



DEPARTMENT OF INFORMATICS

TECHNICAL UNIVERSITY MUNICH

Master Thesis

On the Parameterized Complexity of Semitotal Domination on Graph Classes

Lukas Retschmeier





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Über die Parametrisierte Komplexität des Problems der halbtotalen stabilen Menge auf Graphklassen

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Submission Date:	December 19, 2022



I confirm that this master thesis is my own work and I have documented all sources and material used.

København S
December 19, 2022

Lukas Retschmeier

Acknowledgments

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ABSTRACT

Abstract

For a graph $G = (V, E)$, a set D is called a *semitotal dominating set*, if D is a dominating set and every vertex $v \in D$ is within distance two to another witness $v' \in D$. The MINIMUM SEMITOTAL DOMINATING SET problem is to find a semitotal dominating set of minimum cardinality. The semitotal domination number $\gamma_{t2}(G)$ is the minimum cardinality of a semitotal dominating set and is squeezed between the domination number $\gamma(G)$ and the total domination number $\gamma_t(G)$. Given a graph $G = (V, E)$ and a positive integer k , the SEMITOTAL DOMINATION DECISION problem asks if G has a semitotal dominating set of size at most k .

After the problem was introduced by Goddard, Henning and McMillan in [15], NP-completeness was shown for general graphs [21], *split graphs* [21], *planar graphs* [21], *chordal bipartite graphs* [21], *circle graphs* [24] and *subcubic line graphs of bipartite graphs* [13]. On the other side, there exist polynomial-time algorithms for *AT-free graphs* [24], *graphs of bounded mim-width* [13], *graphs of bounded clique-width* [5], and *interval graphs* [21].

In this thesis, we start the systematic look through the lens of *parameterized complexity* by showing that SEMITOTAL DOMINATING SET is $\omega[2]$ -hard for bipartite graphs and split graphs. By applying the techniques proposed in [1] and [14] for DOMINATING SET and TOTAL DOMINATING SET, we are going to construct a $359k$ kernel for SEMITOTAL DOMINATING SET in planar graphs. This result further complements known linear kernels for other domination problems like PLANAR CONNECTED DOMINATING SET, PLANAR RED-BLUE DOMINATING SET, PLANAR EFFICIENT DOMINATING SET, PLANAR EDGE DOMINATING SET, INDEPENDENT DOMINATING SET and PLANAR DIRECTED DOMINATING SET.

Keywords: Domination; Semitotal Domination; parameterized Complexity; Planar Graphs; Linear Kernel

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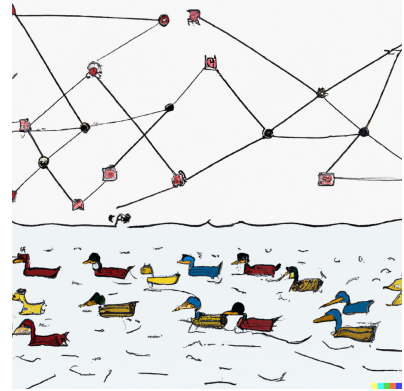
Abstract

Hier kommt noch ein weiterer Abstract rein.

Schlagworte: Stabile Menge; Halbtotale Stabile Menge; Parametrisierte Komplexität; Plättbare Graphen; Linearer Problemkern

CHAPTER 1

TERMINOLOGY AND PRELIMINARIES



“All we have to decide is what to do with the time that is given to us.”

J. R. R. Tolkien, *Gandalf* in *Lord of the Rings*

In this chapter, we will introduce the core definitions used throughout this thesis. Most of the definitions of graph theory are taken from [7]. For definitions in the area of *parameterized complexity*, the book written by Cygan et al. [6] gives an excellent introduction. For standard mathematical notation, the reader is referred to any introductory textbook into discrete mathematics (e.g. [27]).

1.1 Graph Theory

If not explicitly stated otherwise, the following definitions are taken from the book *Graph Theory* written by Reinhard Diestel [8].

1.1.1 Basic Terminology

Definition 1.1.1 (Graph). A simple graph is a pair $G = (V, E)$ of two sets where V denotes the vertices and $E \subseteq V \times V$ the edges of the graph. A vertex $v \in V$ is incident with an edge $e \in E$ if $v \in e$. Two vertices x, y are adjacent, or neighbours, if $\{x, y\} \in E$. By this definition, graph loops and multiple edges are excluded.

1.1 Graph Theory

A multigraph is a pair (V, E) of disjoint sets together with a map $E \rightarrow V \cup [V]^2$ assigning to every edge either one or two vertices, its ends. Multigraphs can have loops and multiple edges.

We usually denote the vertex set by $V(G)$ and its edge set by $E(G)$.

Unless stated otherwise, we usually consider only *simple graphs*, but the notion of *multigraphs* gets important when we later talk about the *underlying multigraph* of a *D-region decomposition*.

Definition 1.1.2 (Subgraph and Induced Subgraph). Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. If $V' \subseteq V$ and $E' \subseteq E$ then G' is a subgraph of G . If G is a subgraph of G' and G' contains all the edges to G with both endpoints in $V(G')$, then G' is an induced subgraph of G and we write $G' = G[V(G')]$.

Definition 1.1.3 (Degrees). Let $G = (V, E)$ be a graph. The degree $d_G(v)$ (shortly $d(v)$ if G is clear from the context) of a vertex $v \in V$ is the number of neighbors of v . We call a vertex of degree 0 as isolated and one of degree 1 as a pendant. If all the vertices of G have the same degree k , then G is k -regular.

Definition 1.1.4 (Closed and Open Neighborhoods [3]). Let $G = (V, E)$ be a (non-empty) graph. The set of all neighbors of v is the open neighborhood of v and denoted by $N(v)$; the set $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of v in G . When G needs to be made explicit, those open and closed neighborhoods are denoted by $N_G(v)$ and $N_G[v]$.

Definition 1.1.5 (isomorphic Graphs). Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. We call G and G' isomorphic, if there exists a bijection $\phi : V \rightarrow V'$ with $\{x, y\} \in E \Leftrightarrow \phi(x)\phi(y) \in E'$ for all $x, y \in V$. Such a map ϕ is called isomorphism.

If a graph G is isomorphic to another graph H , we denote $G \simeq H$.

Definition 1.1.6 (Paths and Cycles). A path is a non-empty graph $P = (V, E)$ of the form $V = \bigcup_{i \in [k]} \{x_i\}$ and $E = \bigcup_{i \in [k-1]} \{x_i x_{i+1}\}$ where the x_i are distinct. The vertices x_0 and x_k are linked by P and are called the ends of P . The length of a path is its number of edges and the path on n vertices is denoted by P_n . We refer to a path P by a natural sequence of its vertices: $P = x_0 x_1 \dots x_k$. Such a path P is a path between x_0 and x_k , or a x_0, x_k -path. If $P = x_0 \dots x_k$ is a path and $k \geq 2$, the graph with vertex set $V(P)$ and edge set $E(P) \cup \{x_k x_0\}$ is a cycle. The cycle on n vertices is denoted as C_n . The distance $d_G(v, w)$ from a vertex v to a vertex w in a graph G is the length of the shortest path between v and w . If v and w are not linked by any path in G , we set $d_G(v, w) = \infty$. Again, if G is clear from the context, we omit the subscripted G and just write $d(v, w)$ instead.

1.1.2 Graph Classes

A *graph class* is a set of graphs \mathfrak{G} that is closed under isomorphism that is if $G \in \mathfrak{G}$ and a $H \simeq G$ then $H \in \mathfrak{G}$ as well.

1 Terminology and Preliminaries

Definition 1.1.7 (Graph Parameters). Let $G = (V, E)$ be a graph. An independent set of G is a set of pairwise non-adjacent vertices. A clique of G is a set of pairwise adjacent vertices. A vertex cover of G is a subset of vertices containing at least one endpoint of every edge. A dominating set is a subset D of vertices such that all vertices not contained in D are adjacent to some vertex in D .

Graph Class 1 (r-partite). Let $r \geq 2$ be an integer. A Graph $G = (V, E)$ is called r-partite if V admits a partition into r classes such that every edge has its ends in different classes: Vertices in the same partition class must not be adjacent. A 2-partite graph is called bipartite.

An r -partite graph in which every two vertices from different partition classes are adjacent is called complete. For the complete bipartite graph on bipartitions $X \uplus Y$ of size m and n , we shortly write $K_{m,n}$.

Graph Class 2 (Complete). If all vertices of a graph $G = (V, E)$ are pairwise adjacent, we say that G is complete. A complete graph on n vertices is a K_n . A K_3 is called a triangle.

Graph Class 3 (Chordal). For a graph $G = (V, E)$, an edge that joins two vertices of a cycle, but is not itself an edge of the cycle is a chord of that cycle.

Furthermore, we say G is chordal (or triangulated) if each of its cycles of length at least four has a chord. In other words, it contains no induced cycle other than triangles.

Graph Class 4 (Split). A split graph is a graph $G = (V, E)$ whose vertices can be partitioned into a clique and an independent set.

Graph Class 5 (Planar). A plane graph is a pair (V, E) of finite sets with the following properties:

- $V \subseteq \mathbb{R}^2$ (Vertices),
- Every edge is an arc between two vertices,
- different edges have different sets of endpoints, and
- The interior of an edge contains no vertex and no point of any other edge

An embedding in the plane, or planar embedding, of an (abstract) graph G is an isomorphism between G and a plane graph H . A plane graph can be seen as a concrete **embedding** of the planar graph into the “plane” \mathbb{R}^2 .

1.2 Computational Complexity Theory

Computational complexity investigates the question of how many computational resources are required to solve a specific problem. We are about to introduce two of the most important classes of problems in classical complexity theory:

The Class P [2]

If we denote **DTIME** as the set of decision problems that are solvable in $\mathcal{O}(n^k)$ time by a deterministic Turing Machine, we can define the class **P** as:

$$\mathbf{P} := \bigcup_{k \in \mathbb{N}} (\text{DTIME}(n^k))$$

The Class NP [2]

A language $L \subseteq \{0,1\}^*$ is in **P** if there exists a polynomial $p : \mathbb{N} \rightarrow \mathbb{N}$ and a polynomial-time Turing Machine M such that for every $x \in \{0,1\}^*$,

$$x \in L \Leftrightarrow \exists u \in \{0,1\}^{p(|x|)} \text{ s.t. } M(x, u) = 1$$

If $x \in L$ and $u \in \{0,1\}^{p(|x|)}$ satisfy $M(x, u) = 1$, then we call u a *certificate* for x .

P denotes the class of all problems that are *efficiently solvable* whereas **NP** contains all problems whose solution can efficiently be verified. Note that $\mathbf{P} \subseteq \mathbf{NP}$, but the opposite is unknown.

1.2.1 NP-Completeness

A major discovery in the early 1970s was the fact that some problems in **NP** are *at least as hard as* any other problem in **NP** by reducing them among each other. This spans a whole “web of reductions” [2] and gives strong evidence that none of these problems can be solved efficiently. The first results in this new field had been published independently by Cook [4] and Levin [25] after Karp [23] had introduced this notion of problem reductions. The Cook-Levin-Theorem [4] proved that the **BOOLEAN SATISFIABILITY** (SAT) problem is **NP-COMPLETE** any problem in **NP** can be reduced to SAT.

as hard as SAT, because a fast algorithm for P would immediately give a fast algorithm for SAT as well. one single algorithm for any of these problems would be enough to efficiently solve all of them. For a comprehensive introduction to classical complexity theory, the reader is referred to [2].

Definition 1.2.1 (Reductions, NP-hardness and NP-COMPLETENESS [2]). We say that a language $A \subseteq \{0,1\}^*$ is polynomial-time Karp reducible to a language $B \subseteq \{0,1\}^*$ (denote $A \leq_p B$) if there is a poly-time computable function $f : \{0,1\}^* \rightarrow \{0,1\}^*$ such that for every $x \in \{0,1\}^*$, $x \in A$ if and only if $f(x) \in B$.

We say that a problem B is **NP-HARD** if $A \leq_p B$ for every $A \in \mathbf{NP}$ and B is **NP-COMPLETE** if additionally $B \in \mathbf{NP}$ holds.

1 Terminology and Preliminaries

There are thousands of **NP-COMPLETE** problems we do not expect to be solvable in polynomial time. The famous question of whether $\mathbf{P} = \mathbf{NP}$ or not is still one of the biggest open questions in mathematics bountied with one million dollars by the *Clay Mathematical Institute* [12]. Most of the domination problems like **DOMINATING SET**, **SEMITOTAL DOMINATING SET**, **TOTAL DOMINATING SET** are **NP-COMPLETE**.

Coping with NP-Completeness Even though we do not expect **NP-COMPLETE** problems to have a polynomial-time algorithm, there are some strategies to cope with them. We can either give up the exactness of a solution to possibly find fast *approximation algorithms* or abandon the search for a polynomial-time algorithm in favor of finding good *Exact Exponential (EEA) Algorithms* instead.

A third technique is using additional structural parameters of a specific problem instance and therefore **restricting the input to special cases**. This idea lead to the development of *parameterized complexity*.

1.2.2 Definitions in Parameterized Complexity

Introduced by Downey and Fellows [9], parameterized complexity extends the classical theory with a framework that allows a more finely-grained analysis of computationally hard problems. The idea is to measure a problem in terms of input size and an additional (structural) parameter k .

We like to find an algorithm that is only exponential in a function $f(k)$, but polynomial in the instance size. k denotes how difficult the problem is:

If k is small then the problem can still be considered tractable although the underlying **NP-HARD** problem counts as intractable in general. Therefore k can be seen as a measure of the difficulty of a given instance. If not marked otherwise, all definitions are taken from [6].

Definition 1.2.2 (Parameterized Problem). A parameterized problem is a $L \subseteq \Sigma^* \times \mathbb{N}$ (Σ finite fixed alphabet) for an instance $(x, k) \in \Sigma^* \times \mathbb{N}$, where k is called the parameter.

The size of an instance of an instance (x, k) of a parameterized problem is $|(x, k)| = |x| + k$ where the parameter k is encoded in unary by convention.

1.2.3 Fixed-Parameter Tractability

We say that a problem is *fixed-parameter tractable (fpt)* if problem instances of size n can be solved in $f(k)n^{\mathcal{O}(1)}$ time for some function f independent of n . Like the class \mathbf{P} can be seen as a notion of *tractability* in classical complexity theory, there is an equivalent in parameterized complexity, which we denote as **FIXED-PARAMETER TRACTABLE (FPT)** and which we can define the following way:

The Class FPT

A parameterized problem $L \subseteq \Sigma^* \times \mathbb{N}$ is called *fixed-parameter tractable* if there exists an algorithm A (called a *fixed-parameter algorithm*), a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ and a constant c such that, given $(x, k) \in \Sigma^* \times \mathbb{N}$, the algorithm A correctly decides whether $(x, k) \in L$ in time bounded by $f(k) \cdot |x|^c$. The complexity class containing all fixed-parameter tractable problems is called **FPT**.

1.2.4 Kernelization

A kernelization algorithm is a natural and intuitive way to approach problems and can be seen as a preprocessing procedure that simplifies parts of an instance already before the actual solving algorithm is run. A visualization of this idea can be seen in Figure 1.2. One can introduce *reduction rules* that iteratively reduce the instance until we are left with a small kernel.

Definition 1.2.3 (Kernelization and Reduction Rules). A kernelization algorithm or kernel is an algorithm \mathfrak{A} for a parameterized problem Q that given an instance (I, k) of Q runs in polynomial time and returns an equivalent instance (I', k') of Q . Moreover, we require that $\text{size}_{\mathfrak{A}}(k) \leq g(k)$ for some computable function $g : \mathbb{N} \rightarrow \mathbb{N}$.

A *reduction rule* is a function $\phi : \Sigma^* \times \mathbb{N} \rightarrow \Sigma^* \times \mathbb{N}$ that maps an instance (x, k) to an equivalent instance (x', k') such that ϕ is computable in time polynomial in $|x|$ and k .

A reduction rule is *sound* (or *safe*) if $(I, k) \in Q \Leftrightarrow (I', k') \in Q$.

We can give a precise definition of the size of the kernel, after a preprocessing algorithm \mathfrak{A} has been executed. $\text{size}_{\mathfrak{A}}$ denotes the largest size of any instance I after \mathfrak{A} has been applied. We consider the size to be infinite if it cannot be bounded by a function in k .

Definition 1.2.4 (Output size of a Preprocessing Algorithm). The output size of a preprocessing algorithms \mathfrak{A} is defined as

$$\text{size}_{\mathfrak{A}}(k) = \sup\{|I'| + k' : (I', k') = \mathfrak{A}(I, k), I \in \Sigma^*\}$$

If we bound $\text{size}_{\mathfrak{A}}$ by a polynomial in k , we say that the problem admits a **polynomial kernel**. Analogous, if the size after the reduction is only linear k , we refer to it as a **linear kernel**.

The following Lemma 1.2.1 shows the relation between the complexity class **FPT** and a kernelization algorithm. If we find a kernelization algorithm \mathfrak{A} for a (decidable) problem P , we immediately obtain an fpt algorithm by first running the \mathfrak{A} on an instance I of P in polynomial time. Assuming that P can be solved by an algorithm \mathfrak{M} running in time $g(n)$ we can use the fact that the kernel is bounded by a function $f(k)$ and apply

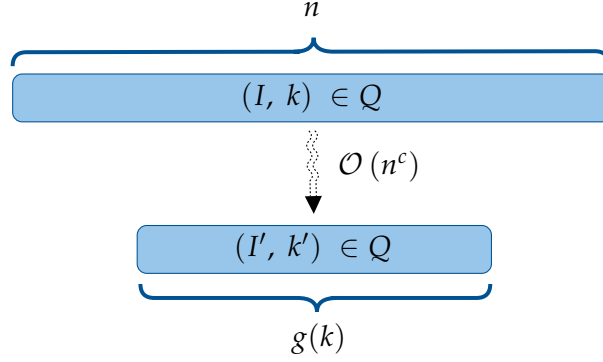


Figure 1.2: Kernelization: Reducing an instance (I, k) of size n to a smaller instance (I', k') in polynomial time. The size of the kernel is a function $g(k)$ only dependent on k .

\mathfrak{M} on the kernel resulting in a total running time of the order $O(g(f(k)) \cdot \text{poly}(n))$ which is fpt. Surprisingly, also the converse is true:

Lemma 1.2.1. *If a parameterized problem Q is **FPT** if and only if it admits a kernelization algorithm.*

In ?? we will use this and by explicitly constructing a kernel for PLANAR SEMITOTAL DOMINATING SET, we show membership of the problem in **FPT**.

1.2.5 Reductions and Parameterized Intractability

It is natural to ask whether all (hard) problems are also fixed-parameter tractable. The answer is no and parameterized complexity has another tool in its toolbox that can be used to show that a problem is unlikely to be in **FPT**. The idea is to transfer the concepts of **NP**-hardness from Section 1.2.1 and reductions from the classical setting to the parameterized world. This raises the need for a new type of reduction that ensures that a reduced instance (I', k') is not only created in fpt time, but the new parameter k' depends only on the size of the parameter in the original instance.

There exists a whole hierarchy of classes $\mathbf{FPT} \subseteq \mathbf{W}[1] \subseteq \mathbf{W}[2] \subseteq \dots \subseteq \mathbf{W}[t] \subseteq \dots$, which is known as the **W**-hierarchy. It is strongly believed that $\mathbf{FPT} \subsetneq \mathbf{W}[t]$ and therefore, we do not expect the existence of an algorithm solving any $\mathbf{W}[t]$ -hard problem in fpt time.

Definition 1.2.5 (Parameterized Reduction). *Let $A, B \subseteq \Sigma^* \times \mathbb{N}$ two parameterized problems. A parameter preserving reduction from A to B is an algorithm that, given an instance (x, k) of A , outputs an instance (x', k') of B such that:*

- (x, k) is a *yes* instance of A *iff* (x', k') is a *yes* instance of B ,
- $k' \leq g(k)$ for some computable function g , and

- runs in *fpt-time* $f(k) \cdot |x|^{\mathcal{O}(1)}$ for some computable function f .

As shown in Lemmas 1.2.2 and 1.2.3 [6] this definition ensures that reductions are transitive and closed under *fpt* reductions.

Lemma 1.2.2 (Closed under *fpt*-reductions). *If there is a parameterized reduction from A to B and $B \in \mathbf{FPT}$, then $A \in \mathbf{FPT}$, too.*

Lemma 1.2.3 (Transitivity). *If there are parameterized reductions from A to B and from B to C , then there is a parameterized reduction from A to C .*

If there exists a parameterized reduction transforming a $\mathbf{W}[t]$ -hard problem A to another problem B , then B is $\mathbf{W}[t]$ -hard as well. We can define the classes $\mathbf{W}[1]$ and $\mathbf{W}[2]$, by giving two problems that are complete for these classes.

The Classes $\mathbf{W}[1]$ [10] and $\mathbf{W}[2]$ [6]

INDEPENDENT SET is $\mathbf{W}[1]$ -complete.

DOMINATING SET is $\mathbf{W}[2]$ -complete.

A problem P is in the class $\mathbf{W}[1]$ (resp. $\mathbf{W}[2]$) if there is a parameterized reduction from P to INDEPENDENT SET (DOMINATING SET).

We have omitted a more precise definition via the WEIGHTED BOOLEAN SATISFIABILITY problem as it is not important for our work. We refer the interested reader to [6, 11] for more details.

CHAPTER 2

ON PARAMETERIZED SEMITOTAL DOMINATION



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In connection with various chessboard problems, the concept of domination can be traced back to the mid-1800s. For example, De Jaenosch attempted in 1862 to solve the minimum number of queens required to fully cover an $n \times n$ -chessboard [22]. Because of the immense amount of publications related to domination, Haynes, Hedetniemi, and Slater started a comprehensive survey of the literature in 1998 [16, 17]. 20 years later, by a series of three more books, Haynes, Henning and Hedetniemi complemented the survey with the latest developments [18, 19, 20].

We are now introducing the problems of DOMINATING SET, SEMITOTAL DOMINATING SET and TOTAL DOMINATING SET and dedicate the rest of the chapter to giving a current status about the complexity status of various graph classes.

2.1 Definition of Domination Problems

We are now going to define the three most important domination problems for this work: DOMINATING SET, SEMITOTAL DOMINATING SET and TOTAL DOMINATING SET. For a specific dominating set D , we say that a vertex d is a *dominating vertex* or *dominator* if $d \in D$. We say that d *dominates* all of its neighbors.

2.1 Definition of Domination Problems

The DOMINATING SET problem asks for a subset D of size at most k of vertices whose set of neighbors covers all the other vertices. In other words: every vertex $v \notin D$ needs to have at least one neighbor in D .

DOMINATING SET [6, p. 586]

Input	Graph $G = (V, E)$, $k \in \mathbb{N}$
Question	Is there a set $X \subseteq V$ of size at most k such that $N[X] = V$?

The TOTAL DOMINATING SET problem adds one additional constraint: Every vertex $v \in D$ in the dominating set must also be dominated by some vertex $v' \in D$ which we call the *witness* of v .

TOTAL DOMINATING SET [6, p. 596]

Input	Graph $G = (V, E)$, $k \in \mathbb{N}$
Question	Does there exist a set $X \subseteq V$ with $ X \leq k$ vertices such that for every $u \in V(G)$ there exists $v \in X$ with $\{u, v\} \in E$

Finally, SEMITOTAL DOMINATION was introduced by Goddard, Henning and McPillan [15] as a relaxation of TOTAL DOMINATION. Assume that we have an arbitrary dominating set D for some (connected) graph $G = (V, E)$ that has size at most two. It is easy to observe that for each $v \in D$ there must be at least one other dominating vertex $v' \in D$ that is at most three steps away because otherwise, this would not be a dominating set. By definition, there is always another neighboring dominating vertex for a total dominating set D . Therefore, it is natural to ask what happens, if we restrict the distance to at most two, which leads us straight to the idea of SEMITOTAL DOMINATION.

In this context, we say that v witnesses v' if $v, v' \in D$ and $d(v, v') \leq 2$ for a semitotal dominating set D .

2 On Parameterized Semitotal Domination

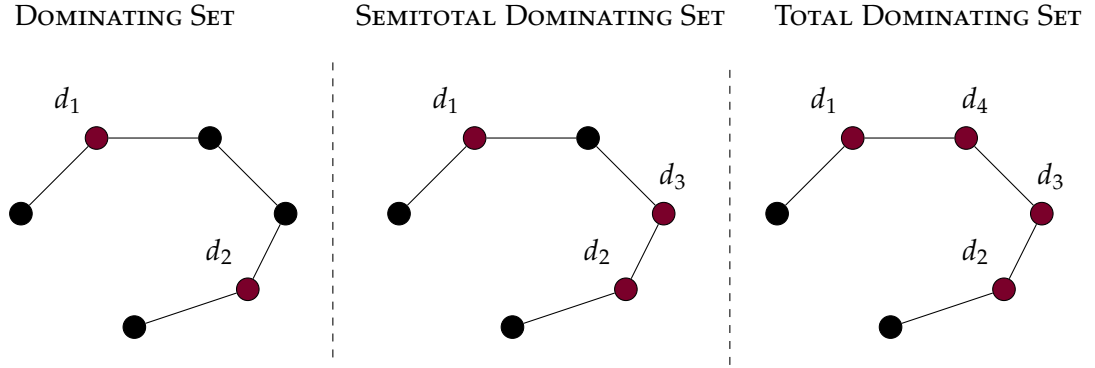


Figure 2.2: An example for a dominating set, semitotal dominating set and a total dominating set, where $\gamma(G) < \gamma_{t2}(G) < \gamma_t(G)$ are strict. In the first case, only two vertices suffice to dominate all others. In the second one, we need a witness between d_1 and d_2 that is at most distance two. In the last case, d_1 and d_2 both need a neighbor in the total dominating set.

SEMITOTAL DOMINATING SET [15]

Input

Graph $G = (V, E)$, $k \in \mathbb{N}$

Question

Is there a subset $X \subseteq V$ with $|X| \leq k$ such that $N[X] = V$ and for all $d_1 \in X$ there exists another $d_2 \in X$ such that $d(d_1, d_2) \leq 2$?

?? shows an examples where the on

Definition 2.1.1 (Domination Parameters). The domination number in a graph G is the minimum cardinality of a dominating set (ds) of G , denoted as $\gamma(G)$. The total domination number is the minimum cardinality of a total dominating set (tds) of G , denoted by $\gamma_t(G)$. The semitotal domination number is the minimum cardinality of a semitotal dominating set (sds) of G , denoted by $\gamma_{t2}(G)$.

We say that a ds D is minimal if no proper subset $S' \subset S$ is a ds and that D is a minimum if it is the smallest ds.

Since every total dominating set is also a semitotal dominating set and every semitotal dominating set is also a dominating set, we have the following fact first observed by Goddard and Henning [15].

Fact 2.1.1. For every graph G with no isolated vertex, $\gamma(G) \leq \gamma_{t2}(G) \leq \gamma_t(G)$

2.2 Complexity Status of SEMITOTAL DOMINATING SET

We can see that the semitotal domination number γ_{t2} is squeezed between the *domination* number and the *total domination* number. It turns out that for some graphs, all of these inequalities can be strict. See Figure 2.2 for an example, where $\gamma(G) < \gamma_{t2} < \gamma_t(G)$.

2.1.1 Preliminaries

* Witness * u pendant of from a vertex c if $N(u) = \{w\}$ * domination

Let D be a dominating set of G and $w \in V(G) \setminus D$. For any neighbor $v \in D \cap N(w)$, we say that d_1 *dominates* w For two dominating vertices d_1, d_2 in D . If

Definition, dominating number

2.2 Complexity Status of Semitotal Dominating Set

2.3 $w[i]$ -Intractibility

Now some $w[i]$ hard classes.

2.3.1 Warm-Up: $W[2]$ -hard on General Graphs

As any bipartite graph with bipartition can be split further into r -partite graph this results also implies the $w[1]$ -hardness of r -partite graphs

2.3.2 $W[2]$ -hard on Bipartite Graphs

We are showing that SEMITOTAL DOMINATING SET is $\omega[2]$ -hard on bipartite graphs by a parameterized reduction from DOMINATING SET on bipartite graphs which is known to be $\omega 2$ -hard ([26, Theorem 1]).

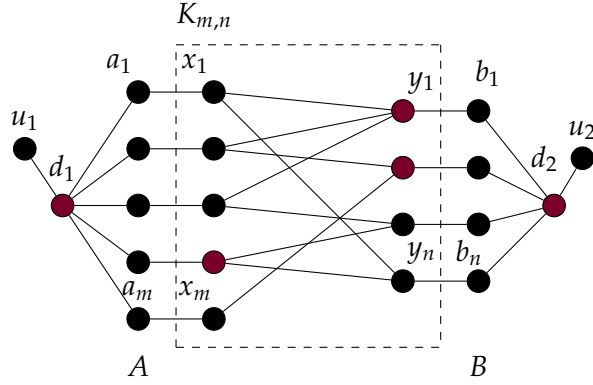


Figure 2.3: Constructing a bipartite G' from the bipartite graph $K_{m,n}$ by duplicating all vertices and adding exactly two forced witnesses.

Theorem 1. SEMITOTAL DOMINATING SET is $\omega[2]$ hard for bipartite Graphs

Proof. Given a bipartite Graph $G = (\{X \cup Y\}, E)$, we construct a bipartite Graph $G' = (\{X' \cup Y'\}, E')$ in the following way:

1. For each vertex $x_i \in X$, we add a new vertex $a_i \in A$ and an edge $\{x_i, a_i\}$ in between.
2. For each vertex $y_j \in Y$, we add a new vertex $b_j \in B$ and an edge $\{y_j, b_j\}$ in between.
3. We add four vertices with edges $\{u_1, d_1\}$ and $\{u_2, d_2\}$, and connect them with all $\{d_1, a_i\}$ and $\{d_2, b_j\}$ ($i \in [m]$ and $j \in [n]$) respectively.

Observation: The constructed G' is bipartite because A and B form an independent set on G' that can be cross-wise attached to one of the previous vertex sets. Setting $X' = X \cup \{u_2, d_1\} \cup B$ and $Y' = X \cup \{u_1, d_2\} \cup A$ form the partitions of the new bipartite G' .

Corollary 2.3.1. G has a dominating set of size k iff G has a semitotal dominating set of size $k' = k + 2$

\Rightarrow Assume a ds in G of size k . We know that $D' = D \cup \{d_1, d_2\}$ is an sds in G' of size $k' = k + 2$, because d_1 dominates u_1 and all $a_i \in A$; d_2 dominates u_2 and all

$b_i \in B$. The rest is dominated by the same vertices as they were in G , but now all of them have either d_1 or d_2 as a witness. More formally, we have by construction of G' that $\forall v \in (D \cap X) : d(v, d_1) = 2$ and $\forall v \in (D \cap Y) : d(v, d_2) = 2$.

\Leftarrow On the contrary, assume any sds D' in G' with size k' . Without loss of generality, we can assume that $u_1, u_2 \notin D'$, because choosing d_1 and d_2 instead is always at least as good and does not violate any witnesses. Therefore, the construction forces $d_1, d_2 \in D'$.

All $a_i \in A$ can only be useful to dominate their partnering x_i ($b_i \in B$ for y_i), because $d_1, d_2 \in D$ is the only second neighbor they have. If $a_i, b_i \in D'$ we replace it with x_i and y_i preserving the size D . As d_1 and d_2 suffice to provide a witness for every vertex in the graph and do not lose any other witnesses, this operation is sound.

In the end, $D = D' \setminus \{d_1, d_2\}$ gives us a ds in G with size $k = k' - 2$

As G' can be constructed in linear time and the parameter k is only blown up by a constant, this reduction is an FPT reduction. Because DOMINATING SET is already $w[2]$ -hard on bipartite graphs ([1]), we imply that SEMITOTAL DOMINATING SET is $w[2]$ -hard as well. \square

2.3.3 $W[2]$ -hard on Split Graphs

TODO Getting started with that.

2.3.4 $W[2]$ -hard on Chordal Graphs

Although the previous result implies $w[2]$ -hardness for chordal graphs, we found another reduction from DOMINATING SET on chordal graphs.

We will introduce the notion of an elimination ordering.

Definition 2.3.1 ([Rose1960]). In a graph $G = (V, E)$ with n vertices, a vertex is called **simplicial** if and only if the subgraph of G induced by the vertex set $\{v\} \cup N(v)$ is a complete graph.

G is said to have a **perfect elimination ordering** if and only if there is an ordering (v_1, \dots, v_n) of the vertices, such that each v_i is simplicial in the subgraph induced by the vertices v_1, \dots, v_i .

The following lemma shows that

Lemma 2.3.1 ([Rose1960]). A graph $G = (V, E)$ is chordal if and only if G has a perfect elimination ordering.

Theorem 2. SEMITOTAL DOMINATING SET restricted to chordal graphs is $\omega[2]$ -hard.

2 On Parameterized Semitotal Domination

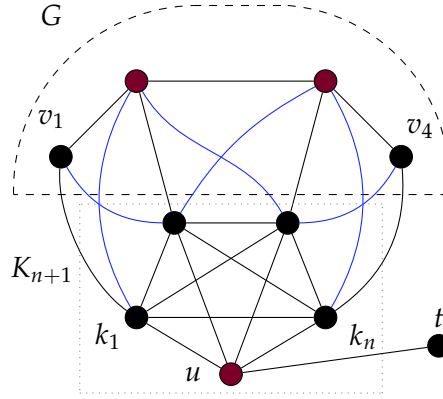


Figure 2.4: Constructing a chordal G' from the chordal graph P_4 by adding a K_5 , connecting its vertices pairwise to G . Adding the (blue) auxiliary vertices are necessary to preserve chordality.

Proof. We will give a reduction from DOMINATING SET on chordal graphs. Given $G = (V, E)$ with vertex set $V = \{v_1, \dots, v_n\}$, we construct a chordal graph G' as described below:

1. Add one complete graph K_{n+1} consisting of the vertices $\{k_1, \dots, k_n, u\}$ and an edge $\{v_i, k_i\}$ to each vertex $v_i \in V$ of G . One vertex of the complete subgraph is not connected to any $v \in V$. Denote it as u .
2. Add one additional vertex t and connect it with u via the edge $\{u, t\}$.
3. For all vertices $v_i \in V$ in G , add a new edge $\{n, k_i\}$ for all neighbors $n \in N(v_i)$.

An example reduction on the graph P_4 is shown in section 2.3.4.

Corollary 2.3.2. $N(v_i) \in G$ forms a clique iff $N(v_i)$ forms a clique in G'

Proof. Assuming that $N(v_i)$ forms a clique in G , we show that it also forms a clique in G' by induction over the number of neighbors $z = \text{abs}(N(v_i))$ in G .

- $z = 0$: Holds trivially as we do not have a neighbor in G and in G' the connected k_i forms a P_1 , hence a clique.
- $z = z + 1$:

By IH, we already know that all neighbors n_1, \dots, n_z form a clique together with their vertices in k_i . As $k_{z+1}, v_{z+1} \in N(v_i)$ now also in G' , we show that $N(v_i)$ still forms clique in G' .

Let k_i be the vertex that was connected with n_i during step 1. All we have to show is that v_{z+1} and k_{z+1} extend our previous clique, hence are fully connected with $N(v_i)$.

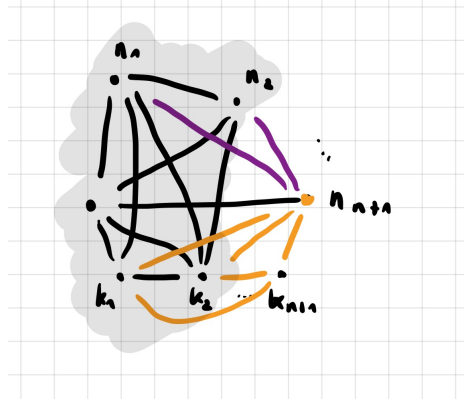


Figure 2.5: Induction Step

v_{z+1} connects to $N(v_i)$ in G by assumption. By our construction, there exists an edge to k_1, \dots, k_z , because we add an edge (n_{z+1}, k_i) if there is an edge from (n_{z+1}, n_i) . (See fig 2.5)

k_{z+1} form a complete subgraph with the other k_i and is connected to all n_i by construction because the edge (n_{z+1}, n_i) exists.

Therefore, $N(v_i)$ will also form a clique in G' .

On the other side, if $N(v_i)$ forms a clique in G' , the vertices of $N(v_i)$ in G form an induced subgraph of G' , hence preserving the clique. ■

Corollary 2.3.3. G is Chordal iff G' is chordal.

Proof. \Rightarrow : Assume G chordal. Then exists a total elimination order $o = (v_1, \dots, v_n)$ in G where removing v_j sequentially returns cliques in $N(v_i)$. Define $o' = (v_1, \dots, v_n, k_1, \dots, k_n, u, t)$. Applying corollary 2.3.2 states that (v_1, \dots, v_n) always gives cliques in G and according to corollary 2.3.2 also in G' . As the rest is directly part of a clique in G' by definition with an additional vertex of degree 1, o' is a total elimination order for G' , hence G' chordal. \Leftarrow : Holds as o' is always a total elimination order in G' and removing the complete subgraph K_{n+1} and u gives a total elimination order in G . ■

Corollary 2.3.4. G has a Dominating Set of size k iff G' has a sds of size $k + 1$

Proof. Assume a ds D of size k in G . $D \cup \{u\}$ is an sds in G' of size $k + 1$, because u dominates t and for each $v \in DS : d(v, u) \leq 2$.

Contrary, assume an sds SD in G' . To dominate t , $u \in SD$ must hold, hence already dominating the complete subgraph K_{n+1} . If a vertex $k_i \in SD$, we exchange it with v_i not losing the domination property. Taking $D = SD - \{u\}$ gives our desired ds of size k . ■

2 On Parameterized Semitotal Domination

As this reduction runs in FPT time and the parameter is only bounded by a function of k , this is an FPT reduction. As DOMINATING SET on Chordal Graphs is $w[2]$ – *hard*, so is SEMITOTAL DOMINATING SET on Chordal Graphs.

□

2.3 $w[i]$ -Intractibility

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