Neodređeni integrali

Definicija. Za funkciju $F : I \to \mathbb{R}$, gde je I interval, kažemo da je *primitivna funkcija* funkcije $f : I \to \mathbb{R}$ ako je

$$F'(x) = f(x)$$

za svako $x \in I$.

Teorema 1. Ako je $F : I \to \mathbb{R}$ primitivna funkcija za $f : I \to \mathbb{R}$, tada je i funkcija $F(x) + C,\ C \in \mathbb{R}$, takođe primitivna funkcija funkcije f.

Definicija. Skup svih primitivnih funkcija funkcije f naziva se *neodređeni integral* funkcije f i označava sa

$$\int f(x)dx.$$

Osnovne osobine neodređenog integrala:

1^o $\big(\int f(x)dx\big)' = f(x)\,,$

2^o $\int F'(x)dx = F(x) + C\,,$

3^o $\int \lambda f(x)dx = \lambda \int f(x)dx,\ \lambda \in \mathbb{R} \setminus \{0\}\,,$

4^o $\int \big(f(x) \pm g(x)\big)dx = \int f(x)dx \pm \int g(x)dx\,.$

Čuvajmo drveće. Nemojte štampati ovaj materijal, ukoliko to nije neophodno.

Tablica integrala

$\int x^n dx = \dfrac{x^{n+1}}{n+1} + C,\ n \neq -1$	$\displaystyle\int \frac{dx}{x} = \ln\lvert x\rvert + C$
$\displaystyle\int \frac{dx}{1+x^2} = \begin{cases} \arctan x + C, \\ -\operatorname{arcctg} x + C \end{cases}$	$\displaystyle\int \frac{dx}{x^2-1} = \frac{1}{2}\ln\left\lvert\frac{x-1}{x+1}\right\rvert + C$
$\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \begin{cases} \arcsin x + C, \\ -\arccos x + C \end{cases}$	$\displaystyle\int \frac{dx}{\sqrt{x^2 \pm 1}} = \ln\lvert x + \sqrt{x^2 \pm 1}\rvert + C$
$\int e^x dx = e^x + C$	$\int a^x dx = \dfrac{a^x}{\ln a} + C,\ a > 0,\ a \neq 1$
$\int \sin x\, dx = -\cos x + C$	$\int \cos x\, dx = \sin x + C$
$\displaystyle\int \frac{dx}{\sin^2 x} = -\cot x + C$	$\displaystyle\int \frac{dx}{\cos^2 x} = \tan x + C$
$\int \sinh x\, dx = \cosh x + C$	$\int \cosh x\, dx = \sinh x + C$
$\displaystyle\int \frac{dx}{\sinh^2 x} = -\coth x + C$	$\displaystyle\int \frac{dx}{\cosh^2 x} = \tanh x + C$

Osnovne metode integracije:

METOD SMENE

Ako je $\int f(x)dx = F(x) + C$, tada je $\int f(u)du = F(u) + C$. Uzmimo da je $x = \varphi(t)$, gde je φ neprekidna funkcija zajedno sa svojim izvodom φ'. Tada

$$\int f\big(\varphi(t)\big)\varphi'(t)dt = F\big(\varphi(t)\big) + C.$$

METOD PARCIJALNE INTEGRACIJE

Ako su u i v diferencijabilne funkcije i funkcija uv' ima primitivnu funkciju, tada je

$$\int udv = uv - \int vdu.$$

Primeri najčešće korišćenih smena:

$dx = \dfrac{1}{a}\, d(ax+b),\ a \neq 0,$	smena: $t = ax+b,\ dt = a\,dx,$
$xdx = \dfrac{1}{2}\, d(x^2),$	smena: $t = x^2,\ dt = 2xdx,$
$xdx = \dfrac{1}{2a}\, d(ax^2+b),\ a \neq 0,$	smena: $t = ax^2+b,\ dt = 2axdx,$
$x^{n-1}dx = \dfrac{1}{n}\, d(x^n),\ n \neq 0,$	smena: $t = x^n,\ dt = nx^{n-1}dx,$
$\dfrac{dx}{\sqrt{x}} = 2\, d(\sqrt{x}),$	smena: $t = \sqrt{x},\ 2t\, dt = dx,$
$\dfrac{dx}{\sqrt{ax+b}} = \dfrac{2}{a}\, d(\sqrt{ax+b}),\ a \neq 0,$	smena: $t = \sqrt{ax+b},\ \dfrac{2}{a} t\, dt = dx,$
$\dfrac{dx}{x} = d(\ln x),$	smena: $t = \ln x,\ dt = \dfrac{dx}{x},$
$\dfrac{dx}{x} = d\big(\ln ax\big),\ a \neq 0,$	smena: $t = \ln ax,\ dt = \dfrac{dx}{x},$
$e^x dx = d\big(e^x\big),$	smena: $t = e^x,\ dt = e^x dx,$
$e^{ax} dx = d\big(\frac{1}{a} e^{ax}\big),\ a \neq 0,$	smena: $t = e^{ax},\ dt = ae^{ax} dx,$
$\sin x\, dx = -d(\cos x),$	smena: $t = \cos x,\ dt = -\sin x\, dx,$
$\cos x\, dx = d(\sin x),$	smena: $t = \sin x,\ dt = \cos x\, dx,$
$\dfrac{dx}{\cos^2 x} = d(\tan x),$	smena: $t = \tan x,\ \dfrac{dt}{1+t^2} = dx,$
$\dfrac{dx}{\sin^2 x} = -d(\cot x),$	smena: $t = \cot x,\ -\dfrac{dt}{1+t^2} = dx,$
$\sinh x\, dx = d(\cosh x),$	smena: $t = \cosh x,\ dt = \sinh x\, dx,$
$\cosh x\, dx = d(\sinh x),$	smena: $t = \sinh x,\ dt = \cosh x\, dx,$
$\dfrac{dx}{\cosh^2 x} = d(\tanh x),$	smena: $t = \tanh x,\ \dfrac{dt}{t^2-1} = dx,$
$\dfrac{dx}{\sinh^2 x} = -d(\coth x),$	smena: $t = \coth x,\ \dfrac{dt}{1-t^2} = dx.$

METOD REKURZIVNIH FORMULA

Određivanje integrala $I_n = \int f_n(x)dx$, funkcija koje zavise od celobrojnog parametra n, moguće je primenom parcijalne integracije ili nekog drugog metoda, svesti na izračunavanja integrala $I_m,\ 0 \le m < n$, istog tipa. Takve relacije, oblika

$$I_n = \Phi(I_{n-1}, \ldots, I_{n-k}),$$

zovemo *rekurzivne formule*. Da bi se na osnovu njih odredila vrednost integrala I_n, neophodno je poznavati uzastopnih k integrala $I_{n-1}, \ldots, I_{n-k}$, kao i prvih k integrala $I_0, \ldots, I_{k-1}$.

INTEGRALI SA KVADRATNIM TRINOMOM

1. $I = \displaystyle\int \frac{mx+n}{ax^2+bx+c}\,dx$

$$m=0: \ I = \int \frac{n\,dx}{ax^2+bx+c} = \frac{n}{a}\int \frac{dx}{x^2+\frac{b}{a}x+\frac{c}{a}} = \frac{n}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2+\frac{c}{a}-\frac{b^2}{4a^2}}$$
$$= \left| t = x+\tfrac{b}{2a},\ h^2 = \left|\tfrac{c}{a}-\tfrac{b^2}{4a^2}\right| \right| = \frac{n}{a}\int \frac{dt}{t^2 \pm h^2}.$$

$$m \ne 0: \ I = m\int \frac{xdx}{ax^2+bx+c} + n\int \frac{dx}{ax^2+bx+c} = mI_1 + nI_2;$$

$$\begin{aligned} I_1 &= \int \frac{xdx}{ax^2+bx+c} = \frac{1}{2a}\int \frac{2ax\,dx}{ax^2+bx+c} \\ &= \frac{1}{2a}\int \frac{2ax+b-b}{ax^2+bx+c}\,dx = \frac{1}{2a}\int \frac{d(ax^2+bx+c)}{ax^2+bx+c} - \frac{b}{2a}I_2 \\ &= \frac{1}{2a}\ln|ax^2+bx+c| - \frac{b}{2a}I_2. \end{aligned}$$

Izračunavanje integrala I_2 opisano je u prethodnom slučaju ($m=0$).

2. $I = \displaystyle\int \frac{mx+n}{\sqrt{ax^2+bx+c}}\,dx$, analogno tipu integrala pod 1.

3. $I = \displaystyle\int \frac{dx}{(mx+n)\sqrt{ax^2+bx+c}}$, smenom $mx+n = \dfrac{1}{t}$ svodi se na slučaj 2.

4. $I = \displaystyle\int \sqrt{ax^2+bx+c}\,dx$, rešava se dovođenjem na potpun kvadrat izraza pod korenom

$$ax^2+bx+c = a\left(\left(x+\frac{b}{2a}\right)^2+\frac{c}{a}-\frac{b^2}{4a^2}\right) = a\left(t^2 \pm h^2\right),\ t = x+\frac{b}{2a},\ h^2 = \left|\frac{c}{a}-\frac{b^2}{4a^2}\right|.$$

INTEGRACIJA RACIONALNIH FUNKCIJA

Funkcija $\dfrac{P(x)}{Q(x)}$, $P(x)$ i $Q(x)$ su polinomi, jeste prava racionalna funkcija ukoliko je stepen polinoma $P(x)$ manji od stepena polinoma $Q(x)$.

Prava racionalna funkcija

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{\prod\limits_{i=1}^{n}(x-a_i)^{s_i}\cdot\prod\limits_{i=1}^{m}\left(x^2+p_ix+q_i\right)^{k_i}},$$

gde su nule kvadratnih trinoma $x^2+p_ix+q_i,\ i=1,\dots,m$ kompleksne, ima razvoj

$$\begin{aligned}
\frac{P(x)}{Q(x)} \quad = \quad & \frac{A_{11}}{x-a_1}+\frac{A_{12}}{(x-a_1)^2}+\cdots+\frac{A_{1s_1}}{(x-a_1)^{s_1}}\\
&+\frac{A_{21}}{x-a_2}+\frac{A_{22}}{(x-a_2)^2}+\cdots+\frac{A_{2s_2}}{(x-a_2)^{s_2}}\\
&+\cdots\\
&+\frac{A_{n1}}{x-a_n}+\frac{A_{n2}}{(x-a_n)^2}+\cdots+\frac{A_{ns_n}}{(x-a_n)^{s_n}}\\
&+\frac{B_{11}x+C_{11}}{x^2+p_1x+q_1}+\frac{B_{12}x+C_{12}}{(x^2+p_1x+q_1)^2}+\cdots+\frac{B_{1k_1}x+C_{1k_1}}{(x^2+p_1x+q_1)^{k_1}}\\
&+\frac{B_{21}x+C_{21}}{x^2+p_2x+q_2}+\frac{B_{22}x+C_{22}}{(x^2+p_2x+q_2)^2}+\cdots+\frac{B_{2k_2}x+C_{2k_2}}{(x^2+p_2x+q_2)^{k_2}}\\
&+\cdots\\
&+\frac{B_{m1}x+C_{m1}}{x^2+p_mx+q_m}+\frac{B_{m2}x+C_{m2}}{(x^2+p_mx+q_m)^2}+\cdots+\frac{B_{mk_m}x+C_{mk_m}}{(x^2+p_mx+q_m)^{k_m}}.
\end{aligned}$$

Konstante $A_{ij},\ B_{ij},\ C_{ij}$ nalazimo metodom neodređenih koeficijenata.

INTEGRACIJA IRACIONALNIH FUNKCIJA:

1. Integral $\displaystyle\int R\Big[x,\Big(\frac{ax+b}{cx+d}\Big)^{p_1/q_1},\Big(\frac{ax+b}{cx+d}\Big)^{p_2/q_2},\dots\Big]dx$, gde je R neka racionalna funkcija i $p_i\in\mathbb{Z},\ q_i\in\mathbb{N}$, uvođenjem smene

$$\frac{ax+b}{cx+d}=t^n,\ n=\mathrm{NZS}\{q_1,q_2,\dots\},$$

svodi se na integral racionalne funkcije.

2. $$\int \frac{P_n(x)dx}{\sqrt{ax^2+bx+c}} = Q_{n-1}(x)\sqrt{ax^2+bx+c} + \lambda \int \frac{dx}{\sqrt{ax^2+bx+c}},$$

   koeficijente polinoma $Q_{n-1}(x)$ i λ određujemo metodom neodređenih koeficijenata nakon diferenciranja gornje jednakosti.

3. Integral $\displaystyle\int \frac{dx}{(x+\alpha)^n\sqrt{ax^2+bx+c}}$ smenom $x+\alpha = \dfrac{1}{t}$ svodi se na slučaj pod 2.

4. Integral $\displaystyle\int R(x, \sqrt{ax^2+bx+c})dx$, gde je R neka racionalna funkcija, primenom Ojlerovih ili trigonometrijskih/hiperboličkih smena svodi se na integral racionalne funkcije.

   Ojlerove smene:

   I) Ukoliko je $a > 0$, uvodimo smenu $\sqrt{ax^2+bx+c} = t \pm \sqrt{a}\,x$;
   II) Kada je $c \geq 0$, smena glasi $\sqrt{ax^2+bx+c} = xt \pm \sqrt{c}$;
   III) U slučaju $ax^2+bx+c = a(x-x_1)(x-x_2),\ x_1, x_2 \in \mathbb{R}$, koristimo smenu $\sqrt{ax^2+bx+c} = t(x-x_1)$.

   Trigonometrijske i hiperboličke smene:

   (a) ako se u integralu javi $\sqrt{a^2-x^2}$, smena: $x = a\sin t$ ili $x = a\cos t$;
   (b) ako se u integralu javi $\sqrt{a^2+x^2}$, smena: $x = a\tan t$ ili $x = a\sinh t$;
   (c) ako se u integralu javi $\sqrt{x^2-a^2}$, smena: $x = \dfrac{a}{\sin t}$ ili $x = \dfrac{a}{\cos t}$ ili $x = a\cosh t$.

5. Integracija binomnog diferencijala:

   $I = \int x^r\big(a+bx^q\big)^p dx$, za $a, b \in \mathbb{R},\ p, q, r \in \mathbb{Q}$

   a) U slučaju $p \in \mathbb{Z},\ q = \dfrac{q_1}{m},\ r = \dfrac{r_1}{m}$, gde su $q_1, r_1 \in \mathbb{Z}$ i $m \in \mathbb{N}$, uvodimo smenu: $x = t^m,\ dx = mt^{m-1}dt$. Polazni integral tada postaje $I = m\displaystyle\int t^{r_1+m-1}\big(a+b\,t^{q_1}\big)^p dt$.
   b) Kada je $\dfrac{r+1}{q} \in \mathbb{Z},\ p = \dfrac{s}{m}$, smena: $a+bx^q = t^m$ ili $x = t^{1/q}$.
   c) Za slučaj $\dfrac{r+1}{q} + p \in \mathbb{Z},\ p = \dfrac{s}{m}$, smena: $x = t^{1/q}$ ili $ax^{-q}+b = t^m$.

INTEGRACIJA TRIGONOMETRIJSKIH FUNKCIJA

Neka je R neka racionalna funkcija. Posmatramo integral oblika

$$I = \int R(\sin x, \cos x)dx.$$

1. U slučaju

   - $R(-\sin x, \cos x) = -R(\sin x, \cos x)$, smena: $t = \cos x$;
   - $R(\sin x, -\cos x) = -R(\sin x, \cos x)$, smena: $t = \sin x$;
   - $R(-\sin x, -\cos x) = R(\sin x, \cos x)$, smena: $t = \tan x$.

   Inače, smenom $t = \tan \frac{x}{2}$, $dx = \frac{2dt}{1+t^2}$, imajući u vidu transformacije

   $$\sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2},$$

   polazni integral postaje integral racionalne funkcije.

2. Transformacije proizvoda u zbir

   $$\begin{aligned} \sin ax \sin bx &= \frac{1}{2}\big(\cos(a-b)x - \cos(a+b)x\big), \\ \sin ax \cos bx &= \frac{1}{2}\big(\sin(a-b)x + \sin(a+b)x\big), \\ \cos ax \cos bx &= \frac{1}{2}\big(\cos(a-b)x + \cos(a+b)x\big), \end{aligned}$$

   svode integrale oblika

   $$\int \sin ax \sin bx\, dx, \quad \int \sin ax \cos bx\, dx, \quad \int \cos ax \cos bx\, dx,$$

   na elementarne integrale.

3. Za $I = \int \sin^m x \cos^n x dx$, gde su $m, n \in \mathbb{Z}$ razlikujemo sledeće slučajeve:

   - $n = 2k+1$ ($m = 2k+1$ analogno):

   $$I = \int \sin^m x (\cos^2 x)^k \cos x dx = \left| \begin{array}{l} t = \sin x \\ dt = \cos x dx \end{array} \right| = \int t^m (1-t^2)^k dt;$$

- $m=2k,\ n=2l$: Snižavamo stepen trigonometrijskih funkcija pod integralom primenom transformacija
$$\sin^2 x=\frac{1-\cos 2x}{2},\qquad \cos^2 x=\frac{1+\cos 2x}{2};$$
- $m=-k,\ n=-l,\ k,l\in\mathbb{N}$:
$$\begin{aligned} I &= \int\frac{dx}{\sin^k x\cos^l x}=\int\frac{1}{\sin^k x\cos^{l-2}x}\,\frac{dx}{\cos^2 x}\\ &= \int\frac{1}{\dfrac{\sin^k x}{\cos^k x}\cos^{l+k-2}x}\,\frac{dx}{\cos^2 x}\\ &= \int\tan^{-k}x\big(1+\tan^2 x\big)^{\frac{l+k-2}{2}}d(\tan x)\end{aligned}$$

4. $\int R\big(\tan x\big)dx$, smena: $t=\tan x,\ dx=\dfrac{dt}{1+t^2}$, takođe

   $\int R\big(\cot x\big)dx$, smena: $t=\cot x,\ dx=-\dfrac{dt}{1+t^2}$.

5. Hiperboličke funkcije analogno, uz napomenu o jednakostima koje važe za ove funkcije:

$$\sinh x=\frac{e^x-e^{-x}}{2},\qquad \cosh x=\frac{e^x+e^{-x}}{2},\qquad \tanh x=\frac{\sinh x}{\cosh x},$$
$$\cosh^2 x-\sinh^2 x=1,\qquad \sinh 2x=2\sinh x\cosh x,\qquad \cosh 2x=\cosh^2 x+\sinh^2 x,$$
$$\sinh\frac{x}{2}=\operatorname{sgn}(x)\sqrt{\frac{\cosh x-1}{2}},\qquad \cosh\frac{x}{2}=\sqrt{\frac{\cosh x+1}{2}},$$
$$\sinh x=\frac{\tanh x}{\sqrt{1-\tanh^2 x}},\qquad \cosh x=\frac{1}{\sqrt{1-\tanh^2 x}},$$
$$\operatorname{arc}\sinh x=\ln\left|x+\sqrt{x^2+1}\,\right|,\qquad \operatorname{arc}\cosh x=\ln\left|x+\sqrt{x^2-1}\,\right|,$$
$$\operatorname{arc}\tanh x=\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|,\qquad \operatorname{arc}\coth x=\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|,$$
$$\operatorname{arc}\tanh\frac{x}{a}=\frac{1}{2}\ln\left|\frac{x+a}{x-a}\right|,\qquad \operatorname{arc}\coth\frac{x}{a}=\frac{1}{2}\ln\left|\frac{x-a}{x+a}\right|,$$

Zadaci

1. Dokazati za $a>0$ i $b\neq c$:

a) $\displaystyle\int\frac{dx}{x+b}=\log|x+b|+C,$

b) $\displaystyle\int\frac{dx}{(x+b)(x+c)}=\frac{1}{c-b}\ln\left|\frac{x+b}{x+c}\right|+C,$

c) $\displaystyle\int\frac{dx}{a^2+x^2}=\frac{1}{a}\arctan\frac{x}{a}+C,$

d) $\displaystyle\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right|+C,$

e) $\displaystyle\int\frac{x^2}{a^2+x^2}\,dx=x-a\arctan\frac{x}{a}+C,$

f) $\displaystyle\int\frac{x^2}{x^2-a^2}\,dx=x-\frac{a}{2}\ln\left|\frac{x+a}{x-a}\right|+C,$

g) $\displaystyle\int\frac{dx}{\sqrt{a^2-x^2}}=\arcsin\frac{x}{a}+C,$

h) $\displaystyle\int\frac{dx}{\sqrt{x^2\pm a^2}}=\ln\left|x+\sqrt{x^2\pm a^2}\right|+C,$

i) $\displaystyle\int\frac{x\,dx}{a^2\pm x^2}=\pm\frac{1}{2}\ln|a^2\pm x^2|+C,$

j) $\displaystyle\int\frac{x\,dx}{\sqrt{a^2\pm x^2}}=\pm\sqrt{a^2\pm x^2}+C,$

k) $\displaystyle\int\frac{dx}{x(a^2\pm x^2)}=\frac{1}{2a^2}\ln\left|\frac{x^2}{a^2\pm x^2}\right|+C,$

l) $\displaystyle\int\frac{dx}{x\sqrt{a^2\pm x^2}}=\frac{1}{2a}\log\left|\frac{\sqrt{a^2\pm x^2}-a}{\sqrt{a^2\pm x^2}+a}\right|+C,$

m) $\displaystyle\int\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}+C,$

n) $\displaystyle\int\sqrt{x^2\pm a^2}dx=\frac{x}{2}\sqrt{x^2\pm a^2}\pm\frac{a^2}{2}\ln\left|x+\sqrt{x^2\pm a^2}\right|+C.$

Rešenje: a) $\displaystyle\int\frac{dx}{x+b}=\int\frac{d(x+b)}{x+b}=\log|x+b|+C.$

b) $$\int\frac{dx}{(x+b)(x+c)}=\frac{1}{c-b}\int\frac{(x+c)-(x+b)}{(x+b)(x+c)}\,dx=\frac{1}{c-b}\left(\int\frac{dx}{x+b}-\int\frac{dx}{x+c}\right)$$

$$\stackrel{1.a)}{=}\frac{1}{c-b}\log\left|\frac{x+b}{x+c}\right|+C.$$

c) Kako je $\displaystyle I=\int\frac{dx}{a^2+x^2}=\frac{1}{a^2}\int\frac{dx}{1+\left(\frac{x}{a}\right)^2}$,

uvođenjem smene $t=\dfrac{x}{a}$, $dx=a\,dt$, polazni integral postaje tablični

$$I=\frac{1}{a}\int\frac{dt}{1+t^2}=\frac{1}{a}\arctan t+C.$$

Vraćanjem smene, konačno dobijamo $I=\dfrac{1}{a}\arctan\dfrac{x}{a}+C.$

d) Integral $I = \displaystyle\int \frac{dx}{a^2 - x^2}$ jeste zapravo specijalan slučaj integrala pod **b)** za $a = a$ i $b = -a$. Tako na osnovu rezulata pod **b)** zaključujemo da je

$$I = \frac{1}{2a} \ln \left| \frac{x+a}{x-a} \right| + C.$$

e) $\displaystyle\int \frac{x^2}{a^2+x^2}\,dx = \int \frac{a^2+x^2-a^2}{a^2+x^2}\,dx = \int dx - a^2 \int \frac{dx}{a^2+x^2} \overset{1.c)}{=} x - a\arctan\frac{x}{a} + C.$

f) $\displaystyle\int \frac{x^2}{x^2-a^2}\,dx = \int \frac{x^2-a^2+a^2}{x^2-a^2}\,dx = \int dx + a^2 \int \frac{dx}{x^2-a^2} \overset{1.d)}{=} x - \frac{a}{2}\ln\left|\frac{x+a}{x-a}\right| + C.$

g) $\displaystyle I = \int \frac{dx}{\sqrt{a^2-x^2}} = \frac{1}{a}\int \frac{dx}{\sqrt{1-\frac{x^2}{a^2}}} = \left| \begin{array}{l} t = \frac{x}{a} \\ dx = a\,dt \end{array} \right| = \int \frac{dt}{\sqrt{1-t^2}}$

$\displaystyle = \arcsin t + C = \arcsin\frac{x}{a} + C.$

h) $\displaystyle I = \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \frac{1}{a}\int \frac{dx}{\sqrt{\frac{x^2}{a^2} \pm 1}} = \left| \begin{array}{l} t = \frac{x}{a} \\ dx = a\,dt \end{array} \right| = \int \frac{dt}{\sqrt{t^2 \pm 1}}$

$\displaystyle = \ln\left|t + \sqrt{t^2 \pm 1}\right| + C_1 = \ln\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} \pm 1}\right| + C_1 = \ln|x + \sqrt{x^2 \pm a^2}| + C,$

gde je $C = C_1 - \ln a$.

i) $\displaystyle I = \int \frac{x dx}{a^2 \pm x^2} = \left| \begin{array}{l} t = a^2 \pm x^2 \\ \pm 2x dx = dt \end{array} \right| = \pm\frac{1}{2}\int \frac{dt}{t} = \pm\frac{1}{2}\ln|t| + C = \pm\frac{1}{2}\ln\left|a^2 \pm x^2\right| + C.$

j) $\displaystyle I = \int \frac{x dx}{\sqrt{a^2 \pm x^2}} = \left| \begin{array}{l} t = a^2 \pm x^2 \\ \pm 2x dx = dt \end{array} \right| = \pm\frac{1}{2}\int t^{-\frac{1}{2}}dt = \pm\frac{1}{2}\,\frac{t^{\frac{1}{2}}}{1/2} + C = \pm\sqrt{a^2 \pm x^2} + C.$

k) $\displaystyle I = \int \frac{dx}{x\big(a^2 \pm x^2\big)} = \frac{1}{a^2}\int \frac{a^2 \pm x^2 \mp x^2}{x\big(a^2 \pm x^2\big)}\,dx = \frac{1}{a^2}\int \frac{dx}{x} \mp \frac{1}{a^2}\int \frac{x\,dx}{a^2 \pm x^2}$

$\displaystyle \overset{1.i)}{=} \frac{1}{a^2}\log|x| - \frac{1}{2a^2}\log\left|a^2 \pm x^2\right| + C = \frac{1}{2a^2}\log\left|\frac{x^2}{a^2 \pm x^2}\right| + C.$

l) $\displaystyle I = \int \frac{dx}{x\sqrt{a^2 \pm x^2}} = \int \frac{x\,dx}{x^2\sqrt{a^2 \pm x^2}} = \left| \begin{array}{l} t = \sqrt{a^2 \pm x^2},\ \ x^2 = \pm\big(t^2 - a^2\big) \\ dt = \frac{\pm x\,dx}{\sqrt{a^2 \pm x^2}} \end{array} \right|$

$\displaystyle = \int \frac{dt}{t^2 - a^2} \overset{1.d)}{=} \frac{1}{2a}\log\left|\frac{t-a}{t+a}\right| + C = \frac{1}{2a}\log\left|\frac{\sqrt{a^2 \pm x^2} - a}{\sqrt{a^2 \pm x^2} + a}\right| + C.$

m) Oblast definisanosti podintegralne funkcije $\sqrt{a^2 - x^2}$ integrala

$$I = \int \sqrt{a^2 - x^2}\,dx$$

je segment $D_f = \big[-a, a\big]$. To opravdava uvođenje smene $x = a\sin t$, pri tom je $dx = a\cos t\,dt$, kada t na primer prolazi segment $t \in [-\pi/2, \pi/2]$. Za takav izbor vrednosti nove nezavisno promenljive t važi $\cos t \geq 0$. Tada je

$$\begin{aligned}
I &= a\int \sqrt{a^2 - a^2\sin^2 t}\cos t\,dt = a^2\int \sqrt{1-\sin^2 t}\,\cos t\,dt \\
&= a^2\int \cos^2 t\,dt = \frac{a^2}{2}\int (1+\cos 2t)dt = \frac{a^2}{2}\Big(\int dt + \int \cos 2t\,dt\Big) \\
&= \frac{a^2}{2}\Big(t + \frac{1}{2}\int \cos 2t\,d(2t)\Big) = \frac{a^2}{2}\Big(t + \frac{1}{2}\sin 2t\Big) + C.
\end{aligned}$$

Primetimo da je $\quad \sin t = \dfrac{x}{a},\ t = \arcsin\dfrac{x}{a}, \quad$ kao i

$$\sin 2t = 2\sin t\cos t = 2\sin t\sqrt{1-\sin^2 t} = \frac{2x}{a}\sqrt{1-\Big(\frac{x}{a}\Big)^2} = \frac{2x}{a^2}\sqrt{a^2-x^2}.$$

Tada imamo

$$I = \frac{a^2}{2}\Big(\arcsin\frac{x}{a} + \frac{x}{a^2}\sqrt{a^2-x^2}\Big) + C = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C.$$

n) $$I = \int \sqrt{x^2 \pm a^2}dx = \left|\begin{array}{ll} u = \sqrt{x^2 \pm a^2}, & du = \frac{x\,dx}{\sqrt{x^2\pm a^2}} \\ dv = dx, & v = \int dx = x \end{array}\right|$$

$$\begin{aligned}
&= x\sqrt{x^2\pm a^2} - \int \frac{x^2\,dx}{\sqrt{x^2\pm a^2}} = x\sqrt{x^2\pm a^2} - \int \frac{x^2 \pm a^2 \mp a^2}{\sqrt{x^2\pm a^2}}\,dx \\
&= x\sqrt{x^2\pm a^2} - \Big(\int \sqrt{x^2\pm a^2}\,dx \mp a^2\int \frac{dx}{\sqrt{x^2\pm a^2}}\Big) \\
&\overset{1.\mathrm{h})}{=} x\sqrt{x^2\pm a^2} - I \pm a^2\ln|x + \sqrt{x^2\pm a^2}|.
\end{aligned}$$

Odatle je

$$2I = x\sqrt{x^2\pm a^2} \pm a^2\ln|x+\sqrt{x^2\pm a^2}| + C_1,$$

odnosno

$$I = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2}\ln|x + \sqrt{x^2 \pm a^2}| + C.$$

Još jedan pogodan način izračunavanja integrala I dat je u nastavku.

$$I = \int \sqrt{x^2 \pm a^2}dx = \int \frac{x^2 \pm a^2}{\sqrt{x^2 \pm a^2}}\,dx = (c_0x + c_1)\sqrt{x^2 \pm a^2} + \lambda \int \frac{dx}{\sqrt{x^2 \pm a^2}}$$

Diferenciranjem ove jednakosti nalazimo

$$\frac{x^2 \pm a^2}{\sqrt{x^2 \pm a^2}} = c_0\sqrt{x^2 \pm a^2} + (c_0x + c_1)\frac{x}{\sqrt{x^2 \pm a^2}} + \lambda\frac{dx}{\sqrt{x^2 \pm a^2}}.$$

Posle množenja sa $\sqrt{x^2 \pm a^2}$ i sređivanja izraza, dobijamo

$$x^2 \pm a^2 = 2c_0x^2 + c_1x \pm c_0a^2 + \lambda.$$

Izjednačavanjem koeficijenata uz iste stepene promenljive x dolazimo do sistema jednačina

$$1 = 2c_0, \qquad 0 = c_1, \qquad \pm a^2 = \pm c_0a^2 + \lambda,$$

čije je rešenje

$$c_0 = 1/2, \qquad c_1 = 0, \qquad \lambda = \pm a^2/2.$$

Time traženi integral postaje

$$I = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2}\int \frac{dx}{\sqrt{x^2 \pm a^2}} \overset{1.\mathrm{h})}{=} \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2}\ln|x + \sqrt{x^2 \pm a^2}| + C.$$

2. Odrediti sledeće integrale:

$$1^0 \int \sqrt{x}(x+2)^3dx \quad 2^0 \int x^2(1+x)^{20}dx \quad 3^0 \int \frac{x\,dx}{(x+9)^{10}}$$

$$4^0 \int \frac{dx}{x^2+4x+4} \quad 5^0 \int \frac{dx}{x^2+x-2} \quad 6^0 \int \sqrt{1-2x^2+x^4}\,dx$$

Rešenje: 1^0 $\int \sqrt{x}(x+2)^3dx = \int x^{1/2}\big(x^3 + 6x^2 + 12x + 8\big)dx$

$$
\begin{aligned}
&= \int \big(x^{7/2}+6x^{5/2}+12x^{3/2}+8x^{1/2}\big)dx\\
&= \sqrt{x}\Big(\frac{2}{9}x^4+\frac{12}{7}x^3+\frac{24}{5}x^2+\frac{16}{3}x\Big)+C.
\end{aligned}
$$

2^0 $\displaystyle\int x^2(1+x)^{20}dx = \left|\begin{array}{l} t=1+x\\ dt=dx\end{array}\right| = \int(t-1)^2t^{20}dt = \int(t^2-2t+1)t^{20}dt$

$$
\begin{aligned}
&= \int\big(t^{22}-2t^{21}+t^{20}\big)dt = \frac{t^{23}}{23}-2\frac{t^{22}}{22}+\frac{t^{21}}{21}+C\\
&= \frac{(1+x)^{23}}{23}-\frac{(1+x)^{22}}{11}+\frac{(1+x)^{21}}{21}+C.
\end{aligned}
$$

3^0 $\displaystyle\int \frac{x\,dx}{(x+9)^{10}} = \left|\begin{array}{l} t=x+9\\ dt=dx\end{array}\right| = \int(t-9)t^{-10}dt = \int\big(t^{-9}-9t^{-10}\big)dt$

$$
= \frac{t^{-8}}{-8}-9\frac{t^{-9}}{-9}+C = \frac{1}{(x+9)^9}-\frac{1}{8(x+9)^8}+C.
$$

4^0 $\displaystyle\int \frac{dx}{x^2+4x+4} = \int\frac{d(x+2)}{(x+2)^2} = \frac{-1}{x+2}+C.$

5^0 $\displaystyle\int \frac{dx}{x^2+x-2} = \int\frac{dx}{(x+2)(x-1)} \stackrel{1.b)}{=} \frac{1}{3}\ln\Big|\frac{x-1}{x+2}\Big|+C.$

6^0 $\displaystyle\int\sqrt{1-2x^2+x^4}\,dx = \int\sqrt{(1-x^2)^2}\,dx = \int\big(1-x^2\big)\,dx = \Big(x-\frac{x^3}{3}\Big)+C,$
za $1-x^2\geq 0$, tj. $x\in[-1,1]$.

Za $x\in(-\infty,-1)$ ili $x\in(1,+\infty)$ važi

$$
\int\sqrt{1-2x^2+x^4}\,dx = -\int\big(1-x^2\big)\,dx = -\Big(x-\frac{x^3}{3}\Big)+C_1.
$$

3. Metodom smene odrediti:

$$1^0 \int \frac{x^3dx}{x^8-2} \qquad 2^0 \int \frac{x^7dx}{(1+x^4)^2} \qquad 3^0 \int \frac{x^8dx}{(x^3-1)^3}$$

$$4^0 \int \frac{x^9}{x^5+1}\,dx \qquad 5^0 \int \frac{dx}{x\big(x^{10}+1\big)^2} \qquad 6^0 \int \frac{1-x^7}{x\big(1+x^7\big)}\,dx$$

$$7^0 \int \frac{xdx}{\sqrt{1+x^4}} \qquad 8^0 \int \frac{dx}{x\sqrt{x^2+1}} \qquad 9^0 \int \Big(x+\frac{1}{2}\Big)\sqrt{x^2+x+1}\,dx$$

Rešenje: 1^0
$$\int \frac{x^3dx}{x^8-2} = \frac{1}{4}\int \frac{d(x^4)}{(x^4)^2-(\sqrt{2})^2} \stackrel{1.d)}{=} \frac{1}{8\sqrt{2}}\ln\left|\frac{x^4-\sqrt{2}}{x^4+\sqrt{2}}\right| + C.$$

2^0
$$\begin{aligned}\int \frac{x^7dx}{(1+x^4)^2} &= \int \frac{x^4}{(1+x^4)^2}\,x^3dx = \left|\begin{array}{l} t=1+x^4 \\ dt=4x^3dx \end{array}\right| = \frac{1}{4}\int \frac{t-1}{t^2}dt \\ &= \frac{1}{4}\int \big(t^{-1}-t^{-2}\big)dt = \frac{1}{4}\big(\ln|t|+1/t\big)+C \\ &= \frac{1}{4}\Big(\ln\big(1+x^4\big)+\frac{1}{1+x^4}\Big)+C.\end{aligned}$$

3^0
$$\begin{aligned}\int \frac{x^8dx}{(x^3-1)^3} &= \int \frac{(x^3)^2}{(x^3-1)^3}x^2dx = \left|\begin{array}{l} t=x^3-1 \\ dt=3x^2dx \end{array}\right| = \frac{1}{3}\int \frac{(t+1)^2}{t^3}\,dt \\ &= \frac{1}{3}\int \Big(\frac{1}{t}+\frac{2}{t^2}+\frac{1}{t^3}\Big)dt = \frac{1}{3}\Big(\ln|t|-\frac{2}{t}-\frac{1}{2t^2}\Big)+C \\ &= \frac{1}{3}\Big(\ln|x^3-1|-\frac{2}{x^3-1}-\frac{1}{2(x^3-1)^2}\Big)+C.\end{aligned}$$

4^0
$$\begin{aligned}\int \frac{x^9}{x^5+1}\,dx &= \frac{1}{5}\int \frac{x^5d\big(x^5\big)}{x^5+1} = \frac{1}{5}\int \frac{x^5+1-1}{x^5+1}\,d\big(x^5\big) \\ &= \frac{1}{5}\left(\int d\big(x^5\big)-\int \frac{d\big(x^5+1\big)}{x^5+1}\right) = \frac{x^5}{5}-\frac{1}{5}\log\big|1+x^5\big|+C.\end{aligned}$$

5^0
$$\int\frac{dx}{x\bigl(x^{10}+1\bigr)^2}=\int\frac{x^9dx}{x^{10}\bigl(x^{10}+1\bigr)^2}=\left|\begin{array}{l} t=x^{10}\\ dt=10x^9dx\end{array}\right|=\frac{1}{10}\int\frac{dt}{t(t+1)^2}$$
$$=\frac{1}{10}\int\frac{t+1-t}{t(t+1)^2}\,dt=\frac{1}{10}\int\frac{dt}{t(t+1)}-\frac{1}{10}\int\frac{d(t+1)}{(t+1)^2}$$
$$\overset{1b)}{=}\frac{1}{10}\log\left|\frac{t}{t+1}\right|+\frac{1}{10(t+1)}+C=\frac{1}{10}\log\left|\frac{x^{10}}{x^{10}+1}\right|+\frac{1}{10\bigl(x^{10}+1\bigr)}+C.$$

6^0
$$\int\frac{1-x^7}{x\bigl(1+x^7\bigr)}\,dx=\int\frac{1-x^7}{x^7\bigl(1+x^7\bigr)}\,x^6dx=\left|\begin{array}{l} t=x^7\\ dt=7x^6dx\end{array}\right|=\frac{1}{7}\int\frac{1-t}{t(1+t)}\,dt$$
$$=\frac{1}{7}\int\frac{1+t-2t}{t(1+t)}\,dt=\frac{1}{7}\log\frac{|x^7|}{\bigl(x^7+1\bigr)^2}+C.$$

7^0
$$\int\frac{xdx}{\sqrt{1+x^4}}=\frac{1}{2}\int\frac{d(x^2)}{\sqrt{1+(x^2)^2}}=\frac{1}{2}\ln\left|x^2+\sqrt{1+x^4}\,\right|+C.$$

8^0
$$\int\frac{dx}{x\sqrt{x^2+1}}=\int\frac{x\,dx}{x^2\sqrt{x^2+1}}=\left|\begin{array}{l} t=\sqrt{x^2+1},\ x^2=t^2-1\\ dt=\frac{x\,dx}{\sqrt{x^2+1}}\end{array}\right|$$
$$=\int\frac{dt}{t^2-1}\overset{1.d)}{=}\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+C_2=\frac{1}{2}\ln\left|\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\right|+C_2.$$

9^0
$$\int\Bigl(x+\frac{1}{2}\Bigr)\sqrt{x^2+x+1}\,dx=\left|\begin{array}{l} t=x^2+x+1\\ dt=(2x+1)dx=2\bigl(x+\frac{1}{2}\bigr)dx\end{array}\right|$$
$$=\frac{1}{2}\int\sqrt{t}\,dt=\frac{1}{2}\,\frac{t^{3/2}}{3/2}+C=\frac{1}{3}\sqrt{(x^2+x+1)^3}+C.$$

4. Metodom smene odrediti:

$$1^0 \int \frac{dx}{\sqrt{5x-2}} \qquad 2^0 \int \frac{x}{\sqrt{1-4x}}\,dx \quad 3^0 \int \sqrt[4]{1-4x}\,dx$$

$$4^0 \int \frac{dx}{\sqrt{x+1}+\sqrt{x-1}} \qquad 5^0 \int \frac{dx}{\sqrt{x+3}-\sqrt{x-2}}$$

$$6^0 \int \frac{x^5}{\sqrt{1-x^2}}\,dx \quad 7^0 \int \frac{x\,dx}{\left(1+x^2\right)^{3/2}} \quad 8^0 \int \frac{x^3dx}{\left(x^2+1\right)^3\sqrt{x^2+1}}$$

$$9^0 \int \frac{2}{(2-x)^2}\sqrt[3]{\frac{2-x}{2+x}}\,dx$$

Rešenje: $1^0 \displaystyle\int \frac{dx}{\sqrt{5x-2}} = \frac{2}{5}\int d(\sqrt{5x-2}) = \frac{2}{5}\sqrt{5x-2} + C.$

$$2^0 \qquad \int \frac{x}{\sqrt{1-4x}}\,dx = \left|\begin{array}{l} t=\sqrt{1-4x},\ x=\frac{1-t^2}{4} \\ dx=-\frac{1}{2}t\,dt \end{array}\right| = \frac{1}{8}\int (t^2-1)dt$$

$$= \frac{1}{8}\left(\frac{t^3}{3}-t\right) + C = \frac{-1}{12}\sqrt{1-4x}\,(1+2x) + C.$$

$$3^0 \qquad \int \sqrt[4]{1-4x}\,dx = \left|\begin{array}{l} t=\sqrt[4]{1-4x},\ t^4=1-4x \\ x=\frac{1-t^4}{4},\ dx=-t^3dt \end{array}\right| = -\int t^4dt = -\frac{t^5}{5} + C$$

$$= \sqrt[4]{1-4x}\,\frac{4x-1}{5} + C.$$

$$4^0 \qquad \int \frac{dx}{\sqrt{x+1}+\sqrt{x-1}} = \int \frac{\sqrt{x+1}-\sqrt{x-1}}{x+1-(x-1)}\,dx = \frac{1}{2}\int \left(\sqrt{x+1}-\sqrt{x-1}\,\right) dx$$

$$= \frac{1}{2}\int \sqrt{x+1}\,d(x+1) - \frac{1}{2}\int \sqrt{x-1}\,d(x-1)$$

$$= \frac{1}{2}\frac{(x+1)^{3/2}}{3/2} - \frac{1}{2}\frac{(x-1)^{3/2}}{3/2} + C = \frac{1}{3}\left((x+1)^{3/2}-(x-1)^{3/2}\right) + C.$$

5^0 $$\int\frac{dx}{\sqrt{x+3}-\sqrt{x-2}}=\int\frac{\sqrt{x+3}+\sqrt{x-2}}{(x+3)-(x-2)}\,dx=\frac{1}{5}\int\left(\sqrt{x+3}+\sqrt{x-2}\right)dx$$
$$=\frac{2}{15}\left((x+3)^{3/2}+(x-2)^{3/2}\right)+C.$$

6^0 $$\int\frac{x^5}{\sqrt{1-x^2}}\,dx=\int\frac{x^4}{\sqrt{1-x^2}}\,xdx=\left|\begin{array}{l} t=\sqrt{1-x^2},\ x^2=1-t^2 \\ dt=-\frac{x\,dx}{\sqrt{1-x^2}} \end{array}\right|$$
$$=-\int(1-t^2)^2dt=-\sqrt{1-x^2}\Big(\frac{1}{5}(1-x^2)^2-\frac{2}{3}(1-x^2)+1\Big)+C.$$

7^0 $$\int\frac{x\,dx}{\left(1+x^2\right)^{3/2}}=\left|\begin{array}{l} t=\frac{1}{\sqrt{1+x^2}} \\ dt=-\frac{x\,dx}{(1+x^2)^{3/2}} \end{array}\right|=-\int dt=-t+C=-\frac{1}{\sqrt{1+x^2}}+C.$$

8^0 $$\int\frac{x^3dx}{\left(x^2+1\right)^3\sqrt{x^2+1}}=\left|\begin{array}{l} t=\sqrt{x^2+1},\ x^2=t^2-1 \\ dt=\frac{x\,dx}{\sqrt{x^2+1}} \end{array}\right|=\int\frac{t^2-1}{t^6}\,dt$$
$$=\int\left(t^{-4}-t^{-6}\right)dt=\frac{t^{-3}}{-3}-\frac{t^{-5}}{-5}+C=\frac{1}{5\sqrt{(x^2+1)^5}}-\frac{1}{3\sqrt{(x^2+1)^3}}+C.$$

9^0 $$\int\frac{2}{(2-x)^2}\sqrt[3]{\frac{2-x}{2+x}}\,dx=\left|\begin{array}{l} t=\sqrt[3]{\frac{2+x}{2-x}} \\ dt=\frac{4/3}{(2-x)^2}\sqrt[3]{\left(\frac{2-x}{2+x}\right)^2}\,dx \end{array}\right|=\frac{3}{2}\int t\,dt$$
$$=\frac{3}{4}t^2+C=\frac{3}{4}\sqrt[3]{\left(\frac{2+x}{2-x}\right)^2}+C.$$

5. Metodom smene izračunati:

$$1^0 \int \frac{dx}{x\ln x} \qquad 2^0 \int \frac{\sqrt{x}+\ln x}{x}\,dx \qquad 3^0 \int \frac{\ln 2x}{x\ln 4x}\,dx$$

$$4^0 \int \frac{dx}{x\sqrt{\ln x}} \qquad 5^0 \int \frac{\ln(1+x)-\ln x}{x(1+x)}\,dx \qquad 6^0 \int x^x(1+\log x)dx$$

$$7^0 \int \frac{x\big(\ln(1+x)+\ln(1-x)\big)^2}{x^2-1}\,dx$$

Rešenje: $1^0 \displaystyle\int \frac{dx}{x\ln x} = \int \frac{d(\ln x)}{\ln x} = \ln|\ln x| + C.$

$$2^0 \quad \int \frac{\sqrt{x}+\ln x}{x}\,dx = \int x^{-1/2}dx + \int \ln x\frac{dx}{x} = \frac{x^{1/2}}{1/2} + \int \ln x\, d(\ln x)$$
$$= 2\sqrt{x} + \frac{1}{2}\ln^2 x + C.$$

3^0 S obzirom da je $d(\ln 4x) = \dfrac{dx}{x}$ i $\ln 4x = \ln 2x + \ln 2$, uvodimo smenu $t = \ln 4x$:

$$\int \frac{\ln 2x}{x\ln 4x}\,dx = \int \frac{t-\ln 2}{t}\,dt = \int dt - \ln 2\int \frac{dt}{t} = t - \ln 2\ \ln|t| + C$$
$$= \ln 4x - \ln 2\ \ln|\ln 4x| + C.$$

$$4^0 \quad \int \frac{dx}{x\sqrt{\ln x}} = \int (\ln x)^{-1/2} d(\ln x) = 2\sqrt{\ln x} + C.$$

$$5^0 \quad \int \frac{\ln(1+x)-\ln x}{x(1+x)}\,dx = \int \frac{\ln\left(1+\frac{1}{x}\right)}{x(1+x)}\,dx = \left|\begin{array}{l} t = \ln\left(1+\frac{1}{x}\right) \\ dt = \frac{1}{1+\frac{1}{x}}\,\frac{-dx}{x^2} = -\frac{dx}{x(1+x)} \end{array}\right|$$
$$= -\int t\,dt = -\frac{t^2}{2} + C = -\frac{1}{2}\ln^2\left(1+\frac{1}{x}\right) + C.$$

$$6^0 \quad \int x^x(1+\log x)dx = \left|\begin{array}{l} t = x^x,\ \log t = x\log x \\ \frac{dt}{t} = (1+\log x)dx \end{array}\right| = \int dt = t + C = x^x + C.$$

7^0 $$\int \frac{x\big(\ln(1+x)+\ln(1-x)\big)^2}{x^2-1}\,dx=\int \ln^2\left(1-x^2\right)\frac{x\,dx}{x^2-1}=\left|\begin{array}{l} t=\ln\left(1-x^2\right) \\ dt=\frac{-2x\,dx}{1-x^2} \end{array}\right|$$
$$=\frac{1}{2}\int t^2 dt=\frac{1}{6}\,t^3+C=\frac{1}{6}\ln^3\left(1-x^2\right)+C.$$

6. Metodom smene odrediti:

1^0 $\displaystyle\int \frac{dx}{e^x+1}$ $\quad 2^0$ $\displaystyle\int \frac{dx}{\sqrt{1+e^{2x}}}$ $\quad 3^0$ $\displaystyle\int \frac{\sqrt{e^x-1}}{1+3e^{-x}}\,dx$

4^0 $\displaystyle\int e^{-x^2-1}x\,dx$ $\quad 5^0$ $\displaystyle\int \frac{x\,e^{-(1+x^2)^{-1/2}}}{\sqrt{(1+x^2)^3}}\,dx$ $\quad 6^0$ $\displaystyle\int \frac{x+1}{x\left(1+x\,e^x\right)}\,dx$

Rešenje: 1^0 $$\int \frac{dx}{e^x+1}=\int \frac{dx}{e^x\left(1+e^{-x}\right)}=-\int \frac{d(1+e^{-x})}{1+e^{-x}}=-\ln\left(1+e^{-x}\right)+C$$
$$=\ln\frac{e^x}{1+e^x}+C=x-\ln\left(1+e^x\right)+C.$$

Isti rezultat možemo da dobijemo i na sledeći način.

$$\int \frac{dx}{e^x+1}=\int \frac{1+e^x-e^x}{e^x+1}\,dx=\int dx-\int \frac{e^x dx}{e^x+1}=x-\int \frac{d\left(e^x+1\right)}{e^x+1}$$
$$=x-\ln\left(e^x+1\right)+C.$$

2^0 $$\int \frac{dx}{\sqrt{1+e^{2x}}}=\int \frac{dx}{e^x\sqrt{e^{-2x}+1}}=\left|\begin{array}{l} t=e^{-x} \\ dt=-e^{-x}dx \end{array}\right|=-\int \frac{dt}{\sqrt{t^2+1}}$$
$$=-\ln|t+\sqrt{t^2+1}|+C=-\ln(e^{-x}+\sqrt{e^{-2x}+1})+C$$
$$=\ln\frac{e^x}{1+\sqrt{1+e^{2x}}}+C=x-\ln\left(1+\sqrt{1+e^{2x}}\right)+C.$$

3^0 $$\int \frac{\sqrt{e^x-1}}{1+3e^{-x}}\,dx=\int \frac{e^x-1}{e^x+3}\,\frac{e^x\,dx}{\sqrt{e^x-1}}=\left|\begin{array}{l} t=\sqrt{e^x-1},\ e^x=t^2+1 \\ dt=\frac{1}{2}\,\frac{e^x\,dx}{\sqrt{e^x-1}} \end{array}\right|$$
$$=2\int \frac{t^2}{t^2+4}\,dt\overset{1.e)}{=}2t-4\arctan\frac{t}{2}+C=2\sqrt{e^x-1}-4\arctan\frac{\sqrt{e^x-1}}{2}+C.$$

4^0 $\displaystyle\int e^{-x^2-1}x\,dx = \left|\begin{array}{l} t=-x^2-1 \\ dt=-2x\,dx \end{array}\right| = -\frac{1}{2}\int e^t dt = -\frac{1}{2}e^{-x^2-1}+C.$

5^0 $\displaystyle\int \frac{xe^{-(1+x^2)^{-\frac{1}{2}}}}{\sqrt{(1+x^2)^3}}\,dx = \left|\begin{array}{l} t=e^{-(1+x^2)^{-\frac{1}{2}}} \\ dt=\frac{e^{-(1+x^2)^{-\frac{1}{2}}}x\,dx}{(1+x^2)^{\frac{3}{2}}} \end{array}\right| = \int dt = t+C = e^{-(1+x^2)^{-\frac{1}{2}}}+C.$

$$6^0 \quad \int \frac{x+1}{x\big(1+x\,e^x\big)}\,dx = \int \frac{(x+1)e^x}{xe^x\big(1+x\,e^x\big)}\,dx = \left|\begin{array}{l} t=xe^x \\ dt=(1+x)e^x dx \end{array}\right|$$

$$= \int \frac{dt}{t(t+1)} \overset{1.b)}{=} \ln\left|\frac{t}{t+1}\right| + C = \ln\left|\frac{xe^x}{xe^x+1}\right| + C.$$

7. Metodom smene odrediti:

$$1^0 \ \int \tan x\,dx \qquad 2^0 \ \int \frac{dx}{\cos x} \qquad 3^0 \ \int \frac{dx}{1+\cos x}$$
$$4^0 \ \int \cot x\,dx \qquad 5^0 \ \int \frac{dx}{\sin x} \qquad 6^0 \ \int \frac{dx}{1+\sin x}$$

Rešenje: 1^0 $\displaystyle\int \tan x\,dx = \int \frac{\sin x\,dx}{\cos x} = -\int \frac{d(\cos x)}{\cos x} = -\ln|\cos x| + C.$

$$2^0 \quad \int \frac{dx}{\cos x} = \int \frac{\cos x\,dx}{\cos^2 x} = \int \frac{d(\sin x)}{1-\sin^2 x} \overset{1.b)}{=} \frac{1}{2}\ln\frac{1+\sin x}{1-\sin x} + C.$$

$$= \frac{1}{2}\ln\frac{(1+\sin x)^2}{1-\sin^2 x} + C = \ln\left|\frac{1+\sin x}{\cos x}\right| + C$$

$$= \log\left|\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right| + C = \log\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| + C.$$

$$3^0 \quad \int \frac{dx}{1+\cos x} = \int \frac{dx}{2\cos^2\frac{x}{2}} = \int \frac{d\left(\frac{x}{2}\right)}{\cos^2\frac{x}{2}} = \tan\frac{x}{2} + C.$$

4^0 $\int \cot x\,dx = \log|\sin x| + C.$

$5^0 \quad \displaystyle\int \frac{dx}{\sin x} = \log\Big|\tan\frac{x}{2}\Big| + C.$

$$6^0 \quad \int \frac{dx}{1+\sin x} = \Big|x = t+\pi/2,\ dx = dt\Big| = \int \frac{dt}{1+\sin(t+\pi/2)} = \int \frac{dt}{1+\cos t}$$

$$\overset{7.3^0}{=} \tan t/2 + C = \tan\frac{x-\frac{\pi}{2}}{2} + C = \frac{\tan\frac{x}{2} - \tan\frac{\pi}{4}}{1+\tan\frac{x}{2}\tan\frac{\pi}{4}} + C$$

$$= \frac{\tan\frac{x}{2}-1}{\tan\frac{x}{2}+1} + C = -\frac{\cos x}{1+\sin x} + C = \frac{2}{1+\cot\frac{x}{2}} + C_1.$$

8. Metodom smene odrediti:

$1^0 \ \int \cos^2 x\,dx \qquad 2^0 \ \int \cos^3 x\,dx \qquad 3^0 \ \int \cos^4 x\,dx$

$4^0 \ \int \sin^2 x\,dx \qquad 5^0 \ \int \sin^3 x\,dx \qquad 6^0 \ \int \sin^4 x\,dx$

Rešenje: 1^0 $\displaystyle\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{1}{2}\int dx + \frac{1}{4}\int \cos 2x\,d(2x)$

$$= \frac{x}{2} + \frac{\sin 2x}{4} + C.$$

$2^0 \quad \displaystyle\int \cos^3 x\,dx = \int \cos^2 x \cos x\,dx = \int \big(1-\sin^2 x\big)d(\sin x) = \sin x - \frac{\sin^3 x}{3} +$ $C.$

Drugi oblik primitivne funkcije može se dobiti snižavanjem stepena trigonometrijske funkcije pod integralom.

$$\int \cos^3 x\,dx = \int \cos^2 x\cos x\,dx = \frac{1}{2}\int \big(1+\cos 2x\big)\cos x\,dx$$

$$= \frac{1}{2}\int \big(\cos x + \cos 2x\cos x\big)dx = \frac{1}{2}\int \Big(\cos x + \frac{1}{2}(\cos 3x + \cos x)\Big)dx$$

$$= \frac{1}{4}\int \big(3\cos x + \cos 3x\big)dx = \frac{3}{4}\sin x + \frac{1}{12}\sin 3x + C.$$

3^0 $$\int \cos^4 x\,dx = \int (\cos^2 x)^2 dx = \int \Big(\frac{1+\cos 2x}{2}\Big)^2 dx = \int \frac{1+2\cos 2x+\cos^2 2x}{4}\,dx$$
$$= \frac{1}{4}\int dx + \frac{1}{4}\int \cos 2x\,d(2x) + \frac{1}{8}\int (1+\cos 4x)dx$$
$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C.$$

4^0 $\int \sin^2 x\,dx = \dfrac{x}{2} - \dfrac{1}{4}\sin 2x + C.$

5^0 $\int \sin^3 x\,dx = -\dfrac{3}{4}\cos x + \dfrac{1}{12}\cos 3x + C = -\cos x + \dfrac{\cos^3 x}{3} + C.$

6^0 $\int \sin^4 x\,dx = \dfrac{3}{8}x - \dfrac{1}{4}\sin 2x + \dfrac{1}{32}\sin 4x + C.$

9. Metodom smene odrediti:

$$\begin{array}{llllll} 1^0 & \displaystyle\int \frac{dx}{\cos^3 x} & 2^0 & \displaystyle\int \frac{dx}{\cos^4 x} & 3^0 & \displaystyle\int \frac{dx}{\sin x\cos^2 x} \\ 4^0 & \displaystyle\int \frac{dx}{\sin^3 x} & 5^0 & \displaystyle\int \frac{dx}{\sin^4 x} & 6^0 & \displaystyle\int \frac{dx}{\sin^2 x\cos x} \\ 7^0 & \int \tan^2 x\,dx & 8^0 & \int \tan^3 x\,dx & 9^0 & \int \tan^4 x\,dx \end{array}$$

$$10^0 \int \frac{\sin 2x}{4\cos^2 x + 9\sin^2 x}\,dx \qquad 11^0 \int \frac{\sin x\cos^3 x}{1+\cos^2 x}\,dx$$

Rešenje: 1^0 Za $x \in \big(-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi\big),\ k \in \mathbb{Z}$, imamo $\cos x > 0$. Tada je

$$\int \frac{dx}{\cos^3 x} = \int \frac{1}{\cos x}\,\frac{dx}{\cos^2 x} = \left| \begin{array}{l} t = \tan x,\ dt = \frac{dx}{\cos^2 x} \\ \cos x = \frac{1}{\sqrt{1+t^2}} \end{array} \right| = \int \sqrt{1+t^2}\,dt$$
$$\overset{1.n)}{=} \Big(\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\ln\big|t + \sqrt{1+t^2}\,\big|\Big) + C$$
$$= \Big(\frac{\tan x}{2\cos x} + \frac{1}{2}\ln\Big|\tan x + \frac{1}{\cos x}\Big|\Big) + C$$
$$= \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\ln\Big|\frac{\sin x + 1}{\cos x}\Big| + C = \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\ln\Big|\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\Big| + C.$$

Integral, takođe možemo odrediti na sledeći način:

$$\int \frac{dx}{\cos^3 x} = \int \frac{\sin^2 x + \cos^2 x}{\cos^3 x}\,dx = \int \frac{\sin^2 x\,dx}{\cos^3 x} + \int \frac{dx}{\cos x}$$

$$= \left| \begin{array}{ll} u = \sin x & du = \cos x\,dx \\ dv = -\frac{d(\cos x)}{\cos^3 x} & v = \frac{1}{2\cos^2 x} \end{array} \right| = \frac{\sin x}{2\cos^2 x} - \frac{1}{2}\int \frac{dx}{\cos x} + \int \frac{dx}{\cos x}$$

$$= \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\int \frac{dx}{\cos x} \overset{7.2^0}{=} \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\log\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| + C.$$

2^0 $$\int \frac{dx}{\cos^4 x} = \int \frac{1}{\cos^2 x}\frac{dx}{\cos^2 x} = \left| \begin{array}{l} t = \tan x,\ dt = \frac{dx}{\cos^2 x} \\ \cos^2 x = \frac{1}{1+\tan^2 x} = \frac{1}{1+t^2} \end{array} \right|$$

$$= \int (1+t^2)dt = t + \frac{t^3}{3} + C = \tan x + \frac{1}{3}\tan^3 x + C.$$

3^0 $$\int \frac{dx}{\sin x\cos^2 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin x\cos^2 x}\,dx = \int \frac{\sin x\,dx}{\cos^2 x} + \int \frac{dx}{\sin x}$$

$$\overset{7.5^0}{=} -\int \frac{d(\cos x)}{\cos^2 x} + \ln\left|\tan\frac{x}{2}\right| = \frac{1}{\cos x} + \ln\left|\tan\frac{x}{2}\right| + C.$$

4^0 $$\int \frac{dx}{\sin^3 x} = \frac{1}{8}\left(\frac{1}{\cos^2\frac{x}{2}} - \frac{1}{\sin^2\frac{x}{2}}\right) + \frac{1}{2}\log\left|\tan\frac{x}{2}\right| + C.$$

5^0 $$\int \frac{dx}{\sin^4 x} = -\cot x - \frac{1}{3}\cot^3 x + C.$$

6^0 $$\int \frac{dx}{\sin^2 x\cos x} = \log\left|\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right| - \frac{1}{\sin x} + C.$$

7^0 $$\int \tan^2 x\,dx = \int \frac{\sin^2 x}{\cos^2 x}\,dx = \int \frac{1-\cos^2 x}{\cos^2 x}\,dx = \int \frac{dx}{\cos^2 x} - \int dx = \tan x - x + C.$$

8^0 $$\int \tan^3 x\,dx = \int \tan x\tan^2 x\,dx = \int \tan x\frac{1-\cos^2 x}{\cos^2 x}\,dx$$

$$= \int \tan x\,d(\tan x) - \int \tan x\,dx \overset{7.1^0}{=} \frac{1}{2}\tan^2 x + \ln|\cos x| + C.$$

9^0 $$\int \tan^4 x\,dx = \int \tan^2 x \frac{1-\cos^2 x}{\cos^2 x}\,dx = \int \tan^2 x\, d(\tan x)\,dx - \int \tan^2 x\,dx$$

$$\overset{9.7^0}{=} \frac{\tan^3 x}{3} - \tan x + x + C.$$

10^0 $$\int \frac{\sin 2x}{4\cos^2 x + 9\sin^2 x}\,dx = \left| \begin{array}{l} t = 4\cos^2 x + 9\sin^2 x \\ dt = 5\sin 2x\,dx \end{array} \right| = \frac{1}{5}\int \frac{dt}{t} = \frac{1}{5}\ln|t| + C$$

$$= \frac{1}{5}\ln\left(4\cos^2 x + 9\sin^2 x\right) + C.$$

11^0 $$\int \frac{\sin x\cos^3 x}{1+\cos^2 x}\,dx = \left| \begin{array}{l} t = 1+\cos^2 x \\ dt = -2\cos x\sin x\,dx \end{array} \right| = \frac{1}{2}\int \frac{1-t}{t}\,dt$$

$$= \frac{1}{2}\left(\int \frac{dt}{t} - \int dt\right) = \frac{1}{2}\ln|t| - \frac{1}{2}t + C$$
$$= \frac{1}{2}\ln\left(1+\cos^2 x\right) - \frac{1+\cos^2 x}{2} + C.$$

10. Metodom smene odrediti:

1^0 $\displaystyle\int \frac{x-\sqrt{\arctan 2x}}{1+4x^2}\,dx$	2^0 $\displaystyle\int \frac{\sin 2x}{\sqrt{\sin x + 1}}\,dx$	3^0 $\displaystyle\int \sqrt{1-\sin 2x}\,dx$
4^0 $\displaystyle\int \cos\frac{1}{x}\,\frac{dx}{x^2}$	5^0 $\displaystyle\int \sqrt{\sin x\cos^2 x}\,dx$	6^0 $\displaystyle\int \cos^2 x\sin 2x\,dx$
7^0 $\displaystyle\int \frac{dx}{\sin x + 2\cos x + 3}$	8^0 $\displaystyle\int \frac{dx}{2\cos x + 3}$	9^0 $\displaystyle\int \frac{dx}{4-3\cos x}$
10^0 $\displaystyle\int \frac{\tan x}{(1+\cos x)^2}\,dx$	11^0 $\displaystyle\int \frac{dx}{(3+\cos x)\sin x}$	12^0 $\displaystyle\int \frac{\sin 3x + \cos 3x}{\sin 3x\cos 3x}\,dx$

Rešenje: 1^0 $\displaystyle\int \frac{x-\sqrt{\arctan 2x}}{1+4x^2}\,dx = \int \frac{x\,dx}{1+4x^2} - \int \frac{\sqrt{\arctan 2x}}{1+4x^2}\,dx$

$$= \frac{1}{8}\int \frac{d(1+4x^2)}{1+4x^2} - \frac{1}{2}\int \sqrt{\arctan 2x}\,d(\arctan 2x)$$
$$= \frac{1}{8}\ln(1+4x^2) - \frac{1}{2}\frac{\arctan^{3/2} 2x}{3/2} + C$$
$$= \frac{1}{8}\ln(1+4x^2) - \frac{1}{3}\arctan^{3/2} 2x + C.$$

2^0 $$\int \frac{\sin 2x}{\sqrt{\sin x+1}}\,dx = 2\int \frac{\sin x\cos x}{\sqrt{\sin x+1}}\,dx = \left|\begin{array}{l} t=\sqrt{\sin x+1},\ \sin x = t^2-1 \\ dt = \frac{1}{2}\frac{\cos x}{\sqrt{\sin x+1}}\,dx \end{array}\right|$$
$$= 4\int \big(t^2-1\big)dt = \frac{4}{3}t^3 - 4t + C = \frac{4}{3}t(t^2-3) + C$$
$$= \frac{4}{3}\sqrt{\sin x+1}\,(\sin x - 2) + C.$$

3^0 $$\int \sqrt{1-\sin 2x}\,dx = \int \sqrt{\sin^2 x + \cos^2 x - 2\sin x\cos x}\,dx$$
$$= \int \sqrt{(\sin x+\cos x)^2}\,dx = \int (\sin x + \cos x)dx$$
$$= \big(\sin x - \cos x\big) + C, \quad \text{za } \sin x + \cos x \geq 0.$$

4^0 $$\int \cos\frac{1}{x}\,\frac{dx}{x^2} = -\int \cos\frac{1}{x}\,d\Big(\frac{1}{x}\Big) = -\sin\frac{1}{x} + C.$$

5^0 $$\int \sqrt{\sin x\cos^2 x}\,dx = \int \sqrt{\sin x}\,\cos x\,dx = \int \sqrt{\sin x}\,d(\sin x) = \frac{(\sin x)^{3/2}}{3/2} + C,$$
za $x \in (2k\pi, 2k\pi + \pi/2)$, $k \in \mathbb{Z}$, gde je $\sin x, \cos x \geq 0$.

6^0 $$\int \cos^2 x\sin 2x\,dx = 2\int \cos^3 x\sin x\,dx = -2\int \cos^3 x\,d(\cos x)$$
$$= -\frac{1}{2}\cos^4 x + C.$$

7^0
$$\int\frac{dx}{\sin x+2\cos x+3}=\left|\begin{array}{ll} t=\tan\frac{x}{2} & dx=\frac{2dt}{1+t^2}\\ \sin x=\frac{2t}{1+t^2} & \cos x=\frac{1-t^2}{1+t^2}\end{array}\right|=2\int\frac{dt}{t^2+2t+5}$$
$$=2\int\frac{d(t+1)}{(t+1)^2+4}=\arctan\frac{t+1}{2}+C=\arctan\frac{\tan\frac{x}{2}+1}{2}+C.$$

8^0
$$\int\frac{dx}{2\cos x+3}=\left|\begin{array}{l} t=\tan\frac{x}{2},\ dx=\frac{2dt}{1+t^2}\\ \cos x=\frac{1-t^2}{1+t^2}\end{array}\right|=\int\frac{2dt}{t^2+5}\overset{1.c)}{=}\frac{2}{\sqrt5}\arctan\frac{\tan\frac{x}{2}}{\sqrt5}+C.$$

9^0
$$\int\frac{dx}{4-3\cos x}=\left|\begin{array}{l} t=\tan\frac{x}{2},\ dx=\frac{2dt}{1+t^2}\\ \cos x=\frac{1-t^2}{1+t^2}\end{array}\right|=\int\frac{2dt}{1+7t^2}$$
$$\overset{1.c)}{=}\frac{2}{\sqrt7}\arctan\left(\sqrt7\tan\frac{x}{2}\right)+C.$$

10^0
$$\int\frac{\tan x}{(1+\cos x)^2}\,dx=\int\frac{\sin x\,dx}{\cos x(1+\cos x)^2}=\left|\begin{array}{l} t=\cos x\\ dt=-\sin x\,dx\end{array}\right|$$
$$=-\int\frac{dt}{t(1+t)^2}=-\int\frac{1+t-t}{t(1+t)^2}\,dt=-\int\frac{dt}{t(1+t)}-\int\frac{d(1+t)}{(1+t)^2}$$
$$\overset{1.b)}{=}\log\left|\frac{1+\cos x}{\cos x}\right|+\frac{1}{1+\cos x}+C.$$

11^0
$$\int\frac{dx}{(3+\cos x)\sin x}=-\int\frac{d(\cos x)}{(3+\cos x)(1-\cos^2 x)}=\left|\begin{array}{l} t=\cos x\\ dt=-\sin x\,dx\end{array}\right|$$
$$=\int\frac{dt}{(3+t)(t^2-1)}=\frac12\int\frac{t+1-(t-1)}{(3+t)(t-1)(t+1)}\,dt$$
$$=\frac12\int\frac{dt}{(3+t)(t-1)}-\frac12\int\frac{dt}{(3+t)(t+1)}$$
$$\overset{1.b)}{=}\frac18\log\left|\frac{t-1}{t+3}\right|+\frac14\log\left|\frac{t+3}{t+1}\right|+C=\frac18\log\left|\frac{(1-\cos x)(3+\cos x)}{(1+\cos x)^2}\right|+C$$

12^0 $$\int \frac{\sin 3x+\cos 3x}{\sin 3x\cos 3x}\,dx=\int\frac{dx}{\cos 3x}+\int\frac{dx}{\sin 3x}=\left|\begin{array}{l} t=3x\\ dx=\frac{dt}{3}\end{array}\right|$$
$$=\frac13\int\frac{dt}{\cos t}+\frac13\int\frac{dt}{\sin t}\overset{7.2^0,7.5^0}{=}\frac13\left(\log\left|\frac{1+\tan\frac t2}{1-\tan\frac t2}\right|+\log\left|\tan\frac t2\right|\right)+C$$
$$=\frac13\log\left|\frac{\left(1+\tan\frac{3x}{2}\right)\tan\frac{3x}{2}}{1-\tan\frac{3x}{2}}\right|+C.$$

11. Metodom smene odrediti:

1^0 $\displaystyle\int\frac{dx}{\sinh x}$ $\qquad$ 2^0 $\displaystyle\int\frac{\sinh x}{\sqrt{\cosh 2x}}\,dx$

3^0 $\displaystyle\int\frac{\sinh x\cosh x}{\sqrt{\sinh^4 x+\cosh^4 x}}\,dx$ $\qquad$ 4^0 $\displaystyle\int\frac{dx}{\cosh^2 x\sqrt[3]{\tanh^2 x}}$

Rešenje: 1^0 $$\int\frac{dx}{\sinh x}=\frac12\int\frac{dx}{\sinh\frac x2\cosh\frac x2}=\frac12\int\frac{dx}{\tanh\frac x2\cosh^2\frac x2}$$
$$=\int\frac{d\left(\tanh\frac x2\right)}{\tanh\frac x2}=\ln\left|\tanh\frac x2\right|+C.$$

2^0 $$\int\frac{\sinh x}{\sqrt{\cosh 2x}}\,dx=\int\frac{\sinh x\,dx}{\sqrt{2\cosh^2 x-1}}=\frac{1}{\sqrt2}\int\frac{d(\cosh x)}{\sqrt{\cosh^2 x-\frac12}}$$
$$\overset{1.h)}{=}\frac{1}{\sqrt2}\ln\left|\cosh x+\sqrt{\cosh^2 x-\frac12}\right|+C.$$

3^0 $$\int\frac{\sinh x\cosh x}{\sqrt{\sinh^4 x+\cosh^4 x}}\,dx=\frac12\int\frac{\sinh 2x\,dx}{\sqrt{(\sinh^2 x)^2+(\cosh^2 x)^2}}$$
$$=\frac12\int\frac{\sinh 2x\,dx}{\sqrt{\left(\frac{\cosh 2x-1}{2}\right)^2+\left(\frac{\cosh 2x+1}{2}\right)^2}}=\int\frac{\sinh 2x\,dx}{\sqrt{2(\cosh^2 2x+1)}}$$
$$=\left|\begin{array}{l} t=\cosh 2x\\ dt=2\sinh 2x\,dx\end{array}\right|=\frac{1}{2\sqrt2}\int\frac{dt}{\sqrt{t^2+1}}=\frac{1}{2\sqrt2}\ln\left|t+\sqrt{t^2+1}\right|+C$$
$$=\frac{1}{2\sqrt2}\ln\left(\cosh 2x+\sqrt{\cosh^2 2x+1}\right)+C.$$

$4^0 \quad \displaystyle\int \frac{dx}{\cosh^2 x \sqrt[3]{\tanh^2 x}} = \int \tanh^{-2/3} x \, d(\tanh x) = 3\sqrt[3]{\tanh x} + C.$

12. Primenom metoda parcijalne integracije odrediti:

$$1^0 \int \frac{dx}{(a^2+x^2)^2} \qquad 2^0 \int \frac{x^2 dx}{(x^2-a^2)^2}$$
$$3^0 \int \frac{x^3}{\sqrt{a^2+x^2}}\, dx \qquad 4^0 \int \frac{x^3}{\sqrt{a^2-x^2}}\, dx$$
$$5^0 \int \frac{dx}{\left(\sqrt{x^2-a^2}\,\right)^5}$$

Rešenje: $1^0 \displaystyle\int \frac{dx}{(a^2+x^2)^2} = \frac{1}{a^2}\int \frac{a^2+x^2-x^2}{(a^2+x^2)^2}\, dx$

$$= \frac{1}{a^2}\int \frac{dx}{a^2+x^2} - \frac{1}{a^2}\int \frac{x^2}{(a^2+x^2)^2}\, dx \overset{1.c)}{=} \frac{1}{a^3}\arctan\frac{x}{a} - \frac{1}{a^2}\int \frac{x^2}{(a^2+x^2)^2}\, dx$$
$$= \left| \begin{array}{ll} u = x & du = dx \\ dv = \frac{x\,dx}{(a^2+x^2)^2} & v = \frac{1}{2}\displaystyle\int \frac{d(a^2+x^2)}{(a^2+x^2)^2} = -\frac{1}{2(a^2+x^2)} \end{array} \right|$$
$$= \frac{1}{a^3}\arctan\frac{x}{a} - \frac{1}{a^2}\left(-\frac{x}{2(a^2+x^2)} + \frac{1}{2}\int \frac{dx}{a^2+x^2} \right)$$
$$= \frac{1}{2a^3}\arctan\frac{x}{a} + \frac{x}{2a^2(a^2+x^2)} + C.$$

$$2^0 \quad \int \frac{x^2 dx}{(x^2-a^2)^2} = \left| \begin{array}{ll} u = x & dv = \frac{x\,dx}{(x^2-a^2)^2} \\ du = dx & v = -\frac{1}{2(x^2-a^2)} \end{array} \right| = -\frac{x}{2(x^2-a^2)} + \frac{1}{2}\int \frac{dx}{x^2-a^2}$$
$$\overset{1.d)}{=} \frac{x}{2(a^2-x^2)} + \frac{1}{4a}\log\left|\frac{x-a}{x+a}\right| + C.$$

3^0 $$\int \frac{x^3}{\sqrt{a^2+x^2}}\,dx = \left| \begin{array}{ll} u = x^2 & du = 2x\,dx \\ dv = \frac{x\,dx}{\sqrt{a^2+x^2}} & v \overset{1.\mathrm{j})}{=} \sqrt{a^2+x^2} \end{array} \right|$$

$$= x^2\sqrt{a^2+x^2} - 2\int x\sqrt{a^2+x^2}\,dx = x^2\sqrt{a^2+x^2} - \int\sqrt{a^2+x^2}\,d\big(a^2+x^2\big)$$

$$= x^2\sqrt{a^2+x^2} - \frac{2}{3}\sqrt{\big(a^2+x^2\big)^3} + C = \frac{x^2-2a^2}{3}\sqrt{a^2+x^2} + C.$$

Drugi način određivanja ovog integrala podrazumeva korišćenje rezultata

$$\int \frac{P_n(x)dx}{\sqrt{ax^2+bx+c}} = Q_{n-1}(x)\sqrt{ax^2+bx+c} + \lambda\int\frac{dx}{\sqrt{ax^2+bx+c}}\ .$$

Tada je $I = \displaystyle\int \frac{x^3}{\sqrt{a^2+x^2}}\,dx = \big(c_0x^2+c_1x+c_2\big)\sqrt{a^2+x^2} + \lambda\int\frac{dx}{\sqrt{a^2+x^2}}$, gde c_0, c_1, c_2 i λ određujemo metodom neodređenih koeficijenata.

Diferenciranjem poslednje jednakosti po x dobijamo

$$\frac{x^3}{\sqrt{a^2+x^2}} = (2c_0x+c_1)\sqrt{a^2+x^2} + \big(c_0x^2+c_1x+c_2\big)\frac{x}{\sqrt{a^2+x^2}} + \frac{\lambda}{\sqrt{a^2+x^2}}.$$

Nakon množenja sa $\sqrt{a^2+x^2}$ i sređivanja, izraz postaje

$$x^3 = 3c_0x^3 + 2c_1x^2 + (2a^2c_0+c_2)x + a^2c_1 + \lambda.$$

Jednačenjem koeficijenata uz iste stepene od x na levoj i desnoj strani jednakosti, dobijamo

$$c_0 = \frac{1}{3},\ c_1 = 0,\ c_2 = -\frac{2a^2}{3},\ \lambda = 0,$$

što ponovo daje vrednost integrala $I = \dfrac{x^2-2a^2}{3}\sqrt{a^2+x^2} + C.$

4^0 $$\int \frac{x^3}{\sqrt{a^2-x^2}}\,dx = \left| \begin{array}{ll} u = x^2 & du = 2x\,dx \\ dv = \frac{x\,dx}{\sqrt{a^2-x^2}} & v \overset{1.\mathrm{j})}{=} -\sqrt{a^2-x^2} \end{array} \right|$$

$$= -x^2\sqrt{a^2-x^2} + 2\int x\sqrt{a^2-x^2}\,dx = -x^2\sqrt{a^2-x^2} - \int\sqrt{a^2-x^2}\,d\big(a^2-x^2\big)$$

$$= -x^2\sqrt{a^2-x^2} - \frac{2}{3}\sqrt{\big(a^2-x^2\big)^3} + C = -\frac{x^2+2a^2}{3}\sqrt{a^2-x^2} + C.$$

Drugi način određivanja :

$$I=\int\frac{x^3}{\sqrt{a^2-x^2}}\,dx=\left(c_0x^2+c_1x+c_2\right)\sqrt{a^2-x^2}+\lambda\int\frac{dx}{\sqrt{a^2-x^2}}\ .$$

Diferenciranjem po x i sređivanjem koeficijenata uz iste stepene od x dobijamo

$$c_0=-\frac{1}{3},\ c_1=0,\ c_2=-\frac{2a^2}{3},\ \lambda=0,$$

tj., $I=-\dfrac{x^2+2a^2}{3}\sqrt{a^2-x^2}+C.$

5^0 $$\int\frac{dx}{\left(\sqrt{x^2-a^2}\right)^5}=\frac{1}{a^2}\int\frac{x^2-(x^2-a^2)}{(x^2-a^2)^{\frac{5}{2}}}\,dx=\frac{1}{a^2}\int\frac{x^2}{(x^2-a^2)^{\frac{5}{2}}}\,dx-\frac{1}{a^2}\int\frac{dx}{(x^2-a^2)^{\frac{3}{2}}}$$

$$=\left|\begin{array}{ll} u=x & dv=\frac{x\,dx}{(x^2-a^2)^{5/2}} \\ du=dx & v=\frac{1}{2}\int\frac{d(x^2-a^2)}{(x^2-a^2)^{5/2}}=-\frac{1}{3(x^2-a^2)^{3/2}}\end{array}\right|$$

$$=\frac{1}{a^2}\left(\frac{-x}{3(x^2-a^2)^{3/2}}+\frac{1}{3}\int\frac{dx}{(x^2-a^2)^{3/2}}\right)-\frac{1}{a^2}\int\frac{dx}{(x^2-a^2)^{3/2}}$$

$$=-\frac{1}{3a^2}\frac{x}{(x^2-a^2)^{3/2}}-\frac{2}{3a^2}\int\frac{dx}{(x^2-a^2)^{3/2}}$$

$$=-\frac{1}{3a^2}\frac{x}{(x^2-a^2)^{3/2}}-\frac{2}{3a^4}\int\frac{x^2-(x^2-a^2)}{(x^2-a^2)^{3/2}}\,dx$$

$$=-\frac{1}{3a^2}\frac{x}{(x^2-a^2)^{3/2}}-\frac{2}{3a^4}\left(\int\frac{x^2}{(x^2-a^2)^{3/2}}\,dx-\int\frac{dx}{\sqrt{x^2-a^2}}\right)$$

$$=\left|\begin{array}{ll} u=x & dv=\frac{x\,dx}{(x^2-a^2)^{3/2}} \\ du=dx & v=\frac{1}{2}\int\frac{d(x^2-a^2)}{(x^2-a^2)^{3/2}}=-\frac{1}{\sqrt{x^2-a^2}}\end{array}\right|$$

$$=\frac{-1}{3a^2}\frac{x}{(x^2-a^2)^{3/2}}-\frac{2}{3a^4}\left(\frac{-x}{\sqrt{x^2-a^2}}+\int\frac{dx}{\sqrt{x^2-a^2}}-\int\frac{dx}{\sqrt{x^2-a^2}}\right)$$

$$=\frac{2}{3a^4}\frac{x}{\sqrt{x^2-a^2}}-\frac{1}{3a^2}\frac{x}{\left(\sqrt{x^2-a^2}\right)^3}+C=\frac{x}{3a^4}\frac{2x^2-3a^2}{\left(\sqrt{x^2-a^2}\right)^3}+C.$$

13. Primenom metoda parcijalne integracije odrediti:

$1^0\ \displaystyle\int \log x\,dx$	$4^0\ \displaystyle\int \frac{\log x}{x^2}\,dx$	$7^0\ \displaystyle\int \log\left(x+\sqrt{a^2+x^2}\right)dx$
$2^0\ \displaystyle\int \log^2 x\,dx$	$5^0\ \displaystyle\int \frac{\log^2 x}{x^2}\,dx$	$8^0\ \displaystyle\int x^2\log\left(x+\sqrt{a^2+x^2}\right)dx$
$3^0\ \displaystyle\int x^2\log^2 x\,dx$	$6^0\ \displaystyle\int x\log\frac{2+x}{2-x}\,dx$	$9^0\ \displaystyle\int \frac{x\log(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}\,dx$

Rešenje: 1^0
$$\int \log x\,dx = \begin{vmatrix} u=\log x & du=\frac{dx}{x} \\ dv=dx & v=x \end{vmatrix} = x\log x - \int dx$$
$$= x\log x - x + C = x(\log x - 1) + C = x\log\frac{x}{e} + C.$$

2^0
$$\int \log^2 x\,dx = \begin{vmatrix} u=\log^2 x & du=2\log x\,\frac{dx}{x} \\ dv=dx & v=x \end{vmatrix} = x\log^2 x - 2\int \log x\,dx$$
$$\overset{13.1^0}{=} x\log^2 x - 2x(\log x - 1) + C = x\Big(\log^2\frac{x}{e} + 1\Big) + C.$$

3^0
$$\int x^2\log^2 x\,dx = \begin{vmatrix} u=\log^2 x & du=\frac{2\log x\,dx}{x} \\ dv=x^2dx & v=\frac{x^3}{3} \end{vmatrix} = \frac{x^3}{3}\log^2 x - \frac{2}{3}\int x^2\log x\,dx$$
$$= \begin{vmatrix} u=\log x & du=\frac{dx}{x} \\ dv=x^2dx & v=\frac{x^3}{3} \end{vmatrix} = \frac{x^3}{3}\log^2 x - \frac{2}{3}\Big(\frac{x^3}{3}\log x - \frac{1}{3}\int x^2dx\Big)$$
$$= \frac{1}{3}x^3\log^2 x - \frac{2}{9}x^3\log x + \frac{2}{27}x^3 + C = \frac{x^3}{27}\Big(9\log^2 x - 6\log x + 2\Big) + C$$
$$= \frac{x^3}{27}\big((3\log x - 1)^2 + 1\big) + C.$$

4^0
$$\int \frac{\log x}{x^2}\,dx = \begin{vmatrix} u=\log x & du=\frac{dx}{x} \\ dv=\frac{dx}{x^2} & v=-\frac{1}{x} \end{vmatrix} = -\frac{1}{x}\log x + \int\frac{dx}{x^2} = -\frac{1}{x}\log x - \frac{1}{x} + C$$
$$= -\frac{1}{x}(\log x + 1) + C = -\frac{1}{x}\log ex + C.$$

5^0 $$\int \frac{\log^2 x}{x^2}\,dx = \left|\begin{matrix} u=\log^2 x & dv=\frac{dx}{x^2} \\ du=\frac{2\log x\,dx}{x} & v=-\frac{1}{x}\end{matrix}\right| = -\frac{1}{x}\log^2 x + 2\int \frac{\log x}{x^2}\,dx$$
$$\overset{13.4^0}{=} -\frac{1}{x}\log^2 x - \frac{2}{x}\log ex + C = -\frac{1}{x}\big(\log^2 ex + 1\big) + C.$$

6^0 $$\int x\log\frac{2+x}{2-x}\,dx = \left|\begin{matrix} u=\log\frac{2+x}{2-x} & dv=x\,dx \\ du=\frac{4\,dx}{4-x^2} & v=\frac{x^2}{2}\end{matrix}\right| = \frac{x^2}{2}\log\frac{2+x}{2-x} + 2\int \frac{x^2}{x^2-4}\,dx$$
$$\overset{1.f)}{=} \frac{x^2}{2}\log\frac{2+x}{2-x} + 2x - 2\log\frac{2+x}{2-x} + C = \frac{x^2-4}{2}\log\frac{2+x}{2-x} + 2x + C.$$

7^0 $$\int \log\big(x+\sqrt{a^2+x^2}\big)\,dx = \left|\begin{matrix} u=\log\big(x+\sqrt{a^2+x^2}\big) & dv=dx \\ du=\frac{dx}{\sqrt{a^2+x^2}} & v=x\end{matrix}\right|$$
$$= x\log\big(x+\sqrt{a^2+x^2}\big) - \int \frac{x\,dx}{\sqrt{a^2+x^2}} \overset{1.j)}{=} x\log\big(x+\sqrt{a^2+x^2}\big) - \sqrt{a^2+x^2} + C.$$

8^0 $$\int x^2\log\big(x+\sqrt{a^2+x^2}\big)\,dx = \left|\begin{matrix} u=\log\big(x+\sqrt{a^2+x^2}\big) & dv=x^2dx \\ du=\frac{dx}{\sqrt{a^2+x^2}} & v=\frac{x^3}{3}\end{matrix}\right|$$
$$= \frac{x^3}{3}\log\big(x+\sqrt{a^2+x^2}\big) - \frac{1}{3}\int \frac{x^3}{\sqrt{a^2+x^2}}\,dx$$
$$\overset{12.3^0}{=} \frac{x^3}{3}\log\big(x+\sqrt{a^2+x^2}\big) - \frac{x^2-2a^2}{9}\sqrt{a^2+x^2} + C.$$

9^0 $$\int \frac{x\log(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}\,dx = \left|\begin{matrix} u=\log\big(x+\sqrt{1+x^2}\big) & dv=\frac{x\,dx}{\sqrt{1+x^2}} \\ du=\frac{dx}{\sqrt{1+x^2}} & v=\sqrt{1+x^2}\end{matrix}\right|$$
$$= \sqrt{1+x^2}\log\big(x+\sqrt{1+x^2}\big) - \int dx = \sqrt{1+x^2}\log\big(x+\sqrt{1+x^2}\big) - x + C.$$

14. Primenom metoda parcijalne integracije odrediti:

1^0 $\int x^3 \sin x\, dx$ $\quad$ 2^0 $\int x \sin^2 x\, dx$ $\quad$ 3^0 $\int x^2 \cos 2x\, dx$

4^0 $\displaystyle\int \frac{dx}{\cos^5 x}\, dx$ $\quad$ 5^0 $\displaystyle\int \tan^7 x\, dx$ $\quad$ 6^0 $\displaystyle\int x \sin\sqrt{x}\, dx$

7^0 $\displaystyle\int (2x-1)\sin 3x\, dx$ $\quad$ 8^0 $\displaystyle\int \sin^3 x \cos^3 x\, dx$ $\quad$ 9^0 $\displaystyle\int \frac{dx}{(\sin x + \cos x)^4}$

Rešenje: 1^0 $\int x^3 \sin x\, dx = \begin{vmatrix} u = x^3 & du = 3x^2 dx \\ dv = \sin x\, dx & v = -\cos x \end{vmatrix}$

$$= -x^3\cos x + 3\int x^2 \cos x\, dx = \begin{vmatrix} u = x^2 & du = 2x\, dx \\ dv = \cos x\, dx & v = \sin x \end{vmatrix}$$

$$= -x^3\cos x + 3\Big(x^2 \sin x - 2\int x \sin x\, dx\Big) = \begin{vmatrix} u = x & du = dx \\ dv = \sin x\, dx & v = -\cos x \end{vmatrix}$$

$$= -x^3\cos x + 3x^2\sin x - 6\Big(-x\cos x + \int \cos x\, dx\Big)$$

$$= -x^3\cos x + 3x^2 \sin x + 6x\cos x - 6\sin x + C.$$

2^0 $$\int x \sin^2 x\, dx = \begin{vmatrix} u = x & dv = \sin^2 x\, dx = \frac{1-\cos 2x}{2}\, dx \\ du = dx & v = \frac{x}{2} - \frac{1}{4}\sin 2x \end{vmatrix}$$

$$= \frac{x^2}{2} - \frac{x}{4}\sin 2x - \int\Big(\frac{x}{2} - \frac{1}{4}\sin 2x\Big)dx$$

$$= \frac{x^2}{2} - \frac{x}{4}\sin 2x - \frac{x^2}{4} - \frac{1}{8}\cos 2x + C = \frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{1}{8}\cos 2x + C.$$

3^0 $$\int x^2 \cos 2x\, dx = \begin{vmatrix} u = x^2 & du = 2x\, dx \\ dv = \cos 2x\, dx & v = \frac{1}{2}\sin 2x \end{vmatrix}$$

$$= \frac{x^2}{2}\sin 2x - \int x \sin 2x\, dx = \begin{vmatrix} u = x & du = dx \\ dv = \sin 2x\, dx & v = -\frac{1}{2}\cos 2x \end{vmatrix}$$

$$= \frac{x^2}{2}\sin 2x + \frac{x}{2}\cos 2x - \frac{1}{2}\int \cos 2x\, dx$$

$$= \frac{x^2}{2}\sin 2x + \frac{x}{2}\cos 2x - \frac{1}{4}\sin 2x + C = \frac{2x^2-1}{4}\sin 2x + \frac{x}{2}\cos 2x + C.$$

4^0 $$\int \frac{dx}{\cos^5 x} = \int \frac{\sin^2 x + \cos^2 x}{\cos^5 x}\, dx = \int \frac{\sin^2 x}{\cos^5 x}\, dx + \int \frac{dx}{\cos^3 x}$$

$$= \left| \begin{matrix} u = \sin x & du = \cos x\, dx \\ dv = \frac{\sin x\, dx}{\cos^5 x} & v = \frac{1}{4\cos^4 x} \end{matrix} \right| = \frac{\sin x}{4\cos^4 x} + \frac{3}{4} \int \frac{dx}{\cos^3 x}$$

$$\overset{9.1^0}{=} \frac{\sin x}{4\cos^4 x} + \frac{3}{8} \frac{\sin x}{\cos^2 x} + \frac{3}{8} \log \left| \frac{\cos \frac{x}{2} + \sin \frac{x}{2}}{\cos \frac{x}{2} - \sin \frac{x}{2}} \right| + C.$$

5^0 $$\int \tan^7 x\, dx = \int \frac{\sin^7 x}{\cos^7 x}\, dx = \left| \begin{matrix} u = \sin^6 x & du = 6\sin^5 x \cos x\, dx \\ dv = \frac{\sin x\, dx}{\cos^7 x} & v = \frac{1}{6\cos^6 x} \end{matrix} \right|$$

$$= \frac{1}{6} \tan^6 x - \int \frac{\sin^5 x}{\cos^5 x}\, dx = \left| \begin{matrix} u = \sin^4 x & du = 4\sin^3 x \cos x\, dx \\ dv = \frac{\sin x\, dx}{\cos^5 x} & v = \frac{1}{4\cos^4 x} \end{matrix} \right|$$

$$= \frac{1}{6} \tan^6 x - \frac{1}{4} \tan^4 x + \int \tan^3 x\, dx$$

$$\overset{9.8^0}{=} \frac{1}{6} \tan^6 x - \frac{1}{4} \tan^4 x + \frac{1}{2} \tan^2 x + \log|\cos x| + C.$$

Integral se može takođe izračunati i primenom metoda smene.

$$\int \tan^7 x\, dx = \int \tan^5 x \tan^2 x\, dx = \int \tan^5 x \frac{1 - \cos^2 x}{\cos^2 x}\, dx$$

$$= \int \tan^5 x\, d(\tan x) - \int \tan^5 x\, dx = \frac{1}{6} \tan^6 x - \int \tan^3 x \frac{1 - \cos^2 x}{\cos^2 x}\, dx$$

$$= \frac{1}{6} \tan^6 x - \frac{1}{4} \tan^4 x + \int \tan^3 x\, dx$$

$$\overset{9.8^0}{=} \frac{1}{6} \tan^6 x - \frac{1}{4} \tan^4 x + \frac{1}{2} \tan^2 x + \log|\cos x| + C.$$

6^0 $$\int x \sin \sqrt{x}\, dx = \left| \begin{matrix} t = \sqrt{x},\ x = t^2 \\ dx = 2t\, dt \end{matrix} \right| = 2 \int t^3 \sin t\, dt$$

$$\overset{14.1^0}{=} 2\big(-t^3 \cos t + 3t^2 \sin t + 6t \cos t - 6 \sin t \big) + C$$

$$= -2\sqrt{x^3} \cos \sqrt{x} + 6x \sin \sqrt{x} + 12\sqrt{x} \cos \sqrt{x} - 12 \sin \sqrt{x} + C.$$

7^0
$$\int (2x-1)\sin 3x\,dx = \begin{vmatrix} u=2x-1 & dv=\sin 3x\,dx \\ du=2dx & v=-\frac{1}{3}\cos 3x \end{vmatrix}$$
$$= -\frac{2x-1}{3}\cos 3x + \frac{2}{3}\int \cos 3x\,dx = \frac{1-2x}{3}\cos 3x + \frac{2}{9}\sin 3x + C.$$

8^0
$$\int \sin^3 x\cos^3 x\,dx = \frac{1}{8}\int \sin^3 2x\,dx = \begin{vmatrix} u=\sin^2 2x & du=4\sin 2x\cos 2x\,dx \\ dv=\sin 2x\,dx & v=-\frac{1}{2}\cos 2x \end{vmatrix}$$
$$= -\frac{1}{2}\sin^2 2x\cos 2x + 2\int \sin 2x\cos^2 2x\,dx$$
$$= -\frac{1}{2}\sin^2 2x\cos 2x - \int \cos^2 2x\,d(\cos 2x)$$
$$= -\frac{1}{2}\sin^2 2x\cos 2x - \frac{1}{3}\cos^3 2x + C.$$

9^0
$$\int \frac{dx}{(\sin x+\cos x)^4} = \int \frac{d(\sin x+\cos x)}{(\cos x-\sin x)(\sin x+\cos x)^4}$$
$$= \begin{vmatrix} u=\frac{1}{\cos x-\sin x} & du=-\frac{\sin x+\cos x}{(\cos x-\sin x)^2}\,dx \\ dv=\frac{d(\sin x+\cos x)}{(\sin x+\cos x)^4} & v=\frac{-1}{3(\sin x+\cos x)^3} \end{vmatrix}$$
$$= \frac{-1}{3(\sin x+\cos x)^3(\cos x-\sin x)} - \frac{1}{3}\int \frac{dx}{(\cos x-\sin x)^2(\sin x+\cos x)^2}$$
$$= \frac{-1}{3(\sin x+\cos x)^2(\cos^2 x-\sin^2 x)} - \frac{1}{3}\int \frac{dx}{(\cos^2 x-\sin^2 x)^2}$$
$$= -\frac{1}{3(1+\sin 2x)\cos 2x} - \frac{1}{3}\int \frac{dx}{\cos^2 2x}$$
$$= -\frac{1}{3(1+\sin 2x)\cos 2x} - \frac{1}{6}\tan 2x + C.$$

15. Primenom metoda parcijalne integracije odrediti:

$$1^0 \int \arcsin x\,dx \qquad 2^0 \int \arccos\frac{x}{2}\,dx \qquad 3^0 \int x^2\arccos x\,dx$$

$$4^0 \int \arcsin^2 x\,dx \qquad 5^0 \int x\arcsin^2 x\,dx \qquad 6^0 \int \frac{\arcsin x}{x^2}\,dx$$

$$7^0 \int \frac{x\arcsin x}{\sqrt{1-x^2}}\,dx \qquad 8^0 \int \arcsin x\arccos x\,dx$$

Rešenje: 1^0 $\displaystyle\int \arcsin x\,dx = \left|\begin{array}{ll} u=\arcsin x & dv=dx \\ du=\frac{dx}{\sqrt{1-x^2}} & v=x \end{array}\right| = x\arcsin x - \int\frac{x\,dx}{\sqrt{1-x^2}}$

$$\overset{1.j)}{=} x\arcsin x + \sqrt{1-x^2} + C.$$

2^0 $\displaystyle\int \arccos\frac{x}{2}\,dx = \left|\begin{array}{ll} u=\arccos\frac{x}{2} & dv=dx \\ du=-\frac{dx}{\sqrt{4-x^2}} & v=x \end{array}\right| = x\arccos\frac{x}{2} + \int\frac{x\,dx}{\sqrt{4-x^2}}$

$$\overset{1.j)}{=} x\arccos\frac{x}{2} - \sqrt{4-x^2} + C.$$

3^0 $\displaystyle\int x^2\arccos x\,dx = \left|\begin{array}{ll} u=\arccos x & dv=x^2dx \\ du=-\frac{dx}{\sqrt{1-x^2}} & v=\frac{x^3}{3} \end{array}\right| = \frac{x^3}{3}\arccos x + \frac{1}{3}\int\frac{x^3}{\sqrt{1-x^2}}\,dx$

$$\overset{12.4^0}{=} \frac{x^3}{3}\arccos x - \frac{x^2+2}{9}\sqrt{1-x^2} + C.$$

4^0 $\displaystyle\int \arcsin^2 x\,dx = \left|\begin{array}{ll} u=\arcsin^2 x & dv=dx \\ du=2\arcsin x\frac{dx}{\sqrt{1-x^2}} & v=x \end{array}\right|$

$$= x\arcsin^2 x - 2\int\arcsin x\frac{x\,dx}{\sqrt{1-x^2}} = \left|\begin{array}{ll} u=\arcsin x & du=\frac{dx}{\sqrt{1-x^2}} \\ dv=\frac{x\,dx}{\sqrt{1-x^2}} & v\overset{1.j)}{=}-\sqrt{1-x^2} \end{array}\right|$$

$$= x\arcsin^2 x + 2\sqrt{1-x^2}\arcsin x - 2\int dx$$

$$= x\arcsin^2 x + 2\sqrt{1-x^2}\arcsin x - 2x + C.$$

5^0 $$\int x\arcsin^2 x\,dx = \left|\begin{array}{ll} u=\arcsin^2 x & dv = x\,dx \\ du = 2\arcsin x\frac{dx}{\sqrt{1-x^2}} & v=\frac{x^2}{2}\end{array}\right|$$
$$= \frac{x^2}{2}\arcsin^2 x - \int \arcsin x\frac{x^2dx}{\sqrt{1-x^2}}$$
$$= \left|\begin{array}{ll} u = x\arcsin x & dv = \frac{x\,dx}{\sqrt{1-x^2}} \\ du = \big(\arcsin x + \frac{x}{\sqrt{1-x^2}}\big)dx & v \overset{1.j)}{=} -\sqrt{1-x^2}\end{array}\right|$$
$$= \frac{x^2}{2}\arcsin^2 x + x\sqrt{1-x^2}\arcsin x - \int\sqrt{1-x^2}\arcsin x\,dx - \int x\,dx$$
$$= \left|\begin{array}{ll} u=\arcsin x & du = \frac{dx}{\sqrt{1-x^2}} \\ dv = \sqrt{1-x^2}\,dx & v \overset{1.m)}{=} \frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\arcsin x\end{array}\right|$$
$$= \frac{x^2-1}{2}\arcsin^2 x + \frac{x}{2}\sqrt{1-x^2}\arcsin x - \frac{x^2}{2} + \frac{1}{2}\int x\,dx$$
$$+\frac{1}{2}\int \arcsin x\,d(\arcsin x)$$
$$= \frac{2x^2-1}{4}\arcsin^2 x + \frac{x}{2}\sqrt{1-x^2}\arcsin x - \frac{x^2}{4} + C.$$

6^0 $$\int\frac{\arcsin x}{x^2}\,dx = \left|\begin{array}{ll} u=\arcsin x & dv=\frac{dx}{x^2} \\ du=\frac{dx}{\sqrt{1-x^2}} & v=-\frac{1}{x}\end{array}\right| = -\frac{\arcsin x}{x} + \int\frac{dx}{x\sqrt{1-x^2}}$$
$$\overset{1.l)}{=} -\frac{1}{x}\arcsin x + \frac{1}{2}\log\left|\frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1}\right| + C.$$

7^0 $$\int\frac{x\arcsin x}{\sqrt{1-x^2}}\,dx = \left|\begin{array}{ll} u=\arcsin x & dv=\frac{x\,dx}{\sqrt{1-x^2}} \\ du=\frac{dx}{\sqrt{1-x^2}} & v\overset{1.j)}{=}-\sqrt{1-x^2}\end{array}\right| = -\sqrt{1-x^2}\arcsin x + \int dx$$
$$= x - \sqrt{1-x^2}\arcsin x + C.$$

8^0 $$\int \arcsin x \arccos x\,dx = \begin{vmatrix} u=\arccos x & dv=\arcsin x\,dx \\ du=-\frac{dx}{\sqrt{1-x^2}} & v\overset{15.1^0}{=} x\arcsin x+\sqrt{1-x^2} \end{vmatrix}$$
$$= x\arcsin x\arccos x+\sqrt{1-x^2}\arccos x+\int \arcsin x\frac{x\,dx}{\sqrt{1-x^2}}+\int dx$$
$$\overset{15.7^0}{=} x\arcsin x\arccos x+\sqrt{1-x^2}\big(\arccos x-\arcsin x\big)+2x+C.$$

Bez korišćenja rezultata 15.1^0 i 15.7^0, integral se može izračunati na sledeći način.

$$\int \arcsin x\arccos x\,dx = \begin{vmatrix} u=\arcsin x\arccos x & dv=dx \\ du=\frac{\arccos x-\arcsin x}{\sqrt{1-x^2}}\,dx & v=x \end{vmatrix}$$
$$= x\arcsin x\arccos x - \int (\arccos x-\arcsin x)\frac{x\,dx}{\sqrt{1-x^2}}$$
$$= \begin{vmatrix} u=\arccos x-\arcsin x & dv=\frac{x\,dx}{\sqrt{1-x^2}} \\ du=-\frac{2dx}{\sqrt{1-x^2}} & v=-\sqrt{1-x^2} \end{vmatrix}$$
$$= x\arcsin x\arccos x+\sqrt{1-x^2}(\arccos x-\arcsin x)+2\int dx$$
$$= x\arcsin x\arccos x+\sqrt{1-x^2}(\arccos x-\arcsin x)+2x+C.$$

16. Primenom metoda parcijalne integracije odrediti:

1^0 $\displaystyle\int x\arctan x\,dx$ $\quad 2^0$ $\displaystyle\int x^2\arctan x\,dx$ $\quad 3^0$ $\displaystyle\int \frac{1}{x^2}\arctan\frac{x}{3}\,dx$

4^0 $\displaystyle\int \arctan\sqrt{x}\,dx$ $\quad 5^0$ $\displaystyle\int \frac{x^2\arctan x}{1+x^2}\,dx$ $\quad 6^0$ $\displaystyle\int \arctan\frac{2x}{1-x^2}\,dx$

7^0 $\displaystyle\int \frac{x\,e^{\arctan x}}{(1+x^2)^{3/2}}\,dx$

Rešenje: 1^0 $$\int x\arctan x\,dx = \begin{vmatrix} u=\arctan x & du=\frac{dx}{1+x^2} \\ dv=x\,dx & v=\frac{x^2}{2} \end{vmatrix}$$
$$= \frac{x^2}{2}\arctan x-\frac{1}{2}\int\frac{x^2}{1+x^2}\,dx \overset{1.e)}{=} \frac{x^2+1}{2}\arctan x-\frac{x}{2}+C.$$

2^0

$$\int x^2 \arctan x\,dx = \left|\begin{array}{ll} u=\arctan x & du=\frac{dx}{1+x^2} \\ dv=x^2\,dx & v=\frac{x^3}{3}\end{array}\right| = \frac{x^3}{3}\arctan x - \frac{1}{3}\int \frac{x^3}{1+x^2}\,dx$$

$$= \left|\begin{array}{l} t=1+x^2 \\ dt=2x\,dx \end{array}\right| = \frac{x^3}{3}\arctan x - \frac{1}{6}\int \frac{t-1}{t}\,dt = \frac{x^3}{3}\arctan x - \frac{1}{6}\big(t-\log|t|\big)+C$$

$$= \frac{x^3}{3}\arctan x - \frac{1}{6}\big(1+x^2-\log(1+x^2)\big)+C.$$

3^0

$$\int \frac{1}{x^2}\arctan\frac{x}{3}\,dx = \left|\begin{array}{ll} u=\arctan\frac{x}{3} & dv=\frac{dx}{x^2} \\ du=\frac{3\,dx}{9+x^2} & v=-\frac{1}{x}\end{array}\right| = -\frac{1}{x}\arctan\frac{x}{3} + \int \frac{3\,dx}{x\big(9+x^2\big)}$$

$$\overset{1.k)}{=} -\frac{1}{x}\arctan\frac{x}{3} + \frac{1}{6}\log\left|\frac{x^2}{x^2+9}\right| + C.$$

4^0

$$\int \arctan\sqrt{x}\,dx = \left|\begin{array}{l} t=\sqrt{x},\ t^2=x \\ 2t\,dt=dx\end{array}\right| = 2\int t\arctan t\,dt$$

$$\overset{16.1^0}{=} \big(t^2+1\big)\arctan t - t + C = (x+1)\arctan\sqrt{x}-\sqrt{x}+C.$$

5^0

$$\int \frac{x^2\arctan x}{1+x^2}\,dx = \left|\begin{array}{ll} u=\arctan x & du=\frac{dx}{1+x^2} \\ dv=\frac{x^2\,dx}{1+x^2} & v\overset{1.e)}{=} x-\arctan x\end{array}\right|$$

$$= x\arctan x - \arctan^2 x - \int \frac{x\,dx}{1+x^2} + \int \arctan x\,d(\arctan x)$$

$$\overset{1.i)}{=} x\arctan x - \frac{1}{2}\arctan^2 x - \frac{1}{2}\log\big(1+x^2\big)+C.$$

6^0

$$\int \arctan\frac{2x}{1-x^2}\,dx = \left|\begin{array}{ll} u=\arctan\frac{2x}{1-x^2} & du=\frac{2dx}{1+x^2} \\ dv=dx & v=x\end{array}\right|$$

$$= x\arctan\frac{2x}{1-x^2} - 2\int\frac{x\,dx}{1+x^2} \overset{1.i)}{=} x\arctan\frac{2x}{1-x^2} - \log\big(1+x^2\big)+C.$$

7^0 $$I=\int \frac{x\,e^{\arctan x}}{(1+x^2)^{3/2}}\,dx = \left| \begin{array}{ll} u=\frac{x}{\sqrt{1+x^2}} & dv=\frac{e^{\arctan x}dx}{1+x^2} \\ du=\frac{dx}{(1+x^2)^{3/2}} & v=e^{\arctan x} \end{array} \right|$$
$$=\frac{xe^{\arctan x}}{\sqrt{1+x^2}}-\int \frac{e^{\arctan x}}{(1+x^2)^{3/2}}\,dx = \left| \begin{array}{ll} u=\frac{1}{\sqrt{1+x^2}} & du=-\frac{x\,dx}{(1+x^2)^{3/2}} \\ dv=\frac{e^{\arctan x}dx}{1+x^2} & v=e^{\arctan x} \end{array} \right|$$
$$=\frac{xe^{\arctan x}}{\sqrt{1+x^2}}-\frac{e^{\arctan x}}{\sqrt{1+x^2}}-\int \frac{x\,e^{\arctan x}}{(1+x^2)^{3/2}}\,dx=\frac{(x-1)\,e^{\arctan x}}{\sqrt{1+x^2}}-I.$$

Odatle je $I=\dfrac{e^{\arctan x}(x-1)}{2\sqrt{1+x^2}}+C.$

17. Primenom metoda parcijalne integracije odrediti:

1^0 $\int x^3e^{2x}\,dx$ 2^0 $\int \frac{x^2e^x}{(x+2)^2}\,dx$ 3^0 $\int e^{\sqrt{x}}dx$

4^0 $\int e^{ax}\cos bx\,dx$ 5^0 $\int e^{ax}\sin bx\,dx$

6^0 $\int e^{2x}\sin^2 x\,dx$ 7^0 $\int e^{2x}\cos^2 x\,dx$

Rešenje: 1^0 $$I=\int x^3e^{2x}dx=\left| \begin{array}{ll} u=x^3 & du=3x^2dx \\ dv=e^{2x}dx & v=\frac{1}{2}e^{2x} \end{array} \right|$$
$$=\frac{x^3}{2}e^{2x}-\frac{3}{2}\int x^2e^{2x}dx=\left| \begin{array}{ll} u=x^2 & du=2x\,dx \\ dv=e^{2x}dx & v=\frac{1}{2}e^{2x} \end{array} \right|$$
$$=\frac{x^3}{2}e^{2x}-\frac{3}{2}\left(\frac{x^2}{2}e^{2x}-\int xe^{2x}dx\right)=\frac{x^3}{2}e^{2x}-\frac{3x^2}{4}e^{2x}+\frac{3}{2}\int xe^{2x}dx$$
$$=\left| \begin{array}{ll} u=x & du=dx \\ dv=e^{2x}dx & v=\frac{1}{2}e^{2x} \end{array} \right|=\frac{x^3}{2}e^{2x}-\frac{3x^2}{4}e^{2x}+\frac{3}{2}\left(\frac{x}{2}e^{2x}-\frac{1}{2}\int e^{2x}dx\right)$$
$$=\frac{x^3}{2}e^{2x}-\frac{3x^2}{4}e^{2x}+\frac{3x}{4}e^{2x}-\frac{3}{8}e^{2x}+C=\frac{e^{2x}}{8}\left(4x^3-6x^2+6x-3\right)+C.$$

2^0 $$\int \frac{x^2e^x}{(x+2)^2}\,dx = \left|\begin{array}{ll} u = x^2e^x & du = x(x+2)e^x dx \\ dv = \frac{dx}{(x+2)^2} & v = -\frac{1}{x+2} \end{array}\right|$$
$$= -\frac{x^2}{x+2}e^x + \int xe^x dx = \left|\begin{array}{ll} u = x & du = dx \\ dv = e^x dx & v = e^x \end{array}\right|$$
$$= -\frac{x^2}{x+2}e^x + xe^x - \int e^x dx = -\frac{x^2}{x+2}e^x + xe^x - e^x + C$$
$$= \frac{x-2}{x+2}\,e^x + C.$$

3^0 $$\int e^{\sqrt{x}}dx = \int \sqrt{x}\frac{e^{\sqrt{x}}dx}{\sqrt{x}} = \left|\begin{array}{ll} u = \sqrt{x} & dv = \frac{e^{\sqrt{x}}dx}{\sqrt{x}} \\ du = \frac{dx}{2\sqrt{x}} & v = 2e^{\sqrt{x}} \end{array}\right| = 2\sqrt{x}e^{\sqrt{x}} - \int \frac{e^{\sqrt{x}}dx}{\sqrt{x}}$$
$$= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C = 2e^{\sqrt{x}}\big(\sqrt{x} - 1\big) + C.$$

4^0 $$I = \int e^{ax}\cos bx\,dx = \left|\begin{array}{ll} u = \cos bx & du = -b\sin bx\,dx \\ dv = e^{ax}dx & v = \frac{1}{a}e^{ax} \end{array}\right|$$
$$= \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\,dx = \left|\begin{array}{ll} u = \sin bx & du = b\cos bx\,dx \\ dv = e^{ax}dx & v = \frac{1}{a}e^{ax} \end{array}\right|$$
$$= \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\Big(\frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bx\,dx\Big)$$
$$= \frac{e^{ax}}{a^2}\big(a\cos bx + b\sin bx\big) - \frac{b^2}{a^2}\,I.$$

Odatle je $\quad \dfrac{a^2+b^2}{a^2}\,I = \dfrac{e^{ax}}{a^2}\big(a\cos bx + b\sin bx\big) + C_1, \quad$ odnosno
$$I = \frac{e^{ax}}{a^2+b^2}\big(a\cos bx + b\sin bx\big) + C.$$

5^0 $$\int e^{ax}\sin bx\,dx = \left|\begin{array}{ll} u = \sin bx & du = b\cos bx\,dx \\ dv = e^{ax}dx & v = \frac{1}{a}e^{ax} \end{array}\right|$$
$$= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bx\,dx \overset{17.4^0}{=} \frac{e^{ax}}{a^2+b^2}\big(a\sin bx - b\cos bx\big) + C.$$

6^0 $$\int e^{2x}\sin^2 x\,dx = \left|\begin{array}{ll} u=\sin^2 x & du = 2\sin x\cos x\,dx = \sin 2x\,dx \\ dv = e^{2x}dx & v=\frac{1}{2}e^{2x} \end{array}\right|$$
$$=\frac{1}{2}\left(e^{2x}\sin^2 x - \int e^{2x}\sin 2x\,dx\right) \overset{17.5^0}{=} \frac{e^{2x}}{8}\big(2-\cos 2x-\sin 2x\big)+C.$$

7^0 $$\int e^{2x}\cos^2 x\,dx = \int e^{2x}\big(1-\sin^2 x\big)\,dx = \int e^{2x}\,dx - \int e^{2x}\sin^2 x\,dx$$
$$\overset{17.6^0}{=} \frac{e^{2x}}{8}\big(2+\cos 2x+\sin 2x\big)+C.$$

18. Odrediti rekurentnu formulu za izračunavanje integrala I_n za $n\in\mathbb{N},\ n\geq 2$ i $a,b\neq 0$.

1^0 $I_n = \int \sqrt{x}\,\log^n x\,dx$	2^0 $I_n = \int \big(1-x^2\big)^{n/2}\,dx$
3^0 $I_n = \int \frac{dx}{(x^2+a^2)^n}$	4^0 $I_n = \int \frac{dx}{(ax^2+bx+c)^n}$
5^0 $I_n = \int x^{2n}\sqrt{x^2\pm a^2}\,dx$	6^0 $I_n = \int x^{2n+1}\sqrt{x^2\pm a^2}\,dx$
7^0 $I_n = \int \sin^n x\,dx$	8^0 $I_n = \int \frac{dx}{\cos^n x}$
9^0 $I_n = \int \tan^n x\,dx$	10^0 $I_n = \int \cot^n x\,dx$
11^0 $I_n = \int \frac{dx}{(a\sin x+b\cos x)^n}$	12^0 $I_n = \int \frac{dx}{(a+b\cos x)^n}$

Rešenje: 1^0 $$I_n = \int \sqrt{x}\,\log^n x\,dx = \left|\begin{array}{ll} u=\log^n x & du = n\log^{n-1}x\,\frac{dx}{x} \\ dv=\sqrt{x}\,dx & v=\frac{2}{3}x\sqrt{x} \end{array}\right|$$
$$=\frac{2}{3}x^{3/2}\log^n x - \frac{2n}{3}\int \sqrt{x}\,\log^{n-1}x\,dx = \frac{2}{3}x^{3/2}\log^n x - \frac{2n}{3}I_{n-1}.$$

Iskoristimo izvedenu relaciju za izračunavanje, na primer, integrala I_3. Kako je

$$I_1=\int\sqrt{x}\,\log x\,dx=\left|\begin{array}{ll} u=\log x & du=\frac{dx}{x} \\ dv=\sqrt{x} & v=\frac{2}{3}x^{3/2}\end{array}\right|=\frac{2}{3}x^{3/2}\log x-\frac{2}{3}\int\sqrt{x}\,dx$$
$$=\frac{2}{3}x^{3/2}\log x-\frac{4}{9}x^{3/2}+C=\frac{2}{9}x^{3/2}\big(3\log x-2\big)+C,$$

na osnovu izvedene rekurentne relacije imamo

$$I_2=\frac{2}{3}x^{3/2}\log^2 x-\frac{4}{3}I_1=\frac{2}{27}x^{3/2}\big(9\log^2 x-12\log x+8\big)+C,$$
$$I_3=\frac{2}{3}x^{3/2}\log^3 x-2I_2=\frac{2}{27}x^{3/2}\big(9\log^3 x-18\log^2 x+24\log x-16\big)+C.$$

2^0 $$I_n=\int\big(1-x^2\big)^{n/2}dx=\int\big(1-x^2\big)\big(1-x^2\big)^{(n-2)/2}dx$$
$$=I_{n-2}-\int x^2\big(1-x^2\big)^{\frac{n-2}{2}}dx=\left|\begin{array}{ll} u=x & dv=x\big(1-x^2\big)^{\frac{n-2}{2}}dx \\ du=dx & v=-\frac{1}{n}(1-x^2)^{n/2}\end{array}\right|$$
$$=I_{n-2}+\frac{x}{n}\big(1-x^2\big)^{n/2}-\frac{1}{n}\int\big(1-x^2\big)^{n/2}dx=-\frac{1}{n}I_n+I_{n-2}+\frac{x}{n}\big(1-x^2\big)^{n/2}.$$

Dakle, $\quad \dfrac{n+1}{n}I_n=I_{n-2}+\dfrac{x}{n}\big(1-x^2\big)^{n/2}, \qquad$ pa je

$$I_n=\frac{n}{n+1}I_{n-2}+\frac{x}{n+1}\big(1-x^2\big)^{n/2}.$$

Znamo da je $I_1=\int\sqrt{1-x^2}\,dx\overset{1.\mathrm{m})}{=}\dfrac{x}{2}\sqrt{1-x^2}+\dfrac{1}{2}\arcsin x+C$. Na osnovu izvedene rekurentne relacije dobijamo

$$I_3=\frac{3}{4}I_1+\frac{x}{4}\big(1-x^2\big)^{3/2}=\frac{3}{8}\arcsin x+\frac{x}{8}\sqrt{1-x^2}\big(5-2x^2\big)+C.$$

Kako je $\quad I_2=\displaystyle\int\big(1-x^2\big)dx=x-\frac{x^3}{3}+C$, možemo odrediti

$$I_4=\frac{4}{5}I_2+\frac{x}{5}\big(1-x^2\big)^2+C=x-\frac{2}{3}x^3+\frac{1}{5}x^5+C.$$

3^0 $$I_n = \int \frac{dx}{(x^2+a^2)^n} = \frac{1}{a^2}\int \frac{x^2+a^2-x^2}{(x^2+a^2)^n}\,dx = \frac{1}{a^2} I_{n-1} - \frac{1}{a^2}\int \frac{x^2}{(x^2+a^2)^n}\,dx$$
$$= \left| \begin{array}{ll} u = x & dv = \frac{x\,dx}{(x^2+a^2)^n} \\ du = dx & v = \frac{1}{(2-2n)(x^2+a^2)^{n-1}} \end{array} \right|$$
$$= \frac{2n-3}{(2n-2)a^2} I_{n-1} + \frac{1}{(2n-2)a^2} \frac{x}{(x^2+a^2)^{n-1}}.$$

Primetimo da su ovaj integral i dobijena rekurentna relacija samo specijalan slučaj integrala i relacije iz narednog primera ($a = 1,\ b = 0,\ c = a^2$).

4^0 $$I_n = \int \frac{dx}{(ax^2+bx+c)^n} = \int \frac{ax^2+bx+c}{(ax^2+bx+c)^{n+1}}\,dx$$
$$= \left| \begin{array}{l} d(ax^2+bx+c) = (2ax+b)dx \\ ax^2+bx+c = \frac{1}{4a}\big((2ax+b)^2+4ac-b^2\big) \end{array} \right|$$
$$= \int \frac{(2ax+b)^2+4ac-b^2}{4a(ax^2+bx+c)^{n+1}}\,dx = \frac{1}{4a}\int \frac{(2ax+b)^2}{(ax^2+bx+c)^{n+1}}\,dx + \frac{4ac-b^2}{4a} I_{n+1}$$
$$= \left| \begin{array}{ll} u = 2ax+b & du = 2a\,dx \\ dv = \frac{d(ax^2+bx+c)}{(ax^2+bx+c)^{n+1}} & v = -\frac{1}{n(ax^2+bx+c)^n} \end{array} \right|$$
$$= \frac{1}{4a}\left(\frac{-(2ax+b)}{n(ax^2+bx+c)^n} + \frac{2a}{n}\int \frac{dx}{(ax^2+bx+c)^n} \right) + \frac{4ac-b^2}{4a} I_{n+1}$$
$$= -\frac{2ax+b}{4an(ax^2+bx+c)^n} + \frac{1}{2n} I_n + \frac{4ac-b^2}{4a} I_{n+1}.$$
$$\Rightarrow\ \frac{4ac-b^2}{4a} I_{n+1} = I_n - \frac{1}{2n} I_n + \frac{2ax+b}{4an\big(ax^2+bx+c\big)^n}$$
$$\Rightarrow\ I_{n+1} = \frac{2a}{4ac-b^2}\frac{2n-1}{n} I_n + \frac{2ax+b}{n\big(4ac-b^2\big)\big(ax^2+bx+c\big)^n}.$$

5^0 $$I_n = \int x^{2n}\sqrt{x^2 \pm a^2}\,dx = \int x^{2n-2}\left(x^2 \pm a^2 \mp a^2\right)\sqrt{x^2 \pm a^2}\,dx$$

$$= \int x^{2n-2}\left(x^2 \pm a^2\right)^{\frac{3}{2}} dx \mp a^2 I_{n-1} = \left|\begin{array}{ll} u = \left(x^2 \pm a^2\right)^{\frac{3}{2}} & dv = x^{2n-2}dx \\ du = 3\sqrt{x^2 \pm a^2}\,x dx & v = \frac{x^{2n-1}}{2n-1} \end{array}\right|$$

$$= \frac{x^{2n-1}}{2n-1}\left(x^2 \pm a^2\right)^{\frac{3}{2}} - \frac{3}{2n-1} I_n \mp a^2 I_{n-1}$$

$$\Rightarrow \frac{2n+2}{2n-1} I_n = \frac{x^{2n-1}}{2n-1}\left(x^2 \pm a^2\right)^{\frac{3}{2}} \mp a^2 I_{n-1}$$

$$\Rightarrow I_n = \frac{x^{2n-1}}{2n+2}\left(x^2 \pm a^2\right)^{\frac{3}{2}} \mp \frac{2n-1}{2n+2} a^2 I_{n-1}.$$

6^0 $$I_n = \int x^{2n+1}\sqrt{x^2 \pm a^2}\,dx = \left|\begin{array}{ll} u = x^{2n} & du = 2nx^{2n-1}dx \\ dv = \sqrt{x^2 \pm a^2}\,x dx & v = \frac{1}{3}\left(x^2 \pm a^2\right)^{\frac{3}{2}} \end{array}\right|$$

$$= \frac{1}{3} x^{2n}\left(x^2 \pm a^2\right)^{\frac{3}{2}} - \frac{2n}{3}\int x^{2n-1}\left(x^2 \pm a^2\right)^{\frac{3}{2}} dx$$

$$= \frac{1}{3} x^{2n}\left(x^2 \pm a^2\right)^{\frac{3}{2}} - \frac{2n}{3}\left(I_n \pm a^2 I_{n-1}\right)$$

$$\Rightarrow \frac{2n+3}{3} I_n = \frac{1}{3} x^{2n}\left(x^2 \pm a^2\right)^{\frac{3}{2}} \mp \frac{2n}{3} a^2 I_{n-1}$$

$$\Rightarrow I_n = \frac{1}{2n+3} x^{2n}\left(x^2 \pm a^2\right)^{\frac{3}{2}} \mp \frac{2na^2}{2n+3} I_{n-1}.$$

7^0 $$I_n = \int \sin^n x\,dx = \int \sin^{n-2} x(1 - \cos^2 x)dx = I_{n-2} - \int \cos^2 x \sin^{n-2} x\,dx$$

$$= \left|\begin{array}{ll} u = \cos x & dv = \sin^{n-2} x\,d(\sin x) \\ du = -\sin x\,dx & v = \frac{1}{n-1}\sin^{n-1} x \end{array}\right|$$

$$= I_{n-2} - \frac{1}{n-1}\sin^{n-1} x\cos x - \frac{1}{n-1} I_n$$

$$\Rightarrow \frac{n}{n-1} I_n = I_{n-2} - \frac{1}{n-1}\sin^{n-1} x\cos x$$

$$\Rightarrow I_n = \frac{n-1}{n} I_{n-2} - \frac{1}{n}\sin^{n-1} x\cos x.$$

8^0
$$I_n = \int \frac{dx}{\cos^n x} = \int \frac{\sin^2 x + \cos^2 x}{\cos^n x}\, dx = \int \frac{\sin^2 x}{\cos^n x}\, dx + \int \frac{dx}{\cos^{n-2} x}$$
$$= \int \frac{\sin^2 x}{\cos^n x}\, dx + I_{n-2} = \left| \begin{array}{ll} u = \sin x & du = \cos x\, dx \\ dv = -\frac{d(\cos x)}{\cos^n x} & v = \frac{1}{(n-1)\cos^{n-1} x} \end{array} \right|$$
$$= \frac{\sin x}{(n-1)\cos^{n-1} x} - \frac{1}{(n-1)} I_{n-2} + I_{n-2}$$
$$= \frac{\sin x}{(n-1)\cos^{n-1} x} + \frac{n-2}{n-1}\, I_{n-2}.$$

9^0
$$I_n = \int \tan^n x\, dx = \int \tan^{n-2} x \frac{1 - \cos^2 x}{\cos^2 x}\, dx = \int \tan^{n-2} x \frac{dx}{\cos^2 x} - I_{n-2}$$
$$= \int \tan^{n-2} x\, d(\tan x) - I_{n-2} = \frac{1}{n-1} \tan^{n-1} x - I_{n-2}.$$

10^0
$$I_n = \int \cot^n x\, dx = \int \cot^{n-2} x \frac{1 - \sin^2 x}{\sin^2 x}\, dx = \int \cot^{n-2} x \frac{dx}{\sin^2 x} - I_{n-2}$$
$$= -\int \cot^{n-2} x\, d(\cot x) - I_{n-2} = -\frac{1}{n-1} \cot^{n-1} x - I_{n-2}.$$

11^0 $$I_n=\int\frac{dx}{(a\sin x+b\cos x)^n}=\int\frac{a\sin x+b\cos x}{(a\sin x+b\cos x)^{n+1}}\,dx$$

$$=\left|\begin{array}{ll} u=\frac{1}{(a\sin x+b\cos x)^{n+1}} & du=-(n+1)\frac{a\cos x-b\sin x}{(a\sin x+b\cos x)^{n+2}}\,dx \\ dv=(a\sin x+b\cos x)dx & v=-(a\cos x-b\sin x) \end{array}\right|$$

$$=-\frac{a\cos x-b\sin x}{(a\sin x+b\cos x)^{n+1}}-(n+1)\int\frac{(a\cos x-b\sin x)^2}{(a\sin x+b\cos x)^{n+2}}\,dx$$

$$=-\frac{a\cos x-b\sin x}{(a\sin x+b\cos x)^{n+1}}-(n+1)\int\frac{a^2+b^2-(a\sin x+b\cos x)^2}{(a\sin x+b\cos x)^{n+2}}\,dx$$

$$=-\frac{a\cos x-b\sin x}{(a\sin x+b\cos x)^{n+1}}-(n+1)\Big(\big(a^2+b^2\big)I_{n+2}-I_n\Big)$$

$$\Rightarrow\ I_{n+2}=\frac{n}{(n+1)\big(a^2+b^2\big)}I_n-\frac{1}{(n+1)\big(a^2+b^2\big)}\frac{a\cos x-b\sin x}{(a\sin x+b\cos x)^{n+1}}.$$

12^0 $$I_n=\int\frac{dx}{(a+b\cos x)^n}=\int\frac{a+b\cos x}{(a+b\cos x)^{n+1}}\,dx$$

$$=a\int\frac{dx}{(a+b\cos x)^{n+1}}+b\int\frac{\cos x\,dx}{(a+b\cos x)^{n+1}}=aI_{n+1}+b\int\frac{\cos x\,dx}{(a+b\cos x)^{n+1}}$$

$$=\left|\begin{array}{ll} u=\frac{1}{(a+b\cos x)^{n+1}} & du=(n+1)b\frac{\sin x\,dx}{(a+b\cos x)^{n+2}} \\ dv=\cos x\,dx & v=\sin x \end{array}\right|$$

$$=aI_{n+1}+b\left(\frac{\sin x}{(a+b\cos x)^{n+1}}-(n+1)b\int\frac{\sin^2 x\,dx}{(a+b\cos x)^{n+2}}\right)$$

$$=aI_{n+1}+\frac{b\sin x}{(a+b\cos x)^{n+1}}+(n+1)b^2\int\frac{1-\cos^2 x}{(a+b\cos x)^{n+2}}\,dx$$

$$=\frac{b\sin x}{(a+b\cos x)^{n+1}}+aI_{n+1}+(n+1)b^2I_{n+2}-\int\frac{(n+1)b^2\cos^2 x}{(a+b\cos x)^{n+2}}\,dx$$

$$=\frac{b\sin x}{(a+b\cos x)^{n+1}}+aI_{n+1}-(n+1)b^2I_{n+2}-(n+1)b^2\int\frac{\left(\frac{a+b\cos x-a}{b}\right)^2dx}{(a+b\cos x)^{n+2}}$$

$$=\frac{b\sin x}{(a+b\cos x)^{n+1}}-(n+1)I_n+a(2n+3)I_{n+1}-(n+1)\big(a^2+b^2\big)I_{n+2}.$$

$$\Rightarrow\ (n+1)\big(a^2+b^2\big)I_{n+2}=\frac{b\sin x}{(a+b\cos x)^{n+1}}-(n+2)I_n+a(2n+3)I_{n+1}$$

$$\Rightarrow\ I_{n+2}=\frac{b}{(n+1)\big(a^2+b^2\big)}\frac{\sin x}{(a+b\cos x)^{n+1}}-\frac{n+2}{(n+1)\big(a^2+b^2\big)}I_n$$
$$+\frac{(2n+3)a}{(n+1)\big(a^2+b^2\big)}I_{n+1}.$$

19. Razlaganjem racionalne funkcije, izračunati sledeće integrale

1^0 $\displaystyle\int\frac{x^3+6}{x^3-5x^2+6x}\,dx$ $\qquad$ 2^0 $\displaystyle\int\frac{dx}{(x-1)^2(x^2+1)^2}$

3^0 $\displaystyle\int\frac{dx}{x(x+1)(x^2+x+1)^3}$ $\qquad$ 4^0 $\displaystyle\int\frac{x}{x^3-3x+2}\,dx$

5^0 $\displaystyle\int\frac{dx}{x^3+1}$ $\qquad$ 6^0 $\displaystyle\int\frac{x+1}{x^3-1}\,dx$ $\qquad$ 7^0 $\displaystyle\int\frac{dx}{x^4+1}$

8^0 $\displaystyle\int\frac{x^2dx}{\big(x^2+2x+2\big)^2}$ $\qquad$ 9^0 $\displaystyle\int\frac{2x^2-5}{x^4-5x^2+6}\,dx$

10^0 $\displaystyle\int\frac{x^3-1}{x^4+x^2}\,dx$ $\qquad$ 11^0 $\displaystyle\int\frac{x+2}{x^3+1}\,dx$ $\qquad$ 12^0 $\displaystyle\int\frac{x+1}{x(x^2+2x+2)}\,dx$

Rešenje: 1^0 $I=\displaystyle\int\frac{x^3+6}{x^3-5x^2+6x}\,dx$; integral najpre svedemo na integral prave racionalne funkcije.

$$\frac{x^3+6}{x^3-5x^2+6x}=\frac{x^3-5x^2+6x+5x^2-6x+6}{x^3-5x^2+6x}=1+\frac{5x^2-6x+6}{x^3-5x^2+6x}.$$

Kako je $x^3-5x^2+6x=x(x^2-5x+6)=x(x-2)(x-3)$, to je

$$I=\int dx+\int\frac{5x^2-6x+6}{x(x-2)(x-3)}\,dx.$$

Nova racionalna podintegralna funkcija dozvoljava razvoj:

$$\frac{5x^2-6x+6}{x(x-2)(x-3)}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x-3}, \tag{1}$$

gde je koeficijente A,B,C potrebno odrediti.

Metod neodređenih koeficijenata podrazumeva sledeće:
pomnožimo jednakost (1) sa $x(x-2)(x-3)$. Ona postaje jednakost dva polinoma u razvijenom obliku

$$\begin{aligned} 5x^2-6x+6 &= A(x-2)(x-3)+Bx(x-3)+Cx(x-2) \\ &= x^2(A+B+C)+x(-5A-3B-2C)+6A. \end{aligned}$$

Izjednačavanjem koeficijenata uz iste stepene promenljive x dolazimo do sistema jednačina

$$5=A+B+C, \quad -6=-5A-3B-2C, \quad 6=6A.$$

Rešavanjem ovog sistema nalazimo $A=1,\ B=-7$ i $C=11$.

Time polazni integral svodimo na zbir četiri elementarna integrala:

$$\begin{aligned} I &= \int dx+\int \frac{dx}{x}-7\int \frac{dx}{x-2}+11\int \frac{dx}{x-3} \\ &= x+\log|x|-7\log|x-2|+11\log|x-3|+C. \end{aligned}$$

2^0 $I=\displaystyle\int \frac{dx}{(x-1)^2(x^2+1)^2}$, podintegralna funkcija dozvoljava razvoj:

$$\frac{1}{(x-1)^2(x^2+1)^2}=\frac{A_1}{x-1}+\frac{A_2}{(x-1)^2}+\frac{B_1x+C_1}{x^2+1}+\frac{B_2x+C_2}{\left(x^2+1\right)^2}. \tag{2}$$

Odredimo A_j, B_j, C_j metodom neodređenih koeficijenata. Nakon množenja (2) sa $(x-1)^2(x^2+1)^2$ dolazimo do jednakosti

$$\begin{aligned} 1 \;=\; & A_1(x-1)\left(x^2+1\right)^2+A_2\left(x^2+1\right)^2+(B_1x+C_1)(x-1)^2\left(x^2+1\right) \\ & +(B_2x+C_2)(x-1)^2 \\ =\; & x^5(A_1+B_1)+x^4(-A_1+A_2-2B_1+C_1)+x^3(2A_1+2B_1+B_2-2C_1) \\ & +x^2(-2A_1+2A_2-2B_1-2B_2+2C_1+C_2) \\ & +x(A_1+B_1+B_2-2C_1-2C_2)-A_1+A_2+C_1+C_2. \end{aligned}$$

Izjednačavanjem koeficijenata dva polinoma uz odgovarajuće stepene od x, dolazimo do sistema jednačina

$$\begin{aligned} &0=A_1+B_1 \\ &0=-A_1+A_2-2B_1+C_1 \\ &0=2A_1+2B_1+B_2-2C_1 \\ &0=-2A_1+2A_2-2B_1-2B_2+2C_1+C_2 \\ &0=A_1+B_1+B_2-2C_1-2C_2 \\ &1=-A_1+A_2+C_1+C_2, \end{aligned}$$

čijim rešavanjem nalazimo vrednosti

$$A_1=-1/2,\ A_2=1/4,\ B_1=1/2,\ C_1=1/4,\ B_2=1/2,\ C_2=0.$$

Polazni integral tada postaje

$$\begin{aligned}
I &= -\frac{1}{2}\int\frac{dx}{x-1}+\frac{1}{4}\int\frac{dx}{(x-1)^2}+\frac{1}{4}\int\frac{2x+1}{x^2+1}\,dx+\frac{1}{2}\int\frac{x\,dx}{\big(x^2+1\big)^2}\\
&= -\frac{1}{2}\log|x-1|-\frac{1}{4(x-1)}+\frac{1}{4}\int\frac{d\big(x^2+1\big)}{x^2+1}+\frac{1}{4}\int\frac{dx}{x^2+1}+\frac{1}{4}\int\frac{d\big(x^2+1\big)}{\big(x^2+1\big)^2}\\
&= \frac{1}{4}\Big(\log\frac{x^2+1}{(x-1)^2}-\frac{x^2+x}{(x-1)\big(x^2+1\big)}+\arctan x\Big)+C.
\end{aligned}$$

3^0 $I=\displaystyle\int\frac{dx}{x(x+1)(x^2+x+1)^3}$; primetimo sledeći razvoj podintegralne funkcije

$$\begin{aligned}
\frac{1}{x(x+1)(x^2+x+1)^3} &= \frac{x^2+x+1-x(x+1)}{x(x+1)(x^2+x+1)^3}\\
&= \frac{1}{x(x+1)(x^2+x+1)^2}-\frac{1}{(x^2+x+1)^3}\\
&= \frac{x^2+x+1-x(x+1)}{x(x+1)(x^2+x+1)^2}-\frac{1}{(x^2+x+1)^3}\\
&= \frac{1}{x(x+1)(x^2+x+1)}-\frac{1}{(x^2+x+1)^2}-\frac{1}{(x^2+x+1)^3}\\
&= \frac{1}{x(x+1)}-\frac{1}{x^2+x+1}-\frac{1}{(x^2+x+1)^2}-\frac{1}{(x^2+x+1)^3}.
\end{aligned}$$

Time polazni integral postaje

$$\begin{aligned}
I &= \int\frac{dx}{x(x+1)}-\int\frac{dx}{x^2+x+1}-\int\frac{dx}{(x^2+x+1)^2}-\int\frac{dx}{(x^2+x+1)^3}\\
&\overset{1.b)}{=} \log\Big|\frac{x}{x+1}\Big|-\int\frac{d(x+\frac{1}{2})}{\big(x+\frac{1}{2}\big)^2+\frac{3}{4}}-\int\frac{d(x+\frac{1}{2})}{\Big(\big(x+\frac{1}{2}\big)^2+\frac{3}{4}\Big)^2}\\
&\qquad -\int\frac{d(x+\frac{1}{2})}{\Big(\big(x+\frac{1}{2}\big)^2+\frac{3}{4}\Big)^3}=\Big|\ t=x+\tfrac{1}{2}\ \Big|\\
&\overset{1.c)}{=} \log\Big|\frac{x}{x+1}\Big|-\frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}-\int\frac{dt}{\big(t^2+\frac{3}{4}\big)^2}-\int\frac{dt}{\big(t^2+\frac{3}{4}\big)^3}\\
&\overset{\substack{12.1^0\\ 18.3^0}}{=} \log\Big|\frac{x}{x+1}\Big|-\frac{14}{3\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}-\frac{2}{3}\,\frac{2x+1}{x^2+x+1}-\frac{2x+1}{6\big(x^2+x+1\big)^2}+C.
\end{aligned}$$

4^0 $\displaystyle\int \frac{x}{x^3-3x+2}\,dx = \frac{2}{9}\log\left|\frac{1-x}{2+x}\right| + \frac{1}{3(1-x)} + C.$

$$5^0 \quad \int \frac{dx}{x^3+1} = \frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + \frac{1}{6}\log\frac{(x+1)^2}{x^2-x+1} + C$$
$$= \frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + \frac{1}{6}\log\left|\frac{(x+1)^3}{x^3+1}\right| + C.$$

6^0 $\displaystyle\int \frac{x+1}{x^3-1}\,dx = \frac{1}{3}\log\left|\frac{(x-1)^2}{x^2+x+1}\right| + C = \frac{1}{3}\log\left|\frac{(x-1)^3}{x^3-1}\right| + C.$

$$7^0 \quad \int \frac{dx}{x^4+1} = \int \frac{dx}{(x^2+1)^2-2x^2} = \frac{1}{2\sqrt{2}}\Big(\arctan(1+\sqrt{2}\,x) - \arctan(1-\sqrt{2}\,x)\Big)$$
$$+\frac{1}{4\sqrt{2}}\log\left|\frac{x^2+\sqrt{2}\,x+1}{x^2-\sqrt{2}\,x+1}\right| + C.$$

8^0 $\displaystyle\int \frac{x^2dx}{\left(x^2+2x+2\right)^2} = \arctan(x+1) + \frac{1}{x^2+2x+2} + C.$

9^0 $\displaystyle\int \frac{2x^2-5}{x^4-5x^2+6}\,dx = \frac{1}{2\sqrt{2}}\log\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right| + \frac{1}{2\sqrt{3}}\log\left|\frac{x-\sqrt{3}}{x+\sqrt{3}}\right| + C.$

10^0 $\displaystyle\int \frac{x^3-1}{x^4+x^2}\,dx = \frac{1}{x} + \frac{1}{2}\log(1+x^2) + \arctan x + C.$

11^0 $\displaystyle\int \frac{x+2}{x^3+1}\,dx = \sqrt{3}\arctan\frac{2x-1}{\sqrt{3}} + \frac{1}{6}\log\left|\frac{(x+1)^3}{x^3+1}\right| + C.$

12^0 $\displaystyle\int \frac{x+1}{x(x^2+2x+2)}\,dx = \frac{1}{2}\arctan(x+1) + \frac{1}{4}\log\frac{x^2}{x^2+2x+2} + C.$

20. Odrediti sledeće integrale iracionalnih funkcija

1^0 $\displaystyle\int \frac{dx}{\sqrt{x+2}+\sqrt[3]{x+2}}$ $\quad 2^0$ $\displaystyle\int \frac{1-\sqrt{x+1}}{1+\sqrt[3]{x+1}}\,dx$ $\quad 3^0$ $\displaystyle\int \frac{dx}{\sqrt[6]{x-1}+\sqrt[3]{x-1}+\sqrt{x-1}}$

4^0 $\displaystyle\int \frac{dx}{(x+1)^5\sqrt{x^2+2x}}$ $\quad 5^0$ $\displaystyle\int \sqrt{\frac{x+1}{x-1}}\,dx$ $\quad 6^0$ $\displaystyle\int \frac{dx}{\sqrt{(x-1)(2-x)}}$

Rešenje: 1^0 $$\int \frac{dx}{\sqrt{x+2}+\sqrt[3]{x+2}} = \begin{vmatrix} t^6 = x+2 \\ dx = 6t^5 dt \end{vmatrix} = 6\int \frac{t^5 dt}{t^3+t^2} = 6\int \frac{t^3 dt}{t+1}$$
$$= 6\int \frac{t^3+1-1}{t+1}\,dt = 6\int (t^2-t+1)dt - 6\int \frac{dt}{t+1}$$
$$= 2t^3 - 3t^2 + 6t - 6\log|t+1| + C$$
$$= 2\sqrt{x+2} - 3\sqrt[3]{x+2} + 6\sqrt[6]{x+2} - 6\log\left|\sqrt[6]{x+2}+1\right| + C.$$

2^0 $$\int \frac{1-\sqrt{x+1}}{1+\sqrt[3]{x+1}}\,dx = \begin{vmatrix} t^6 = x+1 \\ dx = 6t^5 dt \end{vmatrix} = 6\int \frac{(1-t^3)t^5}{1+t^2}\,dt$$
$$= 6\int(-t^6+t^4+t^3-t^2-t+1)dt + 6\int \frac{t-1}{t^2+1}\,dt$$
$$= -\frac{6}{7}t^7 + \frac{6}{5}t^5 + \frac{3}{2}t^4 - 2t^3 - 3t^2 + 6t + 3\int \frac{2t\,dt}{t^2+1} - 6\int \frac{dt}{t^2+1}$$
$$= -\frac{6}{7}t^7 + \frac{6}{5}t^5 + \frac{3}{2}t^4 - 2t^3 - 3t^2 + 6t + 3\log(t^2+1) - 6\arctan t + C$$
$$= -\frac{6}{7}\left(\sqrt[6]{x+1}\right)^7 + \frac{6}{5}\left(\sqrt[6]{x+1}\right)^5 + \frac{3}{2}\left(\sqrt[3]{x+1}\right)^2 - 2\sqrt{x+1}$$
$$-3\sqrt[3]{x+1} + 6\sqrt[6]{x+1} + 3\log(\sqrt[3]{x+1}+1) - 6\arctan\sqrt[6]{x+1} + C.$$

3^0 $$\int \frac{dx}{\sqrt[6]{x-1}+\sqrt[3]{x-1}+\sqrt{x-1}} = \begin{vmatrix} t^6 = x-1 \\ dx = 6t^5 dt \end{vmatrix} = 6\int \frac{t^5}{t+t^2+t^3}\,dt$$
$$= 6\int (t^2-t)dt + 6\int \frac{t\,dt}{t^2+t+1} = 2t^3 - 3t^2 + 3\int \frac{2t+1-1}{t^2+t+1}\,dt$$
$$= 2t^3 - 3t^2 + 3\int \frac{d(t^2+t+1)}{t^2+t+1}\,dt - 3\int \frac{dt}{t^2+t+1}$$
$$= 2t^3 - 3t^2 + 3\log|t^2+t+1| - 3\int \frac{dt}{(t+\frac{1}{2})^2+\frac{3}{4}}$$
$$\overset{1.c)}{=} 2t^3 - 3t^2 + 3\log\left|\frac{t^3-1}{t-1}\right| - \frac{3}{\sqrt{3}/2}\arctan\frac{t+\frac{1}{2}}{\sqrt{3}/2} + C$$
$$= 2\sqrt{x-1} - 3\sqrt[3]{x-1} + 3\log\left|\frac{\sqrt{x-1}-1}{\sqrt[6]{x-1}-1}\right| - 2\sqrt{3}\arctan\frac{2\sqrt[6]{x-1}+1}{\sqrt{3}} + C.$$

4^0 $$I=\int\frac{dx}{(x+1)^5\sqrt{x^2+2x}}=\left|\begin{array}{l} x+1=\frac{1}{t},\ x,t>0\\ dx=-\frac{dt}{t^2}\end{array}\right|=-\int\frac{t^4dt}{\sqrt{1-t^2}}$$
$$=\big(a_0t^3+a_1t^2+a_2t+a_3\big)\sqrt{1-t^2}+\lambda\int\frac{dt}{\sqrt{1-t^2}}.$$

Diferenciranjem po t i množenjem sa $\sqrt{1-t^2}$ dobijamo

$$-t^4=-4a_0t^4-3a_1t^3+(3a_0-2a_2)t^2+(2a_1-a_3)t+a_2+\lambda.$$

Izjednačavanjem koeficijenata uz stepene od t formiramo sistem jednačina

$$\begin{array}{ll} -1=-4a_0 & 0=2a_1-a_3\\ 0=-3a_1 & 0=a_2+\lambda\\ 0=3a_0-2a_2 & \end{array}$$

čije je rešenje

$$a_0=1/4,\quad a_1=0,\quad a_2=3/8,\quad a_3=0,\quad \lambda=-3/8.$$

Nakon sređivanja nalazimo da je polazni integral

$$I=\sqrt{x^2+2x}\Big(\frac{1}{4(x+1)^4}+\frac{3}{8(x+1)^2}\Big)-\frac{3}{8}\arcsin\frac{1}{x+1}+C.$$

Za $x<-2$ integral se izračunava analognim postupkom, vodeći računa da je $\sqrt{t^2}=-t$.

Ovaj integral takođe se može odrediti primenom Ojlerove smene.

$$I=\int\frac{dx}{(x+1)^5\sqrt{x^2+2x}}=\left|\begin{array}{l}\sqrt{x^2+2x}=t\,x,\ x=\frac{2}{t^2-1}\\ dx=-\frac{4t\,dt}{(t^2-1)^2}\end{array}\right|$$
$$=-4\int\frac{(t^2-1)^4}{(t^2+1)^5}\,dt=-4\int\frac{(t^2+1-2)^4}{(t^2+1)^5}\,dt$$
$$=-4\int\frac{(t^2+1)^4-8(t^2+1)^3+24(t^2+1)^2-32(t^2+1)+16}{(t^2+1)^5}\,dt$$
$$=-4\int\frac{dt}{t^2+1}+32\int\frac{dt}{(t^2+1)^2}-96\int\frac{dt}{(t^2+1)^3}+128\int\frac{dt}{(t^2+1)^4}$$
$$-64\int\frac{dt}{(t^2+1)^5}.$$

Koristeći izvedenu rekurentnu relaciju iz zadatka 18.3^0 nalazimo

$$I = -\frac{3}{2}\arctan t + \frac{5t^7 - 3t^5 + 3t^3 - 5t}{2(t^2+1)^4} + C.$$

Vraćanjem smene konačno dobijamo

$$I = \sqrt{x^2+2x}\,\frac{3x^2+6x+5}{4(x+1)^4} - \frac{3}{2}\arctan\frac{\sqrt{x^2+2x}}{x} + C.$$

5^0
$$\begin{aligned}\int\sqrt{\frac{x+1}{x-1}}\,dx &= \left|\begin{array}{l} t=\sqrt{\frac{x+1}{x-1}} \\ dx = \frac{-4t\,dt}{(t^2-1)^2}\end{array}\right| = -4\int\frac{t^2}{(t^2-1)^2}\,dt \\ &\overset{12.2^0}{=} \frac{2t}{t^2-1} + \log\left|\frac{t+1}{t-1}\right| + C \\ &= \sqrt{(x+1)(x-1)} + \log\left|x+\sqrt{(x+1)(x-1)}\right| + C.\end{aligned}$$

6^0
$$\begin{aligned}\int\frac{dx}{\sqrt{(x-1)(2-x)}} &= \left|\begin{array}{l} x\in(1,2)\Rightarrow x = 1+\sin^2 t,\ \ t\in(0,\pi/2) \\ dx = 2\sin t\cos t\,dt, \\ \sqrt{(x-1)(2-x)} = \sqrt{\sin^2 t(1-\sin^2 t)}\end{array}\right| \\ &= 2\int dt = 2t + C = 2\arcsin\sqrt{x-1} + C.\end{aligned}$$

Integral možemo rešiti i primenom treće Ojlerove smene.

$$\begin{aligned} I &= \int\frac{dx}{\sqrt{(x-1)(2-x)}} = \left|\begin{array}{l} \sqrt{(x-1)(2-x)} = t(x-1) \\ x = \frac{t^2+2}{t^2+1} \quad dx = \frac{-2t\,dt}{(t^2+1)^2}\end{array}\right| = -2\int\frac{dt}{t^2+1} \\ &= -2\arctan t + C = -2\arctan\sqrt{\frac{2-x}{x-1}} + C.\end{aligned}$$

21. Svođenjem na integral racionalne funkcije, izračunati sledeće integrale

1^0 $\displaystyle\int\frac{dx}{x+\sqrt{x^2+x+1}}$ 2^0 $\displaystyle\int\frac{dx}{1+\sqrt{1-2x-x^2}}$ 3^0 $\displaystyle\int\frac{x-\sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}}\,dx$

4^0 $\displaystyle\int\sqrt{x^3+x^4}\,dx$ 5^0 $\displaystyle\int\frac{\sqrt{x}}{(1+\sqrt[3]{x})^2}\,dx$ 6^0 $\displaystyle\int\sqrt[3]{3x-x^3}\,dx$

Rešenje: 1^0 $I = \displaystyle\int \frac{dx}{x+\sqrt{x^2+x+1}}$. Potkorena funkcija u integralu odgovara za primenu prve Ojlerove smene:

$$\sqrt{x^2+x+1} = \pm x + t.$$

Znak ispred x biramo tako da pogoduje sređivanju podintegralne funkcije. Kako je

$$t = \mp x + \sqrt{x^2+x+1},$$

posle poređenja sa podintegralnom funkcijom uzimamo $\sqrt{x^2+x+1} = -x+t$. Tada je

$$x^2+x+1 = x^2-2xt+t^2 \;\Rightarrow\; x+1 = t^2-2xt,$$

tj. $x = \dfrac{t^2-1}{1+2t}$, pa je $dx = 2\dfrac{t^2+t+1}{(2t+1)^2}\,dt$. Integral onda glasi

$$I = 2\int \frac{t^2+t+1}{t(2t+1)^2}\,dt.$$

Rastavimo racionalnu funkciju

$$2\frac{t^2+t+1}{t(2t+1)^2} = \frac{A_1}{t} + \frac{A_2}{2t+1} + \frac{A_3}{(2t+1)^2}.$$

Metodom neodređenih koeficijenata nalazimo da je

$$A_1 = 2, \qquad A_2 = -3, \qquad A_3 = -3.$$

Integral se transformiše u elementarne integrale

$$\begin{aligned} I &= 2\int\frac{dt}{t} - 3\int\frac{dt}{2t+1} - \frac{3}{2}\int\frac{d(2t+1)}{(2t+1)^2} = 2\log|t| - \frac{3}{2}\log|2t+1| + \frac{3}{2(2t+1)} + C_1 \\ &= 2\log\left|x+\sqrt{x^2+x+1}\right| - \frac{3}{2}\log\left|2x+2\sqrt{x^2+x+1}+1\right| - x + \sqrt{x^2+x+1} + C, \end{aligned}$$

gde je $C = C_1 - 1/2$.

2^0 $$I=\int\frac{dx}{1+\sqrt{1-2x-x^2}}=\left|\begin{array}{l}\sqrt{1-2x-x^2}=xt-1\\ dx=2\frac{-t^2+2t+1}{(t^2+1)^2}\,dt\end{array}\right|=\int\frac{-t^2+2t+1}{t(t-1)(t^2+1)}\,dt.$$

$$\frac{-t^2+2t+1}{t(t-1)(t^2+1)}=\frac{A_1}{t}+\frac{A_2}{t-1}+\frac{Bt+C}{t^2+1}$$
$$A_1=-1,\quad A_2=1,\quad B=0,\quad C=-2$$

$$I=\int\frac{dt}{t-1}-\int\frac{dt}{t}-2\int\frac{dt}{t^2+1}=\log\left|\frac{t-1}{t}\right|-2\arctan t+C$$
$$=\log\left|\frac{1-x+\sqrt{1-2x-x^2}}{1+\sqrt{1-2x-x^2}}\right|-2\arctan\frac{1+\sqrt{1-2x-x^2}}{x}+C.$$

3^0 $$I=\int\frac{x-\sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}}\,dx=\left|\begin{array}{l}\sqrt{x^2+3x+2}=\sqrt{(x+2)(x+1)}=t(x+1)\\ x=\frac{2-t^2}{t^2-1},\ dx=-\frac{2tdt}{(t^2-1)^2}\end{array}\right|$$
$$=-2\int\frac{t^2+2t}{(t-2)(t-1)(t+1)^3}\,dt.$$

$$-2\frac{t^2+2t}{(t-2)(t-1)(t+1)^3}=-\frac{16}{27(t-2)}+\frac{3}{4(t-1)}-\frac{17}{108(t+1)}+\frac{5}{18(t+1)^2}+\frac{1}{3(t+1)^3}$$

$$I=-\frac{16}{27}\log|t-2|+\frac{3}{4}\log|t-1|-\frac{17}{108}\log|t+1|-\frac{5}{18(t+1)}-\frac{1}{6(t+1)^2}+C$$
$$=-\frac{16}{27}\log\left|\sqrt{\frac{x+2}{x+1}}-2\right|+\frac{3}{4}\log\left|\sqrt{\frac{x+2}{x+1}}-1\right|-\frac{17}{108}\log\left|\sqrt{\frac{x+2}{x+1}}+1\right|$$
$$-\frac{5}{18\left(\sqrt{\frac{x+2}{x+1}}+1\right)}-\frac{1}{6\left(\sqrt{\frac{x+2}{x+1}}+1\right)^2}+C.$$

$$4^0 \quad I=\int \sqrt{x^3+x^4}\,dx = \int x^2(1+x^{-1})^{1/2}\,dx = \left|\begin{array}{l} 1+x^{-1}=t^2 \\ dx=\frac{-2t\,dt}{(t^2-1)^2} \end{array}\right|$$

$$= -2\int \frac{t^2 dt}{(t^2-1)^4} = -2\int \frac{t^2-1+1}{(t^2-1)^4}\,dt$$

$$= -2\int \frac{dt}{(t^2-1)^3} - 2\int \frac{dt}{(t^2-1)^4} = -2I_3 - 2I_4,$$

gde je $I_n = \int \frac{dt}{(t^2-1)^n}$. Rekurentnu relaciju za izračunavanje I_n

$$I_n = \frac{t}{2(n-1)(t^2-1)^{n-1}} - \frac{2n-3}{2(n-1)} I_{n-1}$$

možemo dobiti, na primer, iz 18.4^0 za $a=1, b=0$ i $c=-1$, imajući u vidu da je

$$I_1 = \int \frac{dt}{t^2-1} = \frac{1}{2}\log\left|\frac{t-1}{t+1}\right| + C_1.$$

Tada je

$$I_2 = -\frac{t}{2(t^2-1)} - \frac{1}{4}\log\left|\frac{t-1}{t+1}\right| + C_2.$$

$$I_3 = \frac{3t^3-5t}{8(t^2-1)^2} + \frac{3}{16}\log\left|\frac{t-1}{t+1}\right| + C_3.$$

$$I_4 = -\frac{15t^5-40t^3+33t}{48(t^2-1)^3} - \frac{5}{32}\log\left|\frac{t-1}{t+1}\right| + C_4.$$

$$I = -\frac{3t^5-8t^3-3t}{24(t^2-1)^3} - \frac{1}{16}\log\left|\frac{t-1}{t+1}\right| + C$$

$$= \frac{1}{24}\sqrt{x^2+x}(8x^2+2x-3) + \frac{1}{8}\log\left|\sqrt{x}+\sqrt{1+x}\,\right| + C.$$

5^0
$$\int \frac{\sqrt{x}}{\left(1+\sqrt[3]{x}\right)^2}\, dx = \left| \begin{array}{l} x = t^6 \\ dx = 6t^5 dt \end{array} \right| = 6\int \frac{t^8\, dt}{(t^2+1)^2}$$
$$= 6\int (t^4 - 2t^2 + 3)dt - 6\int \frac{4t^2+3}{(t^2+1)^2}\, dt$$
$$= \frac{6}{5}t^5 - 4t^3 + 18t - 6\int \frac{4(t^2+1)-1}{(t^2+1)^2}\, dt$$
$$= \frac{6}{5}t^5 - 4t^3 + 18t - 24\int \frac{dt}{t^2+1} + 6\int \frac{dt}{(t^2+1)^2}$$
$$\overset{12.1^0}{=} \frac{6}{5}t^5 - 4t^3 + 18t - 21\arctan t + 3\frac{t}{t^2+1} + C$$
$$= \frac{\sqrt[6]{x}}{5(1+\sqrt[3]{x}\,)}\left(6x - 14\sqrt[3]{x^2} + 70\sqrt[3]{x} + 105\right) - 21\arctan\sqrt[6]{x} + C.$$

6^0
$$\int \sqrt[3]{3x-x^3}\, dx = \int x(-1+3x^{-2})^{1/3} dx = \left| \begin{array}{l} -1+3x^{-2} = t^3,\ x = \sqrt{\frac{3}{t^3+1}} \\ dx = -\frac{3\sqrt{3}t^2 dt}{2(t^3+1)^{3/2}} \end{array} \right|$$
$$= -\frac{9}{2}\int \frac{t^3 dt}{(t^3+1)^2} = \left| \begin{array}{ll} u = t & dv = \frac{t^2 dt}{(t^3+1)^2} \\ du = dt & v = -\frac{1}{3(t^3+1)} \end{array} \right| = \frac{3t}{2(t^3+1)} - \frac{3}{2}\int \frac{dt}{t^3+1}$$
$$\overset{19.5^0}{=} \frac{3t}{2(t^3+1)} - \frac{\sqrt{3}}{2}\arctan\frac{2t-1}{\sqrt{3}} - \frac{1}{4}\log\left|\frac{(t+1)^3}{t^3+1}\right| + C$$
$$= \frac{1}{18}\log\frac{\left(\sqrt[3]{3x-x^3}+x\right)^3}{3x} + \frac{x^2}{9} + \frac{1}{9\sqrt{3}}\arctan\frac{2\sqrt[3]{3x-x^3}-x}{\sqrt{3}\,x} + C.$$

22. Svesti sledeće integrale na integral racionalne funkcije.

1^0 $\displaystyle\int \frac{3\tan x+1}{3-\tan x}\, dx$ $\qquad$ 2^0 $\displaystyle\int \frac{dx}{4+3\tan x}$

3^0 $\displaystyle\int \frac{1-2\cos x+3\sin x}{2+3\cos x-4\sin x}\, dx$ $\qquad$ 4^0 $\displaystyle\int \frac{dx}{(\sin x+2\cos x)^3}$

5^0 $\displaystyle\int \frac{dx}{\sin^2 x\cos^4 x}$ $\qquad$ 6^0 $\displaystyle\int \frac{dx}{\left(\sin x+\dfrac{2}{\cos x}\right)^2}$

7^0 $\displaystyle\int \sqrt{\frac{2x-1}{3-x}}\, dx$ $\qquad$ 8^0 $\displaystyle\int \sqrt[3]{\frac{x+2}{2x-1}}\, dx$

Rešenje: 1^0 $\displaystyle\int \frac{3\tan x+1}{3-\tan x}\,dx = \left|\begin{array}{l} t=\tan x \\ dx=\frac{dt}{1+t^2}\end{array}\right| = \int \frac{3t+1}{(3-t)(1+t^2)}\,dt\ .$

2^0 $$\int \frac{dx}{4+3\tan x} = \left|\begin{array}{l} t=\tan x \\ dx=\frac{dt}{1+t^2}\end{array}\right| = \int \frac{dt}{(4+3t)(1+t^2)}\ .$$

3^0 $$\begin{aligned}\int \frac{1-2\cos x+3\sin x}{2+3\cos x-4\sin x}\,dx &= \left|\begin{array}{ll} t=\tan\frac{x}{2} & dx=\frac{2dt}{1+t^2} \\ \sin x=\frac{2t}{1+t^2} & \cos x=\frac{1-t^2}{1+t^2}\end{array}\right| \\ &= 2\int \frac{3t^2+6t-1}{(5-8t-t^2)(1+t^2)}\,dt\ .\end{aligned}$$

4^0 $$\int \frac{dx}{(\sin x+2\cos x)^3} = \left|\begin{array}{ll} t=\tan\frac{x}{2} & dx=\frac{2dt}{1+t^2} \\ \sin x=\frac{2t}{1+t^2} & \cos x=\frac{1-t^2}{1+t^2}\end{array}\right| = \frac{1}{2}\int \frac{(1+t^2)dt}{(1+t-t^2)^3}\ .$$

5^0 $$\begin{aligned}\int \frac{dx}{\sin^2 x\cos^4 x} &= \int \frac{1}{\sin^2 x\cos^2 x}\frac{dx}{\cos^2 x} = \left|\begin{array}{ll} t=\tan x & dt=\frac{dx}{\cos^2 x} \\ \sin^2 x=\frac{t^2}{1+t^2} & \cos^2 x=\frac{1}{1+t^2}\end{array}\right| \\ &= \int \frac{(1+t^2)^2}{t^2}\,dt.\end{aligned}$$

6^0 $$\int \frac{dx}{\left(\sin x+\dfrac{2}{\cos x}\right)^2} = \left|\begin{array}{ll} t=\tan\frac{x}{2} & dx=\frac{2dt}{1+t^2} \\ \sin x=\frac{2t}{1+t^2} & \cos x=\frac{1-t^2}{1+t^2}\end{array}\right| = \frac{1}{2}\int \frac{(1+t^2)(1-t^2)^2dt}{\big(t(1-t^2)+(1+t^2)^2\big)^2}.$$

7^0 $$\int \sqrt{\frac{2x-1}{3-x}}\,dx = \left|\begin{array}{l} t=\sqrt{\frac{2x-1}{3-x}} \\ dx=\frac{10t\,dt}{(2+t^2)^2}\end{array}\right| = 10\int \frac{t^2dt}{(2+t^2)^2}.$$

8^0 $$\int \sqrt[3]{\frac{x+2}{2x-1}}\,dx = \left|\begin{array}{l} t=\sqrt[3]{\frac{x+2}{2x-1}} \\ dx=\frac{-15t^2dt}{(2t^3-1)^2}\end{array}\right| = -15\int \frac{t^3dt}{(2t^3-1)^2}.$$