CS331-HW7-Lukang-Sun

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p1. (i) the method is the same as the one described in class, but with different variance parameters.
(ii)

Lemma. *Assume that functions f_i are convex and L_i-smooth for all i. Let $\mathcal{C}_i \in \mathbb{B}^d(\omega_i)$ for all i. Suppose that $\alpha \leq \frac{1}{\omega_{max}+1}$. Let $h = (h_1, h_2, \ldots, h_n) \in \mathbb{R}^d \times \mathbb{R}^d \cdots \times \mathbb{R}^d = \mathbb{R}^{nd}$ and define $\sigma : \mathbb{R}^{nd} \to [0, \infty)$ and σ^k by*

$$\sigma(h) = \frac{1}{n}\sum_{i=1}^{n} \|h_i - \nabla f_i(x^\star)\|^2 \quad \sigma^k \overset{def}{=} \sigma\left(h^k\right) = \frac{1}{n}\sum_{i=1}^{n} \left\|h_i^k - \nabla f_i(x^\star)\right\|^2$$

Then for all iterations $k \geq 0$ of Algorithm 19 (with different variance parameters) we have

$$\mathrm{E}\left[g^k \mid x^k, h^k\right] = \nabla f\left(x^k\right),$$

$$\mathrm{E}\left[\left\|g^k - \nabla f(x^\star)\right\|^2 \mid x^k, h^k\right] \leq 2\underbrace{\left(\max(L_i(1+\frac{2\omega_i}{n}))\right)}_{A} D_f\left(x^k, x^\star\right) + \underbrace{\frac{2\omega_{max}}{n}}_{B}\sigma^k,$$

$$E\left[\sigma^{k+1} \mid x^k, h^k\right] \leq 2\underbrace{\alpha L_{\max}}_{\tilde{A}} D_f\left(x^k, x^\star\right) + \underbrace{(1-\alpha)}_{\tilde{B}}\sigma^k.$$

Proof. for simplicity, I use $\mathrm{E}[\cdot \mid k]$ to denote $\mathrm{E}[\cdot \mid \mathcal{F}_k]$.$g^k = \sum_i \mathcal{C}_i(\nabla_i(x^k) - h_i^k) + \sum_i h_i^k$, We only prove the second and the third equality, the first one is trivial.

$$\begin{aligned}
&\mathrm{E}\left[||g^k - \nabla f(x^\star)||^2 \mid k\right] = \mathrm{E}\left[||g^k - \nabla f(x_k)||^2 \mid k\right] + ||\nabla f(x_k) - \nabla f(x^\star)||^2 \\
&\leq \frac{1}{n^2}\sum_i \omega_i(2||\nabla f_i(x_k) - \nabla f_i(x^\star)||^2 + 2||\nabla f_i(x^\star - h_i^k)||^2) + \frac{1}{n}\sum_i 2L_i D_{f_i}(x^k, x^\star) \\
&\leq \frac{2\omega_{max}}{n}\sigma^k + \max(2L_i(1+\frac{2\omega_i}{n}))D_f(x^k, x^\star),
\end{aligned} \tag{1}$$

the second inequality is due to the independence of each $\mathcal{C}_i$.

$$\mathrm{E}\left[\sigma^{k+1} \mid k\right] = \frac{1}{n}\mathrm{E}\left[||h_i^k + \alpha m_i^k - \nabla f_i(x^\star)||^2 \mid k\right] \tag{2}$$

if we choose $\alpha \in (0, \frac{1}{\omega_{max}+1}]$, then we have for each index i,

$$\begin{aligned}
&\mathrm{E}\left[||h_i^k + \alpha m_i^k - \nabla f_i(x^\star)||^2 \mid k\right] \\
&= ||h_i^k - \nabla f_i(x^\star)||^2 + 2\alpha\langle \nabla f_i(x^k) - h_i^k, h_i^k - \nabla f_i(x^\star)\rangle + \alpha^2(\omega_i + 1)||\nabla f_i(x^\star) - h_i^k||^2 \\
&\leq ||h_i^k - \nabla f_i(x^\star)||^2 + \alpha(2\langle \nabla f_i(x^k) - h_i^k, h_i^k - \nabla f_i(x^\star)\rangle + ||\nabla f_(x^k) - h_i^k||^2) \\
&= ||h_i^k - \nabla f_i(x^\star)||^2 + \alpha(||\nabla f_i(x^k) - \nabla f_i(x^\star)||^2 - ||h_i^k - \nabla f_i(x^\star)||^2) \\
&\leq (1-\alpha)||h_i^k - \nabla f_i(x^\star)|| + 2\alpha L_i D_{f_i}(x^k, x^\star),
\end{aligned} \tag{3}$$

take summation, we finally get

$$\mathrm{E}\left[\sigma^{k+1} \mid k\right] \leq (1-\alpha)\sigma^k + 2\alpha L_{max} D_f(x^k, x^\star). \tag{4}$$

□

(iii)

Corollary. *Assume that f_i is convex and L_i-smooth for all $i \in [n]$ and f is μ-convex. If the stepsizes satisfy*

$$\alpha \leq \frac{1}{\omega_{max}+1}, \quad \gamma \leq \frac{1}{\max_i(\left(1 + \frac{2\omega_i}{n}\right) L_i) + M L_{\max}\alpha}$$

where $M > \frac{2\omega_{max}}{n\alpha}$, then the iterates of DIANA satisfy

$$\mathrm{E}\left[V^k\right] \leq \max\left\{(1-\gamma\mu)^k, \left(\frac{2\frac{\omega_{max}}{n} + M(1-\alpha)}{M}\right)^k\right\} V^0$$

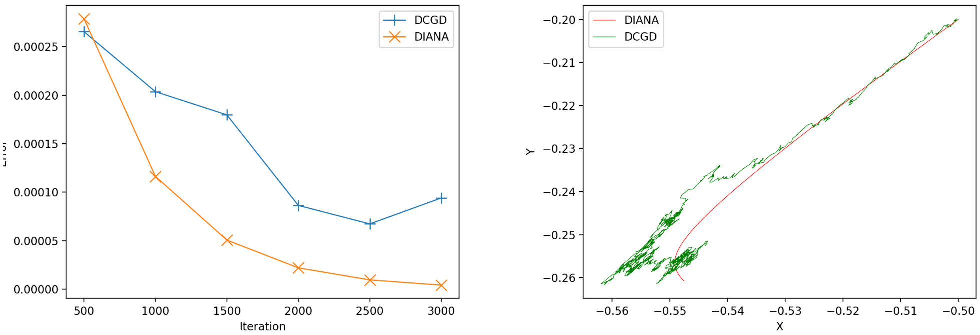
where the Lyapunov function V^k is defined by

$$V^k \overset{def}{=} \left\|x^k - x^\star\right\|^2 + M\gamma^2\sigma^k$$

Proof. use theorem 94 and the last lemma, we get the corollary. □

If each ω_i is identical, this is exactly corollary 101 in the lecture. the interation complexity of DIANA(with different variance)is

$$\max\left\{\frac{1}{\gamma\mu}, \frac{1}{\alpha - \frac{2\omega}{nM}}\right\}\log\frac{1}{\varepsilon} = \max\left\{\kappa + \kappa\frac{6\omega}{n}, 2(\omega+1)\right\}\log\frac{1}{\varepsilon}$$



(a) (b)

Figure 1: (a) shows the error in terms of iteration,(b) shows the trajectories of DINANA and DCGD, you can see that DCGD will converge to the neighborhood of the optimal point with fluctuation while DIANA converges to the optimal point very smoothly, this quite matches the theory's prediction.

(iv) In my experiments(see Figure1.), I set $d = 2, n = 10$, $f(x) = \frac{1}{10}\sum_{i=1}^{10} f_i(x)$, $f_i(x) = \frac{1}{2}\|a_i x - b_i\|_2^2$, $a = [matrix([[0.94884523, 0.31257516]]), matrix([[0.64695759, 0.79089169]]), matrix([[0.70218109, 0.91473775]]), matrix([[0.03035042, 0.21034799]]), matrix([[0.99278455, 0.2554682]]), matrix([[0.16064759, 0.09062056]]), matrix([[0.41438167, 0.77718962]]), matrix([[0.4953842, 0.93027311]]), matrix([[0.7692516, 0.19772597]]), matrix([[0.12430258, 0.03779965]])]$, $b = [matrix([[0.4231786]]), matrix([[0.524466]]), matrix([[0.17981303]]), matrix([[0.50033184]]), matrix([[0.71116473]]), matrix([[0.0604399]]), matrix([[0.37100656]]), matrix([[0.91263449]]), matrix([[0.66417151]]), matrix([[0.65995097]])]$, based on these information, we can get that $x_\star = matrix([[-0.54051746][-0.26890662]]), x_0 = matrix([[-0.5][-0.2]])$, I sample 10 points in the last step to estimate $\mathrm{E}\left[||x_{\text{last step}} - x_\star||^2\right]$. In both setting(DCGD and DIANA), I use Bernoulli compressor(with $p = 0.2$).

p2. (i)

$$g^k = \frac{1}{\tau}\sum_{i\in S^k}(\nabla f_i(x^k) - \nabla f_i(w_i^k)) + \frac{1}{n}\sum_{j=1}^{n}\nabla f_j(w_j^k) \tag{5}$$

$$w_j^{k+1} = \begin{cases} x^k & j \in S^k \\ w_j^k & else \end{cases} \tag{6}$$

(ii)

Lemma. *Assume that for each $i = 1, 2, \ldots, n$, the function f_i is convex and L_i-smooth. Then for SAGA-NICE, we have the following recursions:*

$$\mathrm{E}\left[\left\|g^k - \nabla f\left(x^\star\right)\right\|^2 \mid x^k, w^k\right] \leq 2\underbrace{\left(2\left(\frac{n-\tau}{\tau(n-1)}\max_i L_i + \frac{n(\tau-1)}{\tau(n-1)}L\right)\right)}_{A} D_f\left(x^k, x^\star\right) + \underbrace{2}_{B}\sigma^k$$

and

$$\mathrm{E}\left[\sigma^{k+1} \mid x^k, w^k\right] \leq 2\underbrace{\frac{\tau L_{\max}}{n}}_{\tilde{A}} D_f\left(x^k, x^\star\right) + \underbrace{\left(1 - \frac{\tau}{n}\right)}_{\tilde{B}}\sigma^k$$

where $\sigma(w) \overset{def}{=} \frac{1}{n}\sum_{i=1}^n \left\|\nabla f_i\left(w_i\right) - \nabla f_i\left(x^\star\right)\right\|^2$ *and*

$$\sigma^k \overset{def}{=} \sigma\left(w^k\right) = \frac{1}{n}\sum^n \left\|\nabla f_i\left(w_i^k\right) - \nabla f_i\left(x^\star\right)\right\|^2$$

Proof.

$$\begin{aligned}
&\mathrm{E}\left[||g^k - \nabla f(x^\star)||^2 \mid x^k, w^k\right] \\
&= \mathrm{E}\left[\left\|\frac{1}{\tau}\sum_{i\in S^k}\nabla f_i\left(w^k\right) - \nabla f_i\left(w_i^k\right) + \frac{1}{n}\frac{1}{\tau}\sum_{j=1}^n \nabla f_j\left(w_j^k\right) - \nabla f\left(x^\star\right)\right\|^2 \mid x^k, w^k\right] \\
&= \mathrm{E}\left[\left\|\frac{1}{\tau}\sum_{i\in S^k}\nabla f_i\left(x^k\right) - \nabla f_i\left(x^\star\right) + \nabla f_i\left(x^\star\right) - \nabla f_i\left(w_i^k\right) + \frac{1}{n}\sum_{j=1}^n \nabla f_j\left(w_j^k\right) - \nabla f\left(x^\star\right)\right\|^2 \mid x^k, w^k\right] \\
&\leq E\left[2\left\|\frac{1}{\tau}\sum_{i\in S^k}\nabla f_i\left(x^k\right) - \nabla f_i\left(x^\star\right)\right\|^2 \mid x^k, w^k\right] \\
&+ \mathrm{E}\left[2\left\|\frac{1}{\tau}\sum_{i\in S^k}\nabla f_i\left(x^\star\right) - \nabla f_i\left(w_i^k\right) + \frac{1}{n}\sum_{j=1}^n \nabla f_j\left(w_j^k\right) - \nabla f\left(x^\star\right)\right\|^2 \mid x^k, w^k\right] \\
&= 2\mathrm{E}\left[\left\|\frac{1}{\tau}\sum_{i\in S^k}\nabla f_i\left(x^k\right) - \nabla f_i\left(x^\star\right)\right\|^2 \mid x^k, w^k\right] \\
&+ 2\mathrm{E}\left[\| \underbrace{\frac{1}{\tau}\sum_{i\in S^k}\nabla f_i\left(x^\star\right) - \nabla f_i\left(w_i^k\right)}_{A_i} - \underbrace{\left(\nabla f\left(x^\star\right) - \frac{1}{n}\sum_{j=1}^n \nabla f_j\left(w_j^k\right)\right)}_{\mathrm{E}\left[A_i | x^k, w^k\right]} \|^2 \mid x^k, w^k\right].
\end{aligned} \tag{7}$$

Applying the inequality

$$\mathrm{E}\left[\left\|A_i - \mathrm{E}\left[A_i \mid x^k, w^k\right]\right\|^2 \mid x^k, w^k\right] \leq \mathrm{E}\left[\|A_i\|^2 \mid x^k, w^k\right]$$

we can continue

$$\begin{aligned}
&\mathrm{E}\left[\left\|g^k - \nabla f\left(x^\star\right)\right\|^2 \mid x^k, w^k\right] \\
&\leq 2\mathrm{E}\left[\left\|\frac{1}{\tau}\sum_{i\in S^k} \nabla f_i\left(x^k\right) - \nabla f_i\left(x^\star\right)\right\|^2 \mid x^k, w^k\right] + 2\mathrm{E}\left[\left\|\frac{1}{\tau}\sum_{i\in S^k} \nabla f_i\left(x^\star\right) - \nabla f_i\left(w_i^k\right)\right\|^2 \mid x^k, w^k\right] \\
&= 2(\frac{n-\tau}{\tau(n-1)}\frac{1}{n}\sum_{i=1}^n ||\nabla f_i(x^k) - \nabla f_i(x^\star)||^2 + \frac{n(\tau-1)}{\tau(n-1)}||\nabla f(x^k) - \nabla f(x^\star)||^2) \\
&+ 2(\frac{n-\tau}{\tau(n-1)}\frac{1}{n}\sum_{i=1}^n ||\nabla f_i(x^\star) - \nabla f_i(w_i^k)||^2 + \frac{n(\tau-1)}{\tau(n-1)}||\frac{1}{n}\sum_{i=1}^n \nabla f_i(w_i^k) - \nabla f_i(x^\star)||^2) \\
&\leq 2(\frac{n-\tau}{\tau(n-1)}\frac{1}{n}\sum_{i=1}^n ||\nabla f_i(x^k) - \nabla f_i(x^\star)||^2 + \frac{n(\tau-1)}{\tau(n-1)}||\nabla f(x^k) - \nabla f(x^\star)||^2) \\
&+ 2(\frac{n-\tau}{\tau(n-1)}\frac{1}{n}\sum_{i=1}^n ||\nabla f_i(x^\star) - \nabla f_i(w_i^k)||^2 + \frac{n(\tau-1)}{\tau(n-1)}\frac{1}{n}\sum_{i=1}^n ||\nabla f_i(w_i^k) - \nabla f_i(x^\star)||^2) \\
&= 2(\frac{n-\tau}{\tau(n-1)}\frac{1}{n}\sum_{i=1}^n ||\nabla f_i(x^k) - \nabla f_i(x^\star)||^2 + \frac{n(\tau-1)}{\tau(n-1)}||\nabla f(x^k) - \nabla f(x^\star)||^2) \\
&+ 2\frac{1}{n}\sum_{i=1}^n ||\nabla f_i(x^\star) - \nabla f_i(w_i^k)||^2 \\
&\leq 4\left(\frac{n-\tau}{\tau(n-1)}\max_i L_i + \frac{n(\tau-1)}{\tau(n-1)}L\right) D_f(x^k, x^\star) + 2\sigma^k
\end{aligned} \tag{8}$$

We now proceed to the second recursion. First, note that for every i we have

$$w_i^{k+1} = \begin{cases} w_i^k & \text{with probability} \quad 1 - \frac{\tau}{n} \\ x^k & \text{with probability} \quad \frac{\tau}{n} \end{cases}$$

Therefore,

$$
\begin{aligned}
\mathrm{E}\left[\sigma^{k+1} \mid x^k, w^k\right] &= \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}\left[\left\|\nabla f_i\left(w_i^{k+1}\right) - \nabla f_i\left(x^\star\right)\right\|^2 \mid x^k, w^k\right] \\
&= \frac{1}{n}\sum_{i=1}^{n}\left[\left(1-\frac{\tau}{n}\right)\left\|\nabla f_i\left(w_i^k\right) - \nabla f_i\left(x^\star\right)\right\|^2 + \frac{\tau}{n}\left\|\nabla f_i\left(x^k\right) - \nabla f_i\left(x^\star\right)\right\|^2\right] \\
&= \left(1-\frac{\tau}{n}\right)\sigma^k + \frac{\tau}{n^2}\sum_{i=1}^{n}\underbrace{\left\|\nabla f_i\left(x^k\right) - \nabla f_i\left(x^\star\right)\right\|^2}_{\le 2L_i D_{f_i}\left(x^k, x^\star\right)} \\
&\le \left(1-\frac{\tau}{n}\right)\sigma^k + \frac{2\tau L_{\max}}{n} D_f\left(x^k, x^\star\right)
\end{aligned}
$$

□

σ^k is the same as in the lecture.

(iii)

Corollary. *Assume that for each i, the function f_i is convex L_i-smooth. Further assume f is μ-convex. Choose $\gamma = \frac{1}{6L_{\max}}$. Then SAGA-NICE satisfies:*

$$\mathrm{E}\left[V^k\right] \le \left(1 - \min\left\{\frac{\mu}{6L_{\max}}, \frac{\tau}{2n}\right\}\right)^k V^0$$

where the Lyapunov function is defined by

$$V^k \overset{def}{=} \left\|x^k - x^\star\right\|^2 + \frac{n}{9\tau L_{\max}^2}\sigma^k$$

and

$$\sigma^k \overset{def}{=} \frac{1}{n}\sum_{i=1}^{n}\left\|\nabla f_i\left(w_i^k\right) - \nabla f_i\left(x^\star\right)\right\|^2$$

Proof. In view of (ii), Assumption 6 holds with

$$A \le 2L_{\max}, \quad B = 2, \quad \tilde{A} = \frac{\tau L_{\max}}{n}, \quad \tilde{B} = 1 - \frac{\tau}{n}, \quad C = \tilde{C} = 0$$

We can now apply Theorem 94 with $M = \frac{4n}{\tau} > \frac{2n}{\tau} = \frac{B}{1-\tilde{B}}$. Note that

$$A + M\tilde{A} = (2 + \frac{4}{\tau})L_{\max} > \mu$$

and hence the stepsize bound becomes

$$0 < \gamma \le \min\left\{\frac{1}{\mu}, \frac{1}{A + M\tilde{A}}\right\} = \frac{1}{A + M\tilde{A}} = \frac{1}{6L_{\max}}$$

So, the choice $\gamma = \frac{1}{6L_{\max}}$ is justified. Since $\frac{B+M\tilde{B}}{M} = 1 + \frac{2}{4n} - \frac{1}{n} = 1 - \frac{\tau}{2n}$ and using $\gamma = \frac{1}{6L_{\max}}$, the rate in Theorem 94 becomes

$$\begin{aligned}\max\left\{(1-\gamma\mu)^k, \left(\frac{B+M\tilde{B}}{M}\right)^k\right\} &= \max\left\{\left(1-\frac{\mu}{6L_{\max}}\right)^k, \left(1-\frac{\tau}{2n}\right)^k\right\} \\ &= \left(1-\min\left\{\frac{\mu}{6L_{\max}}, \frac{\tau}{2n}\right\}\right)^k\end{aligned}$$

Finally, the Lyapunov function is

$$V^k = \left\|x^k - x^\star\right\|^2 + M\gamma^2\sigma^k = \left\|x^k - x^\star\right\|^2 \frac{4n}{\tau 36 L_{\max}^2}\sigma^k$$

$\square$

so

$$k \geq \max\left\{\frac{6L_{\max}}{\mu}, \frac{2n}{\tau}\right\}\log\frac{1}{\varepsilon} \Rightarrow \mathrm{E}\left[V^k\right] \leq \varepsilon V^0$$

. (the step size here chosen is not the best).
(iv) when $\tau = 1$, lemma in (ii) is exactly lemma 103. when $\tau = n$, then this is gradient descent, rate from (iii) is

$$k \geq \max\left\{\frac{6L_{\max}}{\mu}, 2\right\}\log\frac{1}{\varepsilon} = \frac{6L_{\max}}{\mu}\log\frac{1}{\varepsilon} \Rightarrow \mathrm{E}\left[V^k\right] \leq \varepsilon V^0$$

, here the rate involves L_{max} instead of L.