$$
\begin{aligned}
\mathbb{E}\left[\|v_s-\tilde{v}_s\|_2^2\right] &\stackrel{(i)}{=} \mathbb{E}\left[\left\|u\int_0^s e^{-2(s-r)}\left(\nabla f(x_r)-\nabla\tilde{f}(x_0)\right)dr\right\|_2^2\right]\\
&= u^2\mathbb{E}\left[\left\|\int_0^s e^{-2(s-r)}\left(\nabla f(x_r)-\nabla\tilde{f}(x_0)dr\right)\right\|_2^2\right]\\
&\stackrel{(ii)}{\leq} su^2\int_0^s \mathbb{E}\left[\left\|e^{-2(s-r)}\left(\nabla f(x_r)-\nabla\tilde{f}(x_0)\right)\right\|_2^2\right]dr\\
&\stackrel{(iii)}{\leq} su^2\int_0^s \mathbb{E}\left[\left\|\left(\nabla f(x_r)-\nabla f(x_0)+\nabla f(x_0)-\nabla\tilde{f}(x_0)\right)\right\|_2^2\right]dr\\
&\stackrel{(iv)}{\leq} 2su^2\int_0^s \mathbb{E}\left[\|(\nabla f(x_r)-\nabla f(x_0))\|_2^2\right]dr + 2su^2\int_0^s \mathbb{E}\left[\left\|\left(\nabla f(x_0)-\nabla\tilde{f}(x_0)\right)\right\|_2^2\right]dr\\
&\stackrel{(v)}{\leq} 2su^2M^2\int_0^s \mathbb{E}\left[\|x_r-x_0\|_2^2\right]dr + \frac{2s^2u^2M^2}{n}\mathbb{E}_{x_0\sim p_0}\left[\|x_0-x^*\|_2^2\right]\\
&\stackrel{(vi)}{=} 2su^2M^2\int_0^s \mathbb{E}\left[\left\|\int_0^r v_w dw\right\|_2^2\right]dr + \frac{2s^2u^2M^2}{n}\mathbb{E}_{x_0\sim p_0}\left[\|x_0-x^*\|_2^2\right]\\
&\stackrel{(vii)}{\leq} 2su^2M^2\int_0^s r\left(\int_0^r \mathbb{E}\left[\|v_w\|_2^2\right]dw\right)dr + \frac{2s^2u^2M^2}{n}\mathbb{E}_{x_0\sim p_0}\left[\|x_0-x^*\|_2^2\right]\\
&\stackrel{(viii)}{\leq} 2su^2M^2\mathcal{E}_K\int_0^s r\left(\int_0^r dw\right)dr + \frac{2s^2u^2M^2}{n}\mathbb{E}_{x_0\sim p_0}\left[\|x_0-x^*\|_2^2\right]\\
&= \frac{2s^4u^2M^2\mathcal{E}_K}{3} + \frac{2s^2u^2M^2}{n}\mathbb{E}_{x_0\sim p_0}\left[\|x_0-x^*\|_2^2\right],
\end{aligned}
$$

Figure 1: copy from "On the Theory of Variance Reduction for Stochastic Gradient Monte Carlo " page 33.

In the paper of "QLSD: Quantised Langevin stochastic dynamics for Bayesian federated learning", they do nothing but deal with the term

$$\int_{\mathrm{X}^b}\left\|\sum_{i=1}^{b}\mathscr{C}\left(F_i^\star\left(\theta,x^{(1,i)}\right),x^{(2,i)}\right)-\nabla U(\theta)\right\|^2\nu^{\otimes b}\left(\mathrm{d}x^{(1:b)}\right),$$

$F_i^\star$ differs in different algorithms.
This is also true when we try to prove "Quantised underdamped langevin stochastic dynamics", corresponding to the red part in the above picture. $\nabla\tilde{f}$ will be replace by $\sum_{i=1}^{b}\mathscr{C}\left(F_i^\star\left(\theta,x^{(1,i)}\right),x^{(2,i)}\right)$, we can deal with this term like in "QLSD" without any difference(actually can be exact the same).
The proof of theorem 4.3 is from "Underdamped Langevin MCMC: A non-asymptotic analysis"

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