On Nonconvex SVGD

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1 Setup

$$F = \frac{1}{n}\sum_{i=1}^{n} f_i, \qquad \pi \sim e^{-F}$$

$$f_n = \frac{1}{n_0}\sum_{i\in I} f_i, \qquad \#I = n_0, \qquad \sigma_n \sim e^{-f_n}$$

$$\phi_t(x) = \left(I - tP_{\mu_n}\nabla \log\left(\frac{\mu_n}{\sigma_n}\right)\right)(x), \qquad \mu_t = (\phi_t)_{\#}\,\mu_n$$

$$\varphi(t) = \int \log\left(\frac{\mu_t}{\pi}\right) d\mu_t$$

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We define the update rule as:

$$S_{\mu_{n+1}}\nabla\log\left(\frac{\mu_{n+1}}{\sigma_{n+1}}\right) = \begin{cases} S_{\mu_{n+1}}\nabla\log\left(\frac{\mu_{n+1}}{\pi}\right) & p, \\ S_{\mu_n}\nabla\log\left(\frac{\mu_n}{\sigma_n}\right) + S_{\mu_{n+1}}\nabla\log\left(\frac{\mu_{n+1}}{\sigma_{I'}}\right) - S_{\mu_n}\nabla\log\left(\frac{\mu_n}{\sigma_{I'}}\right) & 1-p, \end{cases}$$

where $\sigma_{I'} \sim \exp^{-f_{I'}}$, $f_{I'} = \frac{1}{b'}\sum_{i\in I'} f_i$.

3 Main Proof

$$\phi(\gamma) \leq \phi(0) + \gamma\left\langle S_{\mu_n}\nabla\log\frac{\mu_n}{\pi}, S_{mu_n}\nabla\log\left(\frac{\mu_n}{\sigma_n}\right)\right\rangle_H + \frac{C\cdot\gamma^2}{2}\left\langle S_{\mu_n}\nabla\log\frac{\mu_n}{\sigma_n}, S_{\mu_n}\nabla\log\left(\frac{\mu_n}{\sigma_n}\right)\right\rangle_H$$

then by similar calculation as in PAGE, we have,

$$\begin{aligned}\varphi(\gamma) \leqslant \varphi(0) - \frac{\gamma}{2}\left\|S_{\mu_n}\nabla\log\left(\frac{\mu_n}{\pi}\right)\right\|_H^2 - \left(\frac{1}{2\gamma} - \frac{c}{2}\right)\cdot\gamma^2\left\|S_{\mu_n}\nabla\log\left(\frac{\mu_n}{\sigma_n}\right)\right\|_H^2 \\ + \frac{\gamma}{2}\left\|S_{\mu_n}\nabla\log\left(\frac{\mu_n}{\pi}\right) - S_{\mu_n}\log\left(\frac{\mu_n}{\sigma_n}\right)\right\|_H^2\end{aligned}$$

Let

$$G_{n+1} := \mathbf{E}\left[\left\|S_{\mu_{n+1}}\nabla\log\left(\frac{\mu_{n+1}}{\pi}\right) - S_{\mu_{n+1}}\nabla\log\left(\frac{\mu_{n+1}}{\sigma_{n+1}}\right)\right\|_H^2\right]$$

Then a direct calculation now reveals that

$$
\begin{aligned}
G_{n+1} &= p \cdot 0 + (1-p) \cdot \mathrm{E} \left\| S_{\mu_n} \nabla \log \left(\frac{\mu_n}{\sigma_n} \right) + S_{\mu_{n+1}} \nabla \log \left(\frac{\mu_{n+1}}{\sigma_{I'}} \right) - S_{\mu_n} \nabla \log \left(\frac{\mu_n}{\sigma_{I'}} \right) - S_{\mu_{n+1}} \nabla \log \left(\frac{\mu_{n+1}}{\pi} \right) \right\|_H^2 \\
&= (1-p) \cdot \mathrm{E} \left\| S_{\mu_n} \nabla \log \left(\frac{\mu_n}{\sigma_n} \right) + S_{\mu_{n+1}} \nabla \log \left(\frac{\mu_{n+1}}{\sigma_{I'}} \right) - S_{\mu_n} \nabla \log \left(\frac{\mu_n}{\sigma_{I'}} \right) - S_{\mu_{n+1}} \nabla \log \left(\frac{\mu_{n+1}}{\pi} \right) + S_{\mu_n} \nabla \log \left(\frac{\mu_n}{\pi} \right) \right\|_H^2 \\
&\quad + (1-p) \cdot \left\| S_{\mu_n} \nabla \log \left(\frac{\mu_n}{\pi} \right) - S_{\mu_n} \nabla \log \left(\frac{\mu_n}{\sigma_I} \right) \right\|_H^2 .
\end{aligned}
$$

4 Key Inequality

We need to estimate

$$\left\| S_{\mu_{n+1}} \nabla \log \left(\frac{\mu_{n+1}}{\pi} \right) - S_{\mu_n} \nabla \log \left(\frac{\mu_n}{\pi} \right) \right\|_H^2$$

and

$$\left\| S_{\mu_{n+1}} \nabla \log \left(\frac{\mu_{n+1}}{\sigma_{I'}} \right) - S_{\mu_n} \nabla \log \left(\frac{\mu_n}{\sigma_{I'}} \right) \right\|_H^2$$

$S_{\mu_{n+1}} \nabla \log \left(\frac{\mu_{n+1}}{\pi} \right) - S_{\mu_n} \nabla \log \left(\frac{\mu_n}{\pi} \right)$
$= \int k(y,\cdot) \nabla \log(\mu_{n+1}(y)) \mu_{n+1}(y) dy - \int k(\phi_\gamma(x),\cdot) \nabla \log(\mu_n(x)) \mu_n(x) dx$
$+ \int k(\phi_\gamma(x),\cdot) \nabla \log(\pi(x)) \mu_n(x) dx - \int k(y,\cdot) \nabla \log(\pi(y)) \mu_{n+1}(y) dy + \int k(\phi_\gamma(x),\cdot) \nabla \log(\frac{\mu_n(x)}{\pi(x)}) \mu_n(x) dx -$
$\int k(x,\cdot) \nabla \log(\frac{\mu_n(x)}{\pi(x)}) \mu_n(x) dx = I + II + III$

$\left\| II \right\|_H^2 = \int \int k(\phi_\gamma(x), \phi_\gamma(x')) (\nabla \log(x) - \nabla \log(\pi(\phi_\gamma(x)))) (\nabla \log(x') - \nabla \log(\pi(\phi_\gamma(x')))) \mu(x) \mu(x') dx dx'$
since F is L-smooth, so

$|\nabla \log(x') - \nabla \log(\pi(\phi_\gamma(x')))| \le L \cdot |x - \phi_\gamma(x)| = L \cdot \gamma \cdot |P_{\mu_\gamma} \nabla \log(\frac{\mu_n(x)}{\sigma_n(x)})| \le C \cdot \gamma \cdot \left\| S_{\mu_n} \nabla \log(\frac{\mu_n}{\sigma_n}) \right\|_H$

then by Cauchy-Schwartz inequality and μ_n is weekly compact(will see this in the end), we derive

$$\|II\|_H^2 \le C \cdot \gamma^2 \cdot \left\| S_{\mu_n} \nabla \log(\frac{\mu_n}{\sigma_n}) \right\|_H^2$$

$\|I\|_H^2 = \left\| \int k(y,\cdot) \nabla \log(\mu_{n+1}(y)) \mu_{n+1}(y) dy - \int k(\phi_\gamma(x),\cdot) \nabla \log(\mu_n(x)) \mu_n(x) dx \right\|_H^2$
$= \int \int k(y,y') \nabla \log(\mu_{n+1}(y)) \nabla \log(\mu_{n+1}(y')) \mu_{n+1}(y) \mu_{n+1}(y') dy dy'$
$- \int \int k(y,\phi_\gamma(x)) \nabla \log(\mu_{n+1}(y)) \nabla \log(\mu_n(x')) \mu_{n+1}(y) \mu_n(x) dy dx'$
$- \int \int k(\phi_\gamma(x),y') \nabla \log(\mu_{n+1}(y')) \nabla \log(\mu_n(x)) \mu_{n+1}(y') \mu_n(x) dy' dx$
$+ \int \int k(\phi_\gamma(x),\phi_\gamma(x')) \nabla \log(\mu_n(x)) \nabla \log(\mu_n(x')) \mu_n(x) \mu_n(x') dx dx'$
$= \int \int tr \nabla_{1,2}^2 k(y,y') \mu_{n+1}(y) \mu_{n+1}(y') dy dy'$
$- \int \int tr(\nabla_{1,2}^2 k(y,\phi_\gamma(x')) J\phi_\gamma(x')) \mu_{n+1}(y) \mu_n(x') dy dx'$
$- \int \int tr(\nabla_{1,2}^2 k(\phi_\gamma(x),y') J\phi_\gamma(x)) \mu_n(x) \mu_{n+1}(y') dx dy'$
$+ \int \int tr(\nabla_{1,2}^2 K(\phi_\gamma(x),\phi_\gamma(x')) J\phi_\gamma(x) J\phi_\gamma(x')) \mu_n(x) \mu_n(x') dx dx'$
$= \int \int tr(\nabla_{1,2}^2 K(\phi_\gamma(x),\phi_\gamma(x')) (I - J\phi_\gamma(x)) (I - J\phi_\gamma(x'))) \mu_n(x) \mu_n(x') dx dx'$

assume $|\nabla_{1,2}^2 K(\cdot,\cdot)| \le C$, then $tr(\nabla_{1,2}^2 K(\phi_\gamma(x),\phi_\gamma(x'))(I - J\phi_\gamma(x))(I - J\phi_\gamma(x'))) \le C \cdot \gamma^2 \cdot \left\| S_{\mu_n} \nabla \log(\frac{\mu_n}{\sigma_n}) \right\|_H^2$

so $\|I\|_H^2 \le C \cdot \gamma^2 \cdot \left\| S_{\mu_n} \nabla \log(\frac{\mu_n}{\sigma_n}) \right\|_H^2$

$\|III\|_H^2 = \int \int (k(\phi_\gamma(x),\phi_\gamma(x')) - k(\phi_\gamma(x),x') - k(x,\phi_\gamma(x')) + k(x,x')) \nabla \log(\frac{\mu_n(x)}{\pi(x)}) \nabla \log(\frac{\mu_n(x')}{\pi(x')})$
$\mu_n(x) \mu_n(x') dx dx'$.

denote $F(x,x') := k(\phi_\gamma(x),\phi_\gamma(x')) - k(\phi_\gamma(x),x') - k(x,\phi_\gamma(x')) + k(x,x')$

then by integration by part, we have

$\|III\|_H^2 = \int \int (tr(\nabla_{1,2}^2 F(x,x')) + \nabla_2 F(x,x') \nabla \log(\pi(x)) + \nabla_1 F(x,x') \nabla \log(\pi(x')) + F(x,x') \nabla \log(\pi(x)) \nabla \log(\pi(x'))) \mu_n(x) \mu$

We need to know the order of $F(x,x'), \nabla_1 F(x,x'), \nabla_2 F(x,x'), \nabla_{1,2}^2 F(x,x')$

in the following we assume supremum norm the second,third and fourth order derivatives of k are finite, and we miss $O(|\phi_\gamma(x) - x|^2)$ and $O(|\phi_\gamma(x') - x'|^2)$ in the derivation.

$|F(x,x')| = |<\phi_\gamma(x'-x'), \nabla_2 k(\phi_\gamma(x),x')> + <x'-\phi_\gamma(x')), \nabla_2 k(x,x')>|$
$\leq |\nabla^2_{1,2}k(x,x')(\phi_\gamma(x)-x)(\phi_\gamma(x')-x')| \leq C\cdot\gamma^2\cdot\left\|S_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n})\right\|^2_H$
$|\nabla_1 F(x,x')| = |\nabla_1 k(\phi_\gamma(x),\phi_\gamma(x'))J\phi_\gamma(x) - \nabla_1 k(\phi_\gamma(x),x')J\phi_\gamma(x) - \nabla_1 k(x,\phi_\gamma(x')) + \nabla_1 k(x,x')|$
$= |\nabla^2_{1,2}k(\phi_\gamma(x),x')(\phi_\gamma(x'-x'))J\phi_\gamma(x) - \nabla^2_{1,2}k(x,x')(\phi_\gamma(x')-x')|$
$= |\nabla^3_{11,2}k(x,x')(\phi_\gamma(x')-x')(\phi_\gamma(x)-x) + \nabla^2_{1,2}k(\phi_\gamma(x),x)(\phi_\gamma(x')-x')(J\phi_\gamma(x)-I)| \leq C\cdot\gamma^2\left\|S_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n})\right\|^2_H$
similarly $|\nabla_2 F(x,x')| \leq C\cdot\gamma^2\left\|S_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n})\right\|^2_H$
$tr\nabla^2_{1,2}F(x,x') = tr(J\phi_\gamma(x)\nabla^2_{1,2}k(\phi_\gamma(x),\phi_\gamma(x'))J\phi_\gamma(x') - J\phi_\gamma(x)\nabla^2_{1,2}k(\phi_\gamma(x),x') - \nabla^2_{1,2}k(x,\phi_\gamma(x'))J\phi_\gamma(x') + \nabla^2_{1,2}k(x,x'))$,
denote $J\phi_\gamma(x') = I-\gamma JP_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n})(x') = I-\gamma P'$, $J\phi_\gamma(x) = I-\gamma JP_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n})(x) = I-\gamma P$
then
$tr\nabla^2_{1,2}F(x,x') = tr((I-\gamma P)\nabla^3_{1,22}k(\phi_\gamma(x),x')(\phi_\gamma(x')-x') - (I-\gamma P)\nabla^2_{1,2}k(\phi_\gamma(x),\phi_\gamma(x'))(\gamma P')$
$-\nabla^3_{1,22}k(x,x')(\phi_\gamma(x')-x') + (\gamma P')\nabla^2_{1,2}k(x,\phi_\gamma(x')))$
$= tr(\nabla^4_{11,22}k(x,x')(\phi_\gamma(x)-x)(\phi_\gamma(x')-x') - (\gamma P')\nabla^3_{11,2}k(x,\phi_\gamma(x'))(\phi_\gamma(x)-x) - (\gamma P)\nabla^3_{1,22}k(\phi_\gamma(x),x')(\phi_\gamma(x')-x') + (\gamma P)\nabla^2_{1,2}k(x,x')(\gamma P'))$
$|tr\nabla^2_{1,2}F(x,x')| \leq C\cdot\gamma^2\left\|S_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n})\right\|^2_H$
then by weak compactness, we know,
$\|III\|^2_H \leq C\cdot\gamma^2\left\|S_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n})\right\|^2_H$
similarly, since the set of $\{\sigma_{I'}\}\}$ is finite and have similar property with π, so
$\left\|S_{\mu_{n+1}}\nabla\log(\frac{\mu_{n+1}}{\sigma_{I'}}) - S_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_{I'}})\right\|^2_H \leq C\cdot\gamma^2\left\|S_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n})\right\|^2_H$
Finally,
$\mathbf{E}\left\|S_{\mu_{n+1}}\nabla\log(\frac{\mu_{n+1}}{\pi}) - S_{\mu_{n+1}}\nabla\log(\frac{\mu_{n+1}}{\sigma_{n+1}})\right\|^2_H \leq (1-p)\cdot\left\|S_{\mu_n}\nabla\log(\frac{\mu_n}{\pi}) - S_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n})\right\|^2_H + C\cdot\gamma^2\left\|S_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n})\right\|^2_H$
So if γ is small enough(depend on constant C), then
$\mathbf{E}(\Phi_{n+1}) \leq \mathbf{E}(\Phi_n) - \frac{\gamma}{2}\mathbf{E}\left\|S_{\mu_n}\nabla\log(\frac{\mu_n}{\pi})\right\|^2_H$,
where
$\Phi_n := KL(\mu_n|\pi) + \frac{\gamma}{2p}\left\|S_{\mu_n}\nabla\log(\frac{\mu_n}{\pi}) - S_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n})\right\|^2_H$
(you see Φ_n is decreasing, so $KL(\mu_n|\pi) \leq C$, so $\{\mu_n\}$ is weakly compact)
Problem:is $\int |x|^2\mu_n(x)dx$ weakly continuous?

If we cannot prove this, then we need more assumption on π, that is : there exists a constant $a > 0$ and a point x_0 such that $\int e^{ad(x_0,x)^2}\pi(dx) < +\infty$, then by a theorem from Vilani's book "optimal transport old and new" page 592. we have T_1 inequality $W_p(\mu,\nu) \leq \sqrt{\frac{2H_\nu(\mu)}{\lambda}}$, $p=1$,λ is some constant. So we have $W_1(\mu_n,\pi) \leq C$, by L-smoothness of F we have $\int |\nabla\log(\pi(x))|\mu_n(x)dx \leq C\cdot\int |x|\mu_n(x)dx \leq C$

???

5 Setup

$$F = \frac{1}{n}\sum_{i=1}^{n} f_i, \qquad \pi \sim e^{-F}$$

$$f_n = \frac{1}{n_0}\sum_{i\in I} f_i, \qquad \#I = n_0, \qquad \sigma_n \sim e^{-f_n}$$

$$\phi_t(x) = \left(I - tP_{\mu_n}\nabla \log\left(\frac{\mu_n}{\sigma_n}\right)\right)(x), \qquad \mu_t = (\phi_t)_{\#}\,\mu_n$$

$$\varphi(t) = \int \log\left(\frac{\mu_t}{\pi}\right) d\mu_t$$

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We define the update rule as:

$$S_{\mu_{n+1}}\nabla\log\left(\frac{\mu_{n+1}}{\sigma_{n+1}}\right) = \begin{cases} S_{\mu_{n+1}}\nabla\log\left(\frac{\mu_{n+1}}{\pi}\right) & p, \\ S_{\mu_n}\nabla\log\left(\frac{\mu_n}{\sigma_n}\right) + S_{\mu_{n+1}}\nabla\log\left(\frac{\mu_{n+1}}{\sigma_{I'}}\right) - S_{\mu_n}\nabla\log\left(\frac{\mu_n}{\sigma_{I'}}\right) & 1-p, \end{cases}$$

where $\sigma_{I'} \sim \exp^{-f_{I'}}$, $f_{I'} = \frac{1}{b'}\sum_{i\in I'} f_i$.

7 Main Proof

$$\phi(\gamma) \leq \phi(0) + \gamma\left\langle S_{\mu_n}\nabla\log\frac{\mu_n}{\pi}, S_{mu_n}\nabla\log\left(\frac{\mu_n}{\sigma_n}\right)\right\rangle_H + \frac{C\cdot\gamma^2}{2}\left\langle S_{\mu_n}\nabla\log\frac{\mu_n}{\sigma_n}, S_{\mu_n}\nabla\log\left(\frac{\mu_n}{\sigma_n}\right)\right\rangle_H$$

then by similar calculation as in PAGE, we have,

$$\begin{aligned}\varphi(\gamma) &\leqslant \varphi(0) - \frac{\gamma}{2}\left\|S_{\mu_n}\nabla\log\left(\frac{\mu_n}{\pi}\right)\right\|_H^2 - \left(\frac{1}{2\gamma} - \frac{c}{2}\right)\cdot\gamma^2\left\|S_{\mu_n}\nabla\log\left(\frac{\mu_n}{\sigma_n}\right)\right\|_H^2 \\ &\quad + \frac{\gamma}{2}\left\|S_{\mu_n}\nabla\log\left(\frac{\mu_n}{\pi}\right) - S_{\mu_n}\log\left(\frac{\mu_n}{\sigma_n}\right)\right\|_H^2\end{aligned}$$

Let

$$G_{n+1} := \mathbf{E}\left[\left\|S_{\mu_{n+1}}\nabla\log\left(\frac{\mu_{n+1}}{\pi}\right) - S_{\mu_{n+1}}\nabla\log\left(\frac{\mu_{n+1}}{\sigma_{n+1}}\right)\right\|_H^2\right]$$

Then a direct calculation now reveals that

$$\begin{aligned} G_{n+1} &= p\cdot 0 + (1-p)\cdot \mathrm{E}\left\|S_{\mu_n}\nabla\log\left(\frac{\mu_n}{\sigma_n}\right) + S_{\mu_{n+1}}\nabla\log\left(\frac{\mu_{n+1}}{\sigma_{I'}}\right) - S_{\mu_n}\nabla\log\left(\frac{\mu_n}{\sigma_{I'}}\right) - S_{\mu_{n+1}}\nabla\log\left(\frac{\mu_{n+1}}{\pi}\right)\right\|_H^2 \\ &= (1-p)\cdot\mathrm{E}\left\|S_{\mu_n}\nabla\log\left(\frac{\mu_n}{\sigma_n}\right) + S_{\mu_{n+1}}\nabla\log\left(\frac{\mu_{n+1}}{\sigma_{I'}}\right) - S_{\mu_n}\nabla\log\left(\frac{\mu_n}{\sigma_{I'}}\right) - S_{\mu_{n+1}}\nabla\log\left(\frac{\mu_{n+1}}{\pi}\right) + S_{\mu_n}\nabla\log\left(\frac{\mu_n}{\pi}\right)\right\|_H^2 \\ &\quad + (1-p)\cdot\left\|S_{\mu_n}\nabla\log\left(\frac{\mu_n}{\pi}\right) - S_{\mu_n}\nabla\log\left(\frac{\mu_n}{\sigma_I}\right)\right\|_H^2 \end{aligned}$$

8 Key Inequality

There exists stepsize γ_n(in the following for simplicity we write as γ) only depending on $\int |\nabla \log(\pi(x))\mu_n(x)dx|$ and $\int |\nabla \log(\sigma_{I'}(x))|\mu_n(x)dx, such that \mathbf{E}(\Phi_{n+1}) \leq \mathbf{E}(\Phi_n) - \frac{\gamma}{2}\mathbf{E}\left\| S_{\mu_n}\nabla \log(\frac{\mu_n}{\pi})\right\|_H^2$,

where

$\Phi_n := KL(\mu_n|\pi) + \frac{\gamma}{2p}\left\| S_{\mu_n}\nabla\log(\frac{\mu_n}{\pi}) - S_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n})\right\|_H^2$

We need to estimate

$$\left\| S_{\mu_{n+1}}\nabla \log\left(\frac{\mu_{n+1}}{\pi}\right) - S_{\mu_n}\nabla\log\left(\frac{\mu_n}{\pi}\right)\right\|_H^2$$

and

$$\left\| S_{\mu_{n+1}}\nabla \log\left(\frac{\mu_{n+1}}{\sigma_{I'}}\right) - S_{\mu_n}\nabla\log\left(\frac{\mu_n}{\sigma_{I'}}\right)\right\|_H^2$$

$S_{\mu_{n+1}}\nabla\log\left(\frac{\mu_{n+1}}{\pi}\right) - S_{\mu_n}\nabla\log\left(\frac{\mu_n}{\pi}\right)$

$= \int k(y,\cdot)\nabla\log(\mu_{n+1}(y))\mu_{n+1}(y)dy - \int k(\phi_\gamma(x),\cdot)\nabla\log(\mu_n(x))\mu_n(x)dx$

$+ \int k(\phi_\gamma(x),\cdot)\nabla\log(\pi(x))\mu_n(x)dx - \int k(y,\cdot)\nabla\log(\pi(y))\mu_{n+1}(y)dy + \int k(\phi_\gamma(x),\cdot)\nabla\log(\frac{\mu_n(x)}{\pi(x)})\mu_n(x)dx - \int k(x,\cdot)\nabla\log(\frac{\mu_n(x)}{\pi(x)})\mu_n(x)dx = I + II + III$

$\|II\|_H^2 = \int\int k(\phi_\gamma(x),\phi_\gamma(x'))(\nabla\log(x) - \nabla\log(\pi(\phi_\gamma(x))))(\nabla\log(x') - \nabla\log(\pi(\phi_\gamma(x'))))\mu(x)\mu(x')dxdx'$

since F is L-smooth, so

$|\nabla\log(x') - \nabla\log(\pi(\phi_\gamma(x')))| \leq L\cdot|x-\phi_\gamma(x)| = L\cdot\gamma\cdot|P_{\mu_\gamma}\nabla\log(\frac{\mu_n(x)}{\sigma_n(x)})| \leq C\cdot\gamma\cdot\left\|S_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n})\right\|_H$

then by Cauchy-Schwartz inequality and $k(x,x)$ is bounded, we derive

$$\|II\|_H^2 \leq C\cdot\gamma^2\cdot\left\|S_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n})\right\|_H^2$$

$\|I\|_H^2 = \left\|\int k(y,\cdot)\nabla\log(\mu_{n+1}(y))\mu_{n+1}(y)dy - \int k(\phi_\gamma(x),\cdot)\nabla\log(\mu_n(x))\mu_n(x)dx\right\|_H^2$

$= \int\int k(y,y')\nabla\log(\mu_{n+1}(y))\nabla\log(\mu_{n+1}(y'))\mu_{n+1}(y)\mu_{n+1}(y')dydy'$

$-\int\int k(y,\phi_\gamma(x))\nabla\log(\mu_{n+1}(y))\nabla\log(\mu_n(x'))\mu_{n+1}(y)\mu_n(x)dydx'$

$-\int\int k(\phi_\gamma(x),y')\nabla\log(\mu_{n+1}(y'))\nabla\log(\mu_n(x))\mu_{n+1}(y')\mu_n(x)dy'dx$

$+\int\int k(\phi_\gamma(x),\phi_\gamma(x'))\nabla\log(\mu_n(x))\nabla\log(\mu_n(x'))\mu_n(x)\mu_n(x')dxdx'$

$= \int\int tr\nabla^2_{1,2}k(y,y')\mu_{n+1}(y)\mu_{n+1}(y')dydy'$

$-\int\int tr(\nabla^2_{1,2}k(y,\phi_\gamma(x'))J\phi_\gamma(x'))\mu_{n+1}(y)\mu_n(x')dydx'$

$-\int\int tr(\nabla^2_{1,2}k(\phi_\gamma(x),y')J\phi_\gamma(x))\mu_n(x)\mu_{n+1}(y')dxdy'$

$+\int\int tr(\nabla^2_{1,2}K(\phi_\gamma(x),\phi_\gamma(x'))J\phi_\gamma(x)J\phi_\gamma(x'))\mu_n(x)\mu_n(x')dxdx'$

$= \int\int tr(\nabla^2_{1,2}K(\phi_\gamma(x),\phi_\gamma(x'))(I-J\phi_\gamma(x))(I-J\phi_\gamma(x')))\mu_n(x)\mu_n(x')dxdx'$

assume $|\nabla^2_{1,2}K(\cdot,\cdot)| \leq C$, then $tr(\nabla^2_{1,2}K(\phi_\gamma(x),\phi_\gamma(x'))(I-J\phi_\gamma(x))(I-J\phi_\gamma(x'))) \leq C\cdot\gamma^2\cdot\left\|S_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n})\right\|_H^2$

so $\|I\|_H^2 \leq C\cdot\gamma^2\cdot\left\|S_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n})\right\|_H^2$

$\|III\|_H^2 = \int\int(k(\phi_\gamma(x),\phi_\gamma(x')) - k(\phi_\gamma(x),x') - k(x,\phi_\gamma(x')) + k(x,x'))\nabla\log(\frac{\mu_n(x)}{\pi(x)})\nabla\log(\frac{\mu_n(x')}{\pi(x')})$
$\mu_n(x)\mu_n(x')dxdx'$.

denote $F(x,x') := k(\phi_\gamma(x),\phi_\gamma(x')) - k(\phi_\gamma(x),x') - k(x,\phi_\gamma(x')) + k(x,x')$

then by integration by part, we have

$\|III\|_H^2 = \int\int(tr(\nabla^2_{1,2}F(x,x'))+\nabla_2F(x,x')\nabla\log(\pi(x))+\nabla_1F(x,x')\nabla\log(\pi(x'))+F(x,x')\nabla\log(\pi(x))\nabla\log(\pi(x')))\mu_n(x)\mu$

We need to know the order of $F(x,x'), \nabla_1F(x,x'), \nabla_2F(x,x'), \nabla^2_{1,2}F(x,x')$

in the following we assume supremum norm the second,third and fourth order derivatives of k are finite, and we miss $O(|\phi_\gamma(x)-x|^2)$ and $O(|\phi_\gamma(x')-x'|^2)$ in the derivation.

$|F(x,x')| = |<\phi_\gamma(x'-x'),\nabla_2k(\phi_\gamma(x),x')> + <x'-\phi_\gamma(x')),\nabla_2k(x,x')>|$

$\leq |\nabla^2_{1,2}k(x,x')(\phi_\gamma(x)-x)(\phi_\gamma(x')-x')| \leq C\cdot\gamma^2\cdot\left\|S_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n})\right\|_H^2$

$|\nabla_1 F(x,x')| = |\nabla_1 k(\phi_\gamma(x),\phi_\gamma(x'))J\phi_\gamma(x) - \nabla_1 k(\phi_\gamma(x),x')J\phi_\gamma(x) - \nabla_1 k(x,\phi_\gamma(x')) + \nabla_1 k(x,x')|$
$= |\nabla^2_{1,2} k(\phi_\gamma(x),x')(\phi_\gamma(x'-x'))J\phi_\gamma(x) - \nabla^2_{1,2} k(x,x')(\phi_\gamma(x')-x')|$
$= |\nabla^3_{11,2} k(x,x')(\phi_\gamma(x')-x')(\phi_\gamma(x)-x) + \nabla^2_{1,2} k(\phi_\gamma(x),x)(\phi_\gamma(x')-x')(J\phi_\gamma(x)-I)| \le C\cdot\gamma^2 \left\|S_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n})\right\|^2_H$

similarly $|\nabla_2 F(x,x')| \le C\cdot\gamma^2 \left\|S_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n})\right\|^2_H$

$tr\nabla^2_{1,2}F(x,x') = tr(J\phi_\gamma(x)\nabla^2_{1,2}k(\phi_\gamma(x),\phi_\gamma(x'))J\phi_\gamma(x') - J\phi_\gamma(x)\nabla^2_{1,2}k(\phi_\gamma(x),x') - \nabla^2_{1,2}k(x,\phi_\gamma(x'))J\phi_\gamma(x') + \nabla^2_{1,2}k(x,x'))$,

denote $J\phi_\gamma(x') = I - \gamma JP_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n})(x') = I - \gamma P'$, $J\phi_\gamma(x) = I - \gamma JP_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n})(x) = I - \gamma P$

then

$tr\nabla^2_{1,2}F(x,x') = tr((I-\gamma P)\nabla^3_{1,22}k(\phi_\gamma(x),x')(\phi_\gamma(x')-x') - (I-\gamma P)\nabla^2_{1,2}k(\phi_\gamma(x),\phi_\gamma(x'))(\gamma P')$
$-\nabla^3_{1,22}k(x,x')(\phi_\gamma(x')-x') + (\gamma P')\nabla^2_{1,2}k(x,\phi_\gamma(x')))$
$= tr(\nabla^4_{11,22}k(x,x')(\phi_\gamma(x)-x)(\phi_\gamma(x')-x') - (\gamma P')\nabla^3_{11,2}k(x,\phi_\gamma(x'))(\phi_\gamma(x)-x) - (\gamma P)\nabla^3_{1,22}k(\phi_\gamma(x),x')(\phi_\gamma(x')-x') + (\gamma P)\nabla^2_{1,2}k(x,x')(\gamma P'))$

$|tr\nabla^2_{1,2}F(x,x')| \le C\cdot\gamma^2 \left\|S_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n})\right\|^2_H$

then, we know,

$\|III\|^2_H \le (C_1 + C_2\cdot\int|\nabla\log(\pi(x))|\mu_n(x)dx + C_3\cdot(\int|\nabla\log(\pi(x))|\mu_n(x)dx)^2)\cdot\gamma^2\left\|S_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n})\right\|^2_H$

similarly, since the set of $\{\sigma_{I'}\}\}$ is finite and have similar property with π, so

$\left\|S_{\mu_{n+1}}\nabla\log(\frac{\mu_{n+1}}{\sigma_{I'}}) - S_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_{I'}})\right\|^2_H \le (C_1' + C_2'\cdot\int|\nabla\log(\sigma_{I'}(x))|\mu_n(x)dx + C_3'\cdot(\int|\nabla\log(\sigma_{I'}(x))|\mu_n(x)dx)^2)\cdot \gamma^2\left\|S_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n})\right\|^2_H$

Finally,

$\mathbf{E}\left\|S_{\mu_{n+1}}\nabla\log(\frac{\mu_{n+1}}{\pi}) - S_{\mu_{n+1}}\nabla\log(\frac{\mu_{n+1}}{\sigma_{n+1}})\right\|^2_H \le (1-p)\cdot\left\|S_{\mu_n}\nabla\log(\frac{\mu_n}{\pi}) - S_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n})\right\|^2_H + (C_1 + C_2\cdot\int|\nabla\log(\pi(x))|\mu_n(x)dx + C_3\cdot(\int|\nabla\log(\pi(x))|\mu_n(x)dx)^2 + C_1' + C_2'\cdot\int|\nabla\log(\sigma_{I'}(x))|\mu_n(x)dx + C_3'\cdot(\int|\nabla\log(\sigma_{I'}(x))|\mu_n(x)dx)^2)\cdot \gamma^2\left\|S_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n})\right\|^2_H$

So if γ is small enough,s.t $\frac{\gamma}{2p}\cdot(C_1 + C_2\cdot\int|\nabla\log(\pi(x))|\mu_n(x)dx + C_3\cdot(\int|\nabla\log(\pi(x))|\mu_n(x)dx)^2 + C_1' + C_2'\cdot\int|\nabla\log(\sigma_{I'}(x))|\mu_n(x)dx + C_3'\cdot(\int|\nabla\log(\sigma_{I'}(x))|\mu_n(x)dx)^2) - (\frac{1}{2\gamma} - \frac{C}{2}) \le 0$, then

$\mathbf{E}(\Phi_{n+1}) \le \mathbf{E}(\Phi_n) - \frac{\gamma}{2}\mathbf{E}\left\|S_{\mu_n}\nabla\log(\frac{\mu_n}{\pi})\right\|^2_H$,

where

$\Phi_n := KL(\mu_n|\pi) + \frac{\gamma}{2p}\left\|S_{\mu_n}\nabla\log(\frac{\mu_n}{\pi}) - S_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n})\right\|^2_H$

If there exists a constant $a>0$ and a point x_0 such that $\int e^{ad(x_0,x)^2}\pi(dx) < +\infty$, then we have we have T_1 inequality

$$W_1(\mu,\pi) \le C\cdot\sqrt{KL(\mu|\pi)}$$

.

see "optimal transport old and new" page 592.

We assume there exists a constant $a>0$ and a point x_0 such that $\int e^{ad(x_0,x)^2}\pi(dx) < +\infty$, so the last lemma holds.

if we have $\Phi_n \le C$, then we have $\int|\nabla\log(\pi(x))|\mu_n(x)dx \le \tilde{C} \quad KL(\mu_n|\pi) \le \Phi_n \le C$, so $W_1(\mu_n,\pi) \le C\cdot\sqrt{KL(\mu_n,\pi)} \le C_1$, so by L-smoothness of F, we have $\int|\nabla\log(\pi(x))|\mu_n(x)dx \le C\int|x|\mu_n(x)dx = CW_1(\mu_n,\delta_0) \le C(W_1(\mu_n,\pi) + W_1(\pi,\delta)) \le \tilde{C}$

if $\Phi_n \le A$, then we can choose$\gamma_n \ge \gamma_0$, where γ_0 depend on A(as you see in proof of the main theorem, it depends on $\int||\nabla\log(\pi)\mu_n dx$, as you see in the last lemma ,$\int||\nabla\log(\pi)\mu_n dx$ depends on Φ_n), such that $\Phi_{n+1} \le A$

(note $\mu_{n+1} = (I - \gamma_n P_{\mu_n}\nabla\log(\frac{\mu_n}{\sigma_n}))_\sharp\mu_n$, so you see μ_n is independent of γ_n,but depend on γ_{n-1}).from the last lemma we can choose $\gamma_n \ge \gamma_0$, such that $\Phi_{n+1} \le \Phi_n - \frac{\gamma_n}{2}I^2_{stein}(\mu_n|\pi)$, so $KL(\mu_{n+1}|\pi) \le \Phi_{n+1} \le \Phi_n \le A$

if we set a constant A, s.t $\Phi_0 \leq A$, then we can set $\gamma_n = \gamma_0$, for all n, such that,

$\Phi_{n+1} \leq \Phi_n - \frac{\gamma_0}{2} I^2_{Stein}(\mu_n|\pi)$

we can use the last lemma recursively, so finish the proof.