CS331-HW11-Lukang-Sun

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p1.

Proof. if $p = 1$, then $g^k = \nabla f(x^k)$, which is eactly the GD. if $\tau = n$, I will use induction to prove this: for $k = 1$, $g^k = \nabla f(x^k)$. if for $k < K$, this is true, we will prove for $k = K$, $g^K = \nabla f(x^K)$, then by induction, for all k, we have $g^k = \nabla f(x^k)$, which is eactly the GD method. for g^K it could be $\nabla f(x^K)$ or $g^{K-1} + \nabla f(x^K) - \nabla f(x^{K-1})$, since by assumption $g^{K-1} = \nabla f(x^{K-1})$, so $g^{K-1} + \nabla f(x^K) - \nabla f(x^{K-1}) = \nabla f(x^K)$. □

p2.

Theorem. *Assume f is L-smooth, lower bounded by f^{inf} and suppose that Assumption 12 holds* [10]. *Assume $n > 1$, choose minibatch size $\tau \in \{1, 2, \ldots, n\}$, probability $p \in (0, 1]$ and stepsize*

$$0 < \gamma \leq \min\{\frac{p}{2\mu}, \frac{1}{L + L_{\text{avg}}\sqrt{\frac{2(1-p)(n-\tau)}{p(n-1)\tau}}}\} \stackrel{def}{=} \gamma_{p,\tau}$$

Fix $K \geq 1$,then we have

$$E\left[f\left(x^K\right) - f^{\text{inf}} + m\|g^K - \nabla f(x^K)\|^2\right] \leq (1 - \gamma\mu)^K (f(x^0) - f^{\text{inf}}),$$

where $m = \frac{\gamma}{2(p-\gamma\mu)}$.

Proof. A direct calculation now reveals that

$$
\begin{aligned}
G &\overset{\text{def}}{=} \mathrm{E}\left[\left\|g^{k+1}-\nabla f\left(x^{k+1}\right)\right\|^2 \mid x^{k+1}, x^k, g^k, s^k\right] \\
&\overset{(365)}{=} p \underbrace{\left\|\nabla f\left(x^{k+1}\right)-\nabla f\left(x^{k+1}\right)\right\|^2}_{=0} + (1-p)\left\|g^k+\frac{1}{\tau}\sum_{i\in S^k}\left(\nabla f_i\left(x^{k+1}\right)-\nabla f_i\left(x^k\right)\right)-\nabla f\left(x^{k+1}\right)\right\|^2 \\
&= (1-p)\| \underbrace{g^k-\nabla f\left(x^k\right)}_{X} + \frac{1}{\tau}\sum_{i\in S^k}\underbrace{\left(\nabla f_i\left(x^{k+1}\right)-\nabla f_i\left(x^k\right)\right)}_{a_i} - \underbrace{\left(\nabla f\left(x^{k+1}\right)-\nabla f\left(x^k\right)\right)}_{\bar{a}=\frac{1}{n}\sum_i a_i}\|^2 \\
&= (1-p)\|X\|^2 + 2(1-p)\left\langle X, \frac{1}{\tau}\sum_{i\in S^k} a_i-\bar{a}\right\rangle + (1-p)\left\|\frac{1}{\tau}\sum_{i\in S^k} a_i-\bar{a}\right\|^2.
\end{aligned}
$$

Take full expectation, we have

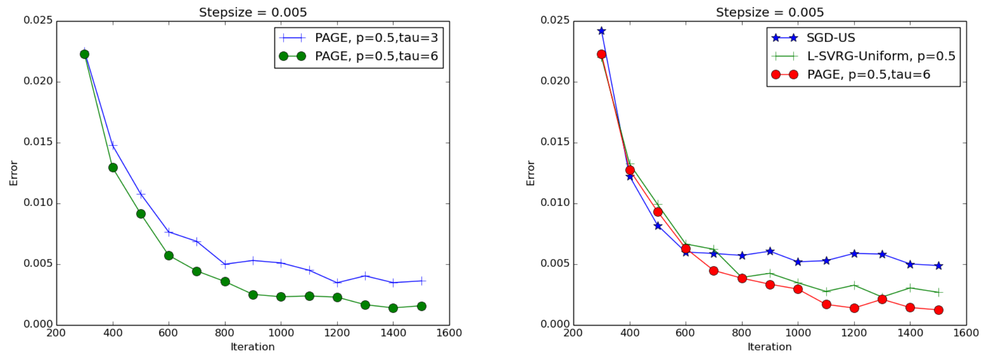
$$
\mathrm{E}\left[\left\|g^{k+1}-\nabla f(x^{k+1})\right\|^2\right] \leq= (1-p)\mathrm{E}\left[\left\|g^k-\nabla f\left(x^k\right)\right\|^2\right] + (1-p)\frac{n-\tau}{(n-1)\tau}L_{\text{avg}}^2\mathrm{E}\left[\left\|x^{k+1}-x^k\right\|^2\right] \tag{1}
$$

Then by lemma 127, we have

$$
\begin{aligned}
&E\left[f\left(x^{k+1}\right)-f^{\inf}+m\|g^{k+1}-\nabla f(x^{k+1})\|^2\right] \\
&\leq \mathrm{E}\left[f\left(x^k\right)-f^{\inf}\right]-\frac{\gamma}{2}\mathrm{E}\left[\left\|\nabla f\left(x^k\right)\right\|^2\right]-\left(\frac{1}{2\gamma}-\frac{L}{2}\right)\mathrm{E}\left[\left\|x^{k+1}-x^k\right\|^2\right]+\frac{\gamma}{2}\mathrm{E}\left[\left\|g^k-\nabla f\left(x^k\right)\right\|^2\right] \\
&+ m\mathrm{E}\left[\|g^{k+1}-\nabla f(x^{k+1})\|^2\right] \\
&\leq \mathrm{E}\left[f\left(x^k\right)-f^{\inf}\right]-\frac{\gamma}{2}\mathrm{E}\left[\left\|\nabla f\left(x^k\right)\right\|^2\right]-\left(\frac{1}{2\gamma}-\frac{L}{2}\right)\mathrm{E}\left[\left\|x^{k+1}-x^k\right\|^2\right]+\frac{\gamma}{2}\mathrm{E}\left[\left\|g^k-\nabla f\left(x^k\right)\right\|^2\right] \\
&+ m\left((1-p)\|g^k-\nabla f\left(x^k\right)\|^2+(1-p)\frac{n-\tau}{(n-1)\tau}L_{\text{avg}}^2\left\|x^{k+1}-x^k\right\|^2\right) \\
&= (1-\gamma\mu)E\left[(f\left(x^k\right)-f^{\inf})+m\|g^k-\nabla f(x^k)\|^2\right] \\
&- \underbrace{\left(\frac{1}{2\gamma}-\frac{L}{2}-m(1-p)\left(\frac{n-\tau}{(n-1)\tau}L_{\text{avg}}^2\right)\right)}_{A}\mathrm{E}\left[\left\|x^{k+1}-x^k\right\|^2\right] \\
&\leq (1-\mu\gamma)E\left[f\left(x^k\right)-f^{\inf}+m\|g^k-\nabla f(x^k)\|^2\right]
\end{aligned} \tag{2}
$$

where $m=\frac{\gamma}{2(p-\gamma\mu)}$, we choose $\gamma\leq\frac{p}{2\mu}$, then $p-\gamma\mu\geq\frac{p}{2}$,the last inequality is due to $\gamma\leq\frac{1}{L+L_{\text{avg}}\sqrt{\frac{21-p)(n-\tau)}{p(n-1)\tau}}}$(due to lemma 128 and the fact $\frac{(1-p)(n-\tau)L_{\text{avg}}^2}{(p-\gamma\mu)(n-1)\tau}\leq\frac{2(1-p)(n-\tau)L_{\text{avg}}^2}{(n-1)\tau p}$ when $\gamma\leq\frac{p}{2\mu}$,we have $A\geq 0$.) Use (2) for K times, then we have

$$
E\left[f\left(x^K\right)-f^{\inf}+m\|g^K-\nabla f(x^K)\|^2\right]\leq(1-\gamma\mu)^K(f(x^0)-f^{\inf})
$$



(a) (b)

Figure 1: (a) shows $\mathrm{E}\left[||\nabla f||^2\right]$ changes in terms of iteration number K with different τ,(b) shows $\mathrm{E}\left[||\nabla f||^2\right]$ changes in terms of iteration number K with different SGD method.

□

p3. (see Figure 1.) a = [matrix([[0.1]]), matrix([[0.424466]]), matrix([[0.77981303]]), matrix([[0.20033184]]), matrix([[0.51116473]]), matrix([[0.2604399]]), matrix([[0.97100656]]), matrix([[0.21263449]]), matrix([[0.26417151]]), matrix([[0.15995097]])]

b = [matrix([[0.4231786]]), matrix([[0.524466]]), matrix([[0.17981303]]), matrix([[0.50033184]]), matrix([[0.71116473]]), matrix([[0.0604399]]), matrix([[0.37100656]]), matrix([[0.91263449]]), matrix([[0.66417151]]), matrix([[0.65995097]])] ,$f = \frac{1}{10}\sum_{i=1}^{10} f_i(x,y)$, $f_i(x,y) = \sin(x+a[i]) + \cos(y+b[i])$, for the SGD method, I use SGD-US, L-SVRG-US and PAGE. Initial point is init = matrix([[-0.5],[-0.2]]). For the batch size of the PAGE algorithm, I choose $\tau = 3 \text{and} 6$ respectively, the results using different batch size are a little different: with larger batch size, the line is smoother but there is not big difference between convergence rate.

The theory predicts that L-SVRG-US and PAGE will converge to the optimal point with rate $\mathcal{O}(\frac{1}{K^2})$, while SGD-US will only converge to a neibourhood of the optimal point, this is verified by the experiments.