DEPARTMENT OF INFORMATICS

TECHNICAL UNIVERSITY MUNICH

Master Thesis

# On the Parametrized Complexity of Semitotal Domination on Graph Classes

Lukas Retschmeier

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|---|---|
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I confirm that this master thesis is my own work and I have documented all sources and material used.

*København S* Lukas Retschmeier
September 16, 2022

Acknowledgments

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# ABSTRACT

*Abstract all the way*

# CHAPTER 1

# INTRODUCTION

Parametrized Complexity emerging branch. Books about that

Semitotal domination introduced by

Idea: Lake with stones, and family of ducks of fixed size wants to occupy the lake so that no other clan tries to take it over. Rules: * A duck can quack freeing up neighboring stones. * Ducks don't like to be alone and want to quack together. So for every duck their must be another duck that is not further than two stones away. Q: Can our ducklings occupy the whole lake while not feeling lonely?

## 1.1 Content of the thesis

In this thesis we continue the systematic analysis of the Semitotal Dominating Set problem by focusing on the parametrized complexity of the problem.

Although the problem already had a lot of attention regarding classical complexity (CITE), only a few results are currently known for the parametrized variant.

As far as we have seen, even the w-hardness of the general case has not been explicitly been proofen in the literature.

In this thesis, we continue the journey towards a systematic analysis by stating some hardness results for specific graph classes for the problem.

**Our contributions** Our main contributations consist of first showing the $w[2]$-hardness of Semitotal Dominating Set for XXXX graphs.
As the Dominating Set problem and the Total Dominating Set problem both admit a linear kernel for planar graphs, it is interesting to analyse wether this results also holds for the Semitotal Dominating Set problem which lays in between these two.

Having these kernels also for other variants like Edge Dominating Set, Efficient Dominating Set, Connected Dominating Set, Planar Red-Blue Dominating Set lent us a great confidence that the result will also work for Semitotal Dominating Set on planar graphs.

Following the approach from ... which alraedy relies on the technique given in, we give some simple data reduction rules for Semitotal Dominating Set on planar graphs leading to a linear kernel. More precisely, we are going to proof the following central theorem of this thesis:

With some modifications we were able to transfer the approach given by Garnero and Stau in [**Garnero2018**] to the Semitotal Dominating Set problem.

**Theorem 1.** *The* Semitotal Dominating Set *problem parametrized by solution size admits a linear kernel on planar graphs. There exists a polynomial-time algorithms that given a planar graph $(G, k)$, either correctly reports that $(G, k)$ is a NO-instance or returns an equivalent instance $(G', k)$ such that $|V(G')| \leq 561 \cdot k$.*

Dominating Set problem and Total Dominating Set problem, both already

# CHAPTER 2

## PRELIMINARIES

We start by recapping some basic notation in Graph Theory and Parametrized Complexity.

Continuing an intensive study of parametrized complexity of that problem.

### 2.1 Graph Theory

We quickly state the following definitions given by **[diestel10]**.

**Definition 1** (Graph **[diestel10]**)**.** *A graph is a pair $G = (V, E)$ of two sets where $V$ denotes the vertices and $E \subseteq V \times V$ the edges of the graph. A vertex $v \in V$ is incident with an edge $e \in E$ if $v \in e$. Two vertices $x, y$ are adjacent, or neighbours, if $\{x, y\} \in E$.*

**Definition 2** (Vertex Degrees)**.** *The degree $d_G(v)$ (If G is clear, also $d(v)$) of a vertex $v$ is the number of neighbors of $v$. We call a vertex of degree 0 as isoliated and one of degree 1 as a pendant.*

**Definition 3** (Special Graph Notations **[diestel10]**)**.** *A simple Graph*
*A directed Graph is a graph*
*A Multi Graph*
*A Planar Graph*

**Definition 4** (Closed and Open Neighborhoods **[Balakrishnan2012]**)**.** *Let $G = (V, E)$ be a (non-empty) graph. The set of all neighbors of $v$ is the open neighborhood of $v$ and denoted by $N(v)$; the set $N[v] = N(v) \uplus \{v\}$ is the closed neighborhood f $v$ in $G$. When $G$ needs to be made explicit, those open and closed neighborhoods are denoted by $N_G(v)$ and $N_G[v]$.*

**Definition 5** (Induced Subgraph)**.** *asd*

**Definition 6** (Isomorphic Graph)**.** *asd*

### Special Graph Classes

We call the class of graphs without any special restrictions "General Graphs".

**Definition 7** (r-partite Graphs)**.** *Let $r \geq 2$ be an integer. A Graph $G = (V, E)$ is called "r-partite" if $V$ admits a parititon into $r$ classes such that every edge has its ends in different classes: Vertices in the same partition class must not be adjacent.*

*For the case $r = 2$ we say that the $G$ is "bipartite"*

**Definition 8** (Chordal Graphs)**.**

**Definition 9** (Split Graphs)**.**

## 2.2 Parametrized Complexity

### 2.2.1 Fixed Parameter Tractability

#### Fixed Parameter Intractability: The $W$ Hierarchy

### 2.2.2 Kernelization

# CHAPTER 3

# ON PARAMETRIZED SEMITOTAL DOMINATION

## 3.1 Semitotal Domination

SEMITOTAL DOMINATING SET

For two dominating vertices $d_1, d_2$ we say that they are wittnesses for each other if $d(d_1, d_2) \leq 2$

Definition, dominating number

**Complexity Status of Semitotal Dominating Set**

## 3.2 *w[i]*-Intractibility

Now some w[i] hard classes.

### 3.2.1 Warm-Up: *W[2]*-hard on General Graphs

As any bipartite graphswith bipartition can be split further into $r$-partite graphsthis results also implies the $w[1]$-hardness of $r$-partite graphs

### 3.2.2 *W[2]*-hard on Bipartite Graphs

**Definition 10** (Bipartite Graph, **[Bondy2008]**)**.** *A bipartite graphs is a Graph G whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y. Such a partition (X,Y) is called a bipartition of G.*

**Theorem 2.** *Semitotal Dominating Set is $\omega[2]$ hard for bipartite Graphs*

*Proof.* Given a bipartite Graph $G = (\{X \cup Y\}, E)$, we construct a bipartite Graph G' in the following way:

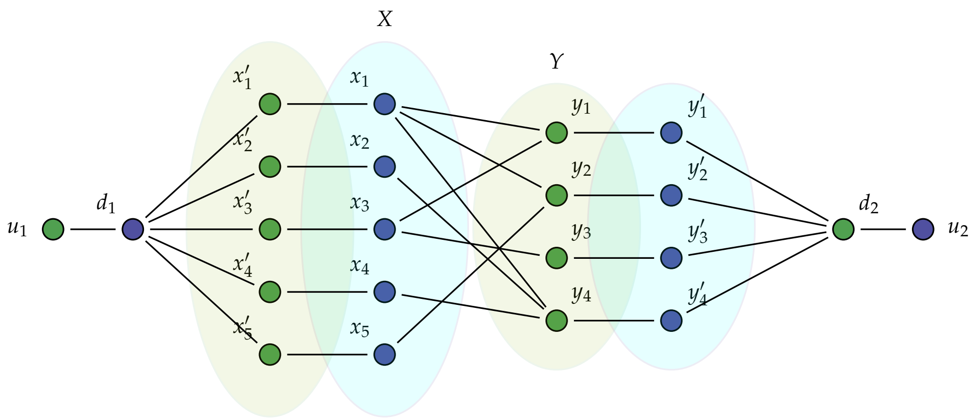


Figure 3.1: Constructing G′ from a bipartite Graph G by duplicating the vertices and adding a dominating tail

1. For each vertex $x_i \in X$ we add a new vertex $x'_i$ and an edge $(x_i, x'_i)$ in between.
2. For each vertex $y_j \in Y$ we add a new vertex $y'_j$ and an edge $(y_j, y'_j)$ in between.
3. We add two $P_1$, namely $(u_1, d_1)$ and $(u_2, d_2)$, and connect them with all $(d_1, x'_i)$ and $(d_2, y'_j)$ respectively.

**Observation:** G′ is clearly bipartite as all $y'_j$ and $x'_i$ form again an Independent Set. Setting $X' = X \cup \{u_2\} \cup \bigcup y'_i$ and $Y' = Y \cup \{u_1\} \cup \bigcup x'_i$ form the partitions of bipartite G′.

**Corollary 1.** *G has a Dominating Set of size k iff G has a Semitotal Dominating Set of size* $k' = k + 2$

⇒: Asume there exists a Dominating Set D in G with size k. $DS = D \cup \{d_1, d_2\}$ is a Semitotal Dominating Set in $G'$ with size $k' = k + 2$ , because $d_1$ dominates $u_1$ and all $x'_i$; $d_2$ dominates $u_2$ and all $y'_i$. Hence, it is a Semitotal Dominating Set, because $\forall v \in (D \cap X) : d(v, d_1) = 2$ and $\forall v \in (D \cap Y) : d(v, d_2) = 2$

⇐: On the contrary, asume any Semitotal Dominating Set $SD$ in $G'$ with size $k'$. WLOG we can asume that $u_1, u_2 \notin DS$.

Our construction forces $d_1$, $d_2 \in DS$. Because all $x'_i$ are only important in dominating $x_i$ ($y'_i$ for $y_i$ resp.) as $d_1, d_2 \in DS$. If $x'_i \in DS$ we simply exchange it with $x_i$ (for $y'_i$ and $y_i$ respectively) in our DS keeping the size of the dominating set. $D = DS \setminus \{d_1, d_2\}$ give us a Dominating Set in G with size $k = k' - 2$

As G′ can be constructed in $\mathcal{O}(n)$ and parameter k is only blown up by a constant, this reduction is a FPT reduction. As Dominating Set is $w[2]$ hard for bipartite Graphs[1] so is Semitotal Dominating Set. □

[1] Citation needed!

### 3.2.3 *W[2]*-hard on Chordal Graphs

### 3.2.4 *W[2]*-hard on Split Graphs

# CHAPTER 4

# A LINEAR KERNEL FOR PLANAR SEMITOTAL DOMINATION

*The best way to explain it is to do it.*
Lewis Caroll, *Alice in Wonderland*

We are going to present a polynomial-time preprocessing procedure which gives a linear kernel for Planar Semitotal Dominating Set parametrized by solution size. Based on the technique first introduced by Alber et al. ([**Alber2004**]) in 2004, an abundance of similar results to other domination problems emerged which gave us the believe that we can also transfer these results to Semitotal Dominating Set. **??** gives an overview about the status of kernels for the planar case on other domination problems. All of these results introduce reduction rules bounding the number of vertices inside so-called "regions" which can be obtained by a special decomposition of the graph.

In the following years, this approach bore fruits in other planar problems as well like Connected Vertex Cover ($11/3k$ in [**Kowalik2013**]), Maximum Triangle Packing ($624k$ in [**Wang2011**]) Induced Matching ($40k$ in [**Kanj2011**]) Full-Degree Spanning Tree (TODO in [**Guo2006**]) Feedback Vertex Set ($13k$ in [**Bonamy2016**]) and Cycle Packing ([**Garnero2019**])

In the upcoming years, many results could generalize the approach to larger graph classes. Fomin and Thilikos started this journey by directly proofing in the same year that the initial reduction rules given by Alber et al. [**Alber2004**] can also be used to obtain a linear kernel on graphs with bounded genus g ([**Fomin2004**]). Alon and Gutner advanced in 2008 with showing that the problem has a linear kernel on $K_{3,h}$-topological-minor-free graph classes and a polynomial kernel for $K_h$-topological-minor-free graph classes ([**Gutner2009**]). In 2007 they extended this result to show that graphs of bounded degeneracy are FPT ([**Alon2007**]). Finally, in 2012 Philip et al.

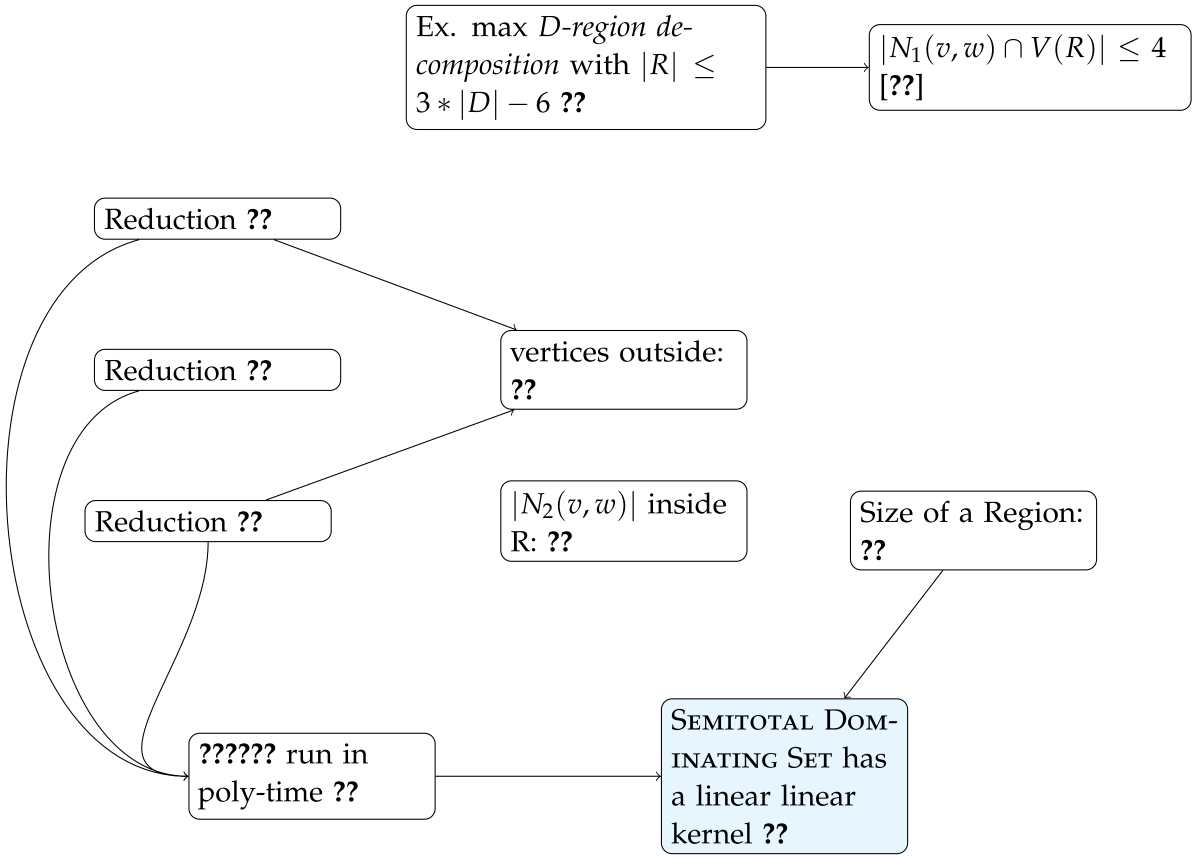


Figure 4.1: *The plan for obtaining a linear kernel for* Planar Semitotal Dominating Set

showed that even $K_{i,j}$-free graph classes admit a polynomial kernel for Dominating Set ([**Philip2012**]). In an attempt to extend these ideas to other problems as well, Bodlaender et al. ([**Bodlaender2016**]) proofed that all problems expressible in counting monadic second-order logic who sastisfy a coverability property admit a polynomial kernel on graphs of bounded genus g.

Although these results are very interesting from a theoretical point of view, the constants for the kernels obtained by these methods are so large that they are not of practical interest. The question is how such a kernel can explicitely and efficiently be constructed.

We are going to show that, with some slight modifications, the kernel described by Garnero and Stau ([**Garnero2014**]) for Planar Total Dominating Set can also be used for Planar Semitotal Dominating Set giving us an explicitely constructed kernel with "reasonable" small constants.

| **Problem** | **Best Known Kernel** | **Source** |
|---|---|---|
| Dominating Set | $67k$ | **[Diekert2005]**[1] |
| Total Dominating Set | $410k$ | **[Garnero2018]**[2] |
| Semitotal Dominating Set | TODO | This Work |
| | | |
| Edge Dominating Set | $14k$ | **[Arge2007]** |
| Efficient Dominating Set | $84k$ | **[Arge2007]** |
| Planar Red-Blue Dominating Set | $43k$ | **[Garnero2017a]** |
| Connected Dominating Set | $130k$ | **[Luo2013]** |
| Directed Dominating Set | ? | **[Alber2006]** |

There is also a masters thesis claiming a bound of 43k **[Halseth2016]**, but a conference or journal version was not found.

Table 4.1: An overview about existing kernels for planar dominating set variants

## 4.1 The Main Idea and The Big Picture

From an algorithmic point of view, all our given reduction rules are local and only concern the neighborhood of at most two vertices and replace them partially with gadgets of constant size. Now checking whether a graph after applying the rule has been changed can be trivially accomplished in constant time.

The main idea is to decompose a graph into so called "regions" where we can bound the number of vertices that are left, after some reduction rules have operated on the graph. [TODO SUM UP THE STRATEGY]

## 4.2 Definitions

We are now giving ....

First, we would like to define what we consider to be a *reduced* graph.

**Definition 11** (Reduced Graph **[Garnero2018]** and **[Garnero2017]**). *A Graph G is reduced under a set of rules if either none of these rules can be applied to G or the application of any of them creates a graph isomorphic to G.*

In our case, we say G is reduced if none of the **??????** are modifying G any more.

This differs from the definition usually given in literature where a graph G is *reduced* under a set of reduction rules, if none of them can be applied to G anymore (Compare e.g. **[Fomin2019]**). The reason is that we are giving reduction rules (see **??** or **??**) that could be applied *ad infinitum* sending us into an endless loop without ever changing G any more. Our definition guarantees termination in that case.

In order to achieve this, we will first split up the neighborhood of one vertice and a pair of vertices into three distinct subsets, give some nice properties about them, and then state the corresponding reduction rules.

In this section, we are giving some key definitions that are used in our reduction rules for obtaining the linear kernel. These as inspired by those given by Garnero and Stau (PLANAR TOTAL DOMINATING SET in [**Garnero2014**] or PLANAR RED-BLUE DOMINATING SET in [**Garnero2017a**]) and already relied on those given by Alber et al. in [**Alber2004**] for PLANAR DOMINATING SET.

The idea is to split up the neighborhood of a single vertex and a pair of vertices into three (disjoint) subsets that make a statement about how strongly the neighborhood is connected with the rest of the graph.

**Definition 12.** *Let $G = (V, E)$ be a graph and let $v \in V$. We denote by $N(v) = \{u \in V : \{u, v\} \in E\}$ the neighborhood of $v$. We split $N(v)$ into three subsets:*

$$N_1(v) = \{u \in N(v) : N(u) \setminus N[v] \neq \emptyset\} \tag{4.1}$$
$$N_2(v) = \{u \in N(v) \setminus N_1(v) : N(u) \cap N_1(v) \neq \emptyset\} \tag{4.2}$$
$$N_3(v) = N(v) \setminus (N_1(v) \cup N_2(v)) \tag{4.3}$$

*In order to inhance future readability, for $i, j \in [1, 3]$, we denote $N_{i,j}(v) := N_i(v) \cup N_j(v)$.*

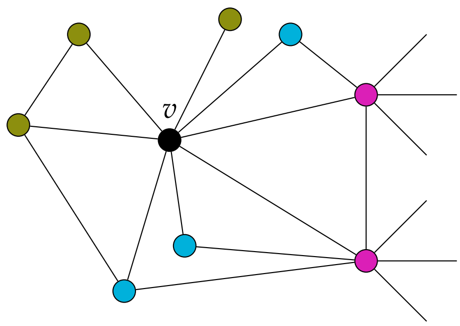


Figure 4.2: *The neighborhood of a vertex split to $N_1(v)$ (blue), $N_2(v)$ (purple), and $N_3(v)$ (green). Note that all purple vertices have at least one blue neighbor making setting them in-between the green and blue vertices.*

Intuitively, these sets are classifying neighbors of $v$ by how much they can interact with the rest of the graph and how much they are locally centered around $v$:
$\mathbf{N_1(v)}$ are all neighbors of $v$ which have at least one adjacent vertex that is outside of $N(v)$ and therefore connect $v$ with the rest of the graph. They could possibly belong to a solution.
$\mathbf{N_2(v)}$ are all neighbors of $v$ that have at least one neighbor from $N_1(v)$. These vertices do not have any function as a dominating vertex and can be seen as a *buffer* bridging

$N_1(v)$-vertices with those from $N_3(v) \cup \{v\}$. Furthermore, they are useless as witnesses, because either we can replace them by $v$ (sharing the same neighborhood) or when being a witness for $v$, we replace it with one $z \in N_1(v)$.
Vertices from $\mathbf{N_3(v)}$ are unmitigated and sealed off from the rest of the graph. They are useless as dominating vertices: For all $z \in N_3(v)$ it holds that $N(z) \subseteq N(v)$ by definition and thus, we would always prefer $v$ as a dominating vertex instead of $z$. Nevertheless, they can be important as a witness for $v$ in the case that $N_1(v) \cup N_2(v) = \emptyset$. We are using this observation in **??** where we shrink $|N_3(v)| \leq 1$

In the following, we are going to further extend this definition to a pair of vertices. Using this, **??** will later try to reduce the neighborhood of two vertice, and similar to **??**, we can deduce some properties.

**Definition 13.** *Let $G = (V, E)$ be a graph and $v, w \in V$. We denote by $N(v, w) = N(v) \cup N(w)$ the neighborhood of the pair $v, w$. We split $N(v, w)$ into three subsets:*

$$N_1(v,w) = \{u \in N(v,w) \mid N(u) \setminus (N(v,w) \cup \{v,w\}) \neq \emptyset\} \tag{4.4}$$

$$N_2(v,w) = \{u \in N(v,w) \setminus N_1(v,w) \mid N(u) \cap N_1(v,w) \neq \emptyset\} \tag{4.5}$$

$$N_3(v,w) = N(v,w) \setminus (N_1(v,w) \cup N_2(v,w)) \tag{4.6}$$

*Again, for $i, j \in [1, 3]$, we denote $N_{i,j}(v, w) = N_i(v, w) \cup N_j(v, w)$.*

Again, $N_1(v, w)$ are those connected with the rest of the graph, $N_2(v, w)$ are a *buffer* between $N_3(v, w) \cup \{v, w\}$ and $N_3(v, w)$ are those isolated from the rest of the graph, but can still be useful as a witness for $v$ or $w$.

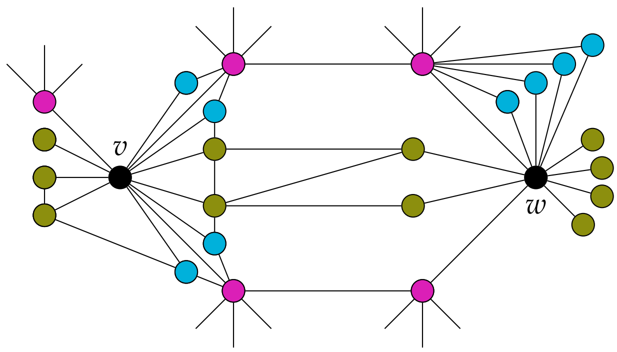


Figure 4.3: *TODO*

Note that for example a vertex $z \in N_1(v) \not\Longrightarrow z \in N_1(v, w)$. Figure **??** gives an example, where $z$ belongs to $N_1(v)$, but not to $N_1(v, w)$.

### 4.2.1 Regions in Planar Graphs

We will introduce a concept that leads towards a new perspective looking at planar graphs, regions

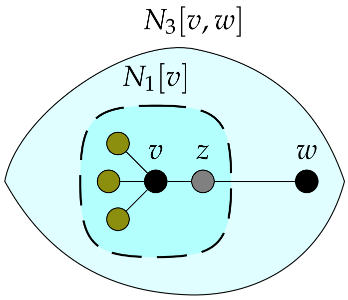


Figure 4.4: *The vertex $z$ is in $N_1(v)$, because there is an edge pointing outside of $N(v)$ to $w$. Contrary, it is not in $N_1(v, w)$, but now belongs to $N_3(v, w)$, because we are considering the "shared" neighborhood*

As it is possible to bound the number of total $vw$-regions in a planar graph, we can analyze the local impacts to these regions from our future reduction rules.

It might be interesting to note that the authors of **[Garnero2017]** have revised their original definitions to set the basis for a more formal analysis.

**Definition 14.** *Two simple paths $p_1, p_2$ in a plane graph $G$ are confluent if:*

1. *they are vertex-disjoint*
2. *they are edge-disjoint and for every common vertex $u$, if $v_i, w_i$ are the neighbors of $u$ in $p_i$, for $i \in [1, 2]$, it holds that $[v_1, w_1, v_2, w_2]$, or*
3. *they are confluent after contracting common edges*

**Definition 15.** *Let $G = (V, E)$ be a plane graph and let $v, w \in V$ be two distinct vertices. A region $R(v, w)$ (also denoted as $vw$-region) is a closed subset of the plane, such that:*

1. *the boundary of $R$ is formed by two confluent simple $vw$-paths with length at most 3*
2. *every vertex in $R$ belongs to $N(v, w)$, and*
3. *the complement of $R$ in the plane is connected.*

*We denote by $\partial R$ the boundary of $R$ and by $V(R)$ the set of vertices which lay (with the plane embedding) in $R$. Furthermore, we call $|V(R)|$ the size of the region.*

*The poles of $R$ are the vertices $v$ and $w$. The boundary paths are the two $vw$-paths that form $\partial R$*

We now have all the definitions ready to formally define a decomposition technique for planar graphs:

**Definition 16.** *Two regions $R_1$ and $R_2$ are non-crossing, if:*

1. $(R_1 \setminus \partial R_1) \cap R_2 = (R_2 \setminus \partial R_2) = \emptyset$, *and*
2. *the boundary paths of* $R_1$ *are pairwise confluent with the ones in* $R_2$

**Definition 17.** *Given a plane graph* $G = (V, E)$ *and* $D \subseteq V$, *a* <u>$D-$ *region Decomposition*</u> *of* $G$ *is a set* $\mathfrak{R}$ *of regions with poles in* $D$ *such that:*

1. *for any vw-region* $R \in \mathfrak{R}$, *it holds that* $D \cap V(R) = \{v, w\}$, *and*
2. *all regions are pairwise non-crossing.*

*We denote* $V(\mathfrak{R}) = \bigcup_{R \in \mathfrak{R}} V(R)$.
*A D-region decomposition is* <u>*maximal*</u> *if there is no region* $R \notin \mathfrak{R}$ *such that* $\mathfrak{R}' = \mathfrak{R} \cup \{R\}$ *is a D-region decomposition with* $V(\mathfrak{R}) \subsetneq V(\mathfrak{R}')$

**??** gives an example of how to decompose a graph into a maximal $D-region$ decomposition with a given Semitotal Dominating Set of size 3.

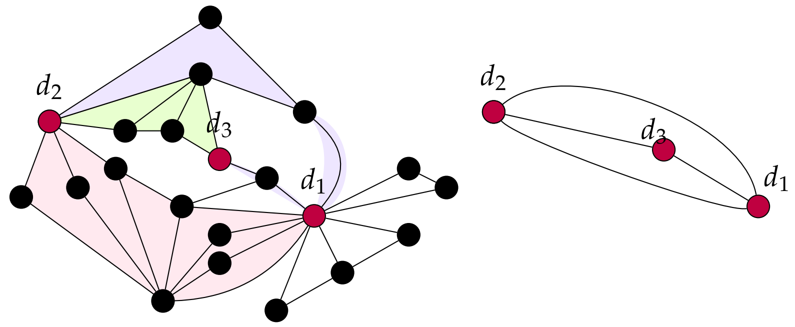


Figure 4.5: *left: A maximal D-region decomposition* $\mathfrak{R}$, *where* $D = \{d_1, d_2, d_3\}$ *form a* Semitotal Dominating Set. *There are two regions between* $d_2$ *and* $d_1$, *one region between* $d_1$ *and* $d_3$ *and one region between* $d_2$ *and* $d_3$. *Observe that some neighbors of* $d_1$) *are not part of any vw-region. For those, our reduction rules are going to take care of that and bound these number of vertices to obtain the kernel. Right: The corresponding underlying multigraph* $G_{\mathfrak{R}}$

We are introducing a special subset of a region, a *simple region* where every vertex is a common neighbor of $v$ and $w$. They will appear in many unexpected astonishing places and are an important tool to operate on small parts of a plane graph. The upcoming **??** will bound the size of these *simple regions*.

**Definition 18.** *A simple vw-region is a vw-region such that:*

1. *its boundary paths have length at most 2, and*
2. $V(R) \setminus \{v, w\} \subseteq N(v) \cap N(w)$.

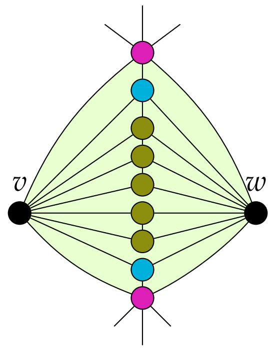


Figure 4.6: *A simple region with two vertices from $N_1(v,w)$ setting the boundary, two vertices from $N_2(v,w)$ and some vertices from $N_3(v,w)$ in between*

**??** shows an example of a simple region containing 9 vertices.

Later we will use properties of the underlying multigraph of a *D-region decomposition*. Refer to **??** for an example.

**Definition 19.** *Let $G = (V, E)$ be a plane graph, let $D \subseteq V$ and let $\mathfrak{R}$ be a $D$-region decomposition of $G$. The underlying multigraph $G_{\mathfrak{R}} = (V_{\mathfrak{R}}, E_{\mathfrak{R}})$ of $\mathfrak{R}$ is such that $V_{\mathfrak{R}} = D$ and there is an edge $\{v, w\} \in E_{\mathfrak{R}}$ for each $vw$-region $R(v, w) \in \mathfrak{R}$*

## 4.3 Reduction Rules

Following the approach by **[Garnero2014]**, we are now stating reduction rules that after exhaustive application will expose a linear kernel.

### 4.3.1 Reduction Rule I: Getting Rid of unneccessary $N_3(v)$ vertices

An exemplarly application of the rule is shown in figure **??**

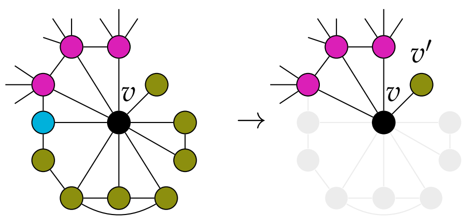


Figure 4.7: *TODO*

**Rule 1.** *Let $G = (V, E)$ be a graph and let $v \in V$. If $|N_3(v)| \geq 1$:*

- *remove $N_3(v)$ from $G$,*

- *add a vertex $v'$ and an edge $\{v, v'\}$*

**Lemma 1.** *Let $G = (V, E)$ be a a graph and let $v \in V$. If $G'$ is the graph obtained by applying* **??** *on $V$, then $G$ has SDS of size $k$ if and only if $G'$ has one.*

*Proof.* This will be the proof for this lemma X □

Note, that we need our definition of a reduced instance given in **??**. If **??** is being applied, it will still leave us with a vertex $z \in N_3(v)$ allowing this rule to be applied again.

### 4.3.2 Reduction Rule II: Shrinking the Size of a Region

Extending the approach for a linear kernel for Dominating Set proposed by Alber et al. in [**Alber2004**], Garnero and Stau transferred these results in [**Garnero2018**] to the Total Dominating Set problem.

Their idea was to strengthen the reduction rules in such a way that the witness properties for total domination are being preserved.

Following their approach in one of the first versions of [**Garnero2014**], we stating reduction rules that. Interestingly, the reduction rules given in the latest version of this paper was not directly be transferable to Semitotal Dominating Set, but an older version giving slightly easier reduction rules could be adjusted to our problem.

which relies on the technique first introduced by Alber et al we try to reduce the neighborhood for two given vertices $v$ and $w$

Before we give the concrete reduction rule, we will define three sets

$$\mathcal{D} = \{\tilde{D} \subseteq N_{2,3}(v, w) \mid N_3(v, w) \subseteq \bigcup_{v \in \tilde{D}} N(v),\ |\tilde{D}| \leq 3\} \tag{4.7}$$

$$\mathcal{D}_v = \{\tilde{D} \subseteq N_{2,3}(v, w) \cup \{v\} \mid N_3(v, w) \subseteq \bigcup_{v \in \tilde{D}} N(v),\ |\tilde{D}| \leq 3,\ v \in \tilde{D}\} \tag{4.8}$$

$$\mathcal{D}_w = \{\tilde{D} \subseteq N_{2,3}(v, w) \cup \{w\} \mid N_3(v, w) \subseteq \bigcup_{v \in \tilde{D}} N(v),\ |\tilde{D}| \leq 3,\ w \in \tilde{D}\} \tag{4.9}$$

**Rule 2.** *Let $G = (V, E)$ be a graph and two distinct $v, w \in V$. If $\mathcal{D} = \emptyset$ we apply the following:*

***Case 1:*** *if $\mathcal{D}_v = \emptyset$ and $D_w = \emptyset$*

- *Remove $N_{2,3}(v, w)$*
- *Add vertices $v'$ and $w'$ and two edges $\{v, v'\}$ and $\{w, w'\}$*

- *If there was a common neighbor of $v$ and $w$ in $N_{2,3}(v,w)$ add another vertex $y$ and two connecting edges $\{v,y\}$ and $\{y,w\}$*

***Case 2:*** *if $\mathcal{D}_v \neq \emptyset$ and $D_w \neq \emptyset$*
*Do nothing*[3]

***Case 3:*** *if $\mathcal{D}_v \neq \emptyset$ and $D_w = \emptyset$*

- *Remove $N_{2,3}(v) \cap N_3(v,w)$*
- *Add $\{v,v'\}$*

***Case 4:*** *if $\mathcal{D}_v = \emptyset$ and $D_w \neq \emptyset$*
*This case is symmetrical to **Case 3**.*

Before proofing **??** we will deduce some *Facts* which are implied by the definitions above.

**Fact 1.** *Let $G = (V,E)$ be a graph, let $v,w \in V$, and let $G'$ be the graph obtained by the application of **??** on $v,w$. If $\mathcal{D} = \emptyset$, then $G$ has a solution if and only if it has a solution containing at least one of the two vertices $\{v,w\}$.*

*Proof.* □

Now we are ready to proof the correctness of **??**

**Fact 2.** *Let $G = (V,E)$ be a graph, let $v,w \in V$, and let $G'$ be the graph obtained by the application of **??** on $v,w$. If $\mathcal{D} = \emptyset$ and $\mathcal{D}_v = \emptyset$ (resp. $\mathcal{D}_w = \emptyset$) then $G'$ has a solution if and only if it has a solution containing $v$ (resp. $w$).*

*Proof.* □

**Lemma 2.** *Let $G = (V,E)$ be a plane graph, $v,w \in V$ and $G' = (V',E')$ be the graph obtained after application of **??** on the pair $\{v,w\}$. Then $G$ has SDS of size $k$ if and only if $G'$ has SDS of size $k$.*

*Proof.* We will proof the claim by analysing the different cases seperately. □

### 4.3.3 Reduction Rule III: Shrinking Simple Regions

**Rule 3.** *Let $G = (V,E)$ be a plane graph, $v,w \in V$ and $R$ be a simple region between $v$ and $w$. If $|V(R) \setminus \{v,w\}| \geq 7$*

- *Remove $N_3(v,w)$*

[3] Originally, reduce Simple Regions [STAU]

- *Add two vertices $h_1$ and $h_2$ and four edges $\{v, h_1\}$, $\{v, h_2\}$, $\{w, h_1\}$ and $\{w, h_2\}$*

**Lemma 3** (Correctness of **??**). *Let $G = (V, E)$ be a plane graph, $v, w \in V$ and $G' = (V', E')$ be the graph obtained after application of* **??** *on the pair $\{v, w\}$. Then $G$ has SDS of size $k$ if and only if $G'$ has SDS of size $k$.*

The application of **??** gives us a bound on the number of vertices inside a simple region.

**Corollary 2.** *Let $G = (V, E)$ be a graph, $v, w \in V$ and $R$ a simple region between $v$ and $w$. If* **??** *has been applied, this simple region has size at most 6.*

*Proof.* Clearly, if $|V(R) \setminus \{v, w\}| < 7$ then the rule would not have changed G and the size of the region would already be bounded by 6. Asuming $|V(R) \setminus \{v, w\}| \geq 7$ we note that every simple region can have at most two distinct vertices from $N_1(v, w)$ and two distinct ones from $N_2(v, w)$ without breaking planarity. These vertices are not touched by the reduction. Adding the two vertices that are being added between $v$ and $w$ gives us the desired upper bound. □

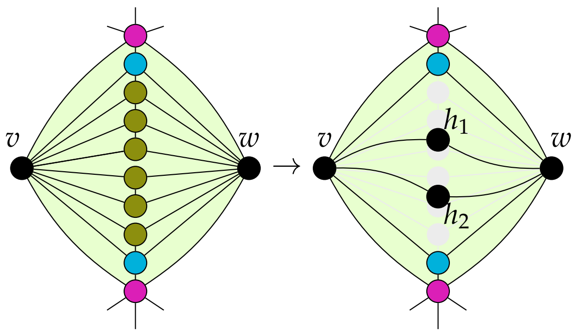


Figure 4.8: *TO BE DONE*

### 4.3.4 Computing Maximal Simple Regions between two vertices

For the sake of completeness, we state an algorithm how a maximal simple regionbetween two vertices $v, w \in V$ can be computed in time $\mathcal{O}(d(v) + d(w))$:

## 4.4 Bounding the Size of the Kernel

We are now putting all our pieces together in order to proof our main result: A linear bound on the kernel size. In order to do so, we distinguish between those vertices that are covered by a maximal *D-region decomposition* and those that are not. In both cases our reduction rules bound the number of vertices to a consant size which means the kernel size does only depend on the number of regions of these decomposition. **??** states that for any solution D, we only have a linear number of regions that cover the whole graph. In particular, we show that given a SEMITOTAL DOMINATING SET D of size k, there exist a maximal *D-region decomposition* $\mathfrak{R}$ such that:

1. $\mathfrak{R}$ has only at most $3|D| - 6$ regions
2. $V(\mathfrak{R})$ covers most vertices of V. There are at most $144 \cdot |D|$ vertices outside of any region.
3. each region of $\mathfrak{R}$ contains at most XX vertices

Combining these three parts will give us a linear kernel.

### 4.4.1 Bounding the Size of a Region

We start are more fine-grained analysis of the impact of the different cases of **??** on a $vw$-region. The main idea is to count the number of simple regions in the $vw$-region and than use the bound on the size of a simple region after **??** was applied exhaustively and which was obtained in **??**.

We start by giving

**Lemma 4.** *Given a plane Graph $G = (V, E)$ and a $vw$-region $R$ $|N_1(v,w) \cap V(R)| \leq 4$ and these vertices lay exactly on the boundary $\partial R$ of $R$.*

*Proof.* □

**Lemma 5.** *[**Garnero2018**] Given a reduced plane graph $G = (V, E)$ and a region $R(v,w)$, $N_2(v,w) \cap N$ can be covered by at most 6 simple regions.*

*Proof.* Let $(v, u_1, u_2, w)$ and $(v, u_3, u_4, w)$ be the two boundary paths of $R(v,w)$. (A shorter path would only lead to a smaller bound). By definition of $N_2(v,w)$, vertices from $N_2(v,w) \cap V(R)$ are common neighbors of $v$ and $w$ and $u_i, i \in [4]$. By planarity, we can cover $N_2(v,w) \cap V(R)$ with at most 6 simple regions. To see this, imagine the graph where edges denote all possible simple $vw$-regions (See fig. **??**). There are at most 8 simple regions possible. but we have to remove at least two of them to maintain planarity.

Furthermore, asuming the graph to be reduced, any intermediate $N_3(v,w)$ which could possible seperate multiple simple regions between $v$ and $u_i$ has been deleted by **??** already. □

We continue by giving a constant bound on the number of simple regions that cover all $N_3(v,w)$ vertices in a given region.

**Lemma 6.** *Given a plane Graph $G = (V, E)$ reduced under **??** and a region R(v, w), if $\mathcal{D}_v \neq$ (resp. $\mathcal{D}_w \neq \emptyset$), $N_3(v,w) \cap V(R)$ can be covered by:*

1. 11 *simple regions if $\mathcal{D}_w \neq \emptyset$,*

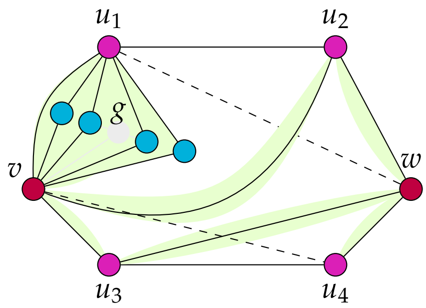


Figure 4.9: *Bounding the maximum number of simple regions inside a region $R(v, w)$. $N_2(v, w)$ is covered by 6 green (simple) regions. A dashed edge would also be an option, but would contradict planarity. Note that the gray vertex $g$ was reduced by* **??** *allowing the formation of exactly one simple region between $v$ and $u_1$*

2. 14 *simple regions if $N_{2,3}(v) \cap N_3(v, w) = \emptyset$*

Note, that the first case applies, when Case 2 & 3 of **??** have been applied and the second one, when Case 4 of **??** was applied.

*Proof.* We will just give some intuition, because the proof of Garnero and Stau in **[Garnero2014]** does not use any special property exposed by the reduction rules. Figure (Add picture about figures) gives a visualization worst case scenarias to cover $N_3(v, w) \cap V(R)$ with simple regions in the relevant cases.[4]

□

**Lemma 7** (#Vertices inside a Region after **??????**)**.** *Let $G = (V, E)$ be a plane graph reduced under* **??????***. Furthermore, let $D$ be a SDS of $G$ and let $v, w \in D$. Any $vw$-region $R$ contains at most 139 vertices distinct from its poles.*

*Proof.* By **????** and **??** to bound the number of vertices inside a simple region, we know that $|N_1(v, w) \cap V(R)| \leq 4$ and $|N_2(v, w) \cap V(R)| \leq 6 \cdot 7 = 42$.

It is still remaining to bound for $|N_3(v, w) \cap V(R)|$, but gladly we have **??**, which took care about them! **??** shows worst case amount of simple regions the indidual cases can have.

**Case 0:** If **??** has **not** been applied
As $\mathcal{D} \neq \emptyset$, there exists a set $\tilde{D} = \{d_1, d_2, d_3\} \in \mathcal{D}$ of at most three vertices dominating $N_3(v, w)$. We observe that vertices from $|N_3(v, w) \cap V(R)|$ are common neighbors of either v or w (by the definition of a $vw$-region) and at least one vertex from $\tilde{D}$. Without violating planarity, we can span at most 6 simple regions. Using **??** and adding $|\tilde{D}| = 3$, we can conclude $|N_3(v, w) \cap V(R)| \leq 6 \cdot 6 + 3 = 39$.

[4]Note: In a newer revision of their paper **[Garnero2018]**, Stau und Garnero removed this proof, because they changed **??** and a more fine-grained analysis was made possible.

**Case 1:** If **??** Case 1 has been applied
In that case $|N_2(v,w) \cap V(R)|$was entirely removed and at $|N_3(v,w) \cap V(R)|$replaced by at most three vertices ($v', w'$ and $y$) added. Hence $|N_3(v,w) \cap V(R)| \leq 3$.

**Case 2:** If **??** Case 2 has been applied
As we know that $\mathcal{D}_v \neq \emptyset$ and $\mathcal{D}_w \neq \emptyset$, we can apply **??** and although **??** has not changed the G, we can cover R with at most 11 simple regions giving as $|N_3(v,w) \cap V(R)| \leq 11 \cdot 6 = 66$ vertices.

**Case 3:** If **??** Case 3 (sym. 4) has been applied
We know that in this case $N_{2,3}(v) \cap N_3(v,w)$ was entirely removed and replaced by a single possible witness. Using **??**, we can cover $(N_3(v,w) \setminus \{v'\} \cap V(R)$ with (at most) 14 simple regions giving us $||N_3(v,w) \cap V(R)|| \leq 14 \cdot 6 + 1 = 85$.

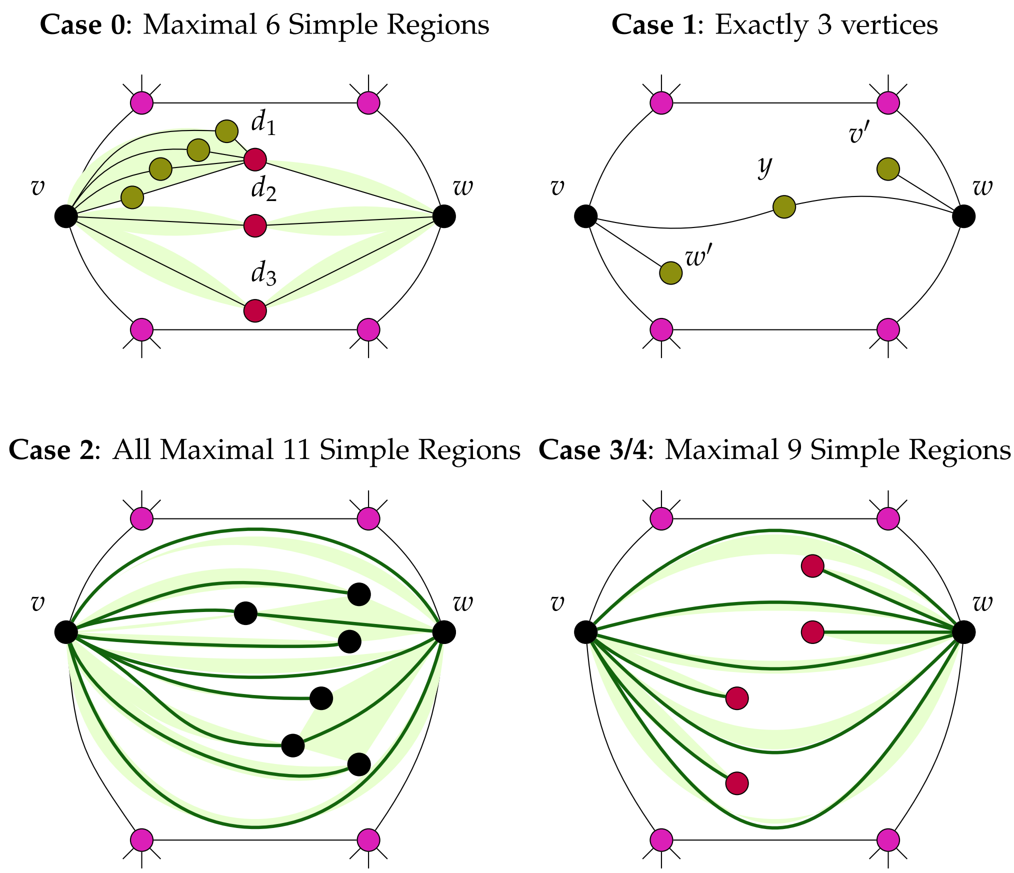


Figure 4.10: *TODO*

All in all, as $V(R) = \{v,w\} \cup (N_1(v,w) \cup N_2(v,w) \cup N_3(v,w)) \cap V(R)$ we get

$$V(R) \leq 2 + 4 + 42 + \max(39, 3, 66, 85) = 139$$

□

### 4.4.2 Number of Vertices outside the Decomposition

We continue to bound the number of vertices that do not lay inside any region of a maximal *D-region decomposition* $\mathfrak{R}$, that is, we bound the size of $V \setminus V(\mathfrak{R})$. **??** ensures that we only have a small amount of $N_3(v)$-pendants. We then try to cover the rest with as few simple regions as possible, because, by application of **??**, these simple regions are of constant size.

**Lemma 8.** *[**Alber2004**] (Deprecated) Every vertex in $u \in V \setminus V(\mathfrak{R})$ is either in D or belongs to a set $N_2(v) \cup N_3(v)$.*

The following lemma states that no vertices from a set $N_1(v)$ will be outside of a maximal *D-region decomposition*.

**Lemma 9.** *[**Alber2004**] Let $G = (V, E)$ be a plane graph and $\mathfrak{R}$ be a maximal D-region decomposition of a DS D. If $u \in N_1(v)$ for some vertex $v \in D$ then $u \in V(\mathfrak{R})$*

In the following, we define $d_{G_\mathfrak{R}}(v) = |\{R(v, w) \in \mathfrak{R}, w \in D\}|$ to be the number of regions in $\mathfrak{R}$ having $v$ as a pole.

**Corollary 3.** *Let $G = (V, E)$ be a graph and D be a set. For any maximal D-region decomposition $\mathfrak{R}$ on G it holds that $\sum_{v \in D} d_{G_\mathfrak{R}}(v) = 2 \cdot |\mathfrak{R}|$.*

*Proof.* The proof follows directly from the handshake lemma applied to the underlying multigraph $G_\mathfrak{R}$. □

**Proposition 1 (#Vertices outside a Region).** *Let $G = (V, E)$ be a plane graph reduced under **????** and let D be a SDS of G. If G has a maximal D-region decomposition, then $|V \setminus (V(\mathfrak{R}) \cup D)| \leq 144|D|$*

With slight modifications, the proof given in **[Garnero2014]** will also work in our case. Note that although assuming the graph to be entirely reduced, the following proof only relies on **????**. The proof uses the observation that vertices from $N_2(v)$ span simple regions between those from $\{v\} \cup N_1(v)$.

*Proof.* Again, we will follow the proof proposed by Alber et al. [**Alber2004**].

The proof does only rely on **????** and we can use the number of vertices in a simple region we have proofen in **??**. In particular, we are going to proof that $V \setminus V(\mathfrak{R}) \leq 48 \cdot |\mathfrak{R}| + 2 \cdot |D|$. Directly placing in **??** will give as the desired bound.

let $\mathfrak{R}$ be a maximal *D-region decomposition* and let $v \in D$. Since D dominates all vertices from $V$, we can consider $V$ as $\bigcup_{v \in D} N(v)$ and thus, we only need to bound the sizes of $N_1(v) \setminus V(\mathfrak{R})$, $N_2(v) \setminus V(\mathfrak{R})$ and $N_3(v) \setminus V(\mathfrak{R})$ separately. In the following, let $v \in D$:

$\mathbf{N_3(v)}$: As we know that **??** has been exhaustively applied, we trivially see that $|N_3(v)| \leq 1$ and hence,

$$\left| \bigcup_{v \in D} N_3(v) \setminus V(\mathfrak{R}) \right| \leq |D|$$

$\mathbf{N_2(v)}$: According to Garnero and Stau ([**Garnero2018**]), we know that $N_2(v) \setminus V(\mathfrak{R})$ can be covered by at most $4d_{G_\mathfrak{R}}(v)$ simple regions between $v$ and some vertices from $N_1(v)$ on the boundary of a region in $\mathfrak{R}$. Figure **??** gives some intuition.

Because G is reduced by assumption, we know by **??** that a simple region can only have at least 6 vertices distinct from its poles and hence,

$$\begin{aligned} \left| \bigcup_{v \in D} N_2(v) \setminus V(\mathfrak{R}) \right| &\leq 6 \sum_{v \in D} 4 \cdot d_{G_\mathfrak{R}}(v) \\ &= 24 \cdot \sum_{v \in D} d_{G_\mathfrak{R}}(v) \\ &\overset{\text{Cor. ??}}{\leq} 48|\mathfrak{R}| \end{aligned} \tag{4.10}$$

$\mathbf{N_1(v)}$: By **??**, we know that $N_1(v) \subseteq V(\mathfrak{R})$ and hence,

$$\left| \bigcup_{v \in D} N_1(v) \setminus V(\mathfrak{R}) \right| = 0$$

Summing up these three upper bounds for each $v \in D$ we obtain the result using the equation from **??**:

$$\begin{aligned} |V \setminus V(\mathfrak{R}) \cup D)| &\leq 48 \cdot |\mathfrak{R}| + |D| && \text{(Lemma ??)} \\ &\leq 48 \cdot (3|D| - 6) + |D| \\ &\leq 144|D| + |D| \\ &= 145|D| \end{aligned} \tag{4.11}$$

□

### 4.4.3 Bounding the Number of Regions

Alber et al. [**Alber2004**] gave a greedy algorithm to construct a maximal *D-region decomposition* for a Dominating Set. Building up on these results, Garnero and Stau gave decomposition procedures for both Planar Red-Blue Dominating Set ([**Garnero2017a**]) and Total Dominating Set ([**Garnero2018**]) relying on the same

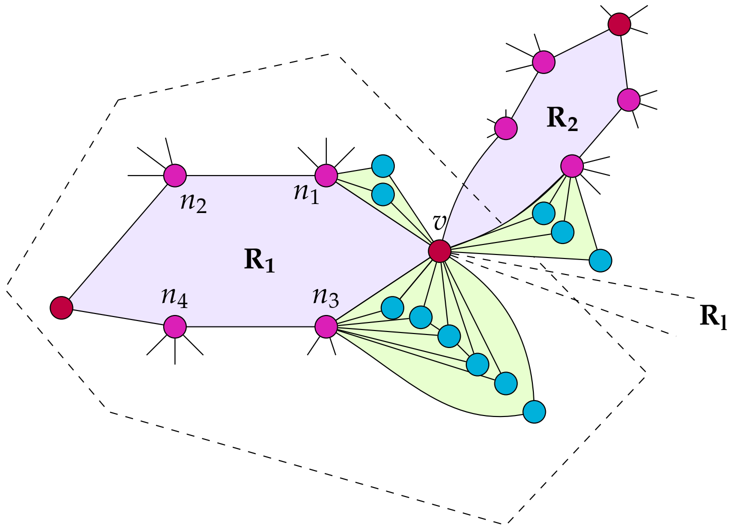


Figure 4.11: *Bounding the number of $N_2(v)$-vertices around a dominating vertex $v$ given a maximal D-region decomposition $\mathfrak{R}$. $v$ is a pole of $R_1, R_2, ...R_j$ and can span simple regions with the help of $N_2(v)$-vertices to at most two $N_1(v)$-vertices per $R_i$. Each region has at most four vertices in $N_1(v,w) \subseteq N_1(v)$ on the boundary of $R_j$, but only at most two can be used for a simple region: Observe that trying to build a simple region between $v$ and $n_2$ in this example would contradict the maximality of $\mathfrak{R}$ Furthermore, the size of these simple regions is bounded after the application of* **??**

technique. This is the core of the linear kernelization, because it states that given a Dominating Set D, we can decompose the graph into a *linear number* of regions.

The following lemma corresponds to [**Alber2004**]. Although the authors gave different reduction rules and require a *reduced* instance as an assumption for the following lemma, they do not use any specific properties exposed by these rules. As any Semitotal Dominating Set is also a Dominating Set, we can safely apply it for our problem as well. For a more detailed and formal proof, one can also refer to [**Garnero2018**].

**Lemma 10.** *Let G be a reduced plane graph and let D be a* Semitotal Dominating Set *with $|D| \geq 3$. There is a maximal D-region decomposition of G such that $|R| \leq 3 \cdot |D| - 6$*

*Proof.* Follows directly from [**Alber2004**] □

**Lemma 11** (Running Time of Reduction Procedure)**.** *TODO Runsi in polynomial Time.*

*Proof.* □

By utilizing all the previous results, we are now finally ready to proof the **??**:

**Theorem ??.** *The* Semitotal Dominating Set *problem parametrized by solution size admits a linear kernel on planar graphs. There exists a polynomial-time algorithms that given a planar graph $(G, k)$, either correctly reports that $(G, k)$ is a NO-instance or returns an equivalent instance $(G', k)$ such that $|V(G')| \leq 561 \cdot k$.*

*Proof.* Let $G = (V, E)$ be the plane input graph and $G' = (V', E')$ be the graph obtained by the exhautive application of the **??????**. As none of our rules change the size of a possible solution D′ in G′, we know by **??**, **??** and **??** that G′ has a Semitotal Dominating Set of size k if and only if G has a Semitotal Dominating Set of size k. In **??**, we have proofen that this preprocessing procedure runs in polynomial time. Asume that G′ admits a solution $D'$.

By taking the size of each region proofen in **??**, the total number of regions in a maximal *D-region decomposition* (**??**) and the number of vertices that can lay outside of any region (**??**), we obtain the following bound:

$$139 \cdot (3k - 6) + 145 \cdot k + k \leq 562 \cdot k \tag{4.12}$$

If $|V(G')| > 562 \cdot k$ G is a NO-instance and we replace G′ by two single disconnected vertices (trivial NO-instance). Then the kernel is of the claimed size.

□

CHAPTER 5

# OPEN QUESTIONS AND FURTHER RESEARCH

* Chordal Bipartite Grap hs a very interesting case. * Improve the Kernel Bound

# LIST OF FIGURES

# LIST OF TABLES