CS331-HW9-Lukang-Sun

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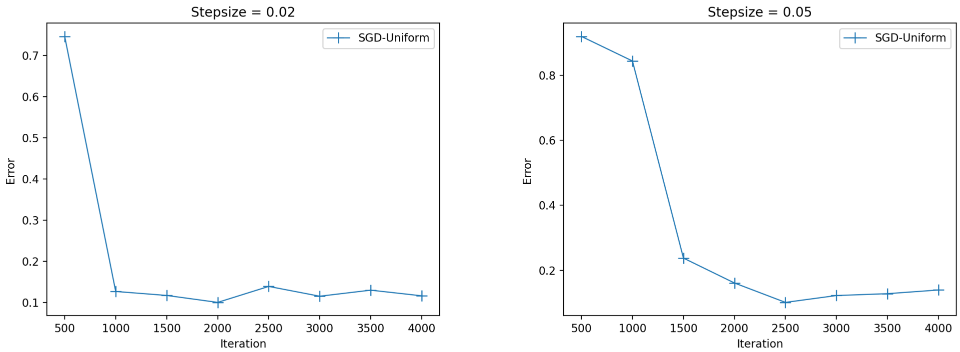
p1. (see Figure 1.) a = [matrix([[0.1]]), matrix([[0.424466]]), matrix([[0.77981303]]), matrix([[0.20033184]]), matrix([[0.51116473]]), matrix([[0.2604399]]), matrix([[0.97100656]]), matrix([[0.21263449]]), matrix([[0.26417151]]), matrix([[0.15995097]])]

b = [matrix([[0.4231786]]), matrix([[0.524466]]), matrix([[0.17981303]]), matrix([[0.50033184]]), matrix([[0.71116473]]), matrix([[0.0604399]]), matrix([[0.37100656]]), matrix([[0.91263449]]), matrix([[0.66417151]]), matrix([[0.65995097]])] ,$f = \frac{1}{10}\sum_{i=1}^{10} f_i(x,y), f_i(x,y) = \sin(x + a[i]) + \cos(y + b[i])$, for the SGD method, I use SGD-Uniform sampling.
(i)each f_i is at most 1-smoothness, since its Hessian is diagonal and bound by $diag(1,1)$, so f is at most 1-smoothness, and it's obvious f_i, f are not convex at all. And this method satisfy $ABC-$ assumption, $\mathrm{E}\left[||g_i(x)||_2^2\right] \leq 2A\left(f(x) - f^{\inf}\right) + B\|\nabla f(x)\|^2 + C$, obviously, C is less 2 since f_i, f bounded by 2, and we can set $A = 0, B = 0(\ \mathrm{E}\left[||g_i||_2^2\right] \leq 2)$.
(ii) I test how the error change in terms of K and the step size. Since in this case, only step size γ and K influence the error.
(iii) First I set γ fixed and change K. Then I set K fixed and change γ. By control one variable, we can clearly find the error change in terms of the other variable.
(iv)Theory states that the error change proportional to $\frac{1}{K}$ in certain error bound when γ is small and fixed and proportional to γ when K is fixed(generally very lage $>> \frac{1}{\gamma}$), these theory statements quite match my experiment results and my experiments shows the error change proportional to $\frac{1}{K}$ in certain error bound when γ is small and fixed and proportional to γ when K is fixed(generally very lage $>> \frac{1}{\gamma}$).

p2.

Lemma. *Assume that f is μ-convex, g^t is unbiased (Assumption 1) and that the AC assumption (Assumption 2) is satisfied. Choose a stepsize satisfying*

$$0 < \gamma_t \leq \frac{1}{A}$$



(a) (b)

Figure 1: (a) shows $\mathrm{E}\left[||\nabla f||^2\right]$ changes in terms of iteration number K when set step size $\gamma = 0.02$,(b) shows $\mathrm{E}\left[||\nabla f||^2\right]$ changes in terms of iteration number K when set step size $\gamma = 0.05$.

Then the iterates $\{x^t\}_{t\geq 0}$ of SGD (Algorithm 3) satisfy

$$\mathrm{E}\left[\left\|x^{t+1} - x^\star\right\|^2\right] \leq (1 - \gamma_t \mu)\mathrm{E}\left[\left\|x^t - x^\star\right\|^2\right] + \gamma_t^2 C$$

Proof. the proof is exactly the same as theorem 30 $\square$

Lemma. *Let $a, b, c \geq 0$ with $0 < a \leq b$. Consider a sequence $\{d_t\}_{t\geq 0}$ satisfying*

$$d_{t+1} \leq (1 - \gamma_t a)\, d_t + \gamma_t^2 c$$

where $\gamma_t \leq \frac{1}{b}$ for all $t \geq 0$. Fix $K > 0$ and let $\theta = \left\lceil \frac{K}{2} \right\rceil$ and $s = \frac{2b}{a}$. Then choosing the stepsize as

$$\gamma_t = \begin{cases} \frac{1}{b}, & \text{if } K \leq 2(s-1) \\ \frac{1}{b}, & \text{if } K > 2(s-1) \text{ and } t < \theta \\ \frac{2}{a(s+t-\theta)} & \text{if } K > 2(s-1) \text{ and } t \geq \theta \end{cases}$$

gives

$$d_K \leq \exp\left(-\frac{aK}{2b}\right) d_0 + \frac{12c}{a^2 K}$$

Proof. this is exactly lemma 117. $\square$

Theorem. *Assume that f is μ-convex, g^t is unbiased(Assumption 1) and that the AC assumption is satisfied. Choose the stepsize as*

$$\gamma_t = \begin{cases} \frac{1}{A}, & \text{if } K \leq 2(s-1) \\ \frac{1}{A}, & \text{if } K > 2(s-1) \text{ and } t < \theta \\ \frac{2}{\mu(s+t-\theta)} & \text{if } K > 2(s-1) \text{ and } t \geq \theta \end{cases},$$

where K is any chosen integer, $\theta = \left\lceil \frac{K}{2} \right\rceil, s = \frac{2A}{\mu}$, then we have

$$\mathrm{E}\left[||x^K - x^\star||_2^2\right] \leq \exp\left(-\frac{\mu K}{2A}\right) ||x^0 - x^\star||^2 + \frac{12C}{\mu^2 K}$$

Proof. this is an corollary of the last lemma. □

p3.

Proof. (i)Let's assume A is a matrix with full row rank.(if not,assume its row rank is l, then left multiply a $m \times m$ full rank matrix L,such that LA's first l row has full rank and the left row is 0, then we can do the same analysis in a affine subspace(ϕ is strongly convex when constrained in a affine space $\left\{Ax + b, x \in \mathbb{R}^d\right\}$). Further if $\phi(x)$ is strongly convex if and only if $\phi_L(x) := \phi(Lx)$ is strongly convex.Since it is easy to see that $D_\phi(x, y) \geq \mu_1 ||x - y||_2^2 \iff D_{\phi_L}(x, y) \geq \mu_2 ||x - y||_2^2$). It's easy to verify that f is convex ($f(\lambda x + (1 - \lambda)y) = \phi(\lambda(Ax + b) + (1 - \lambda)(Ay + b)) \leq \lambda\phi(Ax + b) + \lambda\phi((1 - \lambda)(Ay + b)) = \lambda f(x) + (1 - \lambda)f(y),$) so there is $x^\star$, such that $\nabla f(x^\star) = A^T \nabla\phi(Ax^\star + b) = 0$. Due to $D_\phi(a, y) \leq C||\nabla\phi(a) - \nabla\phi(y)||_2^2$, for any $a, y \in \mathbb{R}^m$, insert $a = Ax+b$ and $Ax^\star+b$, we get $f(x)-f(x^\star)-\langle\nabla\phi(Ax^\star+b), A(x-x^\star)\rangle \leq C||\nabla\nabla\phi(Ax+b) - \nabla\phi(Ax^\star+b)||_2^2 = C||\nabla\phi(Ax+b)||_2^2$ since A^T has full column rank $A^T\nabla\phi(A^\star + b) = 0 \Rightarrow \phi(Ax^\star + b) = 0$, where $\langle\nabla\phi(Ax^\star + b), A(x - x^\star)\rangle (= \langle A^T\nabla\phi(x^\star), x - x^\star\rangle) = 0$, so we have $f(x) - f(x^\star) \leq C||\nabla\phi(Ax + b)||_2^2 \leq CM||\nabla f(x)||_2^2$, where $M = \max_{x \in \mathbb{R}^m} \frac{||x||_2^2}{||A^T x||_2^2} \leq +\infty$,since A^T has full column rank.

(ii)$D_\phi(Ax + b, Ax^\star + b) \geq C_1||A(x - x^\star)||_2^2 \geq C_1 N||x - x^\star||_2^2$, so we have $f(x) - f(x^\star) \geq C_1 N||x - x^\star|_2^2$, where $N := \min_{x \in \mathbb{R}^d} \frac{||Ax||_2^2}{||x||_2^2}$, N could be 0. Actually, let $\phi = \frac{x^2}{2}, A = (1, 0), b = 0$, then $f(x, y) = \frac{x^2}{2}$, $f(0, y) - f(0, 0) = 0 = 0||(0, y) - (0, 0)||_2^2 = 0y^2$. □