Master's Thesis Presentation

On the Parameterized Complexity of Semitotal Dominating Set On Graph Classes

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Creative Introduction

Our Plan for Today

Motivation

DOMINATING SET

Input	Graph $G = (V, E)$, $k \in \mathbb{N}$
Question	Exists $D \subseteq V$ with $\lvert D \rvert \leq k$ such that $N[D] = V$?

- The domination number is the minimum cardinality of a ds of G, denotes as $\gamma(G)$
- **Observation:** In connected G every $v \in D$ has another $z \in D$ with $d(v, z) \leq 3$.

Motivation

TOTAL DOMINATING SET

Input	Graph $G = (V, E)$, $k \in \mathbb{N}$
Question	Exists $D \subseteq V$ with $\lvert D \rvert \leq k$ such that $\forall d_1 \in X : \exists d_2 \in D \setminus \{d_1\}$ with $d(d_1, d_2) \leq 1$?

- The total domination number is the minimum cardinality of a tds of G, denoted as $\gamma_t(G)$.
- We say d_1 witnesses d_2 (and vice versa)

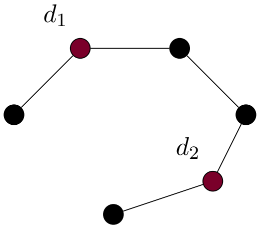
Motivation

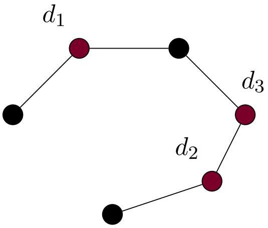
SEMITOTAL DOMINATING SET	
Input	Graph $G = (V, E)$, $k \in \mathbb{N}$
Question	Exists $D \subseteq V$ with $\|D\| \leq k$ such that $\forall d_1 \in X : \exists d_2 \in D \setminus \{d_1\}$ with $d(d_1, d_2) \leq 2$?

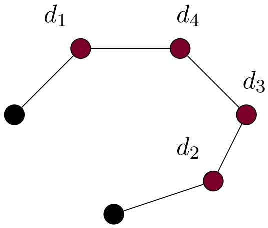
- The semitotal domination number is the minimum cardinality of a sds of G, denoted as $\gamma_{2t}(G)$.
- **Observation**: $\gamma(G) \leq \boldsymbol{\gamma_{2t}(G)} \leq \gamma t(G)$
- We say d_1 witnesses d_2 (and vice versa)

Example: $\gamma(G) < \gamma_{\mathbf{2t}}(\mathrm{G}) < \gamma_t(G)$

Dominating Set	Semitotal Dominating Set	Total Dominating Set







Parameterized Complexity

- NP-hard? We expect problem to be **at least** exponential
- **Idea:** Limit combinatorial explosion to some aspect of the problem
- **Goal:** Find an algorithm running in time $\mathcal{O}(f(k) \cdot n^c)$ for **some** parameter k
- In this work: by solution size
- **Techniques:** Kernelization, Bounded Search Trees, ...

If possible, the problem is **fixed-parameter tractable**.

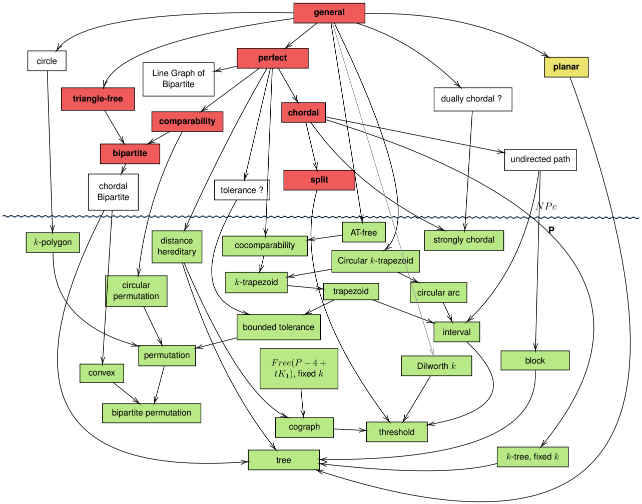
Fixed-Parameter Intractability

- Class NP corresponds to whole hierarchy $W[i]$ in parameterized setting.
- Problems at least $W[1]$-hard considered **fixed-parameter intractable**
- DOMINATING SET is $W[2]$-complete
- **Tool for Proving Hardness**: FPT Reductions, preserving the parameter

Complexity Comparison

Graph Class	Dominating Set		Semitotal Dominating Set		Total Dominatin	
	classical	Parameterized	classical	Parameterized	classical	Paramet
bipartite	NPc **[Bertossi1984]**	W_2 **[Raman2008]**	NPc **[Henning2019]**	W_2 (this)	NPc **[Pfaff1983]**	W_2 (cite
line graph of bipartite	NPc **[Korobitsin1992]**	?	NPc **[Galby2020]**	? (?)	NPc **[McRae1995]**	?
circle	NPc **[Keil1993]**	W_1 **[Bousquet2012]**	NPc **[Kloks2021]**	? (?)	NPc **[McRae1995]**	W_1 **[Bou**
chordal	NPc **[Booth1982]**	W_2 **[Raman2008]**	NPc **[Henning2019]**	W_2 (this)	NPc **[Laskar1983]**	W_1 **[Cha**
s-chordal , $s > 3$	NPc **[Liu2011]**	W_2 **[Liu2011]**	? (?)	? (?)	NPc **[Liu2011]**	W_1 **[Liu**
split	NPc **[Bertossi1984]**	W_2 **[Raman2008]**	NPc **[Henning2019]**	W_2 **this**	NPc **[Laskar1983]**	W_1 **[Cha**
3-claw-free	NPc **[Cygan2011]**	FPT **[Cygan2011]**	Prob. Unk	Prob. Unk	NPc **[McRae1995]**	Unknow
t-claw-free, $t > 3$	NPc **[Cygan2011]**	W_2 **[Cygan2011]**	Prob. Unknown	Unknown	NPc **[McRae1995]**	Prob. Ur
chordal bipartite	NPc **[Mueller1987]**	? (?)	NPc **[Henning2019]**	?	P **[Damaschke1**	
planar	NPc (Sources!)	FPT **[Alber2004]**	NPc	**FPT (this)**	NPc	FPT **[Ga**
undirected path	NPc **[Booth1982]**	FPT **[Figueiredo2022]**	NPc **[Henning2022]**	?	NPc **[Lan2014]**	?
dually chordal	P **[Brandstaedt1998]**		? (attempted **[Galby2020]**)		P **[Kratsch199**	
strongly chordal	P **[Farber1984]**		P **[Tripathi2021]**		NPc **[Farber1984]**	
AT-free	P **[Kratsch2000]**		P **[Kloks2021]**		P **[Kratsch200**	
tolerance	P **[Giannopoulou2016]**		?		?	
block	P **[Farber1984]**		P **[Henning2022]**		P **[Chang198**	
interval	P **[Chang1998a]**		P **[Pradhan2021]**		P **[Bertossi198**	
bounded clique-width	P **[Courcelle1990]**		P **[Courcelle1990]**		P **[Courcelle19**	
bounded mim-width	P **[Belmonte2011, BuiXuan2013]**		P **[Galby2020]**		P **[Belmonte2011, Bui**	

Status SEMITOTAL DOMINATING SET



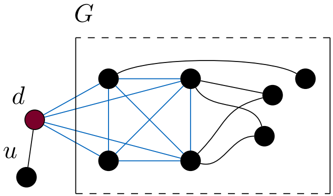
Warmup: Intractability Results

ω_2 hard on split, chordal and bipartite graphs

- **Split Graph:** $G = \texttt{Clique} + \texttt{IndependentSet}$

Split Graphs

SEMITOTAL DOMINATING SET on *split* and *chordal* graphs is ω_2-hard

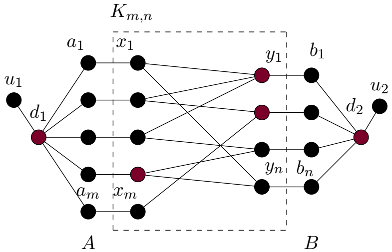


Proof by fpt-reduction from PLANAR DOMINATING SET on split graphs:

1. **Construct** G* by adding v with pendant z to clique. G* split
2. If ds D in G, $D' = D \cup \{v\}$ is sds D'.
3. If sds D' in G', $D \setminus \{v\}$ is D in G
4. Parameter k only changed by constant

Bipartite Graphs

SEMITOTAL DOMINATING SET on *bipartite* graphs is ω_2-hard



Proof by fpt-reduction from PLANAR DOMINATING SET on bipart. graphs:

1. **Construct** Add new neighbor to each vertex and add d_1, d_2, u_1, u_2
2. If ds D in G, then $D' = D \cup \{d_1, d_2\}$ is sds in G'
3. Assume sds D' in G'. If $a_i \in D'$ (b_i), flip. $D = D' \setminus \{d_1, d_2\}$ is ds in G

A Linear Kernel for PLANAR SEMITOTAL DOMINATING SET

Another Explicit kernel for a Dominating Problem

Kernelization

- **Idea:** Preprocess an instance using *Reduction Rules* until hard *kernel* is found.

Kernelization

- **Idea:** Preprocess an instance using *Reduction Rules* until hard *kernel* is found.

$$\overbrace{(I,\ k)\ \in Q}^{n}$$

$$\Big\downarrow\ \mathcal{O}\left(n^{c}\right)$$

$$\underbrace{(I',\ k')\ \in Q}_{g(k)}$$

Related Works

Problem	Size	Source
Planar Dominating Set	$67k$	[Diekert2005]
Planar Total Dominating Set	$410k$	[Garnero2018]
Planar Semitotal Dominating Set	$359k$	**This work**
Planar Edge Dominating Set	$14k$	[Guo2007]
Planar Efficient Dominating Set	$84k$	[Guo2007]
Planar Red-Blue Dominating Set	$43k$	[Garnero2017]
Planar Connected Dominating Set	$130k$	[Luo2013]
Planar Directed Dominating Set	Linear	[Alber2006]

Main Theorem

The Main Theorem

SEMITOTAL DOMINATING SET parameterized by solution size admits a linear kernel on planar graphs. There exists a polynomial-time algorithm that, given a planar graph (G, k), either correctly reports that (G, k) is a NO-instance or returns an equivalent instance (G', k) such that $|V(G')| \leq 359 \cdot k$.

The Big Picture

Given a planar graph $G = (V, E)$, we will:

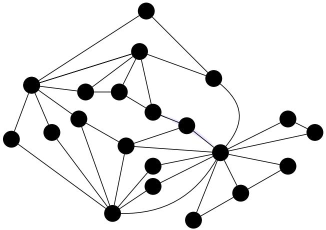
1. Split the neighborhoods of the graph;
2. Define reduction Rules
3. Use the region decomposition to analyse size of each region

The basic Principle: Regions

Region (Simplified)

Given plane G and $v, w \in V$, a region is a closed subset, such that

- there are two non-crossing (but possibly overlapping) boundary paths
- Every vertex in R belongs to $N(v, w)$



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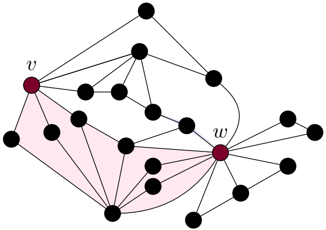


D-region decomposition

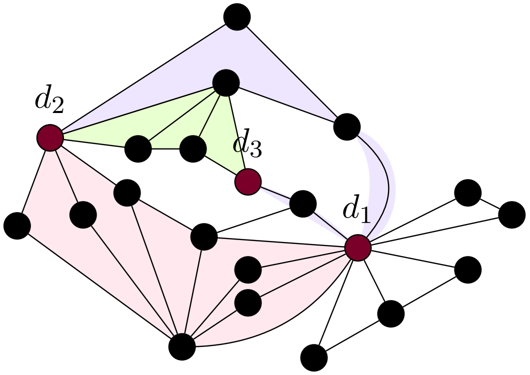
D-region decomposition [Alber2004]

Given $G = (V, W)$ and $D \subseteq V$, a *D-region decomposition* is a set $\mathfrak{R}$ with poles in D such that:

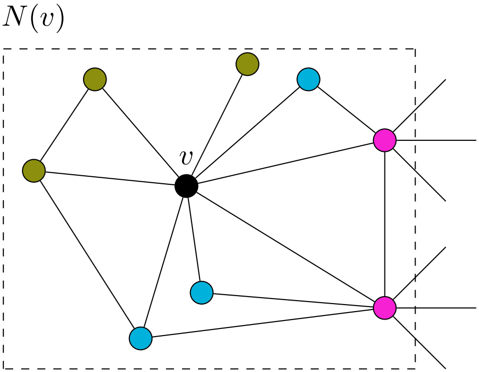
- for any vw-region $R \in \mathfrak{R}$: $D \cap V(R) = \{v, w\}$
- Regions are disjunct, but can share border vertices

A region is **maximal**, if no $R \in \mathfrak{R}$ such that $\mathfrak{R}' = \mathfrak{R} \cup \{R\}$ is a *D-region decomposition* with $V(\mathfrak{R}) \subsetneq V(\mathfrak{R}')$.

Maximal D-region decomposition



Splitting up $N(v)$



Splitting up $N(v)$

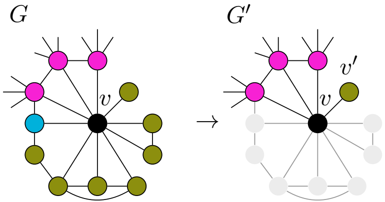
We split $N(v)$ into three subsets:

$$N_1(v) = \{u \in N(v) : N(u) \setminus N[v] \neq \emptyset\} \tag{1}$$
$$N_2(v) = \{u \in N(v) \setminus N_1(v) : N(u) \cap N_1(v) \neq \emptyset\} \tag{2}$$
$$N_3(v) = N(v) \setminus (N_1(v) \cup N_2(v)) \tag{3}$$

For $i, j \in [1, 3]$, we denote $N_{i,j}(v) := N_i(v) \cup N_j(v)$.

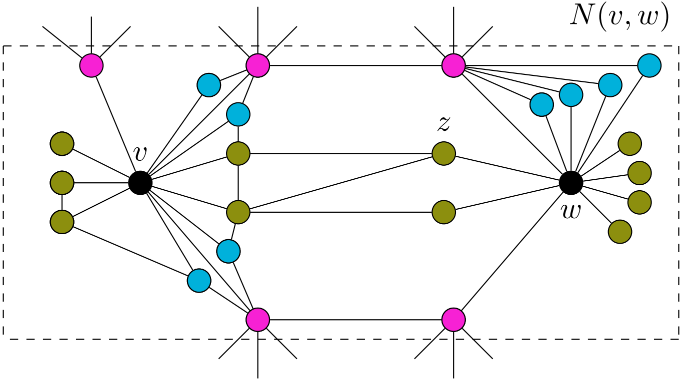
Rule 1, Appetizer: Shrinking $N_3(v)$

Let $G = (V, E)$ be a graph and let $v \in V$. If $|N_3(v)| \geq 1$:
- remove $N_{2,3}(v)$ from G,
- add a vertex v' and an edge $\{v, v'\}$.



- **Idea:** v better choice than $N_{2,3}$

Splitting up $N(v, w)$



Splitting up $N(v, w)$

$$N_1(v,w) = \{u \in N(v,w) \mid N(u) \setminus (N(v,w) \cup \{v,w\}) \neq \emptyset\} \tag{4}$$
$$N_2(v,w) = \{u \in N(v,w) \setminus N_1(v,w) \mid N(u) \cap N_1(v,w) \neq \emptyset\} \tag{5}$$
$$N_3(v,w) = N(v,w) \setminus (N_1(v,w) \cup N_2(v,w)) \tag{6}$$

For $i, j \in [1, 3]$, we denote $N_{i,j}(v,w) = N_i(v,w) \cup N_j(v,w)$.

Rule 2: Setting Up Our Weapons

Key Idea: $N_{2,3}(v, w)$ can **always** be semitotally dominated with 4 vertices.

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Key Idea: $N_{2,3}(v,w)$ can **always** be semitotally dominated with 4 vertices.

$$\mathcal{D} = \{\tilde{D} \subseteq N_{2,3}(v,w) \mid N_3(v,w) \subseteq \bigcup_{v \in \tilde{D}} N(v),\ |\tilde{D}| \leq 3\} \tag{7}$$

$$\mathcal{D}_v = \{\tilde{D} \subseteq N_{2,3}(v,w) \cup \{v\} \mid N_3(v,w) \subseteq \bigcup_{v \in \tilde{D}} N(v),\ |\tilde{D}| \leq 3,\ v \in \tilde{D}\} \tag{8}$$

$$\mathcal{D}_w = \{\tilde{D} \subseteq N_{2,3}(v,w) \cup \{w\} \mid N_3(v,w) \subseteq \bigcup_{v \in \tilde{D}} N(v),\ |\tilde{D}| \leq 3,\ w \in \tilde{D}\} \tag{9}$$

Rule 2

If $\mathcal{D} = \emptyset$ we apply the following:

Case 1: if $\mathcal{D}_v = \emptyset$ and $D_w = \emptyset$

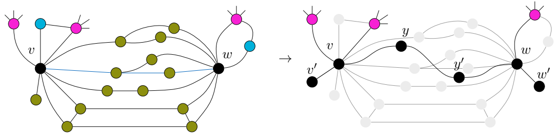
- Remove $N_{2,3}(v,w)$
- Add vertices v' and w' and two edges $\{v, v'\}$ and $\{w, w'\}$
- Preserve $d(v,w)$

Case 2: if $\mathcal{D}_v \neq \emptyset$ and $D_w = \emptyset$

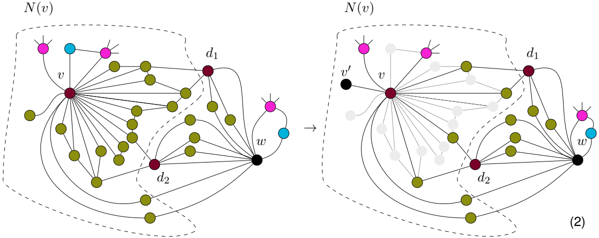
- Remove $N_{2,3}(v)$
- Add $\{v, v'\}$

Case 3: if $\mathcal{D}_v = \emptyset$ and $D_w \neq \emptyset$
Symmetric

Rule 2: Shrinking Regions



(1)



(2)

Rule 3: Shrinking the size of simple regions

Let $G = (V, E)$ be a plane graph, $v, w \in V$ and R be a simple region between v and w. If $|V(R) \setminus \{v, w\}| \geq 5$ apply the following:

Case 1: If $G[R \setminus \partial R] \cong P_3$, then:

- remove $V(R \setminus \partial R)$
- add vertex y with edges $\{v, y\}$ and $\{y, w\}$

Case 2: If $G[R \setminus \partial R] \ncong P_3$, then

- remove $V(R \setminus \partial R)$
- add vertices y, y' and four edges $\{v, y\}$, $\{v, y'\}$, $\{y, w\}$ and $\{y', w\}$

Simple Regions

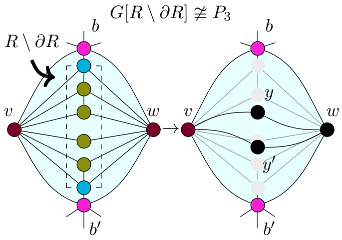
Rule 3: Shrinking the size of simple regions

Case 1: If $G[R \setminus \partial R] \cong P_3$, then:

- **remove $V(R \setminus \partial R)$**
- **add vertex y with edges $\{v, y\}$ and $\{y, w\}$**

Case 2: If $G[R \setminus \partial R] \ncong P_3$, then

- remove $V(R \setminus \partial R)$
- add vertices y, y' and four edges $\{v, y\}$, $\{v, y'\}$, $\{y, w\}$ and $\{y', w\}$



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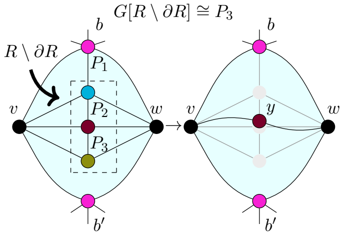
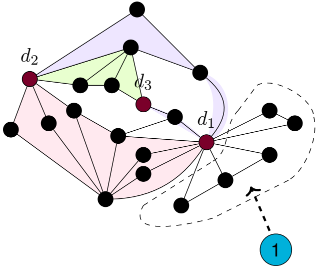
- **add vertices y, y' and four edges $\{v, y\}$, $\{v, y'\}$, $\{y, w\}$ and $\{y', w\}$**



Notes

- All the rule are sound
- and only change the solution size by a constant factor
- they can be applied in pplynomial-time
- Rule 3 is a swiss-army-knife to be found on many surprising places

Bounding the Kernel: Outside



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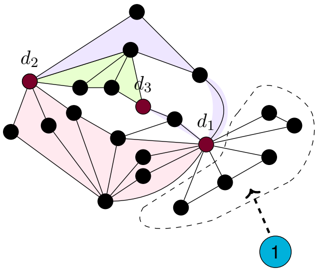
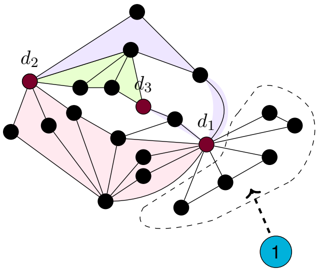


For each d in sds D:

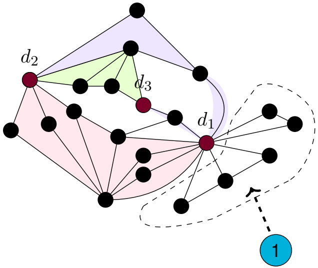
Bounding the Kernel: Outside



For each d in sds D:

1. $|N_1(v) \setminus V(\mathfrak{R})| = 0$ **[Alber2004]**, On Border

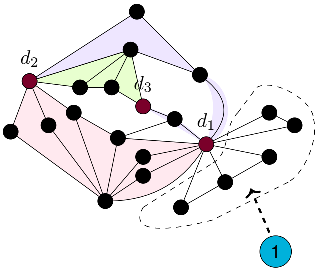
Bounding the Kernel: Outside



For each d in sds D:

1. $|N_1(v) \setminus V(\mathfrak{R})| = 0$ **[Alber2004]**, On Border
2. $|N_2(v) \setminus V(\mathfrak{R})| = 96$ **[Alber2004]**: TODO Reasoning

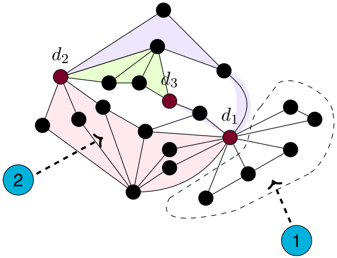
Bounding the Kernel: Outside



For each d in sds D:

1. $|N_1(v) \setminus V(\mathfrak{R})| = 0$ **[Alber2004]**, On Border
2. $|N_2(v) \setminus V(\mathfrak{R})| = 96$ **[Alber2004]**: TODO Reasoning
3. $|N_3(v) \setminus V(\mathfrak{R})| = 1$, by Rule 1

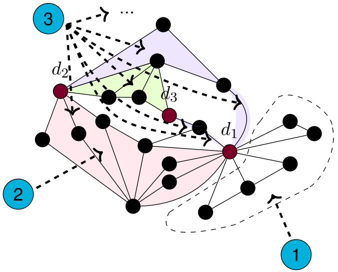
Bounding the Kernel: Idea 2



Bounding the Kernel: Number of Regions

Number of Regions [Alber2004]

Let G be a plane graph and let D be a with $|D| \geq 3$. There is a maximal D-*region decomposition* of G sucht that $|\mathfrak{R}| \leq 3 \cdot |D| - 6$.



Summary: Bounding Kernel Size

Let D be sds of size k. There exists a maximal D*-region decomposition* $\mathfrak{R}$ such that:

1. $\mathfrak{R}$ has only at most $3k - 6$ regions ([**Alber2004**]);
2. There are at most $97 \cdot k$ vertices outside of any region;
3. Each region $R \in \mathfrak{R}$ contains at most 87 vertices.

Hence: $|V| = \bigcup_{v \in D} N(v) = 87 \cdot (3k - 6) + 97 \cdot k + k < 359 \cdot k$

Main Theorem

All reduction rules can be applied in poly/time, hence:

The Main Theorem

The SEMITOTAL DOMINATING SET problem parameterized by solution size admits a linear kernel on planar graphs. There exists a polynomial-time algorithm that, given a planar graph (G, k), either correctly reports that (G, k) is a NO-instance or returns an equivalent instance (G', k) such that $|V(G')| \leq 359 \cdot k$.

Proof: Add Proof here.

Conclusions

Results:

-

Future Work:

- Improve Kernel Size
- Solve complexities for...

References I