



DEPARTMENT OF INFORMATICS

TECHNICAL UNIVERSITY MUNICH

Master Thesis

**On the Parametrized Complexity of  
Semitotal Domination on Graph Classes**

Lukas Retschmeier







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**On the Parametrized Complexity of  
Semitotal Domination on Graph Classes**

**Über die Parametrisierte Komplexität des  
Problems der halbtotalen stabilen Menge  
auf Graphklassen**

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Submission Date: October 26, 2022



I confirm that this master thesis is my own work and I have documented all sources and material used.

*København S*  
October 26, 2022

Lukas Retschmeier

## Acknowledgments



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## **ABSTRACT**

### **Abstrakt**

- Context
- Research Question
- Claim
- approach
- Findings

Classifications?

*Abstract all the way*





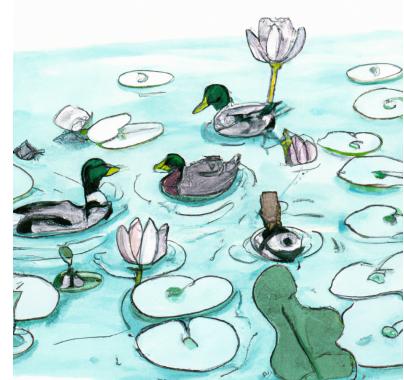
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# CHAPTER 1

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## INTRODUCTION



*To do select another quote*

Lewis Caroll, XXXX

Parametrized Complexity emerging branch. Books about that

Semitotal domination was introduced by

Quack! Quack! Idea: Lake with stones, and a family of ducks of fixed size want to occupy the lake so that no other clan tries to take it over. Rules: \* A duck can quack freeing up neighboring stones. \* Ducks don't like to be alone and want to quack together. So for every duck there must be another duck that is not further than two stones away. Q: Can our ducklings occupy the whole lake while not feeling lonely?

TODO A demo instance next to each other

### Problem Definition

#### DOMINATING SET [7, p. 586]

**Input:**

Graph  $G = (V, E)$  and an integer  $k$

**Question:**

Is there a set  $X \subseteq V$  of size at most  $k$  such that  
 $N[X] = V$ ?

**SEMITOTAL DOMINATING SET [15]**

**Input:** Graph  $G = (V, E)$  and an integer  $k$   
**Question:** Is there a subset  $X \subseteq V$  of size at most  $k$  such that  $N[X] = V$  and for all  $d_1 \in X$  there exists another  $d_2 \in X$  such that  $d(d_1, d_2) \leq 2$ ?

**TOTAL DOMINATING SET [7, p. 596]**

**Input:** Graph  $G = (V, E)$  and an integer  $k$   
**Question:** Does there exist a set  $X \subseteq V$  of at most  $k$  vertices of  $G$  such that for every  $u \in V(G)$  there exists  $v \in X$  with  $\{u, v\} \in E$

## 1.1 Content of the thesis

In this thesis, we continue the systematic analysis of the SEMITOTAL DOMINATING SET problem by focusing on the parametrized complexity of the problem.

Although the problem already had a lot of attention regarding classical complexity (CITE), only a few results are currently known for the parametrized variant.

As far as we have seen, even the  $w$ -hardness of the general case has not been explicitly proven in the literature.

In this thesis, we continue the journey toward a systematic analysis by stating some hardness results for specific graph classes for the problem.

**Our contributions** Our main contributions consist of first showing the  $w[2]$ -hardness of SEMITOTAL DOMINATING SET for XXXX graphs.

As the DOMINATING SET problem and the TOTAL DOMINATING SET problem both admit a linear kernel for planar graphs, it is interesting to analyze whether these results also hold for the SEMITOTAL DOMINATING SET problem which lies in between these two.

Having these kernels also for other variants like EDGE DOMINATING SET, EFFICIENT DOMINATING SET, CONNECTED DOMINATING SET, PLANAR RED-BLUE DOMINATING SET lent us great confidence that the result will also work for SEMITOTAL DOMINATING SET on planar graphs.

Following the approach from ... which already relies on the technique given in, we give some simple data reduction rules for SEMITOTAL DOMINATING SET on planar graphs leading to a linear kernel. More precisely, we are going to prove the following central theorem of this thesis:

## 1 Introduction

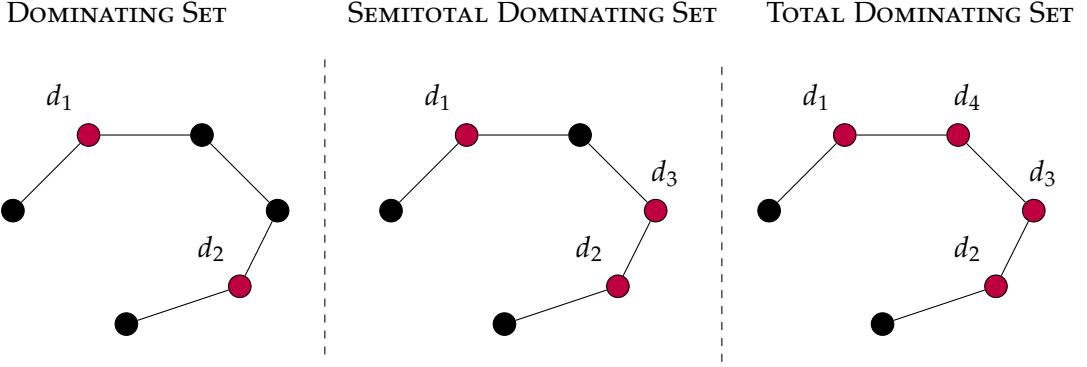


Figure 1.2: An example for a DOMINATING SET, SEMITOTAL DOMINATING SET and TOTAL DOMINATING SET, where  $\gamma(G) < \gamma_{2t}(G) < \gamma_t(G)$  are strict. In the first case, only two vertices suffice to dominate all others. In the second one, we need a witness between  $d_1$  and  $d_2$  that is at most distance two. In the last case,  $d_1$  and  $d_2$  both need a neighbor in the TOTAL DOMINATING SET.

With some modifications, we were able to transfer the approach given by Garnero and Sau in [13] to the SEMITOTAL DOMINATING SET problem.

**Theorem 1.** *The SEMITOTAL DOMINATING SET problem parametrized by solution size admits a linear kernel on planar graphs. There exists a polynomial-time algorithms that given a planar graph  $(G, k)$ , either correctly reports that  $(G, k)$  is a NO-instance or returns an equivalent instance  $(G', k)$  such that  $|V(G')| \leq 561 \cdot k$ .*

DOMINATING SET problem and TOTAL DOMINATING SET problem, both already

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# CHAPTER 2

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## PRELIMINARIES

We start by recapping some basic notation in Graph Theory and Parametrized Complexity.

Continuing an intensive study of parametrized complexity of that problem.

### 2.1 Graph Theory

We quickly state the following definitions given by [9, p. xxx].

**Definition 1 (Graph [9, p. 3]).** A graph is a pair  $G = (V, E)$  of two sets where  $V$  denotes the vertices and  $E \subseteq V \times V$  the edges of the graph. A vertex  $v \in V$  is incident with an edge  $e \in E$  if  $v \in e$ . Two vertices  $x, y$  are adjacent, or neighbours, if  $\{x, y\} \in E$ .

**Definition 2 (Vertex Degrees).** The degree  $d_G(v)$  (If  $G$  is clear, also  $d(v)$ ) of a vertex  $v$  is the number of neighbors of  $v$ . We call a vertex of degree 0 as isolated and one of degree 1 as a pendant.

**Definition 3 (isomorphic Graphs, [9, p. 3]).** Let  $G = (V, E)$  and  $G' = (V', E')$  be two graphs. We call  $G$  and  $G'$  isomorphic, if there exists a bijection  $\phi : V \rightarrow V'$  with  $\{x, y\} \in E \Leftrightarrow \phi(x)\phi(y) \in E'$  for all  $x, y \in V$ . Such a map  $\phi$  is called isomorphism.

If a graph  $G$  is isomorphic to another graph  $H$ , we denote  $G \simeq H$ .

**Definition 4 (Special Graph Notations [9, p. 27]).** A simple Graph

A directed Graph is a graph

A Multi Graph

A Planar Graph

**Definition 5 (Closed and Open Neighborhoods [3]).** Let  $G = (V, E)$  be a (non-empty) graph. The set of all neighbors of  $v$  is the open neighborhood of  $v$  and denoted by  $N(v)$ ; the set  $N[v] = N(v) \cup \{v\}$  is the closed neighborhood of  $v$  in  $G$ . When  $G$  needs to be made explicit, those open and closed neighborhoods are denoted by  $N_G(v)$  and  $N_G[v]$ .

**Definition 6 (Induced Subgraph).** asd

**Definition 7 (Isomorphic Graph).** asd

## Graph Classes

We call the class of graphs without any special restrictions “General Graphs”.

**Definition 8 (r-partite Graphs).** Let  $r \geq 2$  be an integer. A Graph  $G = (V, E)$  is called “ $r$ -partite” if  $V$  admits a partition into  $r$  classes such that every edge has its ends in different classes: Vertices in the same partition class must not be adjacent.

For the case  $r = 2$  we say that the  $G$  is “bipartite”

**Definition 9 (Chordal Graphs).**

**Definition 10 (Split Graphs).**

**Definition 11 (Path Graph  $P_i$ ).**

**Definition 12 (Bipartite Graph, [6, p.5]).** A bipartite graphs is a Graph  $G$  whose vertex set can be partitioned into two subsets  $X$  and  $Y$ , so that each edge has one end in  $X$  and one end in  $Y$ . Such a partition  $(X, Y)$  is called a bipartition of  $G$ .

**Definition 13 (Planar Graphs [9, Chapter 4]).** A plane graph is a pair  $(V, E)$  of finite sets with the following properties:

- $V \subseteq \mathbb{R}^2$  (Vertices)
- Every edge is an arc between two vertices
- different edges have different sets of endpoints
- The interior of an edge contains no vertex and no point of any other edge

An embedding in the plane, or planar embedding, of an (abstract) graph  $G$  is an isomorphism between  $G$  and a plane graph  $H$ . A plane graph can be seen as a concrete **embedding** of the planar graph into the “plane”  $\mathbb{R}^2$ .

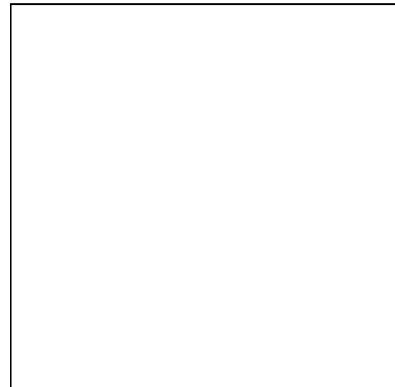
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# CHAPTER 3

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## ON PARAMETRIZED SEMITOTAL DOMINATION



*Todo select another quote*

Lewis Caroll, XXXX

For an introduction into classical complexity theory. Refer to the standard textbooks  
Karp and Cai. Rely on [1]

### 3.1 Parametrized Complexity

\* Decision Problem

**Definition 14 (Parametrized Problem[7, Def 1.1]).** A parametrized problem is a  $L \subseteq \Sigma^* \times \mathbb{N}$  ( $\Sigma$  finite fixed alphabet) for an instance  $(x, k) \in \Sigma^* \times \mathbb{N}$ , where  $k$  is called the parameter.

**Definition 15 (Instance Size).** The size of an instance of an instance  $(x, k)$  of a parametrized problem is  $|(x, k)| = |x| + k$

We will now clarify the basic terminology within Parametrized Complexity. We are now giving a short introduction into the world of parametrized complexity. \* General Introduction

Ways to cope with NP-hard problem.

### 3.1.1 Fixed Parameter Tractability

**Definition 16 (The Class FPT [7, Def 1.2]).** A parametrized problem  $L \subseteq \Sigma^* \times \mathbb{N}$  is called *fixed-parameter tractable* if there exists an algorithm  $A$  (called a *fixed-parameter algorithm*), a computable function  $f : \mathbb{N} \rightarrow \mathbb{N}$  and a constant  $c$  such that, given  $(x, k) \in \Sigma^* \times \mathbb{N}$ , the algorithm  $A$  correctly decides whether  $(x, k) \in L$  in time bounded by  $f(k) \cdot |(x, k)|^c$ . The complexity class containing all fixed-parameter tractable problems is called *FPT*

### 3.1.2 Kernelization

**Definition 17 (kernelization Algorithm[7, Def 2.1]).** A *Kernelization Algorithm* or *kernel* is an algorithm  $\mathfrak{A}$  for a parametrized Problem  $Q$ , that given an instance  $(I, k)$  of  $Q$  works in polynomial time and returns an equivalent instance  $(I', k')$  of  $Q$ . Moreover, we require that  $\text{size}_{\mathfrak{A}}(k) \leq g(k)$  for some computable function  $g : \mathbb{N} \rightarrow \mathbb{N}$

If we bound the size of the kernel by a linear function  $f(m) = \mathcal{O}(k)$ , we say that the problem admits a **linear kernel**.

The main idea, preprocessing algorithm, shrink size as much as possible, sound reduction rules, small output instance

**Definition 18 (Output size of a Preprocessing Procedure [7, p. 18]).** The output size of a preprocessing algorithms  $\mathfrak{A}$  is defined as

$$\text{size}_{\mathfrak{A}}(k) = \sup \{|I'| + l' : (I', k') = \mathfrak{A}(I, k), I \in \Sigma^*\}$$

possibly infinite

Clearly, if there exists a kernelization algorithm for a problem  $L$  and an algorithm  $\mathfrak{A}$  with any runtime to decide  $L$ , the problem is in *FPT* because after the kernelization pre-processing has been applied, the size of the reduced instance is a function merely in  $k$  and independent of the input size  $n$ . In Chapter 4 we will explicitly construct a kernel for PLANAR SEMITOTAL DOMINATING SET and hence showing it to be in *FPT*.

**Definition 19 (Reduction Rules [7, p. 18]).** A **reduction rule** is a function  $\phi : \Sigma^* \times \mathbb{N} \rightarrow \Sigma^* \times \mathbb{N}$  that maps an instance  $(x, k)$  to an equivalent instance  $(x', k')$  such that  $\phi$  is computable in time polynomial in  $|x|$  and  $k$

**Definition 20 (Equivalent Instance [7, p. 18]).** This is a test

**Definition 21.** Soundness of a rule

A **reduction rule** is a function  $\Sigma^* \times \mathbb{N}$  that maps an instance  $(x, k)$  to an equivalent instance  $(x', k')$  such that  $\phi$  is computable in time polynomial in  $|x|$  and  $k$

### 3.1.3 Fixed Parameter Intractability: The $w$ -Hierarchy

### 3.1.4 Compare to classical NP-Hardness theory

#### Parametrized Reductions

**Definition 22 (Parametrized Reduction [7, Def 13.1]).** Let  $A, B \subseteq \Sigma^* \times \mathbb{N}$  two parametrized problems. A Parametrized Reduction from  $A$  to  $B$  is an algorithm that, given an instance  $(x, k)$  of  $A$ , outputs an instance  $(x', k')$  of  $B$  such that

- $(x, k)$  is a yes instance of  $A$  iff  $(x', k')$  is a yes instance of  $B$
- $k' \leq g(k)$  for some computable function  $g$
- the running time is  $f(k) \cdot |x|^{\mathcal{O}(1)}$  (FPT!)

#### The $w$ -hierarchy

## 3.2 Semitotal Domination

### Dominating Set

#### Semitotal Dominating Set

#### Total Dominating Set

we denote  $y$  as the dominating number. Clearly  $y_t < y_s < y_d$ . Notatio of an independent set

### 3.2.1 Preliminaries

#### \* Witness \* domination

Let  $D$  be a dominating set of  $G$  and  $w \in V(G) \setminus D$ . For any neighbor  $v \in D \cap N(w)$ , we say that  $d_1$  dominates  $w$ . For two dominating vertices  $d_1, d_2 \in D$ . If

#### SEMITOTAL DOMINATING SET

#### Definition, dominating number

## 3.3 Complexity Status of Semitotal Dominating Set

### 3.4 $w[i]$ -Intractability

Now some  $w[i]$  hard classes.

#### 3.4.1 Warm-Up: $W[2]$ -hard on General Graphs

As any bipartite graphswith bipartition can be split further into  $r$ -partite graphsthis results also implies the  $w[1]$ -hardness of  $r$ -partite graphs

### 3.4.2 W[2]-hard on Bipartite Graphs

We are showing that SEMITOTAL DOMINATING SET is  $\omega[2]$ -hard on bipartite graphs by a parametrized reduction from DOMINATING SET on bipartite graphs which is known to be  $\omega[2]$ -hard ([24, Theorem 1]).

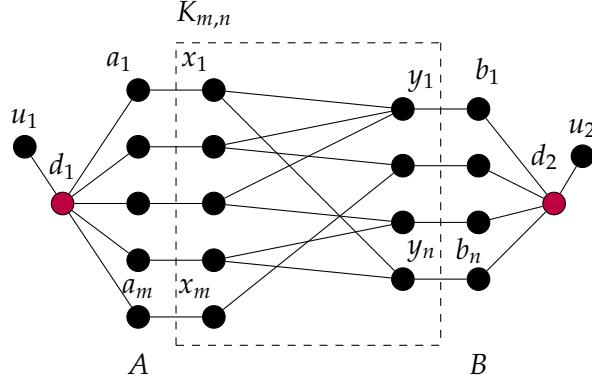


Figure 3.2: Constructing a bipartite  $G'$  from the bipartite graph  $K_{m,n}$  by duplicating all vertices and adding exactly two forced witnesses.

**Theorem 2.** Semitotal Dominating Set is  $\omega[2]$  hard for bipartite Graphs

*Proof.* Given a bipartite Graph  $G = (\{X \cup Y\}, E)$ , we construct a bipartite Graph  $G' = (\{X' \cup Y'\}, E')$  in the following way:

1. For each vertex  $x_i \in X$ , we add a new vertex  $a_i \in A$  and an edge  $\{x_i, a_i\}$  in between.
2. For each vertex  $y_j \in Y$ , we add a new vertex  $b_j \in B$  and an edge  $\{y_j, b_j\}$  in between.
3. We add four vertices with edges  $\{u_1, d_1\}$  and  $\{u_2, d_2\}$ , and connect them with all  $\{d_1, a_i\}$  and  $\{d_2, b_j\}$  ( $i \in [m]$  and  $j \in [n]$ ) respectively.

**Observation:** The constructed  $G'$  is bipartite because  $A$  and  $B$  form an independent set on  $G'$  that can be cross-wise attached to one of the previous vertex sets. Setting  $X' = X \cup \{u_2, d_1\} \cup B$  and  $Y' = X \cup \{u_1, d_2\} \cup A$  form the partitions of the new bipartite  $G'$ .

**Corollary 1.**  $G$  has a DOMINATING SET of size  $k$  iff  $G$  has a SEMITOTAL DOMINATING SET of size  $k' = k + 2$

$\Rightarrow$  Assume a DOMINATING SET in  $G$  of size  $k$ . We know that  $D' = D \cup \{d_1, d_2\}$  is a SEMITOTAL DOMINATING SET in  $G'$  of size  $k' = k + 2$ , because  $d_1$  dominates  $u_1$

and all  $a_i \in A$ ;  $d_2$  dominates  $u_2$  and all  $b_i \in B$ . The rest is dominated by the same vertices as they were in  $G$ , but now all of them have either  $d_1$  or  $d_2$  as a witness. More formally, we have by construction of  $G'$  that  $\forall v \in (D \cap X) : d(v, d_1) = 2$  and  $\forall v \in (D \cap Y) : d(v, d_2) = 2$ .

- ⇐ On the contrary, assume any SEMITOTAL DOMINATING SET  $D'$  in  $G'$  with size  $k'$ . Without loss of generality, we can assume that  $u_1, u_2 \notin D'$ , because choosing  $d_1$  and  $d_2$  instead is always at least as good and does not violate any witnesses. Therefore, the construction forces  $d_1, d_2 \in D'$ .

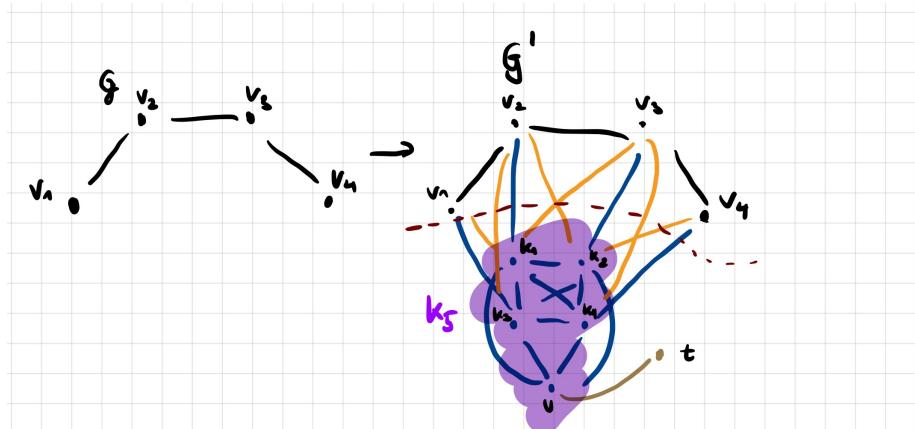
All  $a_i \in A$  can only be useful to dominate their partnering  $x_i$  ( $b_i \in B$  for  $y_i$ ), because  $d_1, d_2 \in D$  is the only second neighbor they have. If  $a_i, b_i \in D'$  we replace it with  $x_i$  and  $y_i$  preserving the size  $D$ . As  $d_1$  and  $d_2$  suffice to provide a witness for every vertex in the graph and do not lose any other witnesses, this operation is sound.

In the end,  $D = D' \setminus \{d_1, d_2\}$  gives us a DOMINATING SET in  $G$  with size  $k = k' - 2$

As  $G'$  can be constructed in linear time and the parameter  $k$  is only blown up by a constant, this reduction is an FPT reduction. Because DOMINATING SET is already  $w[2]$ -hard on bipartite graphs ([]), we imply that SEMITOTAL DOMINATING SET is  $w[2]$ -hard as well.  $\square$

### 3.4.3 $W[2]$ -hard on Chordal Graphs

**Theorem 3.** Semitotal Dominating Set is  $w[2]$  hard on Chordal Graphs



### 3 On Parametrized Semitotal Domination

1. Add a  $K_{n+1}$  consisting of the vertices  $\{k_1, \dots, k_n, u\}$  and add an edge  $(v_i, k_i)$  to each vertex  $v_i$  of  $G$ . One vertex  $u$  in the clique will remain untouched.
2. Add one additional vertex  $t$  and connect it with  $u$ .
3. For all vertices  $v_i$  in  $G$ , add a new edge  $(n, k_i)$  for all  $n \in N(v_i)$ .

**Corollary 2.**  $N(v_i) \in G$  forms a clique iff  $N(v_i)$  forms a clique in  $G'$

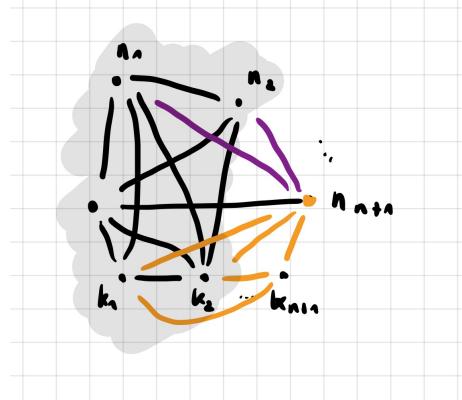


Figure 3.4: Induction Step

*Proof.* Assuming that  $N(v_i)$  forms a clique in  $G$ , we show that it also forms a clique in  $G'$  by induction over the number of neighbors  $z = \text{abs}(N(v_i))$  in  $G$ .

- $z = 0$ : Holds trivially as we do not have a neighbor in  $G$  and in  $G'$  the connected  $k_i$  forms a  $P_1$ , hence a clique.
- $z = z + 1$ :

By IH, we already know that all neighbors  $n_1, \dots, n_z$  form a clique together with their vertices in  $k_i$ . As  $k_{z+1}, v_{z+1} \in N(v_i)$  now also in  $G'$ , we show that  $N(v_i)$  still forms clique in  $G'$ .

Let  $k_i$  be the vertex that was connected with  $n_i$  during step 1. All we have to show is that  $v_{z+1}$  and  $k_{z+1}$  extend our previous clique, hence are fully connected with  $N(v_i)$ .

$v_{z+1}$  connects to  $N(v_i)$  in  $G$  by assumption. By our construction, there exists an edge to  $k_1, \dots, k_z$ , because we add an edge  $(n_{z+1}, k_i)$  if there is an edge from  $(n_{z+1}, n_i)$ . (See fig 3.4)

$k_{z+1}$  form a complete subgraph with the other  $k_i$  and is connected to all  $n_i$  by construction because the edge  $(n_{z+1}, n_i)$  exists.

Therefore,  $N(v_i)$  will also form a clique in  $G'$ .

On the other side, if  $N(v_i)$  forms a clique in  $G'$ , the vertices of  $N(v_i)$  in  $G$  form an induced subgraph of  $G'$ , hence preserving the clique. ■

**Corollary 3.**  $G$  is Chordal iff  $G'$  is chordal.

*Proof.*  $\Rightarrow$ : Assume  $G$  chordal. Then exists a total elimination order  $o = (v_1, \dots, v_n)$  in  $G$  where removing  $v_j$  sequentially returns cliques in  $N(v_i)$ . Define  $o' = (v_1, \dots, v_n, k_1, \dots, k_n, u, t)$ . Applying corollary 2 states that  $(v_1, \dots, v_n)$  always gives cliques in  $G$  and according to corollary 2 also in  $G'$ . As the rest is directly part of a clique in  $G'$  by definition with an additional vertex of degree 1,  $o'$  is a total elimination order for  $G'$ , hence  $G'$  chordal.  
 $\Leftarrow$ : Holds as  $o'$  is always a total elimination order in  $G'$  and removing the complete subgraph  $K_{n+1}$  and  $u$  gives a total elimination order in  $G$ . ■

**Corollary 4.**  $G$  has a Dominating Set of size  $k$  iff  $G'$  has a dominating set of size  $k + 1$

*Proof.* Assume a Dominating Set  $D$  of size  $k$  in  $G$ .  $D \cup \{u\}$  is a Semitotal Dominating Set in  $G'$  of size  $k + 1$ , because  $u$  dominates  $t$  and for each  $v \in DS : d(v, u) \leq 2$ .

Contrary, assume a Semitotal Dominating Set  $SD$  in  $G'$ . In order to dominate  $t$ ,  $u \in SD$  must hold, hence already dominating the complete subgraph  $K_{n+1}$ . If a vertex  $k_i \in SD$ , we exchange it with  $v_i$  still preserving a Dominating Set. Taking  $D = SD - \{u\}$  gives our desired Dominating Set of size  $k$ . ■

As this reduction runs in FPT time and the parameter is only bounded by a function of  $k$ , this is a FPT reduction. As Dominating Set on Chordal Graphs is  $w[2]$  – hard, so is SDS on Chordal Graphs. □

#### 3.4.4 W[2]-hard on Split Graphs

TODO Getting started with that.

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## CHAPTER 4

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### A LINEAR KERNEL FOR PLANAR SEMITOTAL DOMINATION



*The best way to explain it is to do it.*

Lewis Caroll, *Alice in Wonderland*

We are going to present a polynomial-time preprocessing procedure giving a linear kernel for **PLANAR SEMITOTAL DOMINATING SET** parametrized by solution size. Based on the technique first introduced by Alber et al. ([2]) in 2004, an abundance of similar results to other domination problems emerged which gave us the belief we can transfer these results to **SEMITOTAL DOMINATING SET**. Table 4.1 gives an overview of the status of kernels for the planar case on various domination problems. All of these results introduce reduction rules bounding the number of vertices inside so-called “regions” which can be obtained by a special decomposition of the planar graph.

In the following years, this approach bore fruits in other planar problems as well like **CONNECTED VERTEX COVER** ( $11/3k$  in [21]), **MAXIMUM TRIANGLE PACKING** ( $624k$  in [25]) **INDUCED MATCHING** ( $40k$  in [20]) **FULL-DEGREE SPANNING TREE** ([17]) **FEEDBACK VERTEX SET** ( $13k$  in [5]) and **CYCLE PACKING** ([12]).

In the impending years, many results generalized this approach to larger graph classes. Fomin and Thilikos started by proofing that the initial reduction rules given by Alber et al. [2] can also be used to obtain a linear kernel on graphs with bounded genus  $g$  ([11]). Gutner advanced in 2008 by giving a linear kernel on  $K_{3,h}$ -topological-minor-free graph classes and a polynomial kernel for  $K_h$ -topological-minor-free graph classes ([18]). In 2012 Philip, Raman and Sikdar showed that  $K_{i,j}$ -free graph classes

Problem	Best Known Kernel	Source
PLANAR DOMINATING SET	67k	[8] <sup>1</sup>
PLANAR TOTAL DOMINATING SET	410k	[13] <sup>2</sup>
PLANAR SEMITOTAL DOMINATING SET	561k	This work
PLANAR EDGE DOMINATING SET	14k	[16, Th. 2]
PLANAR EFFICIENT DOMINATING SET	84k	[16, Th. 4]
PLANAR RED-BLUE DOMINATING SET	43k	[14]
PLANAR CONNECTED DOMINATING SET	130k	[22]
PLANAR DIRECTED DOMINATING SET	Linear	[1]

<sup>1</sup>There is also a master's thesis claiming a bound of 43k [19], but a conference or journal version was not found.

<sup>2</sup>Improved their own results from first 694k [13, Revision 2012]

Table 4.1: An overview about existing kernels for planar dominating set variants

admit a polynomial kernel for DOMINATING SET ([23]). In an attempt to extend these ideas to other problems as well, Bodlaender et al. proved that all problems expressible in counting monadic second-order logic satisfying a coverability property admit a polynomial kernel on graphs of bounded genus g ([4]).

These meta-results are interesting from a theoretical point of view, but the constants for the kernels obtained by these methods are too large to be of practical interest. The question of how an efficient kernel can be constructed remains. We will transfer the kernel described by Garnero and Sau in their original version of [13, Revision 2014] for PLANAR TOTAL DOMINATING SET can also be “recycled” for PLANAR SEMITOTAL DOMINATING SET giving us an explicitly constructed kernel with “reasonable” small constants.

**The Main Idea** A given a plane graph  $G = (V, E)$  with a given vertex set  $D \subseteq V$  can be decomposed into at most  $3 \cdot |D| - 6$  so-called “regions” (see Definition 29). If  $D$  is a given SEMITOTAL DOMINATING SET of size  $|D|$ , the total number of regions in this decomposition depends linearly on the size of  $D$ . If we define *reduction rules* (Rules 1 to 3) minimizing the number of vertices in and around a region, we can bound the size of a reduced graph. In our case, we give rules to reduce a region down to a constant number of vertices. We can create an equivalent instance of  $G$  whose number of remaining vertices lies linearly in the size of an enquired solution. Such a reduction gives a *Kernel* for PLANAR SEMITOTAL DOMINATING SET.

Interestingly, the reduction rules do not rely on the decomposition itself, but rather consider the neighborhood of every pair of vertices in the graph. The decomposition itself has just used a tool for analyzing the kernel size after the reduction.

## 4.1 Definitions

Before giving the exact reduction rules, we need some definitions which expose some nice properties we are going to exploit. They are inspired by those given by Garnero and Sau (PLANAR TOTAL DOMINATING SET in [13, Revision 2014] or PLANAR RED-BLUE DOMINATING SET in [14]) and reused ideas introduced by Alber et al. in [2] for PLANAR DOMINATING SET.

The idea is to split up the neighborhood of a single vertex and a pair of vertices into three distinct subsets which intuitively give us a level of “confinement” of these vertices and how closely they are related to the rest of the graph ([13, p. 6]).

**Definition 23.** Let  $G = (V, E)$  be a graph and let  $v \in V$ . We denote by  $N(v) = \{u \in V : \{u, v\} \in E\}$  the neighborhood of  $v$ . We split  $N(v)$  into three subsets:

$$N_1(v) = \{u \in N(v) : N(u) \setminus N[v] \neq \emptyset\} \quad (4.1)$$

$$N_2(v) = \{u \in N(v) \setminus N_1(v) : N(u) \cap N_1(v) \neq \emptyset\} \quad (4.2)$$

$$N_3(v) = N(v) \setminus (N_1(v) \cup N_2(v)) \quad (4.3)$$

To enhance future readability, for  $i, j \in [1, 3]$ , we denote  $N_{i,j}(v) := N_i(v) \cup N_j(v)$ . Furthermore, we call a vertex  $v'$  confined by a vertex  $v$ , if  $N(v') \subseteq N[v]$

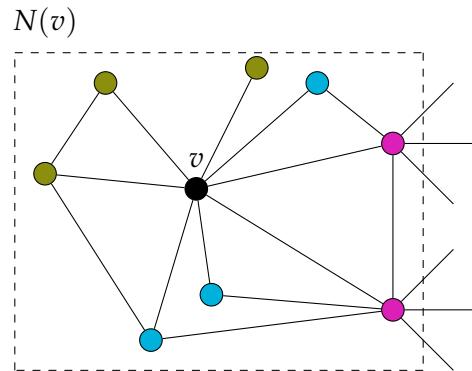


Figure 4.2: The neighborhood of a single vertex  $v$  split to  $N_1(v)$  (purple),  $N_2(v)$  (blue), and  $N_3(v)$  (green).  $N_1(v)$ 's are those having neighbors outside  $N(v)$ ,  $N_2(v)$ 's are a buffer between  $N_1(v)$  and  $N_3(v)$ , and  $N_3(v)$ -vertices are confined in  $N(v)$ .

**N<sub>1</sub>(v)** is all the neighbors of  $v$  which have at least one adjacent vertex that is outside of  $N(v)$  and therefore connects  $v$  with the rest of the graph. They can belong to a SEMITOTAL DOMINATING SET.

- $N_2(v)$  is all neighbors of  $v$  that have at least one neighbor in  $N_1(v)$ . These vertices do not have any function as a dominating vertex and can be seen as a *buffer* bridging  $N_1(v)$ -vertices with those from  $N_3(v) \cup \{v\}$ . Furthermore, they are useless as witnesses, because either we can replace them with  $v$  (sharing the same neighborhood) or when being a witness for  $v$ , we replace it with a  $z \in N_1(v)$ .
- $N_3(v)$  vertices are sealed off from the rest of the graph. They are useless as dominating vertices: For all  $z \in N_3(v)$  it holds that  $N(z) \subseteq N(v)$  by definition and thus, we would always prefer  $v$  as a dominating vertex instead of  $z$ . Nevertheless, they can be important as a witness for  $v$  in the case where it is the only neighbor of  $v$  (this  $N_1(v) \cup N_2(v) = \emptyset$ ). We are using this observation in Rule 1 where we shrink  $|N_3(v)| \leq 1$ .

Next, we are going to extend this notation to a pair of vertices. Using this, Rule 2 will later try to reduce the neighborhood of two vertices, and similar to Definition 23, we observe nice properties. Again, the idea is to classify how strongly the shared neighborhood of two vertices  $N(v) \cup N(w)$  is connected to the rest of the graph.

**Definition 24.** Let  $G = (V, E)$  be a graph and  $v, w \in V$ . We denote by  $N(v, w) := N(v) \cup N(w)$  the shared neighborhood of the pair  $v, w$  and split  $N(v, w)$  into three distinct subsets:

$$N_1(v, w) = \{u \in N(v, w) \mid N(u) \setminus (N(v, w) \cup \{v, w\}) \neq \emptyset\} \quad (4.4)$$

$$N_2(v, w) = \{u \in N(v, w) \setminus N_1(v, w) \mid N(u) \cap N_1(v, w) \neq \emptyset\} \quad (4.5)$$

$$N_3(v, w) = N(v, w) \setminus (N_1(v, w) \cup N_2(v, w)) \quad (4.6)$$

Again, for  $i, j \in [1, 3]$ , we denote  $N_{i,j}(v, w) = N_i(v, w) \cup N_j(v, w)$ .

$N_1(v, w)$  contains those vertices connected with the rest of the graph by at least one vertex not in the neighborhood of  $v$  or  $w$ ,  $N_2(v, w)$ -vertices are a *buffer* between those from  $N_3(v, w) \cup \{v, w\}$  and  $N_1(v, w)$ , and  $N_3(v, w)$  contains vertices isolated from the rest of the graph. You can see an example in Fig. 4.3.

A vertex  $v \in N_i(v)$  is not necessarily also in  $N_i(v, w)$ ! Observe the vertex  $z$  in Fig. 4.3. Unlike  $N_i(v)$  in every  $N_i(v, w)$  ( $i \in [3]$ ) can be in an optimal SEMITOTAL DOMINATING SET (compare Fig. 4.4 for an example in each of the sets).

### 4.1.1 Reduced Graph

Before stating the reduction rules, we want to clarify when we consider a graph to be a *reduced*.

**Definition 25 (Reduced Graph [13, p. 13] and [14]).** A Graph  $G$  is reduced under a set of rules if either none of them can be applied to  $G$  or the application of any of them creates a graph isomorphic to  $G$ .

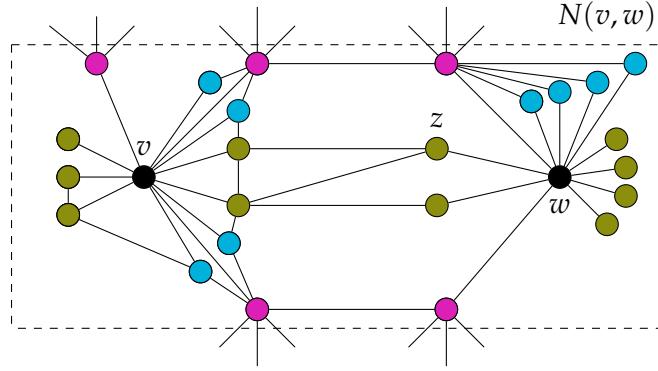


Figure 4.3: The neighborhood of a pair of vertices. Vertices from  $N_3(v, w)$  are colored green,  $N_2(v, w)$ 's blue and  $N_1(v, w)$ 's purple. Note that  $z \in N_1(w)$ , because there is an edge to a neighbor of  $v$ , but  $z \notin N_1(v, w)$  (and rather  $z \in N_3(v, w)$ ).

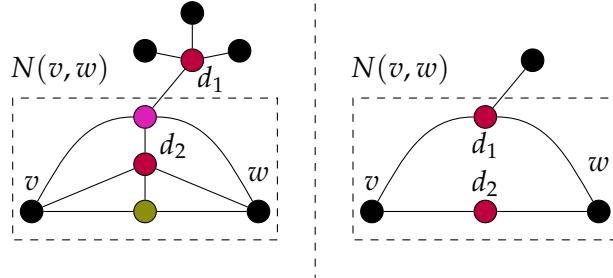


Figure 4.4: Left:  $\{d_1, d_2\}$  with  $d_2 \in N_2(v, w)$  form the only minimal SEMITOTAL DOMINATING SET. Right:  $d_1 \in N_1(v, w)$  and  $d_2 \in N_3(v, w)$  optimal.

This definition differs from the definition usually given in literature where a graph  $G$  is *reduced* under a set of reduction rules if none of them can be applied to  $G$  anymore (compare e.g. [10]). Some of our reduction rules (Rule 1 or Rule 2) could be applied *ad infinitum* creating an endless loop that does not change  $G$  any more. Our definition guarantees termination in that case. All of the given reduction rules are local and only need the neighborhood of at most two vertices and replace them partially with gadgets of constant size. Now checking whether a graph after applying the rule has been isomorphically changed can be trivially accomplished in constant time.

In our case, we say  $G$  is reduced if all of the Rules 1 to 3 have exhaustively been applied.

#### 4.1.2 Regions in Planar Graphs

Alber et al. gave in [2] a novel approach to look at planar graphs. In their analysis, they stated a constructive algorithm that decomposes a planar graph into local “regions”.

## 4.1 Definitions

Vaguely said, a region is a set of vertices that are enclosed by a boundary path in a fixed planar embedding of the graph.

The following definitions are based on those given by Garnero and Sau in [13, Revision 2014] and will lead toward a clean definition of a *region* and what we understand as a *D-region decomposition*. More detailed explanations and concrete examples can be found in their paper.

**Definition 26.** Two simple paths  $p_1, p_2$  in a plane graph  $G$  are confluent if:

1. they are vertex-disjoint
2. they are edge-disjoint and for every common vertex  $u$ , if  $v_i, w_i$  are the neighbors of  $u$  in  $p_i$ , for  $i \in [1, 2]$ , it holds that  $[v_1, w_1, v_2, w_2]$ , or
3. they are confluent after contracting common edges

**Definition 27.** Let  $G = (V, E)$  be a plane graph and let  $v, w \in V$  be two distinct vertices. A region  $R(v, w)$  (synonymy denoted as  $vw$ -region) is a closed subset of the plane, such that:

1. the boundary of  $R$  is formed by two confluent simple  $vw$ -paths with length at most 3
2. every vertex in  $R$  belongs to  $N(v, w)$ , and
3. the complement of  $R$  in the plane is connected.

We denote with  $\partial R$  the boundary of  $R$  and by  $V(R)$  the set of vertices laying (on the plane embedding) in  $R$ . Furthermore, we call  $|V(R)|$  the size of the region.

The poles of  $R$  are the vertices  $v$  and  $w$ . The boundary paths are the two  $vw$ -paths that form  $\partial R$ .

**Definition 28.** Two regions  $R_1$  and  $R_2$  are non-crossing, if:

1.  $(R_1 \setminus \partial R_1) \cap R_2 = (R_2 \setminus \partial R_2) = \emptyset$ , and
2. the boundary paths of  $R_1$  are pairwise confluent with the ones in  $R_2$

We now have all the definitions ready to formally a maximal *D-region decomposition* on planar graphs:

**Definition 29.** Given a plane graph  $G = (V, E)$  and  $D \subseteq V$ , a  $D$  – region Decomposition of  $G$  is a set  $\mathfrak{R}$  of regions with poles in  $D$  such that:

1. for any  $vw$ -region  $R \in \mathfrak{R}$ , it holds that  $D \cap V(R) = \{v, w\}$ , and
2. all regions are pairwise non-crossing.

We denote  $V(\mathfrak{R}) = \bigcup_{R \in \mathfrak{R}} V(R)$ .

A *D-region decomposition* is maximal if there is no region  $R \notin \mathfrak{R}$  such that  $\mathfrak{R}' = \mathfrak{R} \cup \{R\}$  is a *D-region decomposition* with  $V(\mathfrak{R}) \subsetneq V(\mathfrak{R}')$ .

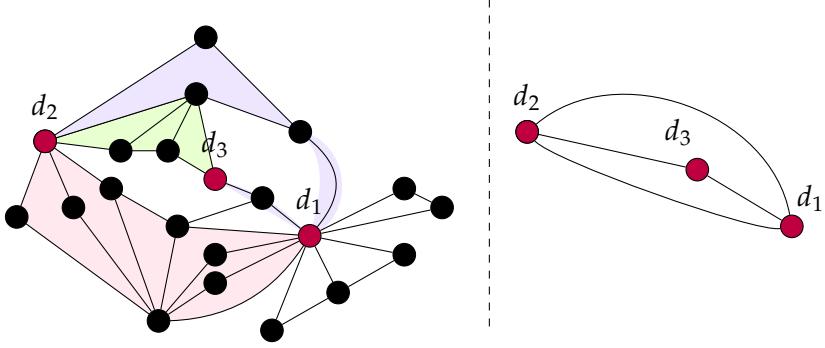


Figure 4.5: Left: A maximal  $D$ -region decomposition  $\mathfrak{R}$ , where  $D = \{d_1, d_2, d_3\}$  form a SEMITOTAL DOMINATING SET. There are two regions between  $d_2$  and  $d_1$  (purple and pink), one region between  $d_1$  and  $d_3$  (green) and one region between  $d_2$  and  $d_3$  (purple). Observe that some neighbors of  $d_1$  are not part of any  $vw$ -region. Our reduction rules are going to take care of them and bound these number of vertices to obtain the kernel. Right: The corresponding underlying multigraph  $G_{\mathfrak{R}}$ . Every edge denotes a region between  $d_i$  and  $d_j$ .

Fig. 4.5 gives an example of how to decompose a graph into a maximal  $D$  – region decomposition with a given SEMITOTAL DOMINATING SET  $D$  of size 3.

We are introducing a special subset of a region, namely *simple region* where every vertex is a common neighbor of  $v$  and  $w$ . They will appear in many unexpected astonishing places and are an important tool to operate on small parts of a plane graph. The upcoming Rule 3 will bound the size of these *simple regions*. Interestingly, in the first version of the paper about the linear kernel for PLANAR TOTAL DOMINATING SET ([13, Revision 2014]), they were not given independently but covered by one of their reduction rules (Rule 2). As it turned out, the analysis is getting simpler if we treat them in a separate rule (In our case: Rule 3) and so did Garnero and Sau in a revised version of their paper four years later ([13]).

**Definition 30.** A simple  $vw$ -region is a  $vw$ -region such that:

1. its boundary paths have length at most 2, and
2.  $V(R) \setminus \{v, w\} \subseteq N(v) \cap N(w)$ .

Fig. 4.6 shows an example of a simple region containing 9 distinct vertices.

In the analysis, we will also use properties of the *underlying multigraph* of a  $D$ -region decomposition  $\mathfrak{R}$ . Refer to Fig. 4.5 for an example.

**Definition 31.** Let  $G = (V, E)$  be a plane graph, let  $D \subseteq V$  and let  $\mathfrak{R}$  be a  $D$ -region decomposition of  $G$ . The underlying multigraph  $G_{\mathfrak{R}} = (V_{\mathfrak{R}}, E_{\mathfrak{R}})$  of  $\mathfrak{R}$  is such that  $V_{\mathfrak{R}} = D$  and there is an edge  $\{v, w\} \in E_{\mathfrak{R}}$  for each  $vw$ -region  $R(v, w) \in \mathfrak{R}$

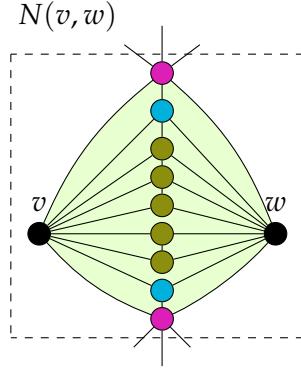


Figure 4.6: A simple region with two vertices from  $N_1(v,w)$  (purple) setting the boundary, two vertices from  $N_2(v,w)$  (blue) and some vertices from  $N_3(v,w)$  (green) in between.

## 4.2 The Big Picture

Fig. 4.7 gives a high-level view of how we are going to obtain the linear kernel for PLANAR SEMITOTAL DOMINATING SET. We will first derive three different reduction rules (Rules 1 to 3 are green in the overview), prove that they preserve the solution size  $k$  and run in polynomial-time. Then we use the existence of a maximal *D-region decomposition*  $\mathfrak{R}$  on planar graphs to bound the number of vertices that fly around a given region  $R \in \mathfrak{R}$ . This will lead us towards a bound on the number of vertices inside  $R$ . Furthermore, we observe that the number of vertices that are not enclosed in  $R$ , but lie outside the border is bounded, too. We will often encounter hidden simple regions which are reduced by Rule 3 and therefore of constant size by Corollary 5. As we know that the total number of regions  $R$  in the *D-region decomposition* is linear in  $k$ , we obtained a linear kernel for the PLANAR SEMITOTAL DOMINATING SET as well.

## 4.3 The Reduction Rules

Following the ideas proposed by Garnero and Sau in [13, Revision 2014], we state reduction rules that after exhaustive application will expose a linear kernel. We note that especially for Rule 2, we relied on the first revision of the paper because in the following years their kernel size was improved for the cost of making them more specific to PLANAR TOTAL DOMINATING SET and we saw that the latest rules stated will not work for PLANAR SEMITOTAL DOMINATING SET. The main challenge in all of those rules is to maintain the witness properties of the graph because a vertex inside a region can be important as a witness for vertices in another region and can not always simply be reduced. Therefore a more sophisticated analysis is required.

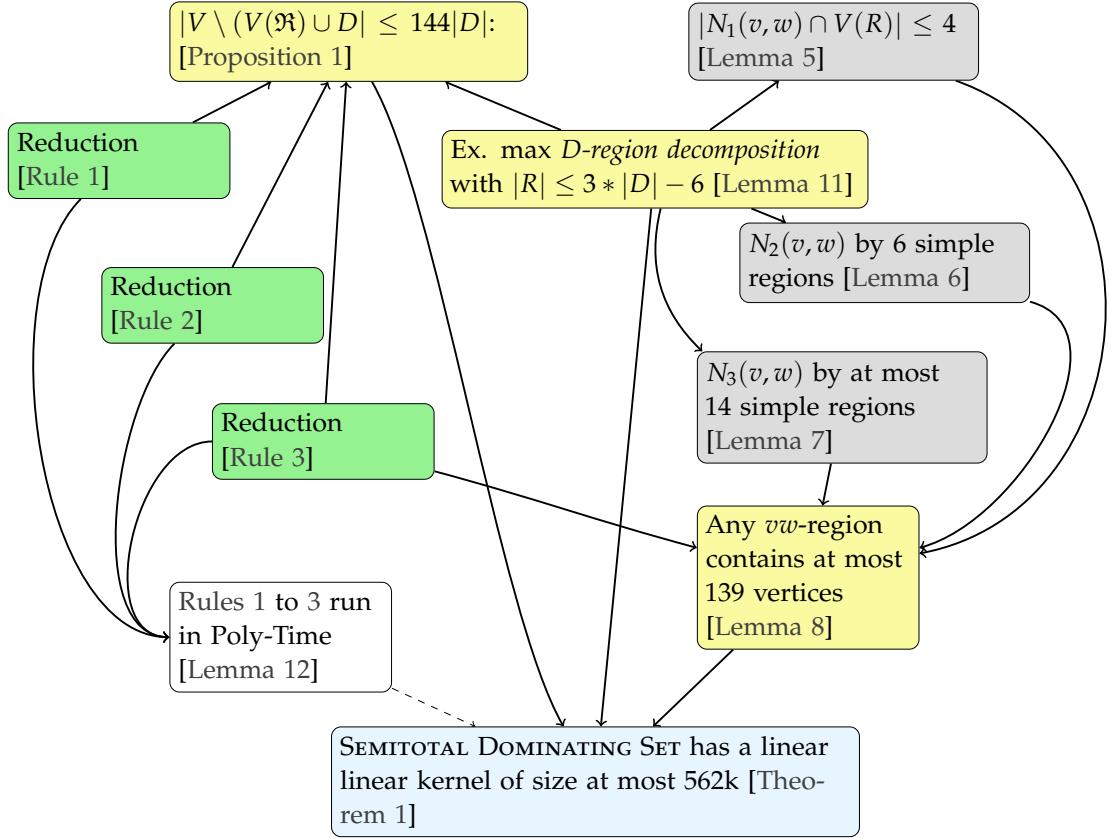


Figure 4.7: The plan for obtaining a linear kernel for PLANAR SEMITOTAL DOMINATING SET. Starting with the reduction rules (Green) we will derive the number of vertices inside and outside of a vw-region.

### 4.3.1 Reduction Rule I: Getting Rid of unnecessary $N_3(v)$ vertices

The idea of the first rule is the observation that a vertex  $v' \in N_{2,3}(v)$  dominates  $v$  and possibly vertices from  $N_2(v)$  and  $N_3(v)$ . As  $N(v') \subseteq N(v)$  and the Fact 1 that a witness for  $v'$  is also witness for  $v$ , we can use  $v$  instead of  $v'$  as a dominating vertex. Therefore, we can remove  $N_{2,3}$  from the graph. Nevertheless,  $v'$  can be a witness for  $v$  itself and might be required in a solution. Our rule ensures that at least one  $N_3(v)$ -vertex is preserved. An example for this rule is shown in Fig. 4.8.

**Fact 1.** Let  $G = (V, E)$ ,  $v \in V$  and  $v' \in N_{2,3}(v)$ . Any witness  $w \neq v$  for  $v'$  is also a witness for  $v$ .

*Proof.* By assumption  $v'$  is witnessed by a vertex  $w \neq v$  with  $d(v', w) \leq 2$ . It follows directly from the definition of  $v' \in N_{2,3}(v)$  that  $N(v') \subseteq N[v]$  and hence  $v'$  is confined

### 4.3 The Reduction Rules

inside the neighborhood of  $v$ . Every path  $P = (v', n, w)$  from  $v'$  to possible witness  $w$  within two steps must pass at least one vertex  $n \in N[v]$  as all neighbors. This implies that there exists also a path from  $v$  to  $w$  with a length of at most two and  $v$

□

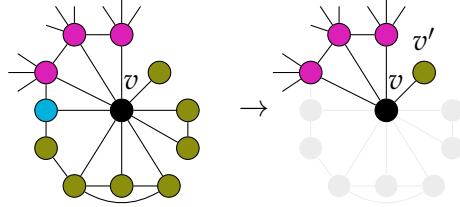


Figure 4.8: *Simplifying  $N_{23}(v)$ : As  $|N_3(v)| \geq 1$ , we remove  $N_{23}(v)$  and add a new witness  $v'$ .  $N_1(v)$  remains untouched.*

**Rule 1.** Let  $G = (V, E)$  be a graph and let  $v \in V$ . If  $|N_3(v)| \geq 1$ :

- remove  $N_{2,3}(v)$  from  $G$ ,
- add a vertex  $v'$  and an edge  $\{v, v'\}$

We can now prove the correctness of this rule.

**Lemma 1.** Let  $G = (V, E)$  be a graph and let  $v \in V$ . If  $G'$  is the graph obtained by applying Rule 1 on  $G$ , then  $G$  has a SEMITOTAL DOMINATING SET of size  $k$  if and only if  $G'$  has a SEMITOTAL DOMINATING SET of size  $k$ .

*Proof.*  $\Rightarrow$  Let  $D$  be a SEMITOTAL DOMINATING SET in  $G$  of size  $k$ . Because Rule 1 has been applied, we can assume  $N_{2,3}(v) \neq \emptyset$  in  $G$ .

To dominate all vertices from  $N_3(v)$  we either need  $v \in D$  or at least one other vertex  $d \in N_{2,3}(v) \cap D$ . In the latter case, we can replace this vertex by  $v$  directly. By Fact 1, we know that a witness for  $d$  (which is not  $v$ ) is also a witness for  $v$  and therefore, the replacing preserves the SEMITOTAL DOMINATING SET. Henceforth, we assume  $v \in D$  and  $N_{2,3}(v)$  is already dominated by  $v$ .

If Rule 1 has removed at least one dominating vertex from  $N_{2,3}(v)$ , we set  $D' = D \setminus N_{2,3}(v) \cup \{v'\}$  otherwise  $D = D'$ . In the first case this vertex in  $N_{2,3}(v) \cap D$  could have possibly been a witness for  $v$ , so we select  $v' \in D'$  as a witness for  $v$  ensuring  $D'$  to be a SEMITOTAL DOMINATING SET. In both cases  $v'$  is by assumption dominated by  $v$  and  $|D| \leq |D'|$ .

$\Leftarrow$  Assume  $D'$  to be a SEMITOTAL DOMINATING SET in  $G'$ . We can assume that  $v \in D'$ , because  $v'$  has to be dominated and it is always better to choose  $v$  instead of  $v'$ .

## 4 A Linear Kernel for Planar Semitotal Domination

If  $v' \in D'$  is a witness for  $v$  in  $G'$ , we have to preserve a witness in  $G$  as well. As we know that  $N_3(v) \neq \emptyset$ , we can replace it by an arbitrary vertex  $d \in N_3(v)$  in  $G$ .

In the second case, the witness for  $v$  came either from a vertex  $o \in N_1(v)$  or some neighbor  $N(o) \setminus N(v)$  outside the neighborhood of  $v$  which has not been touched by this reduction rule. In summary, if  $v' \in D'$ , we set  $D = D' \cup \{d\} \setminus \{v'\}$  for any  $d \in N_3(v)$  and otherwise  $D = D'$ . In both cases,  $N_{2,3}(v)$  is dominated by  $v$  and  $|D| = |D'|$ .

□

**Lemma 2.** *A plane graph  $G$  of  $n$  vertices is reduced under Rule 1 in time  $\mathcal{O}(n)$*

*Proof.* As Rule 1 stayed the same, the proof directly follows [2, Lemma 2]. □

Note that we need our definition of a reduced instance given in 25. If Rule 3 is being applied, it will still leave us with a vertex  $z \in N_3(v)$  allowing this rule to be applied again.

### 4.3.2 Reduction Rule II: Shrinking the Size of a Region

The second rule is the heart of the whole reduction and it aims to minimize the neighborhood of two distinct vertices. The rule follows Garnero and Sau's approach in one of the early versions of [13, Revision 2014] for PLANAR TOTAL DOMINATING SET. The reduction rules given in the latest version ([13, Revision 2018]) were not directly transferable to PLANAR SEMITOTAL DOMINATING SET, because they heavily rely on the property of a TOTAL DOMINATING SET that a witness  $w$  for  $v$  **must** be a direct neighbor of  $w$ . In the case of the more relaxed SEMITOTAL DOMINATING SET, the witness is allowed to have more range.

It can be observed that in the worst case four vertices are needed to semi-totally dominate  $N(v, w)$  of two vertices  $v, w \in V$ :  $v, w$  and two witnesses for them. Exemplary, observe the graph consisting of two distinct  $K_{1,m}$  with  $m \in \mathbb{N}$  with centers  $v$  and  $w$ .

Before we give the concrete reduction rule, we need to define three sets. Intuitively, we first try to find a set  $\tilde{D} \subseteq N_{2,3}(v, w)$  of size at most three dominating  $N_3(v, w)$  without using  $v$  or  $w$ . If no such set exists, we allow  $v$  (resp.  $w$ ) and try to find one again. If we now find such a set, we can conclude that  $v$  ( $w$ ) must be part of a solution.

**Definition 32.** *Let  $G = (V, E)$  be a graph and let  $v, w \in V$ . We now consider all the sets that can dominate  $N_3(v, w)$ :*

$$\mathcal{D} = \{\tilde{D} \subseteq N_{2,3}(v, w) \mid N_3(v, w) \subseteq \bigcup_{v \in \tilde{D}} N(v), |\tilde{D}| \leq 3\} \quad (4.7)$$

$$\mathcal{D}_v = \{\tilde{D} \subseteq N_{2,3}(v, w) \cup \{v\} \mid N_3(v, w) \subseteq \bigcup_{v \in \tilde{D}} N(v), |\tilde{D}| \leq 3, v \in \tilde{D}\} \quad (4.8)$$

$$\mathcal{D}_w = \{\tilde{D} \subseteq N_{2,3}(v, w) \cup \{w\} \mid N_3(v, w) \subseteq \bigcup_{v \in \tilde{D}} N(v), |\tilde{D}| \leq 3, w \in \tilde{D}\} \quad (4.9)$$

### 4.3 The Reduction Rules

Furthermore, we shortly denote  $\bigcup \mathcal{D}_v = \bigcup_{D \in \mathcal{D}_v} D$  and  $\bigcup \mathcal{D}_w = \bigcup_{D \in \mathcal{D}_w} D$ .

Assuming that  $v$  and  $w$  are closely connected with  $d(v, w) \leq 2$ , it might suffice to consider only sets of size at most three, because an intermediate vertex could witness  $v$  and  $w$  at the same time. In the later analysis, the  $D$ -region decomposition exactly creates regions around  $N(v, w)$  requiring at least one path from  $v$  to  $w$  lengthed two. As the following rule is only used to locally investigate such regions, we could add the requirement of a distance of two to it and work with sets of size at most three. We believe that this could further improve the kernel.

We are now ready to state Rule 2:

**Rule 2.** Let  $G = (V, E)$  be a graph and  $v, w$  be two distinct vertices from  $V$ . If  $\mathcal{D} = \emptyset$  we apply the following:

**Case 1:** if  $\mathcal{D}_v = \emptyset$  and  $\mathcal{D}_w = \emptyset$

- Remove  $N_{2,3}(v, w)$
- Add vertices  $v'$  and  $w'$  and two edges  $\{v, v'\}$  and  $\{w, w'\}$
- If there was a common neighbor of  $v$  and  $w$  in  $N_{2,3}(v, w)$ , add another vertex  $y$  and two connecting edges  $\{v, y\}$  and  $\{y, w\}$
- If there was no common neighbor of  $v$  and  $w$  in  $N_{2,3}(v, w)$ , but at least one path of length three from  $v$  to  $w$  via only vertices from  $N_{2,3}(v, w)$ , add two vertices  $y$  and  $y'$  and connecting edges  $\{v, y\}$ ,  $\{y, y'\}$  and  $\{y', w\}$

**Case 2:** if  $\mathcal{D}_v \neq \emptyset$  and  $\mathcal{D}_w = \emptyset$

- Remove  $N_{2,3}(v)$
- Add  $\{v, v'\}$

**Case 3:** if  $\mathcal{D}_v = \emptyset$  and  $\mathcal{D}_w \neq \emptyset$

This case is symmetrical to case (2).

In case (1), we know by Fact 3 that  $v$  and  $w$  must be in  $D$ . Therefore, we introduce two forcing vertices  $v'$  and  $w'$  in  $G'$  and remove  $N_{2,3}(v, w)$  as these vertices are dominated by  $v$  and  $w$ . But if we remove  $N_{2,3}(v, w)$  entirely, we could lose solutions: First, the case that  $v$  is a direct witness of  $w$  ( $d(v, w) = 2$ ) and that there is one intermediate witness on a path of length three from  $v$  to  $w$  via vertices in  $N_{2,3}(v, w)$ , which could be a witness for both  $v$  and  $w$  at the same time.

Note that if we would not distinguish between these two cases and had added one intermediate vertex in both of them, we would possibly have generated some wrong solutions, because  $v$  could always witness  $w$ .

#### 4 A Linear Kernel for Planar Semitotal Domination

Again by Fact 3 we know for cases (2) and (3) that  $v \in D$  and similar to Rule 1 we can simplify the neighborhood  $N_{2,3}(v)$ . Fact 2 states, that these vertices are only useful for witnessing  $v$ , but do not go beyond what  $v$  already witnesses. Observe that removing  $N_{2,3}$  can not break any connectivity as all vertices in  $N_{2,3}(v)$  are confined in  $v$ . The cases where  $\mathcal{D}_v \neq \emptyset$  and  $\mathcal{D}_w \neq \emptyset$  are not required and in a later analysis only the existence of these two sets is required and the application of Rule 3 will be critical.

Before proofing Rule 2 we will deduce some facts which are implied by the definitions above. These facts justify the definition of the sets  $\mathcal{D}$ ,  $\mathcal{D}_v$  and  $\mathcal{D}_w$ .

**Fact 2.** *Let  $G = (V, E)$  be a graph, let  $v, w \in V$ , and let  $G'$  be the graph obtained by the application of Rule 2 on  $v, w$ . If  $\mathcal{D} = \emptyset$ , then  $G$  has a solution if and only if it has a solution containing at least one of the two vertices  $\{v, w\}$ .*

*Proof.* Because  $\mathcal{D} = \emptyset$ , any SDS of  $G$  has to contain  $v$  or  $w$ , or at least four vertices from  $N_{2,3}(v, w)$ . In the second case, these four vertices can be replaced with  $v, w$  and two neighbors of  $v$  and  $w$  still forming a SEMITOTAL DOMINATING SET.  $\square$

The second fact states that if  $\mathcal{D}_v$  (resp.  $\mathcal{D}_w$ ) is empty, too, we only need to consider solutions containing  $w$  (or  $v$ ):

**Fact 3.** *Let  $G = (V, E)$  be a graph, let  $v, w \in V$ , and let  $G'$  be the graph obtained by the application of Rule 2 on  $v, w$ . If  $\mathcal{D} = \emptyset$  and  $\mathcal{D}_w = \emptyset$  (resp.  $\mathcal{D}_v = \emptyset$ ) then  $G'$  has a solution if and only if it has a solution containing  $v$  (resp.  $w$ ).*

*Proof.* As  $\mathcal{D}_v = \emptyset$ , no set of the form  $\{v\}$ ,  $\{v, u\}$  or  $\{v, u, u'\}$  with  $u, u' \in N_{2,3}(v, w)$  can dominate  $N_3(v, w)$ . Since also  $\mathcal{D} = \emptyset$  any SDS of  $G$  has to contain  $v$  or at least four vertices by Fact 2. In the last case, we again replace these four vertices with  $v, w$  and two neighbors respectively and we can conclude that  $v$  belongs to the solution.  $\square$

Now we are ready to prove the correctness of Rule 2

**Lemma 3.** *Let  $G = (V, E)$  be a plane graph,  $v, w \in V$  and  $G' = (V', E')$  be the graph obtained after application of Rule 2 on the pair  $\{v, w\}$ . Then  $G$  has SDS of size  $k$  if and only if  $G'$  has SDS of size  $k$ .*

*Proof.* We will prove the claim by analyzing the different cases of the rule independently.

$\Rightarrow$  Consider a SEMITOTAL DOMINATING SET  $D$  in  $G$ . We show that  $G'$  also has a SDS with  $|D'| \leq |D|$ . By assumption, we have  $\mathcal{D} = \emptyset$ .

a)  $\mathcal{D}_v = \emptyset \wedge \mathcal{D}_w = \emptyset$ : By applying Fact 3 twice, we know that both  $v, w \in D$ . Therefore,  $v', w'$ , and potentially  $y$  and  $y'$  are dominated by  $v$  or  $w$  in  $G'$ .

We now have three cases: Either  $v$  and  $w$  are so far away that both have their witnesses themselves; they are of distance three, such that potentially one witness on a path from  $v$  to  $w$  (which could go through  $N_{2,3}(v, w)$  and will

### 4.3 The Reduction Rules

be removed) is required or they can be of distance less than three, such that they already witness each other directly.

We will now build  $D'$  depending on which vertices from  $D \cap N_{2,3}(v, w)$  have been removed.

- If the rule has not removed any  $d \in D$ , we simply set  $D' = D$ . If  $v$  was a witness for  $w$  (and vice versa), Rule 2 will preserve it by introducing the vertex  $y$ . Otherwise, these witnesses are preserved.
- If  $d(v, w) > 3$ , then  $v$  and  $w$  are not sharing any common witnesses. If the rule has removed a vertex from  $D \cap N(v)$ , we set  $D' = D \setminus N_{2,3}(v, w) \cup \{v'\}$ . If the rule has removed a vertex from  $D \cap N(w)$ , we set  $D' = D \setminus N_{2,3}(v, w) \cup \{w'\}$ . If the rule has removed a vertex from  $(D \cap N(v))$  and a vertex from  $(D \cap N(w))$ , we set  $D' = D \setminus N_{2,3}(v, w) \cup \{v', w'\}$ .
- If  $d(v, w) = 3$ , then there could possibly be a path via  $N_{2,3}(v, w)$  vertices containing a witness for both  $v$  and  $w$ . If the rule removed a vertex  $D \cap N_{2,3}(v, w)m$ , we set  $D' = D \setminus N_{2,3}(v, w) \cup \{y\}$ . Note that we could also choose  $y' \in D'$ , because  $y$ 's only function is to be a single witness for  $v$  and  $w$  and everything it could witness too, will also be witnessed by  $v, w \in D'$  (Fact 2).
- If  $d(v, w) \leq 2$ , then  $v$  directly witnesses  $w$  (and vice versa). even if the rule has removed a vertex  $z \in D \cap N_{2,3}(v, w)$ , we can ignore that, because Fact 2 states that  $v$  and  $w$  will witness the same vertices as  $z$  did. Hence, we set  $D' = D \setminus N_{2,3}(v, w)$ .

In all of the cases, it follows that  $D'$  is a SDS of  $G'$  with  $|D'| \leq |D|$

- b)  $\mathcal{D}_v \neq \emptyset \wedge \mathcal{D}_w = \emptyset$ : As  $\mathcal{D}_w = \emptyset$  and Fact 3, we know that  $v \in D$  and  $v$  dominates  $N_{2,3}(v)$ . If a vertex  $d \in D \cap N_{2,3}(v)$  was removed, we set  $D' = D \setminus N_{2,3}(v) \cup \{v'\}$ , else  $D' = D$ . Deleting dominating vertices  $d \in D \cap N_{2,3}(v)$  does not destroy the witness properties of the graph, because by Fact 2 we know that everything  $d$  could witness, is also witnessed by  $v$ . If  $d$  was a witness for  $v$ , we have replaced it with  $v'$  in  $G'$ . Note that otherwise a vertex from  $N_1(v) \cup \{p \in (N(z) \setminus N(v)) | z \in N_1(v)\}$  is a witness for  $v$  that is not touched by this reduction. Clearly,  $|D'| \leq |D|$  holds.
- c)  $\mathcal{D}_v = \emptyset \wedge \mathcal{D}_w \neq \emptyset$ : Symmetrical to previous case.

$\Leftarrow$  Let  $D'$  be a SEMITOTAL DOMINATING SET in  $G'$  and  $\mathcal{D} = \emptyset$ . We show that  $G$  has a SDS  $D$  with  $|D| \leq |D'|$  by distinguishing the different cases again.

- a)  $\mathcal{D}_v = \emptyset \wedge \mathcal{D}_w = \emptyset$ : In any case we know that  $v, w \in D$  to dominate  $v'$  and  $w'$  and therefore also dominating  $N_{2,3}(v, w)$  in  $G$ . To preserve the distance  $d(v, w)$  the rule might have introduced additional vertices  $y$  and  $y'$ .

- If only  $y$  was introduced we know that there was a common neighbor  $n \in N(v) \cap N(w)$  of  $v$  and  $w$ .  $y$  allows  $v$  to witness  $w$  (and vice versa) and is not part of a solution itself. (assuming  $y \notin D'$ ). Hence, we set  $D = D'$ .
  - If  $y$  and  $y'$  were added, a solution could use one of them to provide a single witness for  $v$  and  $w$ . There exists a path  $p = (v, n_1, n_2, w)$  from  $v$  to  $w$  in  $G$  only using vertices from  $N_{2,3}(v, w)$ . As  $n_1$  and  $n_2$  both witness  $v$  and  $w$ , we put one of them in  $D$  if at least one of  $y$  or  $y'$  are dominating vertices in  $G'$ . Hence, if  $y \in D'$  or  $y' \in D'$ , we set  $D = D' \setminus \{y, y'\} \cup \{n_1\}$ .
- b)  $\mathcal{D}_v \neq \emptyset \wedge \mathcal{D}_w = \emptyset$ : Clearly,  $v \in D'$  to dominate  $v'$ . If  $v \in D'$ , we set  $D = D' \setminus \{v'\} \cup d$  for some vertex  $d \in N_{2,3}(v, w)$  and otherwise  $D = D'$ . If  $v'$  was the witness of  $v$ , it is now replaced by  $d$  and  $D$  is an SDS with  $|D| \leq |D'|$ .
- c)  $\mathcal{D}_v = \emptyset \wedge \mathcal{D}_w \neq \emptyset$ : Symmetrical to previous case.

In all cases we have shown that  $|D| \leq |D'|$  and  $D$  is a SEMITOTAL DOMINATING SET of  $G$ .

□

#### 4.3.3 Reduction Rule III: Shrinking Simple Regions

If two vertices  $v$  and  $w$  share enough neighbors with each other, we can sometimes conclude at least one of them to be in a solution.

By planarity, a *simple region* has at most two vertices from  $N_1(v, w)$  (namely the border  $\partial$ ), two vertices from  $N_2(v, w)$  connected to the border and unlimited  $N_3(v, w)$  vertices squeezed in the middle.

With a more sophisticated analysis, we were able to modify this rule such that the bound given in [13] remained valid although we might have to add some more vertices as originally described.

**Rule 3.** Let  $G = (V, E)$  be a plane graph,  $v, w \in V$  and  $R$  be a simple region between  $v$  and  $w$ . If  $|V(R) \setminus \{v, w\}| \geq 5$  apply the following:

**Case 1:** If  $V(R \setminus \partial R) \cong P_3$  then

- Remove  $V(R \setminus \partial R)$
- add vertex  $y$  with edges  $\{v, y\}$  and  $\{y, w\}$

**Case 2:** If  $V(R \setminus \partial R) \not\cong P_3$  then

- Remove  $V(R \setminus \partial R)$

### 4.3 The Reduction Rules

- add vertices  $y, y'$  and four edges  $\{v, y\}, \{v, y'\}, \{y, w\}$  and  $\{y', w\}$

**Lemma 4 (Correctness of Rule 3).** Let  $G = (V, E)$  be a plane graph,  $v, w \in V$  and  $G' = (V', E')$  be the graph obtained after application of Rule 3 on the pair  $\{v, w\}$ . Then  $G$  has SDS of size  $k$  if and only if  $G'$  has SDS of size  $k$ .

*Proof.*  $\Rightarrow$  Consider a SEMITOTAL DOMINATING SET  $D$  in  $G$ . We show that  $G'$  also has a SDS with  $|D'| \leq |D|$ . By assumption, we have  $D = \emptyset$ .

$\Rightarrow$  Consider a SEMITOTAL DOMINATING SET  $D$  in  $G'$ . We show that  $G$  has a SDS with  $|D| \leq |D'|$ . By assumption, we have  $D = \emptyset$ .  $\square$

The application of Rule 3 gives us a bound on the number of vertices inside a simple region.

**Corollary 5.** Let  $G = (V, E)$  be a graph,  $v, w \in V$  and  $R$  a simple region between  $v$  and  $w$ . If Rule 3 has been applied, this simple region has a size of at most 6.

*Proof.* If  $|V(R) \setminus \{v, w\}| < 7$  then the rule would not have changed  $G$  and the size of the region would already be bounded by 6. Assuming  $|V(R) \setminus \{v, w\}| \geq 7$  we note that every simple region can have at most two distinct vertices from  $N_1(v, w)$  and two distinct ones from  $N_2(v, w)$  without breaking planarity. These vertices are not touched by the reduction. Adding the two vertices that are being added between  $v$  and  $w$  gives us the desired upper bound.  $\square$

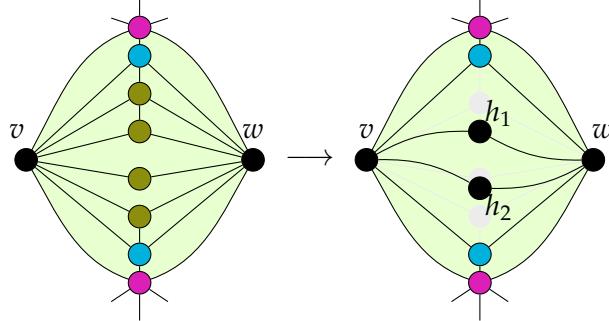


Figure 4.9: TO BE DONE

#### 4.3.4 Computing Maximal Simple Regions between two vertices

For the sake of completeness, we state an algorithm how a maximal simple region between two vertices  $v, w \in V$  can be computed in time  $\mathcal{O}(d(v) + d(w))$ :

## 4.4 Bounding the Size of the Kernel

We are now putting all our pieces together to prove the main result: A kernel bounded linear in the solution size  $k$ . To do so, we distinguish between those vertices that are covered inside a region in a maximal *D-region decomposition* and those that are not. In both cases, our reduction rules bound the number of vertices to a constant size which means the kernel size does only depend on the number of regions of these decompositions. As Fig. 4.5 states that for any solution (vertex set)  $D$ , we only need a linear number of regions that cover the whole graph. In particular, we show that given a SEMITOTAL DOMINATING SET  $D$  of size  $k$ , there exists a maximal *D-region decomposition*  $\mathfrak{R}$  such that:

1.  $\mathfrak{R}$  has only at most  $3|D| - 6$  regions
2.  $V(\mathfrak{R})$  covers most vertices of  $V$ . There are at most  $144 \cdot |D|$  vertices outside of any region.
3. each region of  $\mathfrak{R}$  contains at most 139 vertices

Combining these three statements will give us a linear kernel. Figure 4.7 shows them highlighted in *yellow*.

### 4.4.1 Bounding the Size of a Region

We start with a more fine-grained analysis of the impact of the different cases of Rule 2 on a  $vw$ -region. The main idea is to count the number of simple regions in the  $vw$ -region and then use the bound on the size of a simple region after Rule 3 was applied exhaustively and which was obtained in Corollary 5.

**Lemma 5.** *Given a plane Graph  $G = (V, E)$  and a  $vw$ -region  $R$   $|N_1(v, w) \cap V(R)| \leq 4$  and these vertices lay exactly on the boundary  $\partial R$  of  $R$ .*

*Proof.*

□

**Lemma 6.** *[13, See Fact 5] Given a reduced plane graph  $G = (V, E)$  and a region  $R(v, w)$ ,  $N_2(v, w) \cap V(R)$  can be covered by at most 6 simple regions.*

*Proof.* Let  $(v, u_1, u_2, w)$  and  $(v, u_3, u_4, w)$  be the two boundary paths of  $R(v, w)$ . (A shorter path would only lead to a smaller bound). By definition of  $N_2(v, w)$ , vertices from  $N_2(v, w) \cap V(R)$  are common neighbors of  $v$  and  $w$  and  $u_i, i \in [4]$ . By planarity, we can cover  $N_2(v, w) \cap V(R)$  with at most 6 simple regions. To see this, imagine the graph where edges denote all possible simple  $vw$ -regions (See fig. 4.10). There are at most 8 simple regions possible. but we have to remove at least two of them to maintain planarity.

Furthermore, assuming the graph to be reduced, any intermediate  $N_3(v, w)$  which could separate multiple simple regions between  $v$  and  $u_i$  has been deleted by Rule 1 already.

□

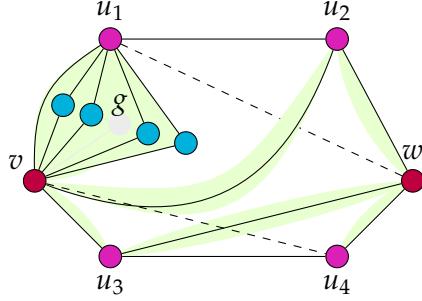


Figure 4.10: *Bounding the maximum number of simple regions inside a region  $R(v,w)$ .  $N_2(v,w)$  is covered by 6 green (simple) regions. A dashed edge would also be an option but would contradict planarity. Note that the gray vertex  $g$  was reduced by Rule 1 allowing the formation of exactly one simple region between  $v$  and  $u_1$*

We continue by giving a constant bound on the number of simple regions that cover all  $N_3(v,w)$  vertices in a given region.

**Lemma 7.** *Given a plane Graph  $G = (V, E)$  reduced under Rule 2 and a region  $R(v, w)$ , if  $\mathcal{D}_v \neq (\text{resp. } \mathcal{D}_w \neq \emptyset)$ ,  $N_3(v, w) \cap V(R)$  can be covered by:*

1. 11 simple regions if  $\mathcal{D}_w \neq \emptyset$ ,
2. 14 simple regions if  $N_{2,3}(v) \cap N_3(v, w) = \emptyset$

Note, that the first case applies, when Case 2 & 3 of Rule 2 have been applied and the second one, when Case 4 of Rule 2 was applied.

*Proof.* We will just give some intuition because the proof of Garnero and Sau in [13, Revision 2014, Fact 6] does not use any special property exposed by the reduction rules. Figure (Add picture about figures) gives a visualization of the worst-case scenarios to cover  $N_3(v, w) \cap V(R)$  with simple regions in the relevant cases.<sup>1</sup>

□

**Lemma 8 (#Vertices inside a Region after Rules 1 to 3).** *Let  $G = (V, E)$  be a plane graph reduced under Rules 1 to 3. Furthermore, let  $D$  be an SDS of  $G$  and let  $v, w \in D$ . Any  $vw$ -region  $R$  contains at most 139 vertices distinct from its poles.*

*Proof.* By Lemmas 5 and 6 and Corollary 5 to bound the number of vertices inside a simple region, we know that  $|N_1(v, w) \cap V(R)| \leq 4$  and  $|N_2(v, w) \cap V(R)| \leq 6 \cdot 7 = 42$ .

---

<sup>1</sup>Note: In a newer revision of their paper [13], Garnero and Sau removed this proof, because they changed Rule 2 and a more fine-grained analysis was made possible.

#### 4 A Linear Kernel for Planar Semitotal Domination

It is remaining to bound for  $|N_3(v, w) \cap V(R)|$ , but gladly we have Rule 2, which took care of them! Fig. 4.11 shows the worst-case amount of simple regions the individual cases can have.

**Case 0:** Rule 2 has **not** been applied in the following two cases: Either  $\mathcal{D} \neq \emptyset$  or ( $\mathcal{D} = \emptyset \wedge \mathcal{D}_v \neq \emptyset \wedge \mathcal{D}_w \neq \emptyset$ ):

1. If  $\mathcal{D} \neq \emptyset$ , there exists a set  $\tilde{\mathcal{D}} = \{d_1, d_2, d_3\} \in \mathcal{D}$  of at most three vertices dominating  $N_3(v, w)$ . We observe that vertices from  $|N_3(v, w) \cap V(R)|$  are common neighbors of either  $v$  or  $w$  (by the definition of a  $vw$ -region) and at least one vertex from  $\tilde{\mathcal{D}}$ . Without violating planarity, we can span at most 6 simple regions. Using Corollary 5 and adding  $|\tilde{\mathcal{D}}| = 3$ , we can conclude  $|N_3(v, w) \cap V(R)| \leq 6 \cdot 6 + 3 = 39$ .
2. If  $\mathcal{D} = \emptyset$ ,  $\mathcal{D}_v \neq \emptyset$  and  $\mathcal{D}_w \neq \emptyset$ , we can apply Lemma 7 and although Rule 2 has not changed the  $G$ , we can cover  $R$  with at most 11 simple regions giving us  $|N_3(v, w) \cap V(R)| \leq 11 \cdot 6 = 66$  vertices.

**Case 1:** If Rule 2 Case (1) has been applied

In that case,  $|N_2(v, w) \cap V(R)|$  was entirely removed and at  $|N_3(v, w) \cap V(R)|$  replaced by at most three vertices ( $v'$ ,  $w'$  and  $y$ ) added. Hence  $|N_3(v, w) \cap V(R)| \leq 3$ .

**Case 2:** If Rule 2 Cases (2) and (3) have been applied

We know that in this case  $N_{2,3}(v) \cap N_3(v, w)$  was entirely removed and replaced by a single possible witness. Using Lemma 7, we can cover  $(N_3(v, w) \setminus \{v'\}) \cap V(R)$  with (at most) 14 simple regions giving us  $||N_3(v, w) \cap V(R)|| \leq 14 \cdot 6 + 1 = 85$ .

All in all, as  $V(R) = \{v, w\} \cup (N_1(v, w) \cup N_2(v, w) \cup N_3(v, w)) \cap V(R)$  we get

$$V(R) \leq 2 + 4 + 42 + \max(39, 3, 66, 85) = 139$$

□

#### 4.4.2 Number of Vertices outside the Decomposition

We continue to bound the number of vertices that do not lay inside any region of a maximal  $D$ -region decomposition  $\mathfrak{R}$ , that is, we bound the size of  $V \setminus V(\mathfrak{R})$ . Rule 1 ensures that we only have a small amount of  $N_3(v)$ -pendants. We then try to cover the rest with as few simple regions as possible, because, by application of Rule 3, these simple regions are of constant size.

**Lemma 9.** [2] (Deprecated) Every vertex in  $u \in V \setminus V(\mathfrak{R})$  is either in  $D$  or belongs to a set  $N_2(v) \cup N_3(v)$ .

The following lemma states that no vertices from a set  $N_1(v)$  will be outside of a maximal  $D$ -region decomposition.

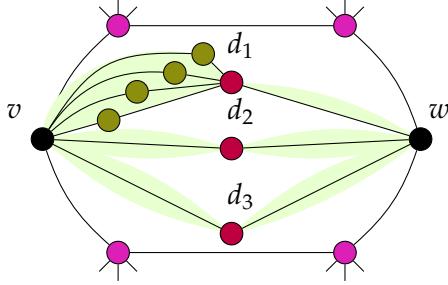
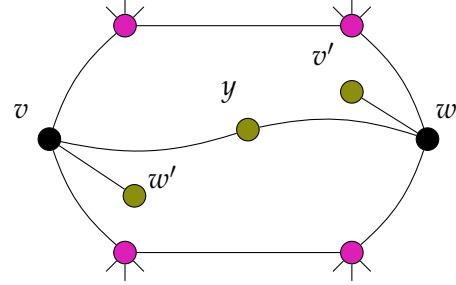
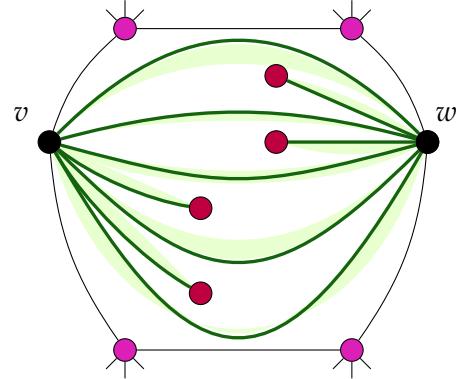
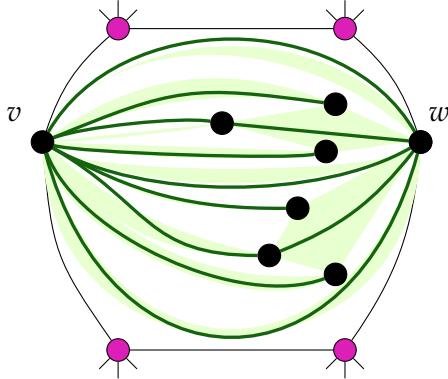
**Case 0:** Maximal 6 Simple Regions

**Case 1:** Exactly 3 vertices

**Case 2:** All Maximal 11 Simple Regions    **Case 3/4:** Maximal 9 Simple Regions


Figure 4.11: TODO

**Lemma 10.** [2, Lemma 6] Let  $G = (V, E)$  be a plane graph and  $\mathfrak{R}$  be a maximal  $D$ -region decomposition of a DS  $D$ . If  $u \in N_1(v)$  for some vertex  $v \in D$  then  $u \in V(\mathfrak{R})$

In the following, we define  $d_{G_{\mathfrak{R}}}(v) = |\{R(v, w) \in \mathfrak{R}, w \in D\}|$  to be the number of regions in  $\mathfrak{R}$  having  $v$  as a pole.

**Corollary 6.** Let  $G = (V, E)$  be a graph and  $D$  be a set. For any maximal  $D$ -region decomposition  $\mathfrak{R}$  on  $G$  it holds that  $\sum_{v \in D} d_{G_{\mathfrak{R}}}(v) = 2 \cdot |\mathfrak{R}|$ .

*Proof.* The proof follows directly from the handshake lemma applied to the underlying multigraph  $G_{\mathfrak{R}}$ .  $\square$

**Proposition 1 (#Vertices outside a Region).** Let  $G = (V, E)$  be a plane graph reduced under Rules 1 and 2 and let  $D$  be a SDS of  $G$ . If  $G$  has a maximal  $D$ -region decomposition, then  $|V \setminus (V(\mathfrak{R}) \cup D)| \leq 144|D|$

#### 4 A Linear Kernel for Planar Semitotal Domination

With slight modifications, the proof given in [13, Revision 2014] will also work in our case. Note that although assuming the graph to be entirely reduced, the following proof only relies on Rules 1 and 3. The proof uses the observation that vertices from  $N_2(v)$  span simple regions between those from  $\{v\} \cup N_1(v)$ .

*Proof.* Again, we will follow the proof proposed by Alber et al. [2, Proposition 2].

The proof does only rely on Rules 1 and 3 and we can use the number of vertices in a simple region we have proven in Corollary 5. In particular, we are going to proof that  $V \setminus V(\mathfrak{R}) \leq 48 \cdot |\mathfrak{R}| + 2 \cdot |D|$ . Directly placing in Lemma 11 will give the desired bound.

let  $\mathfrak{R}$  be a maximal *D-region decomposition* and let  $v \in D$ . Since  $D$  dominates all vertices from  $V$ , we can consider  $V$  as  $\bigcup_{v \in D} N(v)$  and thus, we only need to bound the sizes of  $N_1(v) \setminus V(\mathfrak{R})$ ,  $N_2(v) \setminus V(\mathfrak{R})$  and  $N_3(v) \setminus V(\mathfrak{R})$  separately. In the following, let  $v \in D$ :

**N<sub>3</sub>(v):** As we know that Rule 1 has been exhaustively applied, we trivially see that  $|N_3(v)| \leq 1$  and hence,

$$\left| \bigcup_{v \in D} N_3(v) \setminus V(\mathfrak{R}) \right| \leq |D|$$

**N<sub>2</sub>(v):** According to Garnero and Sau ([13, Proposition 2]), we know that  $N_2(v) \setminus V(\mathfrak{R})$  can be covered by at most  $4d_{G_{\mathfrak{R}}}(v)$  simple regions between  $v$  and some vertices from  $N_1(v)$  on the boundary of a region in  $\mathfrak{R}$ . Figure 4.12 gives some intuition.

Because  $G$  is reduced by assumption, we know by Corollary 5 that a simple region can only have at least 6 vertices distinct from its poles and hence,

$$\begin{aligned} \left| \bigcup_{v \in D} N_2(v) \setminus V(\mathfrak{R}) \right| &\leq 6 \sum_{v \in D} 4 \cdot d_{G_{\mathfrak{R}}}(v) \\ &= 24 \cdot \sum_{v \in D} d_{G_{\mathfrak{R}}}(v) \\ &\stackrel{\text{Cor. 4.4.2}}{\leq} 48|\mathfrak{R}| \end{aligned} \tag{4.10}$$

**N<sub>1</sub>(v):** By Lemma 10, we know that  $N_1(v) \subseteq V(\mathfrak{R})$  and hence,

$$\left| \bigcup_{v \in D} N_1(v) \setminus V(\mathfrak{R}) \right| = 0$$

Summing up these three upper bounds for each  $v \in D$  we obtain the result using the equation from Lemma 11:

#### 4.4 Bounding the Size of the Kernel

$$\begin{aligned}
|V \setminus V(\mathfrak{R}) \cup D)| &\leq 48 \cdot |\mathfrak{R}| + |D| && \text{(Lemma 11)} \\
&\leq 48 \cdot (3|D| - 6) + |D| \\
&\leq 144|D| + |D| \\
&= 145|D|
\end{aligned} \tag{4.11}$$

□

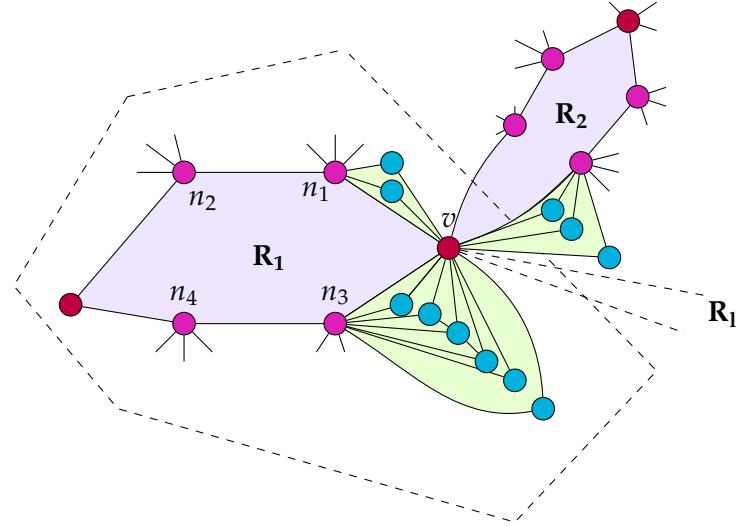


Figure 4.12: *Bounding the number of  $N_2(v)$ -vertices around a dominating vertex  $v$  given a maximal  $D$ -region decomposition  $\mathfrak{R}$ .  $v$  is a pole of  $R_1, R_2, \dots, R_j$  and can span simple regions with the help of  $N_2(v)$ -vertices to at most two  $N_1(v)$ -vertices per  $R_i$ . Each region has at most four vertices in  $N_1(v, w) \subseteq N_1(v)$  on the boundary of  $R_i$ , but only at most two can be used for a simple region: Observe that trying to build a simple region between  $v$  and  $n_2$  in this example would contradict the maximality of  $\mathfrak{R}$ . Furthermore, the size of these simple regions is bounded after the application of Rule 3*

#### 4.4.3 Bounding the Number of Regions

Alber et al. [2, Proposition 1] gave a greedy algorithm to construct a maximal  $D$ -region decomposition for a DOMINATING SET. Building upon these results, Garnero and Sau gave decomposition procedures for both PLANAR RED-BLUE DOMINATING SET ([14]) and TOTAL DOMINATING SET ([13]) relying on the same technique. This is the core of our linear kernelization because it states that given a DOMINATING SET  $D$ , we can decompose the graph into a *linear number* of regions.

#### 4 A Linear Kernel for Planar Semitotal Domination

The following lemma corresponds to [2, Proposition 1 and Lemma 5]. Although the authors gave different reduction rules and require a *reduced* instance as an assumption for the following lemma, they do not use any specific properties exposed by these rules. As any SEMITOTAL DOMINATING SET is also a DOMINATING SET, we can safely apply it to our problem as well. For a more detailed and formal proof, one can also refer to [13, Proposition 1].

**Lemma 11.** *Let  $G$  be a reduced plane graph and let  $D$  be a SEMITOTAL DOMINATING SET with  $|D| \geq 3$ . There is a maximal  $D$ -region decomposition of  $G$  such that  $|R| \leq 3 \cdot |D| - 6$*

*Proof.* Follows directly from [2, Proposition 1 and Lemma 5] □

**Lemma 12 (Running Time of Reduction Procedure).**

*Proof.* □

By utilizing all the previous results, we are now finally ready to proof the Theorem 1:

**Theorem 1.** *The SEMITOTAL DOMINATING SET problem parametrized by solution size admits a linear kernel on planar graphs. There exists a polynomial-time algorithms that given a planar graph  $(G, k)$ , either correctly reports that  $(G, k)$  is a NO-instance or returns an equivalent instance  $(G', k)$  such that  $|V(G')| \leq 561 \cdot k$ .*

*Proof.* Let  $G = (V, E)$  be the plane input graph and  $G' = (V', E')$  be the graph obtained by the exhaustive application of the Rules 1 to 3. As none of our rules change the size of a possible solution  $D'$  in  $G'$ , we know by Lemma 1, Lemma 3 and Lemma 4 that  $G'$  has a SEMITOTAL DOMINATING SET of size  $k$  if and only if  $G$  has a SEMITOTAL DOMINATING SET of size  $k$ . In Lemma 12, we have proven that this preprocessing procedure runs in polynomial time.

Assume that  $G'$  admits a solution  $D'$ .

By taking the size of each region proven in Proposition 1, the total number of regions in a maximal  $D$ -region decomposition (Lemma 11) and the number of vertices that can lay outside of any region (Proposition 1), we obtain the following bound:

$$139 \cdot (3k - 6) + 145 \cdot k + k \leq 561 \cdot k \quad (4.12)$$

If  $|V(G')| > 561 \cdot k$   $G$  is a NO-instance and we replace  $G'$  by two single disconnected vertices (trivial NO-instance). Then the kernel is of the claimed size. □

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## CHAPTER 5

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### OPEN QUESTIONS AND FURTHER RESEARCH



*To do select another quote*

Lewis Caroll, XXXX

\* Chordal Bipartite Graph has a very interesting case. \* Improve the Kernel Bound

## *5 Open Questions and Further Research*

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## BIBLIOGRAPHY

- [1] J. Alber, B. Dorn, and R. Niedermeier. “A General Data Reduction Scheme for Domination in Graphs.” In: *SOFSEM 2006: Theory and Practice of Computer Science, 32nd Conference on Current Trends in Theory and Practice of Computer Science, Merin, Czech Republic, January 21-27, 2006, Proceedings*. Ed. by J. Wiedermann, G. Tel, J. Pokorný, M. Bieliková, and J. Stuller. Vol. 3831. Lecture Notes in Computer Science. Springer, 2006, pp. 137–147. doi: [10.1007/11611257\\\_{1}\{1}](https://doi.org/10.1007/11611257_{1}_{1}).
- [2] J. Alber, M. R. Fellows, and R. Niedermeier. “Polynomial-time data reduction for dominating set.” In: (May 2004), pp. 363–384. doi: [10.1145/990308.990309](https://doi.org/10.1145/990308.990309).
- [3] R. Balakrishnan and K. Ranganathan. *A textbook of graph theory*. English. 2nd ed. Universitext. New York, NY: Springer, 2012. ISBN: 978-1-4614-4528-9; 978-1-4614-4529-6. doi: [10.1007/978-1-4614-4529-6](https://doi.org/10.1007/978-1-4614-4529-6).
- [4] H. L. Bodlaender, F. V. Fomin, D. Lokshtanov, E. Penninkx, S. Saurabh, and D. M. Thilikos. “(Meta) Kernelization.” In: *J. ACM* 63.5 (2016), 44:1–44:69. doi: [10.1145/2973749](https://doi.org/10.1145/2973749).
- [5] M. Bonamy and Ł. Kowalik. “A 13k-kernel for planar feedback vertex set via region decomposition.” en. In: *Theoretical Computer Science* 645 (Sept. 2016), pp. 25–40. issn: 0304-3975. doi: [10.1016/j.tcs.2016.05.031](https://doi.org/10.1016/j.tcs.2016.05.031).
- [6] J. A. Bondy and U. S. R. Murty. *Graph Theory*. Graduate Texts in Mathematics. Springer, 2008. ISBN: 978-1-84628-970-5. doi: [10.1007/978-1-84628-970-5](https://doi.org/10.1007/978-1-84628-970-5).
- [7] M. Cygan, F. V. Fomin, Ł. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. *Parameterized Algorithms*. Berlin, Heidelberg: Springer, 2015. ISBN: 978-3-319-21275-3.
- [8] V. Diekert and B. Durand, eds. *STACS 2005, 22nd Annual Symposium on Theoretical Aspects of Computer Science, Stuttgart, Germany, February 24-26, 2005, Proceedings*. Vol. 3404. Lecture Notes in Computer Science. Springer, 2005. ISBN: 3-540-24998-2. doi: [10.1007/b106485](https://doi.org/10.1007/b106485).
- [9] R. Diestel. *Graph Theory*. Fourth. Vol. 173. Graduate Texts in Mathematics. Heidelberg; New York: Springer, 2010. ISBN: 9783642142789 3642142788 9783642142796 3642142796.
- [10] F. V. Fomin, D. Lokshtanov, S. Saurabh, and M. Zehavi. *Kernelization. Theory of parameterized preprocessing*. English. Cambridge: Cambridge University Press, 2019. ISBN: 978-1-107-05776-0; 978-1-107-41515-7. doi: [10.1017/9781107415157](https://doi.org/10.1017/9781107415157).

## Bibliography

- [11] F. V. Fomin and D. M. Thilikos. “Fast Parameterized Algorithms for Graphs on Surfaces: Linear Kernel and Exponential Speed-Up.” en. In: *Automata, Languages and Programming*. Ed. by J. Díaz, J. Karhumäki, A. Lepistö, and D. Sannella. Lecture Notes in Computer Science. Berlin, Heidelberg: Springer, 2004, pp. 581–592. ISBN: 9783540278368. doi: [10.1007/978-3-540-27836-8\\_50](https://doi.org/10.1007/978-3-540-27836-8_50).
- [12] V. Garnero, C. Paul, I. Sau, and D. M. Thilikos. “Explicit Linear Kernels for Packing Problems.” en. In: *Algorithmica* 81.4 (Apr. 2019), pp. 1615–1656. ISSN: 1432-0541. doi: [10.1007/s00453-018-0495-5](https://doi.org/10.1007/s00453-018-0495-5).
- [13] V. Garnero and I. Sau. “A Linear Kernel for Planar Total Dominating Set.” In: *Discrete Mathematics & Theoretical Computer Science* Vol. 20 no. 1 (May 2018). doi: [10.23638/DMTCS-20-1-14](https://doi.org/10.23638/DMTCS-20-1-14). eprint: [1211.0978](https://arxiv.org/abs/1211.0978).
- [14] V. Garnero, I. Sau, and D. M. Thilikos. “A linear kernel for planar red-blue dominating set.” In: *Discret. Appl. Math.* 217 (2017), pp. 536–547. doi: [10.1016/j.dam.2016.09.045](https://doi.org/10.1016/j.dam.2016.09.045).
- [15] W. Goddard, M. A. Henning, and C. A. McPillan. “Semitotal domination in graphs.” In: *A Canadian journal of applied mathematics, computer science and statistics* 94 (June 2014).
- [16] J. Guo and R. Niedermeier. “Linear Problem Kernels for NP-Hard Problems on Planar Graphs.” In: *Automata, Languages and Programming*. Ed. by L. Arge, C. Cachin, T. Jurdziński, and A. Tarlecki. Berlin, Heidelberg: Springer Berlin Heidelberg, 2007, pp. 375–386. ISBN: 978-3-540-73420-8.
- [17] J. Guo, R. Niedermeier, and S. Wernicke. “Fixed-Parameter Tractability Results for Full-Degree Spanning Tree and Its Dual.” en. In: *Parameterized and Exact Computation*. Ed. by H. L. Bodlaender and M. A. Langston. Lecture Notes in Computer Science. Berlin, Heidelberg: Springer, 2006, pp. 203–214. ISBN: 9783540391012. doi: [10.1007/11847250\\_19](https://doi.org/10.1007/11847250_19).
- [18] S. Gutner. “Polynomial Kernels and Faster Algorithms for the Dominating Set Problem on Graphs with an Excluded Minor.” In: *Parameterized and Exact Computation: 4th International Workshop, IWPEC 2009, Copenhagen, Denmark, September 10-11, 2009, Revised Selected Papers*. Berlin, Heidelberg: Springer-Verlag, Dec. 2009, pp. 246–257. ISBN: 9783642112683.
- [19] J. T. Halseth. “A 43k Kernel for Planar Dominating Set using Computer-Aided Reduction Rule Discovery.” In: *University of Bergen, University Library* (Feb. 2016).
- [20] I. Kanj, M. J. Pelsmajer, M. Schaefer, and G. Xia. “On the induced matching problem.” en. In: *Journal of Computer and System Sciences* 77.6 (Nov. 2011), pp. 1058–1070. ISSN: 0022-0000. doi: [10.1016/j.jcss.2010.09.001](https://doi.org/10.1016/j.jcss.2010.09.001).
- [21] Ł. Kowalik, M. Pilipczuk, and K. Suchan. “Towards optimal kernel for connected vertex cover in planar graphs.” en. In: *Discrete Applied Mathematics* 161.7 (May 2013), pp. 1154–1161. ISSN: 0166-218X. doi: [10.1016/j.dam.2012.12.001](https://doi.org/10.1016/j.dam.2012.12.001).

## Bibliography

- [22] W. Luo, J. Wang, Q. Feng, J. Guo, and J. Chen. "Improved linear problem kernel for planar connected dominating set." In: *Theor. Comput. Sci.* 511 (2013), pp. 2–12. doi: [10.1016/j.tcs.2013.06.011](https://doi.org/10.1016/j.tcs.2013.06.011).
- [23] G. Philip, V. Raman, and S. Sikdar. "Polynomial kernels for dominating set in graphs of bounded degeneracy and beyond." In: *ACM Transactions on Algorithms* 9.1 (Dec. 2012), 11:1–11:23. issn: 1549-6325. doi: [10.1145/2390176.2390187](https://doi.org/10.1145/2390176.2390187).
- [24] V. Raman and S. Saurabh. "Short Cycles Make W-hard Problems Hard: FPT Algorithms for W-hard Problems in Graphs with no Short Cycles." In: *Algorithmica* 52.2 (2008), pp. 203–225. issn: 1432-0541. doi: [10.1007/s00453-007-9148-9](https://doi.org/10.1007/s00453-007-9148-9).
- [25] J. Wang, Y. Yang, J. Guo, and J. Chen. "Linear Problem Kernels for Planar Graph Problems with Small Distance Property." en. In: *Mathematical Foundations of Computer Science 2011*. Ed. by F. Murlak and P. Sankowski. Lecture Notes in Computer Science. Berlin, Heidelberg: Springer, 2011, pp. 592–603. ISBN: 9783642229930. doi: [10.1007/978-3-642-22993-0\\_53](https://doi.org/10.1007/978-3-642-22993-0_53).

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