Data Mining Assignment 3

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1 Choosing r, b

Description:

Consider computing LSH using $t = 160$ hash functions. We want to find all object pairs which have Jaccard similarity above $\tau = 0.85$.

A: Use the trick mentioned in class and the notes to estimate the best values of hash functions b within each of r bands to provide the S-curve

$$f(s) = 1 - (1 - s^b)^r$$

with good separation at τ. Report these values.

B: Consider the 4 objects A, B, C, D, with the following pair-wise similarities:

	A	B	C	D
A	1	0.75	0.25	0.35
B	0.75	1	0.1	0.45
C	0.25	0.1	1	0.92
D	0.35	0.45	0.92	1

Using your choice of r and b and $f(\cdot)$, what is the probability of each pair of the four objects for being estimated to having similarity greater that $\tau = 0.85$? Report 6 numbers. (*Show your work.*)

Solution:

A: For this we will use the "rule of thumb" mentioned in class. That is take

$$b = -\log_\tau(t)$$
$$r = t/b$$

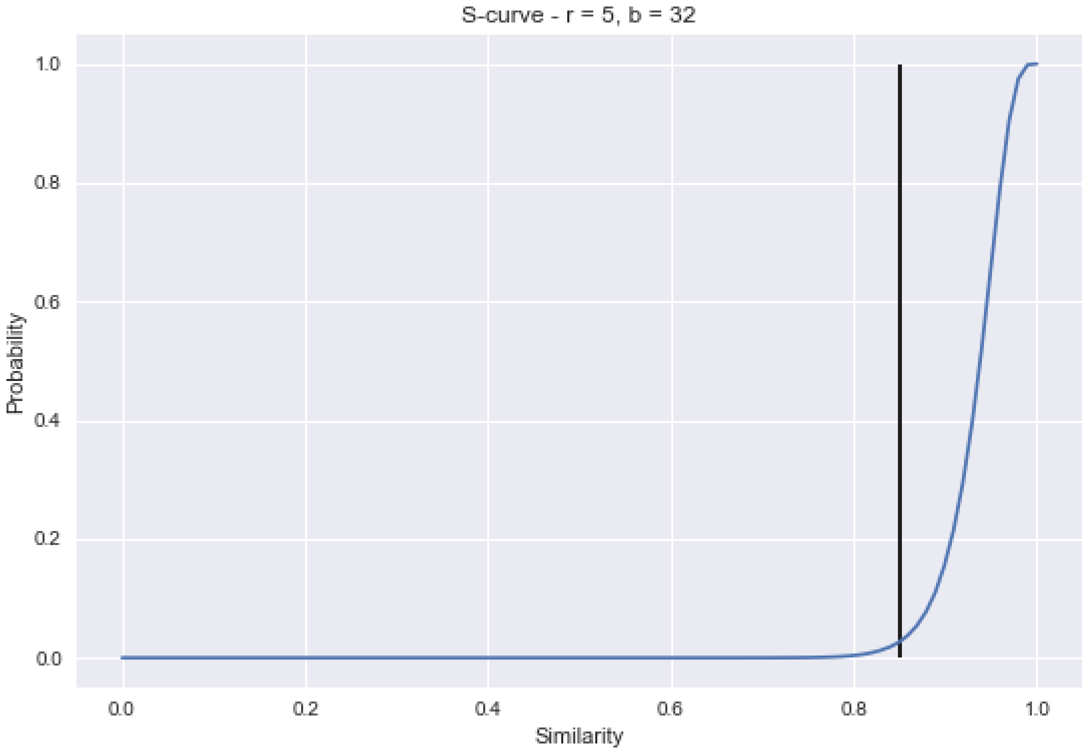
Since we know τ and t we can easily compute the best values of r, b that make the steepest slope at τ. We will also show a plot so that we can see the curve that is produced.

$$b = -\frac{\ln t}{\ln \tau} \approx 32$$

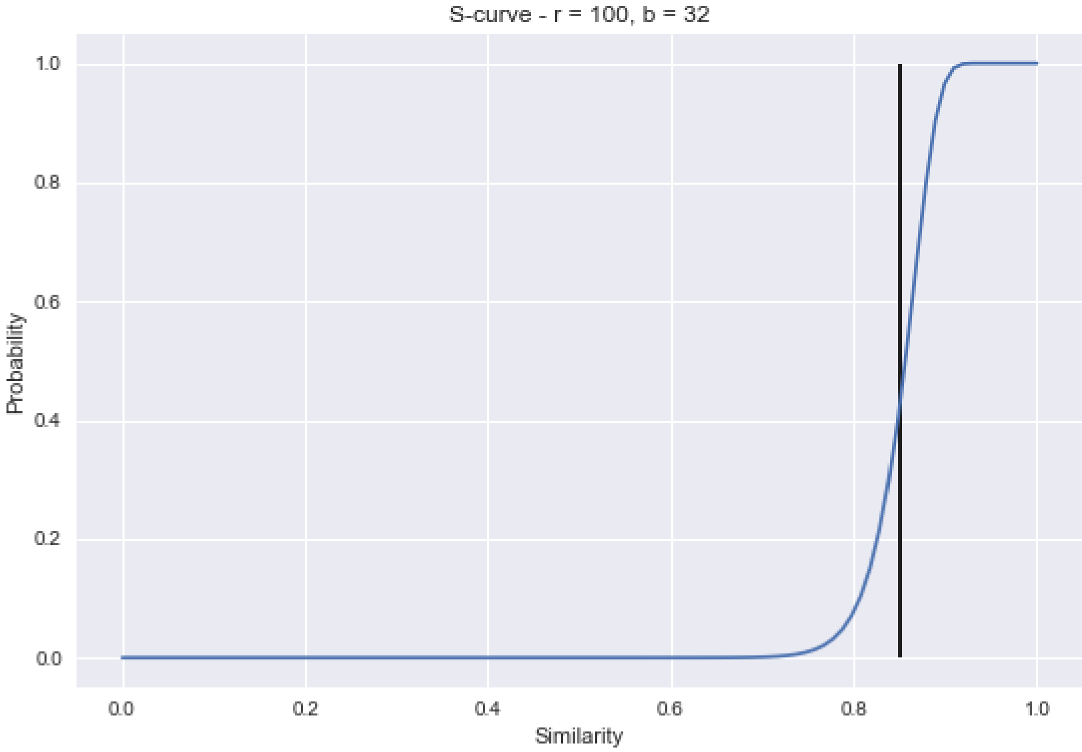
and

$$r = \frac{t}{b} = \frac{160}{31} \approx 5$$

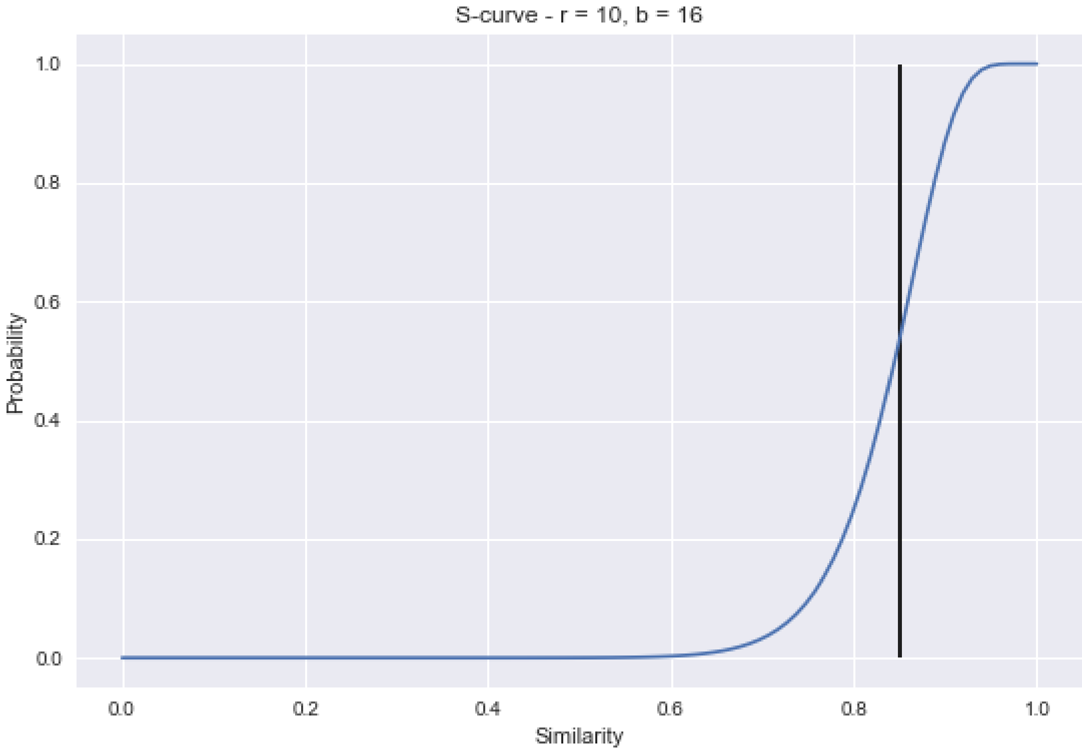
Now we can plot f and see if our choice of r, b is any good.



As we can see the black line is $\tau = 0.85$ and the curve is not capturing the desired threshold well. So now we experiment, a lot, with r, b and we find that $b = 31$ and $r = 100$ produces a better curve for our threshold τ.



However this does not satisfy our budget of $t = 160$. So we must find a better one that fits our budget. After more experimenting we need r to be bigger while not making b too small. The sweet spot ended up being $b = 16$ and $r = 16$.



B: For this we can ignore the diagonal since the probability is 1 for the similarity of a set with itself. Notice that the pair-wise matrix above is symmetric (as it should be) so we only need to compute the probability for the upper half or lower half.

Here is all of the work

$$\begin{aligned}
s_1 &= \hat{\mathrm{JS}}_t(A, B) = 0.75 \\
s_2 &= \hat{\mathrm{JS}}_t(A, C) = 0.25 \\
s_3 &= \hat{\mathrm{JS}}_t(A, D) = 0.35 \\
s_4 &= \hat{\mathrm{JS}}_t(B, C) = 0.1 \\
s_5 &= \hat{\mathrm{JS}}_t(B, D) = 0.45 \\
s_6 &= \hat{\mathrm{JS}}_t(C, D) = 0.92
\end{aligned}$$

Now for the computation

$$\begin{aligned}
f(s_1) &= 0.0958 \\
f(s_2) &= 2.33 \times 10^{-9} \\
f(s_3) &= 5.07 \times 10^{-7} \\
f(s_4) &= 1.11 \times 10^{-15} \\
f(s_5) &= 2.83 \times 10^{-5} \\
f(s_6) &= 0.9530
\end{aligned}$$

And our choice works well for s_1 we can see that it is close to 0 even for how close 0.75 is to our threshold. For s_6 we suspected that it should be close to one and it is. Note that if we kept our original "rule of thumb" r, b our probability for s_6 would be $f(s_6) = 0.32$ which is not good enough.

2 Generating Random Directions

Description:

A: Describe how to generate a single random unit vector in $d = 10$ dimensions using only the operation $u \leftarrow \mathrm{unif}(0, 1)$ which generates a uniform random variable between 0 and 1.

B: Generate $t = 160$ unit vectors in $\mathbb{R}^d$ for $d = 100$. Plot the cdf of their pairwise dot products.

Solution:

A: We can use the wonderful property that a gaussian RV's coordinates are independent of each other. If we were to just use two uniform RV's and normalize them then we would have a bias and the vector generation would not be reasonable. Similar issues arise when you try to generate uniformly random numbers with in a circle/sphere. So first we generate two $\mathrm{unif}(0, 1)$ u_1, u_2 and then transform (Box-Muller) them in such a way that they land on the unit ball in whatever dimension we need. The transformation is as follows.

$$\begin{aligned}
y_1 &= \sqrt{-2 \ln u_1} \cos 2\pi u_2 \\
y_2 &= \sqrt{-2 \ln u_1} \sin 2\pi u_2
\end{aligned}$$

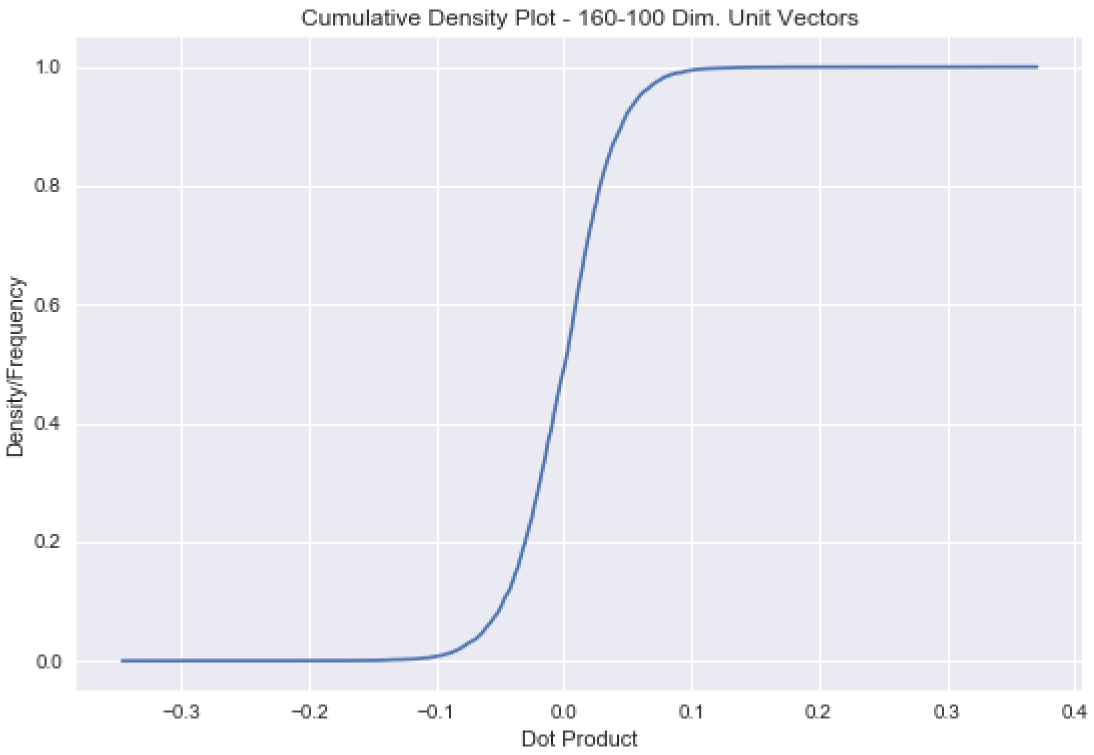
Then we have a random unit vector in $\mathbb{R}^2$. In the case where we want to generate random unit vectors in $\mathbb{R}^{10}$ we need to generate 5 pairs and they will all be independent. In other words use a $\text{unif}(0, 1)$ and the transformation above to generate 5 pairs of points and combine them into a single *d*-dimensional vector.

Alternatively it is possible to generate *d*-dimensional vectors using *d* gaussian random variables with mean 0.

B: We will use two methods and plot both dot product CDF's to see that they are in fact very similar if not the same. Note: we will use the same plotting technique from **A2**.

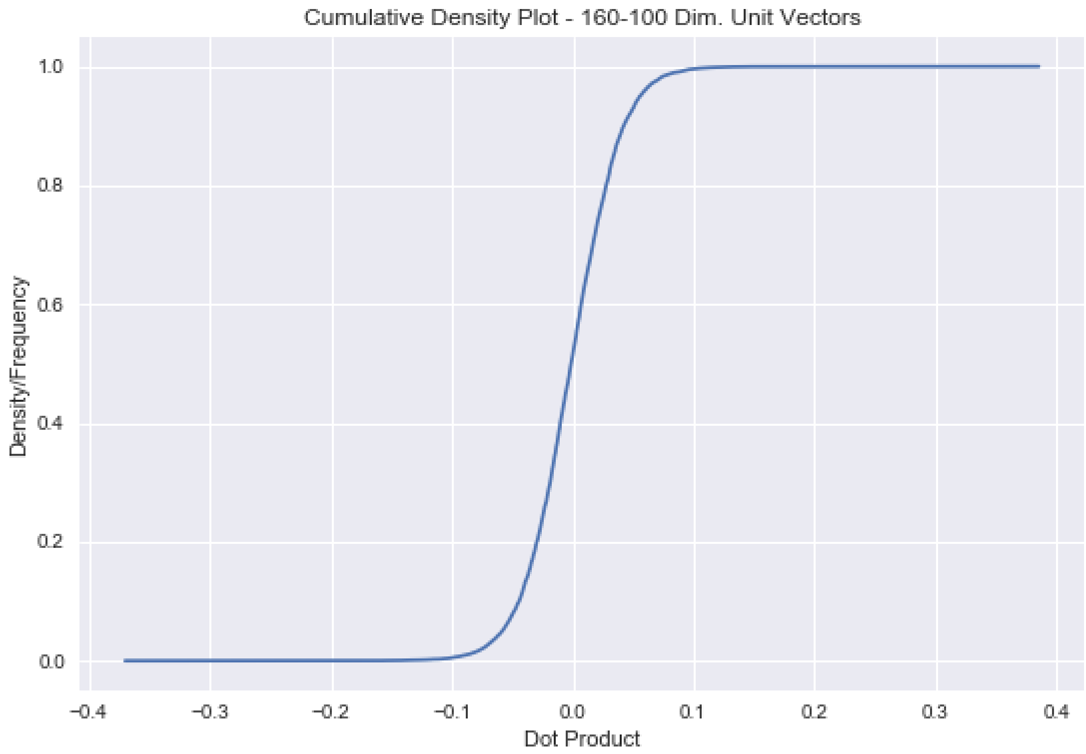
First we simply generate $d = 100$ iid $N(0, 1)$ random numbers and that is one of the t vectors. Then we compute the $\binom{t}{2}$ dot products and plot a CDF of the values.

Here is the result:

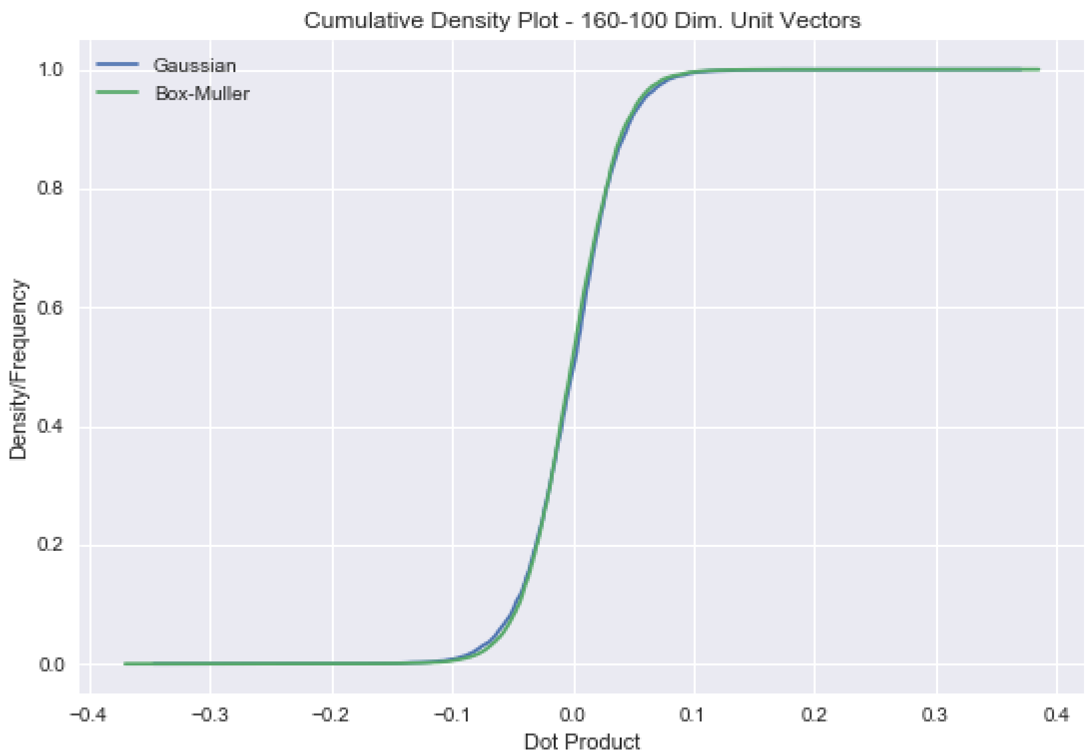


Second we use the Box-Muller transformation as described above except $d = 100$. So we generate 50 pairs and combine them into a single 100-dimensional vector.

Here are the results:



And here is the comparison of the two graphs layered on top of one another:



As we can see these are nearly identical so it is reasonable to assume that they are distributed properly.

3

Description:

Consider the $n = 500$ data points in $\mathbb{R}^d$ for $d = 100$ in data set `R`, given at the top. We will use the angular similarity between two vectors $a, b \in \mathbb{R}^d$:

$$\mathbf{s}_{\text{ang}}(a, b) = 1 - \frac{1}{\pi}\arccos(\langle \bar{a}, \bar{b} \rangle).$$

If a, b are not unit vectors, then we covert them to $\bar{a} = a/||a||_2$ and $\bar{b} = a/||b||_2$. The definition of $\mathbf{s}_{\text{ang}}(a, b)$ assumes that the input are unit vectors, and takes a value between 0 and 1, with 1 meaning most similar.

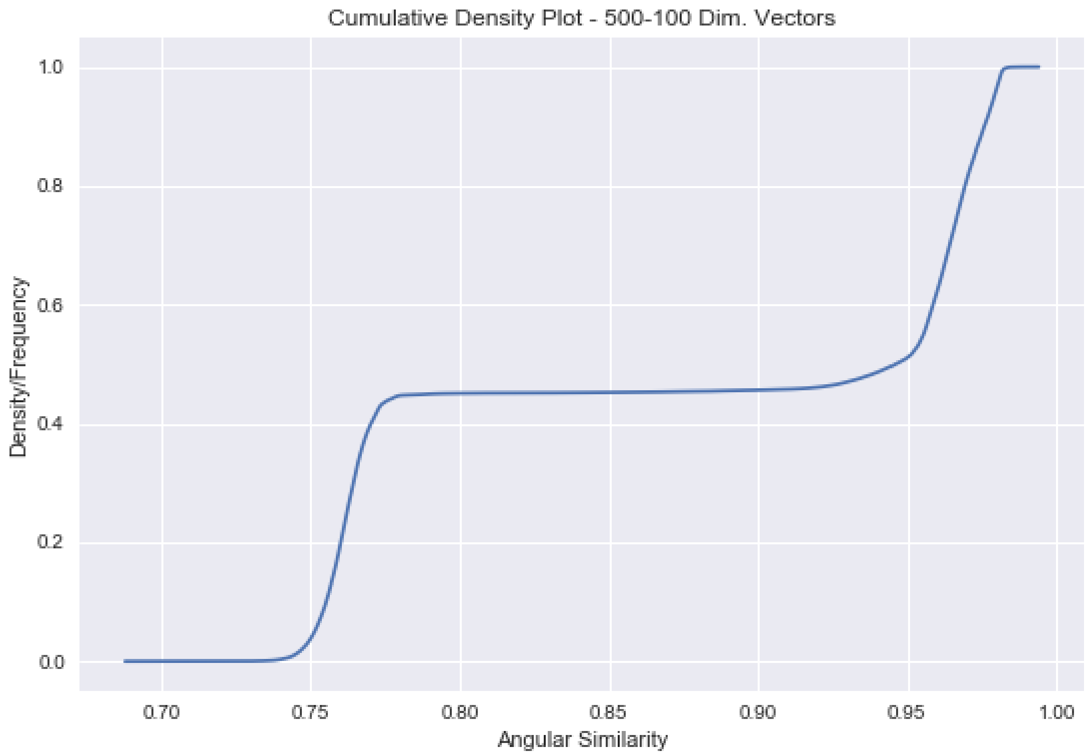
A: Compute all pairs of dot products, and plot a CDF of their angular similarities. Report the number with similarity greater than $\tau = 0.85$.

B: Now compute the dot products and angular similarities among $\binom{t}{2}$ pairs of the t random vectors unit vectors from **Q2.B**. Again plot the CDF, and report the number with angular similarity above $\tau = 0.85$.

Solution:

A: We read in the file and compute the angular similarity defined above. Then we compute the CDF of this similarity.

Here is the CDF.

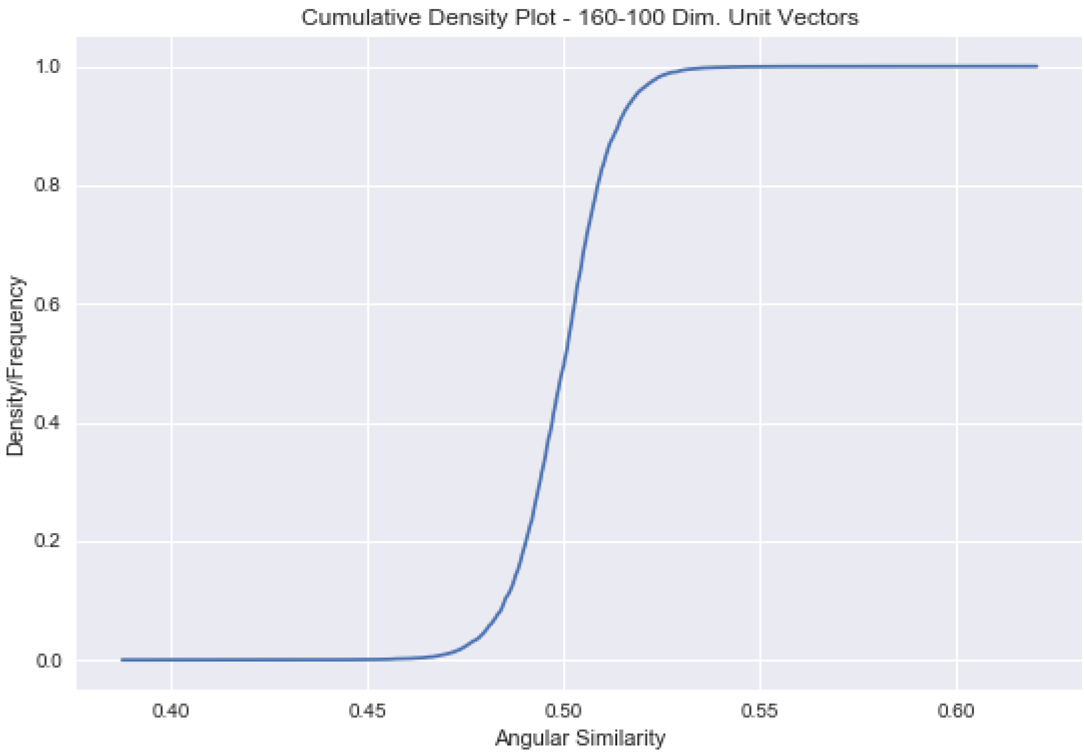


Using a simple filter we can find the number of combinations that have a similarity above $\tau = 0.85$. It is also possible to report the pairs that have this similarity or greater.

We found there to be 67299 pairs that have a similarity greater than $\tau = 0.85$. This is a little over 50% of the $\binom{n}{2}$ combinations.

B: Now we repeat the same process with the random unit vectors generated in **Q2.B**.

Here is the CDF:



The number of similarities above $\tau = 0.85$ is 0.

End