Exercise 6

May 8, 2022

1 Exercise 6

1.1 a)

$$f(x) = \begin{cases} N\mathrm{e}^{-x/\tau} & 0 \leq x < \infty \\ 0 & else \end{cases}$$

Determination of normilization constant N: $\int_0^\infty N\mathrm{e}^{-x/\tau}\mathrm{d}x \overset{!}{=} 1$

$\Leftrightarrow N[-\tau \cdot \mathrm{e}^{-x/\tau}]_0^\infty = N \cdot \tau = 1$

$\Leftrightarrow N = \frac{1}{\tau}$

Transformation of uniform Distribution u to x with propability density $f(x)$: $u = F(x) = [-\mathrm{e}^{-y/\tau}]_0^x = -\mathrm{e}^{-x/\tau} + 1$

$\Leftrightarrow 1 - u = \mathrm{e}^{-x/\tau} \qquad |\ln(...)$

$1 - u \in [0, 1]$, because $u \in [0, 1]$

$\Leftrightarrow \ln(1 - u) = -x/\tau$

$\Leftrightarrow x = -\ln(1 - u) \cdot \tau$

1.2 b)

$$f(x) = \begin{cases} Nx^{-n} & x_{\mathrm{min}} \leq x \leq x_{\mathrm{max}} \\ 0 & else \end{cases}$$

Determination of normilization constant N: $\int_{x_{\mathrm{min}}}^{x_{\mathrm{max}}} Nx^{-n}\mathrm{d}x \overset{!}{=} 1$

$\Leftrightarrow N[\frac{1}{1-n}x^{1-n}]_{x_{\mathrm{min}}}^{x_{\mathrm{max}}} = 1$

$\Leftrightarrow N\frac{1}{1-n}\left(x_{\mathrm{max}}^{1-n} - x_{\mathrm{min}}^{1-n}\right) = 1$

$\Leftrightarrow N = \frac{1-n}{x_{\mathrm{max}}^{1-n} - x_{\mathrm{min}}^{1-n}}$

Transformation of uniform Distribution u to x with propability density $f(x)$: $u = F(x) = N[\frac{y^{1-n}}{1-n}]_{x_{\mathrm{min}}}^x$

$\Leftrightarrow \frac{u(1-n)}{N} = x^{1-n} - x_{\mathrm{min}}^{1-n}$

$\Leftrightarrow x^{1-n} = \frac{u(1-n)}{N} + x_{\min}^{1-n}$

$\Leftrightarrow x = \sqrt[1-n]{\frac{u(1-n)}{N} + x_{\min}^{1-n}} \qquad |N = \frac{1-n}{x_{\max}^{1-n} - x_{\min}^{1-n}}$

$\Leftrightarrow x = \sqrt[1-n]{u(x_{\max}^{1-n} - x_{\min}^{1-n}) + x_{\min}^{1-n}}$

$\Leftrightarrow x = x_{\min} \sqrt[1-n]{u\left((\frac{x_{\max}}{x_{\min}})^{1-n} - 1\right) + 1}$

1.3 c)

$f(x) = \frac{1}{\,} \frac{1}{1+x^2}$

Transformation of uniform Distribution u to x with propability density $f(x)$: $u = F(x) = \frac{1}{\,} \int_{-\infty}^{x} \frac{1}{1+y^2} \mathrm{d}y$

$\Leftrightarrow \;\; \cdot u = \arctan(x) + \frac{\,}{2}$

$\Leftrightarrow x = \tan\left(\;\; (u - \frac{1}{2})\right)$