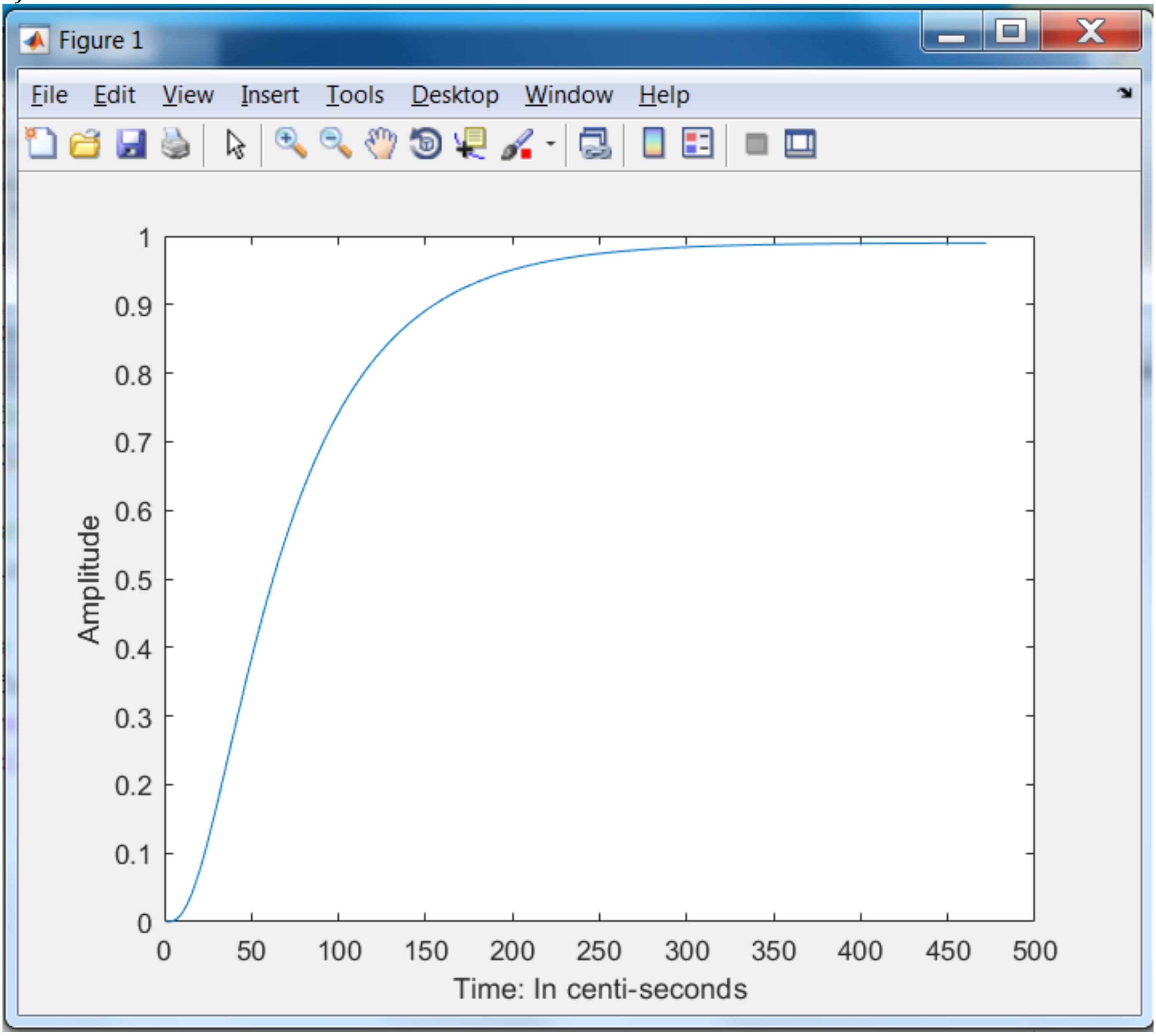
<u>Explanation of differences in the results:</u>
For 2C questions:
a) Generally speaking all of them have very identical looking changes in results, however it is important to look at the scales. The axis really highlights the big differences and based on these results, it boils down to amplitude. The results for:
x = [0; 0: 1; 0; 0]^T have very a 10 times smaller amplitude than those of x = [0 1: 0886 0 0]^T and x = [0; 1: 1; 0; 0]^T which themselves are almost identical with x = [0; 1: 1; 0; 0]^T having very slightly greater amplitude.

Results for x = [0; 0: 5; 0; 0]^T on the other hand have half the amplitude as the results for x = [0 1: 0886 0 0]^T and x = [0; 1: 1; 0; 0]^T.
Everything else LOOKS similar as far as variations over time.

For 2D questions:
Amplitude increases as well but so does the rebounding overshoot of all curves that start by descending into a global minima. All curves also tend to converge more aggressively and rapidly onto steady state values.

3.

c)



The pole s = -2 dominates because poles closer to the imaginary axis are more dominant than those further to the left from it. The other 2 poles are very far away as they both have -10 real values.

4. The final gains I chose were:

Proportional Gain Value: 550

Differential Gain Value: 345

Integral Gain Value: 35

Here is the plot of the step response: