1. а) $$I=\int_0^{\sqrt3}\frac{4x+\operatorname{arctg}^7x+e^{\operatorname{arctg}x}}{2x^2+2}\,dx=\underbrace{\int_0^{\sqrt3}\frac{4x}{2(x^2+1)}\,dx}_{I_1}+\underbrace{\int_0^{\sqrt3}\frac{\operatorname{arctg}^7x+e^{\operatorname{arctg}x}}{2(x^2+1)}\,dx}_{I_2}$$

$$I_1=\int_0^{\sqrt3}\frac{2x}{x^2+1}\,dx=\left(\begin{array}{l}t=x^2+1\\ dt=2x\,dx\end{array}\quad\begin{array}{c|c|c}x&0&\sqrt3\\\hline t&1&4\end{array}\right)=\int_1^4\frac{dt}{t}=\ln|t|\Big|_1^4=\ln4=2\ln2$$

$$I_2=\left(\begin{array}{l}t=\operatorname{arctg}x\\ dt=\dfrac{dx}{x^2+1}\end{array}\quad\begin{array}{c|c|c}x&0&\sqrt3\\\hline t&0&\frac{\pi}{3}\end{array}\right)=\int_0^{\frac{\pi}{3}}\frac{t^7+e^t}{2}\,dt=\frac12\left(\frac{t^8}{8}+e^t\right)\Big|_0^{\frac{\pi}{3}}=$$

$$=\frac1{16}\cdot\left(\frac{\pi}{3}\right)^8+\frac12\cdot e^{\frac{\pi}{3}}-\left(\frac12\cdot\frac{0^8}{8}+\frac12\cdot e^0\right)=\frac1{16}\left(\frac{\pi}{3}\right)^8+\frac12e^{\frac{\pi}{3}}-\frac12$$

$$\Rightarrow\quad I=2\ln2+\frac1{16}\left(\frac{\pi}{3}\right)^8+\frac12e^{\frac{\pi}{3}}-\frac12$$

б) $$\int\frac{x^2}{\sqrt{x^2+1}}\,dx=\left(\begin{array}{l}x=\operatorname{sh}t\\ dx=\operatorname{ch}t\,dt\end{array}\quad\begin{array}{l}t=\operatorname{arcsh}x=\ln\left(x+\sqrt{x^2+1}\right)\\ \sqrt{x^2+1}=\sqrt{\operatorname{ch}^2t}=\operatorname{ch}t\end{array}\right)=$$

$$=\int\frac{\operatorname{sh}^2t}{\operatorname{ch}t}\cdot\operatorname{ch}t\,dt=\int\operatorname{sh}^2t\,dt=\int\left(\frac{e^t-e^{-t}}{2}\right)^2dt=$$

$$=\frac14\int\left(e^{2t}-2\underbrace{e^t\cdot e^{-t}}_{1}+e^{-2t}\right)dt=\frac14\int\left(e^{2t}-2+e^{-2t}\right)dt=$$

$$=\frac14\left(\frac12e^{2t}-2t-\frac12e^{-2t}\right)=\frac18\left(e^{2t}-e^{-2t}\right)-\frac{t}{2}=$$

$$=\frac18\left(e^{2\ln\left(x+\sqrt{x^2+1}\right)}-e^{-2\ln\left(x+\sqrt{x^2+1}\right)}\right)-\frac{\ln\left(x+\sqrt{x^2+1}\right)}{2}+C$$

овај израз може и да се среди

$$e^{2t}=\left(e^t\right)^2=\left(e^{\ln\left(x+\sqrt{x^2+1}\right)}\right)^2=\left(x+\sqrt{x^2+1}\right)^2=x^2+2x\sqrt{x^2+1}+x^2+1=$$
$$=2x^2+1+2x\sqrt{x^2+1}$$

$$e^{-2t}=\frac1{e^{2t}}=\left(\frac1{e^t}\right)^2=\left(\frac1{x+\sqrt{x^2+1}}\right)^2=\left(\frac1{x+\sqrt{x^2+1}}\cdot\frac{x-\sqrt{x^2+1}}{x-\sqrt{x^2+1}}\right)^2=$$

$$=\left(\frac{x-\sqrt{x^2+1}}{x^2-(x^2+1)}\right)^2=\left(\sqrt{x^2+1}-x\right)^2=x^2+1-2x\sqrt{x^2+1}+x^2=2x^2+1-2x\sqrt{x^2+1}$$

$$\Rightarrow \quad e^{2t} - e^{-2t} = 4x\sqrt{x^2+1}$$

$$\Rightarrow \quad \int \frac{x^2}{\sqrt{x^2+1}}\,dx = \frac{1}{8}\cdot 4x\sqrt{x^2+1} - \frac{\ln\left(x+\sqrt{x^2+1}\right)}{2} = \frac{x\sqrt{x^2+1} - \ln\left(x+\sqrt{x^2+1}\right)}{2} + C$$

2. начин:

$$\int \frac{x^2}{\sqrt{x^2+1}}\,dx = \left(\begin{array}{ll} x = \operatorname{tg} t & \sqrt{x^2+1} = \sqrt{\frac{\sin^2 t}{\cos^2 t} + 1} = \sqrt{\frac{1}{\cos^2 t}} \\ dx = \frac{dt}{\cos^2 t} & = \frac{1}{\cos t} \end{array} \right) =$$

$$= \int \frac{\operatorname{tg}^2 t}{\frac{1}{\cos t}} \cdot \frac{dt}{\cos^2 t} = \int \frac{\frac{\sin^2 t}{\cos^2 t}}{\cos t}\,dt = \int \frac{\sin^2 t}{\cos^3 t}\,dt \overset{\text{непарна по } \cos t}{=}$$

$$= \int \frac{\sin^2 t \cos t}{\cos^4 t}\,dt = \left(\begin{array}{l} u = \sin t \\ du = \cos t\,dt \end{array} \right) = \int \frac{u^2\,du}{(1-u^2)^2} =$$

$$= \int \frac{u^2}{(u-1)^2(u+1)^2}\,du = \int \frac{A}{u-1}\,du + \int \frac{B}{(u-1)^2}\,du + \int \frac{C}{u+1}\,du + \int \frac{D}{(u+1)^2}\,du \quad \ldots$$

б)

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x\ln(x^2+1) + \cos^3 x}{4|\sin x| + \cos^2 x - 5}\,dx = \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x\ln(x^2+1)}{4|\sin x| + \cos^2 x - 5}\,dx}_{\text{непарна}} + \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^3 x}{4|\sin x| + \cos^2 x - 5}\,dx}_{\text{парна}}$$

непарна, парна, парна, парна, парна

$$= 0 + 2\int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{4|\sin x| + \cos^2 x - 5}\,dx \overset{\sin x \geq 0 \text{ на } [0, \frac{\pi}{2}]}{=} 2\int_0^{\pi/2} \frac{\cos^3 x}{4\sin x + \cos^2 x - 5}\,dx \overset{\text{непарна по } \cos x}{=}$$

$$= \left(\begin{array}{l} t = \sin x \\ dt = \cos x\,dx \end{array} \quad \begin{array}{c|c|c} x & 0 & \pi/2 \\ \hline t & 0 & 1 \end{array} \right) = 2\int_0^1 \frac{(1-t^2)\,dt}{4t + 1 - t^2 - 5} = 2\int_0^1 \frac{1-t^2}{-(t^2-4t+4)}\,dt$$

$$= 2\int_0^1 \frac{t^2-1}{t^2-4t+4}\,dt = 2\int_0^1 \left(1 + \frac{4t-5}{t^2-4t+4}\right) dt = \underbrace{2\int_0^1 dt}_{2t\big|_0^1 = 2} + 2\int_0^1 \frac{4t-5}{(t-2)^2}\,dt$$

$$(t^2-1) : (t^2-4t+4) = 1$$

$$\frac{-(t^2-4t+4)}{4t-5}$$

$$\int_0^1 \frac{4t-5}{(t-2)^2}\,dt = \int_0^1 \frac{A}{t-2}\,dt + \int_0^1 \frac{B}{(t-2)^2}$$

$$\frac{4t-5}{(t-2)^2} = \frac{A}{t-2} + \frac{B}{(t-2)^2} \quad \Big/ \cdot (t-2)^2$$

$$4t-5 = A(t-2)+B$$

$$4t-5 = At-2A+B$$

$$A=4$$

$$-2A+B=-5 \implies B=3$$

$$\implies \int_0^1 \frac{4t-5}{(t-2)^2}\,dt = 4\int_0^1 \frac{1}{t-2}\,dt + 3\int_0^1 \frac{dt}{(t-2)^2} = \left(\begin{matrix} s=t-2 \\ ds=dt \end{matrix} \quad \begin{array}{c|c|c} t & 0 & 1 \\ \hline s & -2 & -1 \end{array} \right) =$$

$$= 4\int_{-2}^{-1} \frac{ds}{s} + 3\int_{-2}^{-1} \frac{ds}{s^2} = 4\cdot \ln|s|\Big|_{-2}^{-1} + 3\cdot \frac{-1}{s}\Big|_{-2}^{-1} =$$

$$= 4(\ln 1 - \ln 2) - 3\cdot\left(\frac{1}{-1} - \frac{1}{-2}\right) = -4\ln 2 - 3\cdot\left(-\frac{1}{2}\right) = \frac{3}{2} - 4\ln 2$$

$$\implies I = 2 + 2\left(\frac{3}{2} - 4\ln 2\right) = 5 - 8\ln 2$$

2. б)

$$I = \int_0^\pi \underbrace{\frac{(\pi-x)\cdot\sqrt[3]{1-\cos x}}{\sqrt{2x+x^3}\cdot\sin x}}_{f(x)} = \underbrace{\int_0^1 f(x)\,dx}_{I_1} + \underbrace{\int_1^\pi f(x)\,dx}_{I_2}$$

сингуларитети $\quad 2x+x^3=0 \quad , \quad \sin x = 0$

$$\Downarrow \qquad\qquad \Downarrow$$

$$x(2+x^2)=0 \qquad x \in \{0,\pi\}$$

$$\Downarrow$$

$$x=0 \qquad \implies \boxed{0 \text{ и } \pi}$$

I_1: $\quad \pi - x \sim \overset{=\text{const}\neq 0}{\pi}, \; x\to 0+$

$$1-\cos x \sim 1-\left(1-\frac{x^2}{2}\right) = \frac{x^2}{2}, \; x\to 0+$$

$$\sqrt[3]{1-\cos x} \sim \sqrt[3]{\frac{x^2}{2}} = \underset{\neq 0}{const} \cdot x^{\frac{2}{3}}, \quad x \to 0+$$

$$\sqrt{2x+x^3} = \sqrt{x\underbrace{(2+x^2)}_{\downarrow 2}} \sim \sqrt{2x} = \underset{\neq 0}{const} \cdot x^{\frac{1}{2}}, \quad x \to 0+$$

$$\sin x \sim x, \quad x \to 0+$$

$$\Rightarrow \quad f(x) \sim \underset{\neq 0}{const} \cdot \frac{x^{\frac{2}{3}}}{x^{\frac{1}{2}} \cdot x} = \underset{\neq 0}{const} \cdot \frac{1}{x^{5/6}}, \quad x \to 0+$$

$$\int_0^1 \frac{1}{x^{5/6}}\,dx \quad (K) \quad \left(\frac{5}{6}<1\right) \overset{\text{II } \bar{u}.\text{-k.}}{\Rightarrow} \quad I_1 \quad (K)$$

$$I_2 = \left(\begin{array}{l} t = \pi - x \\ dt = -dx \end{array} \quad \begin{array}{c|c|c} x & 1 & \pi \\ \hline t & \pi-1 & 0 \end{array} \right) = \int_{\pi-1}^{0} \frac{t\sqrt[3]{1-\cos(\pi-t)}}{\sqrt{2(\pi-t)+(\pi-t)^3} \cdot \sin(\pi-t)}(-dt) =$$

$$= \int_0^{\pi-1} \underbrace{\frac{t \cdot \sqrt[3]{1+\cos t}}{\sqrt{2(\pi-t)+(\pi-t)^3} \cdot \sin t}}_{g(t)}\,dt$$

$$\sqrt[3]{1+\cos t} \sim \sqrt[3]{1+1} = \sqrt[3]{2} \overset{= const \neq 0}{}, \quad t \to 0+$$

$$\sqrt{2(\pi-t)+(\pi-t)^3} \sim \sqrt{2\pi+\pi^3} = const \neq 0, \quad t \to 0+$$

$$\sin t \sim t, \quad t \to 0+$$

$$g(t) \sim \underset{\neq 0}{const} \cdot \frac{t}{t} = \underset{\neq 0}{const} \cdot \frac{1}{t^0}, \quad t \to 0+$$

$$\int_0^{\pi-1} \frac{1}{t^0}\,dt \quad (K) \quad (0<1) \overset{\text{II } \bar{u}.\text{-k.}}{\Rightarrow} \quad I_2 \quad (K)$$

$$I_1 \ (K) \ \text{u} \ I_2 \ (K) \quad \Rightarrow \quad I \ (K)$$

a) $$I = \int_1^{+\infty} \frac{\ln(x+1)}{x^3}\,dx$$

$$\ln(x+1) = \ln\left(x\left(1+\frac{1}{x}\right)\right) = \ln x + \underbrace{\ln\left(1+\frac{1}{x}\right)}_{\downarrow 0} \sim \ln x, \quad x \to +\infty$$

$$\Rightarrow \quad \frac{\ln(x+1)}{x^3} \sim \frac{1}{x^3 \ln^{-1} x}, \; x \to +\infty \quad \text{u} \quad \int \frac{dx}{x^3 \ln^{-1} x} \; Ⓚ \quad (3>1)$$

II ū-k.

$$\Rightarrow \quad I \; Ⓚ$$

$$I = \lim_{\beta\to+\infty} \int_1^\beta \frac{\ln(x+1)}{x^3}\,dx = \left(\begin{array}{ll} u = \ln(x+1) & d\vartheta = \frac{dx}{x^3} = x^{-3}dx \\ du = \frac{dx}{x+1} & \vartheta = \int x^{-3}\,dx = -\frac{1}{2x^2} \end{array} \right) =$$

$$= \lim_{\beta\to+\infty} \left(-\frac{\ln(x+1)}{2x^2}\Big|_1^\beta + \frac{1}{2}\int_1^\beta \frac{dx}{x^2(x+1)} \right)$$

$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} \;\Big/ \cdot x^2(x+1)$$

$$1 = Ax(x+1) + B(x+1) + Cx^2$$

$$1 = (A+C)x^2 + (A+B)x + B$$

$$A+C=0$$
$$A+B=0 \qquad \Rightarrow C=1$$
$$B=1 \qquad \Rightarrow A=-1$$

$$\Rightarrow \quad I = \lim_{\beta\to+\infty} \left(-\frac{\ln(x+1)}{2x^2}\Big|_1^\beta + \frac{1}{2}\left(-\int_1^\beta \frac{dx}{x} + \int_1^\beta \frac{dx}{x^2} + \int_1^\beta \frac{dx}{x+1} \right) \right) =$$

$$= \lim_{\beta\to+\infty} \left(-\frac{\ln(\beta+1)}{2\beta^2} - \frac{-\ln 2}{2} + \frac{1}{2}\left(-\ln|x|\Big|_1^\beta - \frac{1}{x}\Big|_1^\beta + \ln|x+1|\Big|_1^\beta \right) \right) =$$

$$= \lim_{\beta\to+\infty} \left(-\frac{\ln(\beta+1)}{2\beta^2} + \frac{\ln 2}{2} + \frac{1}{2}\left(-\ln\beta + \cancel{\ln 1}^{\,0} - \frac{1}{\beta} + 1 + \ln(\beta+1) - \ln 2 \right) \right) =$$

$$= \lim_{\beta\to+\infty} \left(-\underbrace{\frac{\ln(\beta+1)}{2\beta^2}}_{\to 0} + \frac{\ln 2}{2} + \frac{1}{2}\left(\underbrace{\ln \underbrace{\frac{\beta+1}{\beta}}_{\to 1}}_{\to 0} + 1 - \ln 2 \right) \right) =$$

(β^2 jače od $\ln(\beta+1)$)

$$= 0 + \frac{\ln 2}{2} + \frac{1-\ln 2}{2} = \frac{1}{2}$$

3. a) $$\underbrace{\sum_{n=1}^{\infty} \left(\frac{\ln n}{\ln(n+1)} \right)^{\frac{1}{n^2}}}$$ $\qquad a_n > 0 \Rightarrow |a_n| = a_n$

a_n

$$\ln(n+1) = \ln\left(n\left(1+\frac{1}{n}\right)\right) = \ln n + \overbrace{\ln\left(1+\frac{1}{n}\right)}^{\to 0} \sim \ln n, \; n\to\infty$$

$$\Rightarrow \quad \frac{\ln n}{\ln(n+1)} \sim 1, \; n\to\infty$$

$$\frac{1}{n^2} \sim 0, \; n\to\infty \qquad \Rightarrow \quad a_n \sim 1^0 = 1 \neq 0, \; n\to\infty$$

$$\Rightarrow \quad a_n \not\to 0, \; n\to\infty$$

$\Rightarrow \quad \sum a_n$ (Д) и апсолутно дивергира, па не конвергира ни условно

б) $$\sum_{n=1}^{\infty} \underbrace{\frac{n \operatorname{arctg}\frac{1}{n^2}}{\ln(n+8)}}_{a_n} \qquad a_n > 0 \Rightarrow |a_n| = a_n$$

$$\frac{1}{n^2} \to 0 \quad \Rightarrow \quad \operatorname{arctg}\frac{1}{n^2} \sim \frac{1}{n^2}, \; n\to\infty$$

$$\ln(n+8) \sim \ln n, \; n\to\infty$$

$$\Rightarrow \quad a_n \sim \frac{n\cdot\frac{1}{n^2}}{\ln n} = \frac{1}{n^1\cdot \ln^1 n}, \; n\to\infty$$

$\Rightarrow \quad \sum a_n$ (Д) и апс. дивергира, па не конв. ни условно

в) $$\sum_{n=1}^{\infty} \underbrace{(-1)^n \cdot \frac{2^{n+3}\, n!}{(n+5)^n}}_{a_n}, \qquad |a_n| = \frac{2^{n+3}\cdot n!}{(n+5)^n}$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{\overset{2}{\cancel{2^{n+4}}}\cdot \overset{n+1}{\cancel{(n+1)!}}}{(n+6)^{n+1}}}{\frac{\cancel{2^{n+3}}\cdot \cancel{n!}}{(n+5)^n}} = \lim_{n\to\infty} \frac{2(n+1)(n+5)^n}{(n+6)^{n+1}} =$$

$$= \lim_{n\to\infty} \frac{2(n+1)\cdot(n+5)^n}{(n+6)\cdot(n+6)^n} = \lim_{n\to\infty} 2\cdot \overset{\to 1}{\frac{n+1}{n+6}} \cdot \left(\frac{n+5}{n+6}\right)^n =$$

$$= 2\lim_{n\to\infty}\left(1-\frac{1}{n+6}\right)^n = 2\lim_{n\to\infty}\left(\left(1+\frac{1}{-(n+6)}\right)^{-(n+6)}\right)^{-\frac{n}{n+6}} =$$

$$= 2\lim_{n\to\infty} e^{-\frac{n}{n+6}} = 2e^{-1} = \frac{2}{e} < 1 \overset{\text{Даламбер}}{\Longrightarrow} \sum_{n=1}^{\infty} |a_n| \text{ (К)}$$

$\Rightarrow \sum a_n$ (АК) $\Rightarrow \sum a_n$ (К) $\Rightarrow \sum a_n$ не конв. условно

2. начин: Стирлинг $n! = n^n e^{-n}\sqrt{2\pi n}$, па Коши: $L = \frac{2}{e} < 1$

4. а) $$\sum_{n=1}^{\infty} \underbrace{\frac{n^2}{n^3+9}}_{a_n}(x+4)^n \qquad x_0 = -4$$

$$R = \frac{1}{\limsup\limits_{n\to\infty}\sqrt[n]{|a_n|}} = 1$$

$$\limsup_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\frac{n^2}{n^3+9}} = \lim_{n\to\infty}\sqrt[n]{\frac{n^2}{n^3(1+\underset{\to 0}{\frac{9}{n^3}})}} =$$

$$= \lim_{n\to\infty}\frac{1}{\sqrt[n]{n}} = \frac{1}{1} = 1$$

$$\Rightarrow (-5,-3) \subseteq D \subseteq [-5,-3]$$

$x=-3$: $$\sum_{n=1}^{\infty}\frac{n^2}{n^3+9}\cdot 1^n = \sum_{n=1}^{\infty}\frac{n^2}{n^3+9} \text{ (Д)} \quad \left(\frac{n^2}{n^3+9} \sim \frac{1}{n}\right)$$

$$\Rightarrow -3 \notin D$$

$\underline{x=-5:}$ $\quad \sum_{n=1}^{\infty} \underbrace{\frac{n^2}{n^3+9}}_{a_n} \cdot (-1)^n \qquad a_n \sim \frac{1}{n} \to 0,\ n\to\infty$

a_n опада?

$$f(x)=\frac{x^2}{x^3+9}, \quad f:[1,+\infty)\to\mathbb{R}$$

$$f'(x)=\frac{2x(x^3+9)-x^2\cdot 3x^2}{(x^3+9)^2}=\frac{18x-x^4}{(x^3+9)^2}=$$

$$=\frac{\overset{>0}{x}(18-x^3)}{\underbrace{(x^3+9)^2}_{>0}} < 0 \quad \text{за} \quad x > \sqrt[3]{18}$$

$\Rightarrow (a_n) \downarrow$ почев. од 3. члана

Лајбниц $\Rightarrow$ $\sum_{n=1}^{\infty} (-1)^n a_n$ (К) $\Rightarrow -5 \in D$

$\Rightarrow D=[-5,-3)$

б) $\sum_{n=1}^{\infty} \underbrace{\frac{e^{-2nx^2}}{n(n+x)}}_{f_n(x)}$

$n\in\mathbb{N}$ и $x\in[0,+\infty)$ произвољни:

$$|f_n(x)| = \frac{e^{-2nx^2}}{n(n+x)} \leq \frac{1}{n^2} =: c_n$$

$$\uparrow$$

$-2nx^2 \leq 0 \Rightarrow e^{-2nx^2} \leq e^0 = 1$

$n+x \geq n$

и $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ (К) $(2>1)$

Вајерштрас $\Rightarrow$ $\sum_{n=1}^{\infty} f_n(x)$ РК на $[0,+\infty)$

$$\lim_{x\to 0+} f_n(x) = \lim_{x\to 0+} \frac{e^{-2nx^2}}{n(n+x)} = \frac{e^{-2n\cdot 0}}{n\cdot n} = \frac{1}{n^2}$$

$$\lim_{x\to +\infty} f_n(x) = \lim_{x\to +\infty} \frac{e^{\overbrace{-2nx^2}^{\to -\infty}}}{\underbrace{n(n+x)}_{\to +\infty}} = \frac{e^{-\infty}}{n\cdot\infty} = \frac{0}{\infty} = 0$$

T.

$$\Rightarrow \lim_{x\to 0+} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \lim_{x\to 0+} f_n(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\lim_{x\to +\infty} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \lim_{x\to +\infty} f_n(x) = \sum_{n=1}^{\infty} 0 = 0$$