16-642 Fall 2017: Reference Notes for Euler-Lagrange Dynamics

George Kantor

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Now we move into the topic of dynamics, which can be though of as the relationship between applied joint torques/forces and manipulator motion. Let's dive in:

1 Dynamics: Euler Lagrange Equations

The Euler-Lagrange equations give the equations of motion for a manipulator as follows:

$$\boxed{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = \tau_k}$$

for $k = 1, 2, \ldots, n$. Here q is the vector of joint angles, τ is the vector of joint torques, and $\mathcal{L}$ is called the Lagrangian. It is a function of both q and $\dot{q}$. It is defined to be

$$\mathcal{L}(q, \dot{q}) = \mathcal{K}(q, \dot{q}) - \mathcal{P}(q)$$

where $\mathcal{K}(q, \dot{q})$ is the kinetic energy of the robot and $\mathcal{P}(q)$ is the potential energy.

I know this all seems confusing and abstract at this point. We will make it more concrete over the course of the next few days. Here is what you should get now:

- the stuff in the box defines n equations
- the left side of each equations is a function of all of the joint variables q and their first and second derivatives ($\dot{q}$ and $\ddot{q}$).
- the right side of the kth equation is the torque applied to the kth joint
- we call this "a system of n coupled ordinary differential equations (or ODEs)"
- given the following:
  - the initial joint positions $q(0)$
  - the initial joint velocities $\dot{q}(0)$
  - the joint torques as function of time $\tau(t)$, defined for some interval of time $t \in [0, t_f]$.

  we can solve the Euler-Lagrange equations to get $q(t)$, $\dot{q}(t)$, and $\ddot{q}(t)$ for the time interval $t \in [0, t_f]$. In other words *we can solve for the motion that results from applying the torques in* $\tau(t)$.

- it is hard to solve these equations!
- it is easy to find approximate solutions using numerical methods. (we'll learn how to do this).

For this class, we will just take these equations as gospel truth. They can be derived in a number of ways, all of which are beyond the scope of this class. There is a very nice discussion in the book (section 7.1.2) that derives the Euler-Lagrange equations using the principle of virtual work. You won't be responsible for this material on an exam, but reading it and understanding will make you a better person.

2 Lagrange Eqs for Planar Systems

Euler Lagrange equations work exactly the same for 2D systems as they do for 3D systems, however the 2D case is dramatically less tedious.

The kinetic energy for a 2D rigid body with position (x, y) and orientation ϕ is

$$\mathcal{K} = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}\mathcal{I}\dot{\phi}^2$$

where m is the mass of the body and $\mathcal{I}$ is the moment of inertia. The variables x, y, and ϕ must be defined with respect to some non-moving frame (aka an "inertial frame"). The moment of inertia is defined to be

$$\mathcal{I} = \int\int (x^2 + y^2)\rho(x, y)dx\,dy$$

where $\rho(x, y)$ is the mass density function. The value of I depends on what frame x and y are expressed in, so when it is not obvious we will indicate the frame with a superscript (e.g., I^0). Actually, I only depends on the location of the origin of the frame, not on the orientation of the frame.

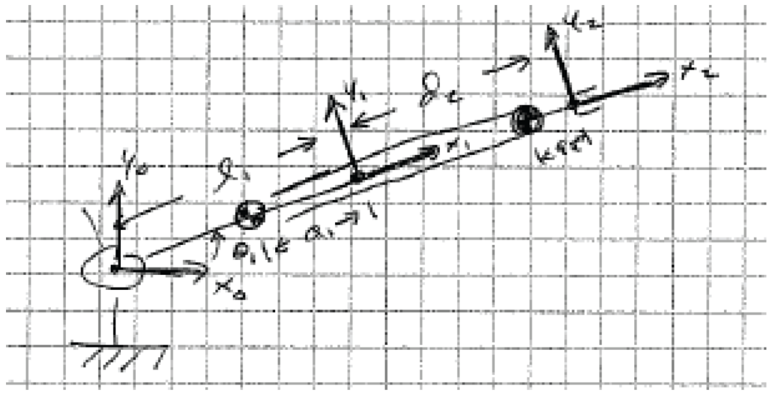
The moment of inertia resists changes in angular velocity kind of like mass resists changes in linear velocity.

For us, potential energy comes exclusively from gravity. Assume y points up, and let (x_{cm}, y_{cm}) denote the location of the center of mass. Then

$$\mathcal{P} = mgy_{cm}.$$

where g is the acceleration due to gravity, (usually 9.8m/s^2).

3 Example: Planar RP Manipulator



Let m_1,$\mathcal{I}_1$,m_2, and $\mathcal{I}_2$ be the mass for link 1, the moment of inertia for link 1, the mass for link 2, and the moment of inertia for link 2, respectively. The moments of inertia for each link are computed with respect to the frame attached to the center of mass of that link.

joint variables:

$$q = \begin{bmatrix} \theta_1 \\ d_2 \end{bmatrix}$$

joint torques/forces:

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

first link: We start by attaching a frame to the center of mass of the first link whose orientation is the same as the DH frame 1 at the end of the link. The pose of a this frame can be easily found to be

$$\begin{bmatrix} x_{c1} \\ y_{c1} \\ \phi_{c1} \end{bmatrix} = \begin{bmatrix} (\ell_1 - a_1)\cos\theta_1 \\ (\ell_1 - a_1)\sin\theta_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} (\ell_1 - a_1)\cos q_1 \\ (\ell_1 - a_1)\sin q_1 \\ q_1 \end{bmatrix}$$

which means the velocity of the frame at the COM of the first link is

$$v_{c1}^0 = \begin{bmatrix} -(\ell_1 - a_1)\dot{q}_1 \sin q_1 \\ (\ell_1 - a_1)\dot{q}_1 \cos q_1 \\ \dot{q}_1 \end{bmatrix}$$

which means the kinetic energy of the first link is

$$\mathcal{K}_1(q,\dot{q}) = \frac{1}{2}m_1\left((-(\ell_1 - a_1)\sin q_1)^2 + ((\ell_1 - a_1)\cos q_1)^2\right) + \frac{1}{2}\mathcal{I}_1\dot{q}_1^2$$

$$= \frac{1}{2}m_1(\ell_1 - a_1)^2\dot{q}_1^2 + \frac{1}{2}\mathcal{I}_1\dot{q}_1^2$$

The potential energy is determined by the height of center of mass:

$$\mathcal{P}_1(q) = m_1 g(\ell_1 - a_1)\sin q_1$$

second link: We repeat a similar process to get the kinetic and potential energies of the second link:

$$\begin{bmatrix} x_{c2} \\ y_{c2} \\ \phi_{c2} \end{bmatrix} = \begin{bmatrix} (\ell_1 + d_2 - a_2)\cos\theta_1 \\ (\ell_1 + d_2 - a_2)\sin\theta_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} (\ell_1 - a_2 + q_2)\cos q_1 \\ (\ell_1 - a_2 + q_2)\sin q_1 \\ q_1 \end{bmatrix} \triangleq f(q)$$

which means the velocity of the second frame is

$$\begin{aligned} v_{c2}^0 &= \frac{\partial f}{\partial q}\dot{q} \\ &= \begin{bmatrix} -(\ell_1 - a_2 + q_2)\sin q_1 & \cos q_1 \\ (\ell_1 - a_2 + q_2)\cos q_1 & \sin q_1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \\ &= \begin{bmatrix} -(\ell_1 - a_2 + q_2)\dot{q}_1 \sin q_1 + \dot{q}_2 \cos q_1 \\ (\ell_1 - a_2 + q_2)\dot{q}_1 \cos q_1 + \dot{q}_2 \sin q_1 \\ \dot{q}_1 \end{bmatrix} \end{aligned}$$

which means the kinetic energy of the second link is

$$\begin{aligned} \mathcal{K}_2(q,\dot{q}) &= \frac{1}{2}m_2\left((-(\ell_1 - a_2 + q_2)\dot{q}_1 \sin q_1 + \dot{q}_2\cos q_1)^2 + ((\ell_1 - a_2 + q_2)\dot{q}_1\cos q_1 + \dot{q}_2 \sin q_1)^2\right) + \frac{1}{2}\mathcal{I}_2\dot{q}_1^2 \\ &= \frac{1}{2}m_2\left((\ell_1 - a_2 + q_2)^2\dot{q}_1^2\sin^2 q_1 - 2(\ell_1 - a_2 + q_2)\dot{q}_1\dot{q}_2\sin q_1\cos q_1 + \dot{q}_2^2\cos^2 q_1 + (\ell_1 - a_2 + q_2)^2\dot{q}_1^2\cos^2 q_1\right. \\ &\quad \left. +2(\ell_1 - a_2 + q_2)\dot{q}_1\dot{q}_2\cos q_1 \sin q_1 + \dot{q}_2^2\sin^2 q_1\right) + \frac{1}{2}\mathcal{I}_2\dot{q}_1^2 \\ &= \frac{1}{2}m_2\left((\ell_1 - a_2 + q_2)^2\dot{q}_1^2 + \dot{q}_2^2\right) + \frac{1}{2}\mathcal{I}_2\dot{q}_1^2 \end{aligned}$$

And the potential energy of the second link is

$$\mathcal{P}_2(q) = m_2 g(\ell_1 + q_2 - a_2)\sin q_1$$

the Lagrangian: Now we can put the energies together to get the Lagrangian:

$$\begin{aligned} \mathcal{L}(q,\dot{q}) &= \mathcal{K}_1(q,\dot{q}) + \mathcal{K}_2(q,\dot{q}) - \mathcal{P}_1(q) - \mathcal{P}_2(q) \\ &= \frac{1}{2}m_1(\ell_1 - a_1)^2\dot{q}_1^2 + \frac{1}{2}\mathcal{I}_1\dot{q}_1^2 + \frac{1}{2}m_2\left((\ell_1 - a_2 + q_2)^2\dot{q}_1^2 + \dot{q}_2^2\right) + \frac{1}{2}\mathcal{I}_2\dot{q}_1^2 - m_1 g(\ell_1 - a_1)\sin q_1 - m_2 g(\ell_1 + q_2 - a_2)\sin q_1 \end{aligned}$$

Grouping terms gives:

$$= \frac{1}{2}\left[\underbrace{\left(m_1(\ell_1 - a_1)^2 + \mathcal{I}_1 + m_2(\ell_1 - a_2 + q_2)^2 + I_2\right)}_{\triangleq \Gamma_1(q_2)}\dot{q}_1^2 + m_2\dot{q}_2^2\right] - \underbrace{\left(m_1 g(\ell_1 - a_1) + m_2 g(\ell_1 + q_2 - a_2)\right)}_{\triangleq \Gamma_2(q_2)}\sin q_1$$

So we write the Lagrangian as

$$\boxed{\mathcal{L}(q,\dot{q}) = \frac{1}{2}\Gamma_1(q_2)\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2 - \Gamma_2(q_2)\sin q_1.}$$

Euler Lagrange Equation for first link

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} - \frac{\partial \mathcal{L}}{\partial q_1} = \tau_1$$

doing the math:

$$\frac{d}{dt}\left(\Gamma_1(q_2)\dot{q}_1\right) - \left(-\Gamma_2(q_2)\cos q_1\right) = \tau_1$$

Using the product rule to expand the first term, this becomes:

$$\frac{d\Gamma_1(q_2)}{dt}\dot{q}_1 + \Gamma_1(q_2)\ddot{q}_1 + \Gamma_2(q_2)\cos q_1 = \tau_1$$

Now using the chain rule:

$$\frac{\partial \Gamma_1}{\partial q_2}\dot{q}_2\dot{q}_1 + \Gamma_1(q_2)\ddot{q}_1 + \Gamma_2(q_2)\cos q_1 = \tau_1$$

Finally, evaluating $\frac{\partial \Gamma_1}{\partial q_2}$ yields the first equation of motion:

$$\boxed{2m_2(\ell_1 - a_2 + q_2)\dot{q}_2\dot{q}_1 + \Gamma_1(q_2)\ddot{q}_1 + \Gamma_2(q_2)\cos q_1 = \tau_1}$$

Euler Lagrange Equation for second link

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2} - \frac{\partial \mathcal{L}}{\partial q_2} = \tau_2$$

doing the math:

$$\frac{d}{dt}\left(m_2\dot{q}_2\right) - \left(\frac{1}{2}\frac{\partial \Gamma_1}{\partial q_2}\dot{q}_1^2 - \frac{\partial \Gamma_2}{\partial q_2}\sin q_1\right) = \tau_2$$

$$\frac{d}{dt}\left(m_2\dot{q}_2\right) - \left(\frac{1}{2}2m_2(\ell_1 - a_2 + q_2)\dot{q}_1^2 - m_2 g\sin q_1\right) = \tau_2$$

$$\boxed{m_2\ddot{q}_2 - m_2(\ell_1 - a_2 + q_2)\dot{q}_1^2 + m_2 g\sin q_1 = \tau_2}$$

The last two boxed equations are the “equations of motion” for the RP arm. They can be written lots of different ways, here is one we call “standard form”:

$$\underbrace{\begin{bmatrix} \Gamma_1(q_2) & 0 \\ 0 & m_2 \end{bmatrix}}_{\triangleq M(q)} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 2m_2(\ell_1 - a_2 + q_2)\dot{q}_1 \\ -m_2(\ell_1 - a_2 + q_2)\dot{q}_1 & 0 \end{bmatrix}}_{\triangleq C(q,\dot{q})} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \Gamma_2(q_2)\cos q_1 \\ m_2 g\sin q_1 \end{bmatrix}}_{\triangleq G(q)} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

The matrix $M(q)$ is called the *mass matrix*. The matrix $C(q, \dot{q})$ is called the *Coriolis and centrifugal matrix*. (note that C is not unique!) The vector $G(q)$ is called the *gravity vector*.

4 Quick Review

Euler-Lagrange equations again:

$$\boxed{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = \tau_k}$$

for $k = 1, 2, \dots, n$. $\mathcal{L}$ is called the Lagrangian. and is defined to be

$$\mathcal{L}(q, \dot{q}) = \mathcal{K}(q, \dot{q}) - \mathcal{P}(q)$$

where $\mathcal{K}(q, \dot{q})$ is the kinetic energy of the robot (which is something like the $\frac{1}{2}mv^2$ you probably remember from your physics class) and $\mathcal{P}(q)$ is the potential energy (which is something like the mgh you probably also remember from physics).

These equations can always be put into standard mechanical form:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau,$$

Where the matrix $M(q)$ is called the *mass matrix*. The matrix $C(q, \dot{q})$ is called the *Coriolis and centrifugal matrix*. (note that C is not unique!) The vector $G(q)$ is called the *gravity vector*.

The mass matrix is always square and invertible, which means we can always solve for $\ddot{q}$:

$$\ddot{q} = M(q)^{-1}\left(\tau - C(q, \dot{q})\dot{q} - G(q)\right).$$

This formulation will be useful for generating approximate numerical solutions. Note that is is a system of n coupled second-order ODEs.

5 State Space Representation

We can transform the above system of n coupled second-order into a system of $2n$ coupled first-order ODES as follows. Define the *state vector* to be

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

Note that this is a $2n$ dimensional vector. Define vectors $x_1 = q$ and $x_2 = \dot{q}$ to the the "top" and "bottom" of x. Now we can write $\dot{x}$:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} x_2 \\ M(q)^{-1}\left(\tau - C(q, \dot{q})\dot{q} - G(q)\right) \end{bmatrix} = \begin{bmatrix} x_2 \\ M(x_1)^{-1}\left(\tau - C(x)x_2 - G(x_1)\right) \end{bmatrix} \triangleq f(x, \tau)$$

One more thing – usually inputs in state space systems are represented with the letter u, so we define $u = \tau$, which yields the set of $2n$ coupled first order ODEs:

$$\dot{x} = f(x, u)$$

6 A More General View of Euler-Lagrange

So far, we have seen Euler-Lagrange used to generate equations of motions for robotic manipulators, but it is actually a very general technique that can be applied to any frictionless mechanical system. Generally, the vector q is called the *generalized configuration* and τ is called the *generalized force*. For a general mechanical system (e.g., a rock flying through the air), q can be any description of the configuration (e.g., for the rock, q could be (x, y)). The definition of q also defines τ: there is a one-to-one correspondence between the elements of q and the elements of τ. If $q_1 = x$, i.e., q_1 measures linear displacement in the x direction, the τ_1 must be F_x, a linear force in the same direction. If $q_2 = \theta_y$, i.e., q_2 measures rotation about the y axis, then τ_2 must be a torque about the y axis.

7 Friction

The equations of motion we have derived so far ar for frictionless systems. This is unfortunate since friction usually plays a significant role in the dynamics of mechanical system. For example, there is no real pendulum that oscillates forever at the same amplitude because friction is always taking energy away.

It is not hard to come up with reasonable models for friction (though it is really hard to come up with exact models!). We start by assuming that friction can be modeled as a force applied at the joint. This means that the standard form equation becomes:

$$M(q)\ddot{q} - C(q, \dot{q})\dot{q} - G(q) = \tau + F(q, \dot{q})$$

where $F(q, \dot{q})$ is the vector of friction forces. Here are a few standard friction models:

7.1 Viscous Friction

The easiest friction model to use is the viscous friction model. It assumes that the friction force applied at the ith joint has

1. magnitude proportional to the velocity of the ith joint
2. direction opposing the velocity

This gives us an expression for the friction on the ith joint:

$$F_i = -\gamma_{vi}\dot{q}_i$$

where $\gamma_{vi} > 0$ is called the viscous friction coefficient.

7.2 Coulomb Friction

Coulomb friction is more difficult to use than viscous friction. It is also usually more accurate. Coulomb friction assumes that the frictional force at the ith joint:

1. opposes velocity

2. has constant magnitude

This results in the following expression:

$$F_i = -\gamma_{ci}\text{sign}(\dot{q}_i)$$

7.3 Static Friction

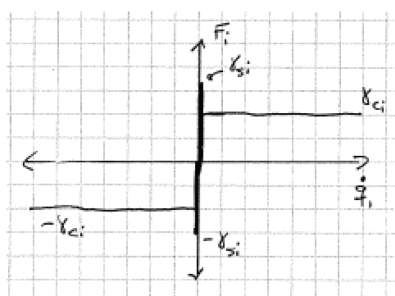
Static friction models the fact that is requires a larger force to get something moving than it does to keep it moving. Static friction is sometimes called stiction. It is usually combined with Coulomb friction. The fricional force at the ith joint has the following properites:

- it opposes velocity or, if there is no velocity, it opposes the applied torque τ_i.
- if the velocity is zero, then
  1. if the magnitude of the applied force is below some threshold γ_{si}, then the frictional force exactly cancels the applied force.
  2. is the magnitude of the applied force is greater than γ_{si} then the magnitude of the frictional force is γ_{si}.
- if the velocity is nonzero, then the Coulomb friction model is used.

This can be written as

$$F_i = \begin{cases} -\min(|\tau_i|, \gamma_{si})\text{sign}(\tau_i) & \text{if} \quad \dot{q}_i \neq 0 \\ -\gamma_{ci}\text{sign}(\dot{q}_i) & \text{if} \quad \dot{q}_i > 0 \end{cases}$$

Usually $\gamma_{si} > \gamma_{ci}$ so that the plot of friction force vs. velocity looks like this:



8 Euler Lagrange Dynamics in $SE(3)$

This section describes and derives the "cookbook" equations for Euler-Lagrange dynamics in $SE(3)$. The cookbook equations are useful to know about, however will not be covered in detail in class.

The Euler-Lagrange equations are a system of n coupled second order ordinary differential equations (ODEs).

We've worked through this for planar robots, now we'll look at the same thing for robots that live in 3D. Conceptually it is the same, but the extra dimension and the two extra rotational degrees of freedom make things a lot more tedious.

So this leads us to another "cookbook" set of equations. The equations that follow are very complicated, but they are also very straightforward to use.

8.1 Kinetic Energy

The kinetic energy of a rigid body in $SE(3)$ moving with velocity

$$\xi = \begin{bmatrix} v \\ \omega \end{bmatrix}$$

is

$$\mathcal{K} = \frac{1}{2} m v^T v + \frac{1}{2} \omega^T \mathcal{I} \omega$$

where m is the mass of the body and $\mathcal{I}$ is a 3×3 matrix called the inertia tensor.

The inertia tensor is defined to be

$$\mathcal{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

where

$$I_{xx} = \int\int\int (y^2 + z^2)\rho(x, y, z) dx\, dy\, dz$$

$$I_{yy} = \int\int\int (x^2 + z^2)\rho(x, y, z) dx\, dy\, dz$$

$$I_{zz} = \int\int\int (x^2 + y^2)\rho(x, y, z) dx\, dy\, dz$$

$$I_{xy} = I_{yx} = -\int\int\int xy\rho(x, y, z) dx\, dy\, dz$$

$$I_{xz} = I_{zx} = -\int\int\int xz\rho(x, y, z) dx\, dy\, dz$$

$$I_{yz} = I_{zy} = -\int\int\int yz\rho(x, y, z) dx\, dy\, dz$$

see page 252 in the Spong et al. book.

Note that $\mathcal{I}$ is always a symmetric matrix.

Note also that the definition of $\mathcal{I}$ depends on the choice of coordinates, so it will be different for different frames.

In particular, in the kinetic energy equations, everything will usually be expressed in some "inertial" (fixed) frame, usually frame 0. Since the rigid body is moving around, the components of the inertia tensor will all change with time, making them very difficult to compute. We'll figure out how to deal with this next.

8.2 Computing $\mathcal{I}^0$

If we compute the inertia with respect to the body frame, then it is constant (since the body does not move in the body frame!). This makes is easier to compute. But how can we relate the body frame inertia tensor $\mathcal{I}^b$ (in the book they just call it I) with the fixed frame inertia tensor $\mathcal{I}^0$ (which is just called $\mathcal{I}$ in the book).

Intuitively, the energy due to rotation should be the same regardless of how the frame is oriented, so we must have

$$\omega^{bT}\mathcal{I}^b\omega^b = \omega^{0T}\mathcal{I}^0\omega^0$$

If R_b^0 describes the orientation of the body then

$$\omega^0 = R_b^0\omega^b$$

or

$$\omega^b = R_b^{0T}\omega^0$$

Subsituting gives

$$\omega^{0T}R_b^0\mathcal{I}^b R_b^{0T}\omega^0 = \omega^{0T}\mathcal{I}^0\omega^0,$$

which means that

$$\mathcal{I}^0 = R_b^0\mathcal{I}^b R_b^{0T}$$

This is good: we can easily compute $\mathcal{I}^b$, then easily turn it into $\mathcal{I}^0$ using the rotation matrix.

8.3 n-link Manipulator

Now lets look at the kinetic energy of an n link manipulator. The total kinetic energy is equal to the sum of the kinetic energies of each link:

$$\mathcal{K}(q,\dot{q}) = \sum_{i=1}^{n}\mathcal{K}_i(q,\dot{q})$$

So we need to figure out $\mathcal{K}_i$ for each link, then we can add them together. There are lots of ways to do this. Here's how they do it in the book:

1. put a frame at the center of mass of link i, parallel to the DH frame i. We'll call this new frame ci.
2. compute the forward kinematics between frame 0 and frame ci, H_{ci}^0.
3. compute the linear and angular velocity Jacobians (J_{vi} and $J_{\omega i}$) that map from joint velocities to velocities at the frame ci.
4. compute the inertia tensor $\mathcal{I}^i$ of the ith body with respect to the frame ci.
5. the kinetic energy $\mathcal{K}_i$ can then be derived as follows:

$$K_i(q,\dot{q}) = \frac{1}{2}m_i(v_{ci}^0)^T v_{ci}^0 + \frac{1}{2}(\omega_{ci}^0)^T\mathcal{I}^0\omega_{ci}^0$$

using the facts that $v_{ci}^0 = J_{vi}\dot{q}$, $\omega_{ci}^0 = J_{\omega i}\dot{q}$, and $\mathcal{I}^0 = R_{ci}^0\mathcal{I}^i(R_{ci}^0)^T$, we get

$$= \frac{1}{2}m_i\dot{q}^T J_{vi}^T J_{vi}\dot{q}\frac{1}{2}\dot{q}^T J_{\omega i}^T R_{ci}^0\mathcal{I}^i(R_{ci}^0)^T J_{\omega i}\dot{q}$$

Summing the individual $\mathcal{K}_i$s and factoring out the common terms gives and expression for the total kinetic energy:

$$\mathcal{K}(q, \dot{q}) = \frac{1}{2}\dot{q}^T \underbrace{\left[\sum_{i=1}^{n} \left(m_i J_{vi}^T J_{vi} + J_{\omega i}^T R_{ci}^0 \mathcal{I}^i (R_{ci}^0)^T J_{\omega i}\right)\right]}_{D(q)} \dot{q}$$

Note that all of the stuff in the middle is a big matrix that depends only on q (not $\dot{q}$). We define all of that stuff to be $D(q)$. $D(q)$ is called the inertia matrix. Sometimes it is called the mass matrix. (But it is not to the same thing as the inertia tensor!).

Note that $D(q)$ is symmetric, i.e., $d_{ij}(q) = d_{ji}(q)$.

With D defined as above, we can express $\mathcal{K}$ simply as

$$\mathcal{K}(q, \dot{q}) = \frac{1}{2}\dot{q}^T D(q)\dot{q}$$

It can also be written as

$$\mathcal{K}(q, \dot{q}) = \frac{1}{2}\sum_{i,j} d_{ij}(q)\dot{q}_i\dot{q}_j,$$

where d_{ij} is the i, jth element of D.

8.4 Potential Energy

In the systems we care about the only source of potential energy is gravity. Namely, the potential energy of the ith link is

$$\mathcal{P}_i(q) = m_i g^T o_{ci}$$

where g is the gravity vector, usually

$$g = \begin{bmatrix} 0 \\ 0 \\ -9.8\text{m/s} \end{bmatrix}$$

The total potential energy is then

$$\mathcal{P}(q) = \sum_{i=1}^{n} m_i g^T o_{ci}$$

8.5 Equations of Motion

Using the expressions for $\mathcal{K}$ and $\mathcal{P}$ derived above we get the Lagrangian for an n-link manipulator:

$$\mathcal{L} = \mathcal{K} - \mathcal{P} = \frac{1}{2}\sum_{i,j} d_{ij}(q)\dot{q}_i\dot{q}_j - P(q)$$

We want to apply the Euler-Lagrange equations to this $\mathcal{L}$ to get the equations of motion. Recall that

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = \tau_k$$

So lets work our way through the terms. The derivative is linear operator so we can move it inside the summation:

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \frac{1}{2}\sum_{i,j}\frac{\partial}{\partial \dot{q}_k} d_{ij}(q)\dot{q}_i\dot{q}_j$$

All of the terms in the summation are zero except when $j = k$ or $i = k$, so

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \frac{1}{2}\sum_{i} d_{ik}\dot{q}_i + \frac{1}{2}\sum_{j} d_{kj}\dot{q}_j.$$

Both i and j are dummy variables, so we can call them whatever we want to. So we can write the first term as a sum over j instead of i:

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \frac{1}{2}\sum_{j} d_{jk}\dot{q}_i + \frac{1}{2}\sum_{j} d_{kj}\dot{q}_j.$$

Finally, since D is symmetric ($d_{jk} = d_{kj}$) we get

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \sum_{j} d_{kj}\dot{q}_j$$

Now take the time derivative:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \sum_{j}\left(\frac{dd_{kj}}{dt}\dot{q}_j + d_{kj}\ddot{q}_j\right)$$

Separating terms and using the chain rule to get $\frac{dd_{kj}}{dt} = \frac{\partial d_{kj}}{\partial q}\dot{q} = \sum_i \frac{\partial d_{kj}}{\partial q_i}\dot{q}_i$ the above expression can be written:

$$= \sum_{j} d_{kj}\ddot{q}_j + \sum_{i,j}\frac{\partial d_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j$$

Now compute

$$\frac{\partial \mathcal{L}}{\partial q_k} = \frac{1}{2}\sum_{i,j}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j - \sum_{i=1}^{n} m_i g^T \frac{\partial o_{ci}}{\partial q_k}$$

Putting it all together, we get n equations of motion:

$$\sum_{j} d_{kj}\ddot{q}_j + \sum_{i,j}\left(\frac{\partial d_{ij}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right)\dot{q}_i\dot{q}_j + \sum_{i=1}^{n} m_i g^T \frac{\partial o_{ci}}{\partial q_k} = \tau_k$$

This gives a set of n equations ($k = 1, 2, \ldots, n$). We can rewrite this set of equations in the *matrix form* as follows:

$$\boxed{D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau}$$

where the elements of the matrix C are

$$C_{kj} = \sum_i \left(\frac{\partial d_{ij}}{\partial q_i} - \frac{1}{2} \frac{\partial d_{ij}}{\partial q_k} \right) \dot{q}_i$$

and G is defined to be

$$G = \left(\frac{\partial \mathcal{P}}{\partial q} \right) = g^T \begin{bmatrix} \sum_{i=1}^n m_i \frac{\partial o_{ci}}{\partial q_1} \\ \sum_{i=2}^n m_i \frac{\partial o_{ci}}{\partial q_2} \\ \vdots \\ \sum_{i=k}^n m_i \frac{\partial o_{ci}}{\partial q_k} \\ \vdots \\ m_n \frac{\partial o_{cn}}{\partial q_n} \end{bmatrix}$$

Christoffel Symbols:

It can be shown that

$$\sum_{i,j} \left(\frac{\partial d_{ij}}{\partial q_i} - \frac{1}{2} \frac{\partial d_{ij}}{\partial q_k} \right) = \sum_{i,j} \underbrace{\frac{1}{2} \left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right)}_{\triangleq c_{ijk}}$$

Where the c_{ijk} are called the *Christoffel symbols*. So an alternate way of writing the Coriolis/centrifugal terms in the equations of motion is

$$C(q, \dot{q})\dot{q} = \sum_{i=1}^n \sum_{j=1}^n c_{ijk}(q) \dot{q}_i \dot{q}_j.$$

Mass matrix Note that the inertia matrix $D(q)$ is exactly the same thing that we called the mass matrix $M(q)$ in the 2D case. The two can be used interchangably, though the terminology "mass matrix" and the use of the letter M is a bit more standard.

8.6 Summary

We now have a very systematic method to find the dynamics of an n-link manipulator. Here is a summary of the process:

1. assign DH frames $\{i\}$, and COM frames $\{ci\}$
2. compute forward kinematics for each COM frame $H_{ci}^0(q)$
3. compute Jacobians for each COM frame J_{vi} and $J_{\omega i}$
4. compute mass matrix $D(q)$ (aka $M(q)$):

$$D(q) = \sum_{i=1}^n \left(m_i J_{vi}^T J_{vi} + J_{\omega i}^T R_{ci}^0 \mathcal{I}^i (R_{ci}^0)^T J_{\omega i} \right)$$

5. compute Christoffel symbols c_{ijk}, $i, j, k = 1, \ldots, n$:

$$c_{ijk} = \frac{1}{2}\left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right)$$

6. compute centrifugal/Coriolis vector $\bar{C}(q, \dot{q})$

$$\bar{C}(q, \dot{q}) = \begin{bmatrix} \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij1}(q)\dot{q}_i\dot{q}_j \\ \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij2}(q)\dot{q}_i\dot{q}_j \\ \vdots \\ \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ijn}(q)\dot{q}_i\dot{q}_j \end{bmatrix}$$

7. compute gravity vector $G(q)$

$$G = \left(\frac{\partial \mathcal{P}}{\partial q}\right) = g^T \begin{bmatrix} \sum_{i=1}^{n} m_i \frac{\partial o_{ci}}{\partial q_1} \\ \sum_{i=2}^{n} m_i \frac{\partial o_{ci}}{\partial q_2} \\ \vdots \\ \sum_{i=k}^{n} m_i \frac{\partial o_{ci}}{\partial q_k} \\ \vdots \\ m_n \frac{\partial o_{cn}}{\partial q_n} \end{bmatrix}$$

8. put it all together to get equations of motion in standard form:

$$D(q)\ddot{q} + \bar{C}(q, \dot{q}) + G(q) = \tau$$