Assignment 1

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I worked on this problem set with Shabab "Spike Mamba" Ahmed, Seyoung "Crowbird" Won, and Terrence "Tianqi" Dai.

My code is comprised of one Matlab file (`run_dynare_ps1.m`) and Dynare files for all of the individual parts of Section 2. The Dynare files are all very similar; they define parameters, shocks, variables, and the model in the same way, and then stochastically solve. The Matlab file runs the Dynare files in sequence, and also exports the computed welfare values to the console.

1 Optimal Monetary Policy

1. In the case with no commitment, the central bank will pick all the time t variables every period. Thus, the optimization problem simplifies to

$$\min x_t, \pi_t \; \frac{(\pi_t - \pi^*)^2}{2} + \frac{\gamma(x_t - x^*)^2}{2}$$

We can then set up Lagrange:

$$\begin{aligned}\mathcal{L} = &\frac{(\pi_t - \pi^*)^2}{2} + \frac{\gamma(x_t - x^*)^2}{2} \\ &+ \mu_1(\pi_t - \beta E_t[\pi_{t+1}] - \lambda x_t - \varepsilon_{\pi t}) \\ &+ \mu_2(x_t - E_t[x_{t+1}] + \frac{1}{\sigma}[i_t - E_t[\pi_{t+1}]] - \epsilon_{xt})\end{aligned}$$

This has first order conditions

$$\begin{aligned}[x_t] &: \gamma(x_t - x^*) - \lambda\mu + \mu_2 = 0 \\ [\pi_t] &: \pi_t - \pi^* + \mu_1 = 0 \\ [i_t] &: \frac{\mu_2}{\sigma} = 0 \\ [\mu_1] &: \pi_t = \beta E_t[\pi_{t+1}] + \lambda x_t + \varepsilon_{\pi t}\end{aligned}$$

Combining yields

$$
\begin{aligned}
&\gamma(x_t - x^*) = \lambda\mu \\
\implies &\gamma(x_t - x^*) = -\lambda(\pi_t - \pi^*) \\
\implies &\pi_t = -\frac{\gamma}{\lambda}(x_t - x^*) + \pi^* \\
\implies &\pi_t = -\frac{\gamma}{\lambda}\left(\left(\frac{\pi_t - \beta E_t[\pi_{t+1}] - \varepsilon_{\pi t}}{\lambda}\right) - x^*\right) + \pi^* \\
\implies &\pi_t\left(\frac{\lambda^2 + \gamma}{\lambda^2}\right) = \frac{\beta\gamma}{\lambda^2}E_t[\pi_{t+1}] + \frac{\gamma}{\lambda^2}\varepsilon_{\pi t} + \frac{\gamma}{\lambda}x^* + \pi^* \\
\implies &\pi_t = \frac{\beta\gamma}{\lambda^2+\gamma}E_t[\pi_{t+1}] + \frac{\gamma}{\lambda^2+\gamma}\varepsilon_{\pi t} + \frac{\lambda\gamma}{\lambda^2+\gamma}x^* + \frac{\lambda^2}{\lambda^2+\gamma}\pi^*
\end{aligned}
$$

We now need $E_t[\pi_{t+1}]$. Iterating forward, we get

$$E_t[\pi_{t+1}] = \frac{\beta\gamma}{\lambda^2+\gamma}E_t[\pi_{t+2}] + \frac{\gamma}{\lambda^2+\gamma}\rho_\pi\varepsilon_{\pi t} + \frac{\lambda\gamma}{\lambda^2+\gamma}x^* + \frac{\lambda^2}{\lambda^2+\gamma}\pi^*$$

Thus, we have an infinite geometric series, where each term besides the error has ratio

$$R = \frac{\beta\gamma}{\lambda^2+\gamma} \implies \frac{1}{1-R} = \frac{\lambda^2+\gamma}{\lambda^2+\gamma-\beta\gamma}$$

and the error term has ratio

$$R_\rho = \frac{\rho\beta\gamma}{\lambda^2+\gamma} \implies \frac{1}{1-R_\rho} = \frac{\lambda^2+\gamma}{\lambda^2+\gamma-\rho\beta\gamma}$$

Note that by the given, $0 < R_\rho < R < 1$, so this is a valid geometric series. Then, we can simply multiply the first terms by the relevant values:

$$\boxed{\pi_t = \frac{\gamma\varepsilon_{\pi t}}{\lambda^2+\gamma-\rho_\pi\beta\gamma} + \frac{\lambda\gamma x^* + \lambda^2\pi^*}{\lambda^2+\gamma-\beta\gamma}}$$

We substitute into the condition for x_t:

$$
\begin{aligned}
x_t &= x^* - \frac{\lambda}{\gamma}(\pi_t - \pi^*) \\
&= \boxed{x^* - \frac{\lambda}{\gamma}\left(\frac{\gamma\varepsilon_{\pi t}}{\lambda^2+\gamma-\rho_\pi\beta\gamma} + \frac{\lambda\gamma x^* + \lambda^2\pi^*}{\lambda^2+\gamma-\beta\gamma} - \pi^*\right)}
\end{aligned}
$$

To get i_t, we need $E[x_{t+1}]$. We have

$$
\begin{aligned}
\gamma(x_t - x^*) &= -\lambda(\pi_t - \pi^*) \\
\implies x_t &= -\frac{\lambda}{\gamma}(\pi_t - \pi^*) + x^* \\
\implies x_{t+1} &= -\frac{\lambda}{\gamma}(\pi_{t+1} - \pi^*) + x^* \\
\implies E_t[x_{t+1}] &= -\frac{\lambda}{\gamma}(E_t[\pi_{t+1}] - \pi^*) + x^*
\end{aligned}
$$

We now note that the only difference between π_t and π_{t+1} is the latter will begin with an additionally discounted ρ term, indicating that we have

$$
\begin{aligned}
E_t[\pi_{t+1}] &= \frac{\gamma\rho_\pi\varepsilon_{\pi t}}{\lambda^2 + \gamma - \rho_\pi\beta\gamma} + \frac{\lambda\gamma x^* + \lambda^2\pi^*}{\lambda^2 + \gamma - \beta\gamma} \\
E_t[x_{t+1}] &= x^* - \frac{\lambda}{\gamma}\left(\frac{\gamma\rho_\pi\varepsilon_{\pi t}}{\lambda^2 + \gamma - \rho_\pi\beta\gamma} + \frac{\lambda\gamma x^* + \lambda^2\pi^*}{\lambda^2 + \gamma - \beta\gamma} - \pi^*\right)
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
i_t &= \sigma(E_t[x_{t+1}] - x_t + \epsilon_{xt}) + E_t[\pi_{t+1}] \\
&= \sigma\left(x^* - \frac{\lambda}{\gamma}\left(\frac{\gamma\rho_\pi\varepsilon_{\pi t}}{\lambda^2 + \gamma - \rho_\pi\beta\gamma} + \frac{\lambda\gamma x^* + \lambda^2\pi^*}{\lambda^2 + \gamma - \beta\gamma} - \pi^*\right)\right. \\
&\quad \left. - \left(x^* - \frac{\lambda}{\gamma}\left(\frac{\gamma\varepsilon_{\pi t}}{\lambda^2 + \gamma - \rho_\pi\beta\gamma} + \frac{\lambda\gamma x^* + \lambda^2\pi^*}{\lambda^2 + \gamma - \beta\gamma} - \pi^*\right)\right)\right. \\
&\quad \left. + \epsilon_{xt} + \frac{\gamma\rho_\pi\varepsilon_{\pi t}}{\lambda^2 + \gamma - \rho_\pi\beta\gamma} + \frac{\lambda\gamma x^* + \lambda^2\pi^*}{\lambda^2 + \gamma - \beta\gamma}\right. \\
&= \sigma\left(\epsilon_{xt} + \frac{\lambda}{\gamma}\left(\frac{\gamma\varepsilon_{\pi t}}{\lambda^2 + \gamma - \rho_\pi\beta\gamma} - \frac{\gamma\rho_\pi\varepsilon_{\pi t}}{\lambda^2 + \gamma - \rho_\pi\beta\gamma}\right)\right) \\
&\quad + \frac{\gamma\rho_\pi\varepsilon_{\pi t}}{\lambda^2 + \gamma - \rho_\pi\beta\gamma} + \frac{\lambda\gamma x^* + \lambda^2\pi^*}{\lambda^2 + \gamma - \beta\gamma}
\end{aligned}
$$

2. We set up the Lagrangian (removing expectations as the model is now deterministic)

$$
\begin{aligned}
\mathcal{L} = \sum_{j=0}^{\infty} &\beta^j \frac{(\pi_{t+j} - \pi^*)^2 + \gamma(x_{t+j} - x^*)^2}{2} \\
&+ \beta^j \mu_{1t+j}(\pi_{t+j} - \beta\pi_{t+j+1} - \lambda x_{t+j}) \\
&+ \beta^j \mu_{2t+j}\left(x_{t+j} - x_{t+j+1} + \frac{1}{\sigma}[i_{t+j} - \pi_{t+j+1}]\right)
\end{aligned}
$$

This has first order conditions

$$
\begin{aligned}
[\pi_{t+j}] &: \beta^j(\pi_{t+j} - \pi^*) + \beta^j \mu_{1t+j} - \beta^j \mu_{1t+j-1} - \beta^{j-1}\frac{\mu_{2t+j-1}}{\sigma} = 0 \\
[x_{t+j}] &: \gamma\beta^j(x_{t+j} - x^*) - \beta^j \lambda \mu_{1t+j} + \beta^j \mu_{2t+j} - \beta^{j-1}\mu_{2t+j-1} = 0 \\
[i_{t+j}] &: \beta^j \frac{\mu_{2t+j}}{\sigma} = 0
\end{aligned}
$$

The third condition implies that $\mu_2 = 0$ in all time periods. Then, we have

$$
\begin{aligned}
[\pi_{t+j}] &: \pi_{t+j} - \pi^* = \mu_{1t+j-1} - \mu_{1t+j} \\
[x_{t+j}] &: \gamma(x_{t+j} - x^*) = \lambda \mu_{1t+j}
\end{aligned}
$$

Taking the previous time period and multiplying by β, we also have

$$\gamma(x_{t+j-1} - x^*) = \lambda \mu_{1t+j-1}$$

Differencing yields

$$\gamma(x_{t+j} - x_{t+j-1}) = \lambda(\mu_{1t+j} - \mu_{1t+j-1})$$

We have from the π_{t+j} condition that

$$\mu_{1t+j} - \mu_{1t+j-1} = -\pi_{t+j} + \pi^*$$

Combining yields the first result:

$$\boxed{x_{t+j} - x_{t+j-1} = -\frac{\lambda}{\gamma}(\pi_{t+j} - \pi^*)}$$

To get the second condition, we note that we have

$$
\begin{aligned}
x_{t+j} &= x^* + \frac{\lambda \mu_{t+j}}{\gamma} \\
\pi_{t+j} &= \pi^* + \mu_{t+j-1} - \mu_{t+j}
\end{aligned}
$$

Using the New Keynesian Phillips curve, then

$$
\begin{aligned}
& \pi_{t+j} = \beta \pi_{t+j+1} + \lambda x_{t+j} \\
\implies & \pi^* + \mu_{t+j-1} - \mu_{t+j} = \beta\left(\pi^* + \mu_{t+j} - \mu_{t+j+1}\right) + \lambda x_{t+j} \\
\implies & \beta \mu_{t+j+1} - \beta \mu_{t+j} - \mu_{t+j} + \mu_{t+j-1} = (\beta - 1)\pi^* + \lambda x_{t+j} \\
\implies & \beta \mu_{t+j+1} - \beta \mu_{t+j} - \mu_{t+j} + \mu_{t+j-1} = (\beta - 1)\pi^* + \lambda\left(x^* + \frac{\lambda \mu_{t+j}}{\gamma}\right) \\
\implies & \boxed{\beta \mu_{t+j+1} - \left(1 + \beta + \frac{\lambda^2}{\gamma}\right)\mu_{t+j} + \mu_{t+j-1} = (\beta - 1)\pi^* + \lambda x^*}
\end{aligned}
$$

This policy is not time-consistent. If it were, then the solution that we found in the first part would satisfy the given conditions we've derived. However, as the x_t solution is invariant in time, the result only holds if

$$0 = \frac{\lambda}{\gamma}(\pi_t - \pi^*) \implies \pi_t = \pi^*$$

This is not the result that we derived, so this policy is not time-consistent (the central bank would deviate in a given period if they could).

3. We substitute in the guess to the latter condition we derived:

$$\beta(a + bz^{t+j+1}) - \left(1 + \beta + \frac{\lambda^2}{\gamma}\right)(a + bz^{t+j}) + a + bz^{t+j-1} = (\beta - 1)\pi^* + \lambda x^*$$

We can rewrite this as

$$bz^{t+j-1}\left(\beta z^2 - \left(1 + \beta + \frac{\lambda^2}{\gamma}\right)z + 1\right) - \frac{a\lambda^2}{\gamma} = (\beta - 1)\pi^* + \lambda x^*$$

We then must have

$$a = -\frac{\gamma}{\lambda^2}\left((\beta - 1)\pi^* + \lambda x^*\right)$$

We then can use the quadratic equation to find z:

$$z = \frac{1 + \beta + \frac{\lambda^2}{\gamma} \pm \sqrt{\left(1 + \beta + \frac{\lambda^2}{\gamma}\right)^2 - 4\beta}}{2\beta}$$

We note that the negative root of this condition is stable, i.e. positive and less than 1, as the coefficient on the z term is greater than 1 and the discriminant is positive. The we can use the initial condition to find b:

$$\mu_{-1} = a + \frac{b}{z} = 0$$

$$\implies b = -az = \left(\frac{\gamma}{\lambda^2}\left((\beta - 1)\pi^* + \lambda x^*\right)\right)\left(\frac{1 + \beta + \frac{\lambda^2}{\gamma} - \sqrt{\left(1 + \beta + \frac{\lambda^2}{\gamma}\right)^2 - 4\beta}}{2\beta}\right)$$

Finally, we note that as the root is stable, so $z < 1$, we have that

$$\lim_{t\to\infty} a + bz^t = a$$

Then we have

$$\lim_{t\to\infty} \mu_{t-1} - \mu_t = a - a = 0$$

Thus, by our first order condition for π_{t+j}, we have that

$$\lim_{t\to\infty} \pi_t = \pi^*$$

2 Dynare

1. Refer to Dynare script `problem_2_1.mod`. The resulting welfare is $\boxed{-0.003278}$.

2. Refer to Dynare script `problem_2_2.mod`. The resulting welfare is $\boxed{-0.001512}$.

   Commitment, then, does improve welfare.

3. (a) Refer to Dynare script `problem_2_3a.mod`. The resulting welfare is $\boxed{-0.001641}$.

   (b) Refer to Dynare script `problem_2_3b.mod`. The resulting welfare is $\boxed{-0.132443}$.

   (c) Refer to Dynare script `problem_2_3c.mod`. The resulting welfare is $\boxed{-0.012845}$.

   We note that all of these cases are not as good as commitment, although inflation targeting is close. Output targeting is very ineffective, because it neglects to use the value of the γ weighting by eliminating the variance of x_t altogether at the expense of variance in π_t. The Taylor rule is not as effective as discretionary or commitment monetary policy.

4. The discretionary responses are below

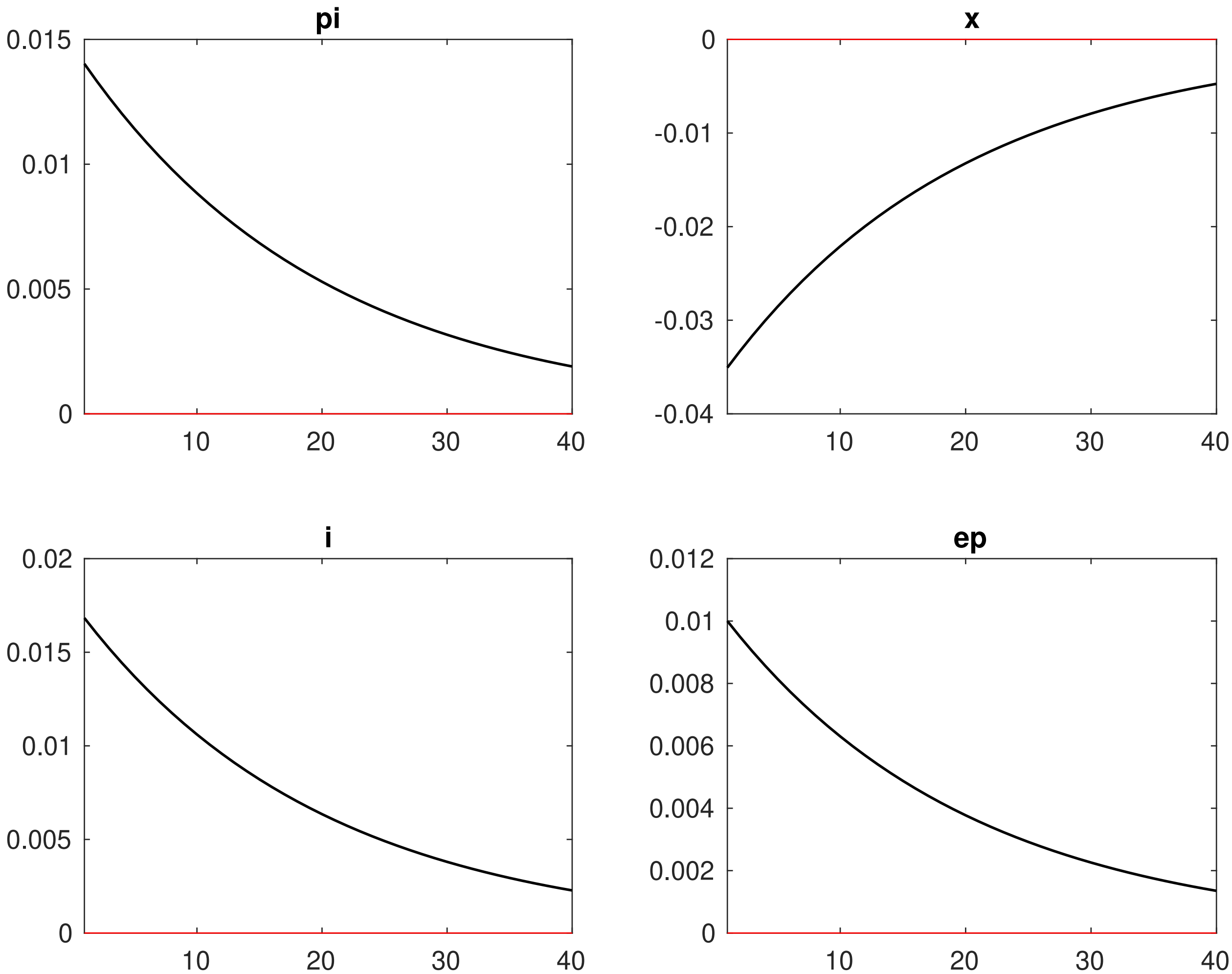

Figure 1: Responses to Price Shock (Discretionary)

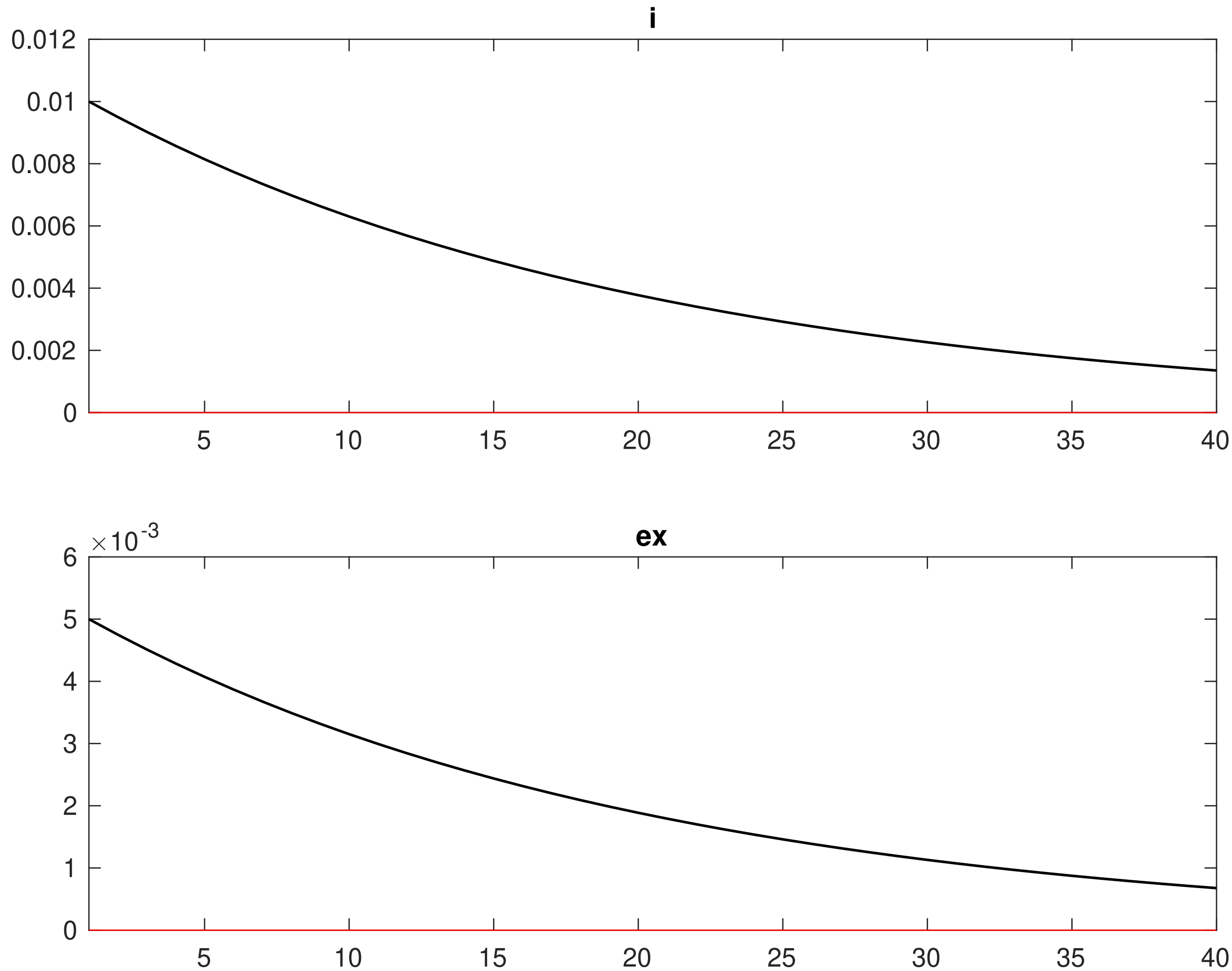

Figure 2: Responses to Demand Shock (Discretionary)

We note that the only variable that moves with a demand shock is the interest rate, which offsets it exactly; π_t and x_t are unaffected. For the commitment case, we have:

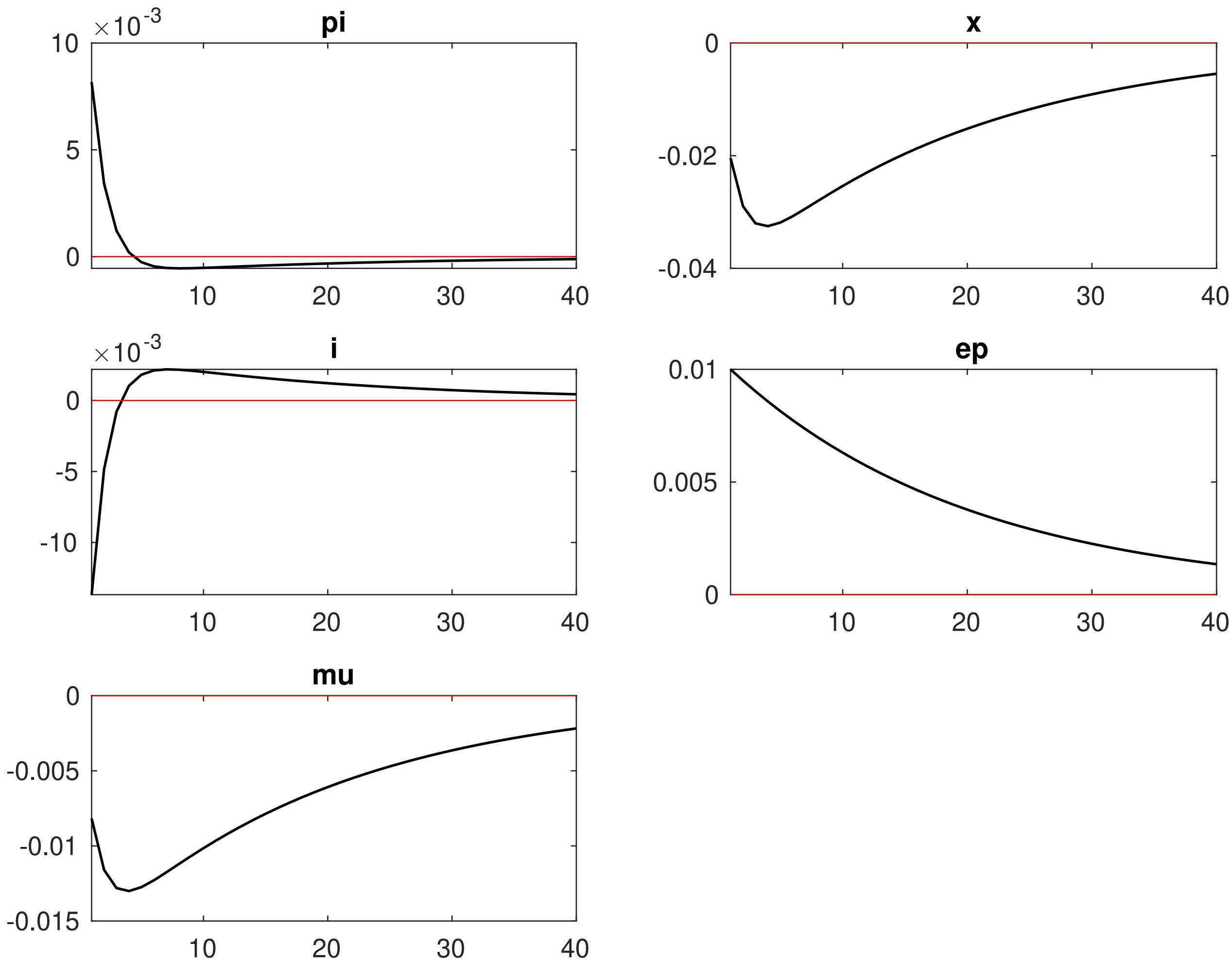

Figure 3: Responses to Price Shock (Commitment)

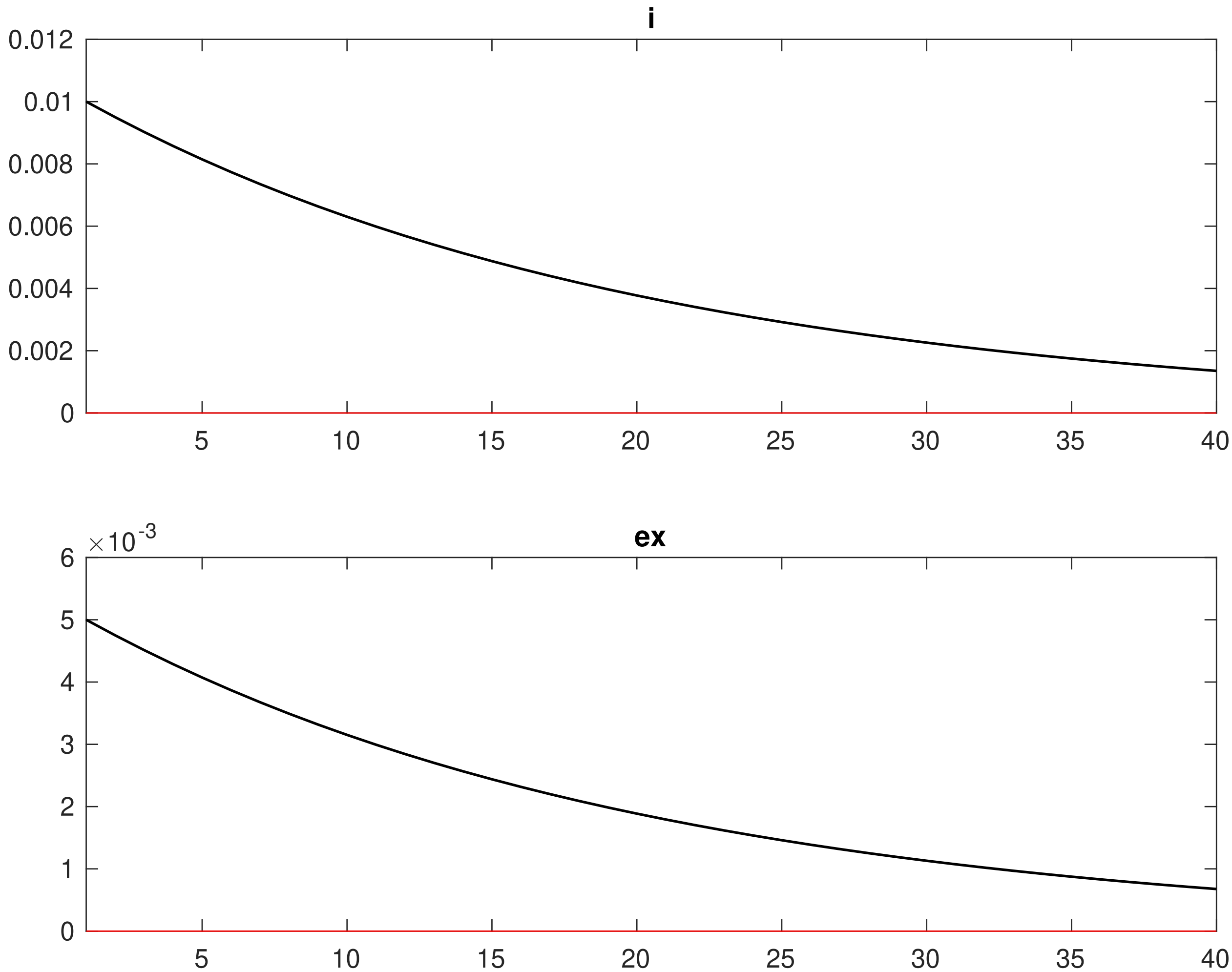

Figure 4: Responses to Demand Shock (Commitment)

Again, there is no response in x_t and π_t to a demand shock. With commitment, because the central bank cannot change, all of the variables exceed steady state after the shock, and then have to return to the steady state path to reconverge to the steady state. This means that the convergence is nonlinear and will result in values of π which are lower than steady state, which does not occur in the discretionary case.