DEPARTMENT OF INFORMATICS

TECHNICAL UNIVERSITY MUNICH

Master Thesis

On the Parameterized Complexity of Semitotal Domination on Graph Classes

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Über die Parametrisierte Komplexität des Problems der halbtotalen stabilen Menge auf Graphklassen

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I confirm that this master thesis is my own work and I have documented all sources and material used.

København S Lukas Retschmeier
January 20, 2023

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CONTENTS

ABSTRACT

Abstract

For a graph $G = (V, E)$, a set D is called a *semitotal dominating set*, if D is a dominating set and every vertex $v \in D$ is within distance two to another witness $v' \in D$. The MINIMUM SEMITOTAL DOMINATING SET problem is to find a semitotal dominating set of minimum cardinality. The semitotal domination number $\gamma_{t2}(G)$ is the minimum cardinality of a semitotal dominating set and is squeezed between the domination number $\gamma(G)$ and the total domination number $\gamma_t(G)$. Given $G = (V, E)$ and a positive integer k, the SEMITOTAL DOMINATION DECISION problem asks if $\gamma_{t2} \leq k$.

After introduced by Goddard, Henning and McPillan [37], **NP**-completeness of the problem was shown for various graph classes like general graphs, *split*, *planar*, *chordal bipartite* and *circle* graphs [48, 53]. Contrary, there exist polynomial-time algorithms for *AT-free*, *block* and *interval* graphs as well as for *graphs of bounded mim-width*, *graphs of bounded clique-width* [18, 31, 47, 48, 53]. After giving a status about the complexity of the problem, we start a systematic look through the lens of *parameterized complexity* by showing that SEMITOTAL DOMINATING SET is W_2-hard for bipartite graphs and split graphs when parameterized by solution size. On the positive side, we extend a technique proposed by Alber et al. [2] for DOMINATING SET to construct a linear kernel of size $359k$ for SEMITOTAL DOMINATING SET on planar graphs by giving introducing reduction rules.

This result complements known linear kernels for other domination problems like CONNECTED DOMINATING SET, PLANAR RED-BLUE DOMINATING SET, EFFICIENT DOMINATING SET, EDGE DOMINATING SET, INDEPENDENT DOMINATING SET, and DIRECTED DOMINATING SET on planar graphs.

Keywords: Domination; Semitotal Domination; Parameterized Complexity; Planar Graphs; Linear Kernel; Problem Reduction; Graph Theory

Abstract

Ein Graph $G = (V, E)$ und eine Menge D wird als *halbtotale stabile Menge* bezeichnet, falls D eine stabile Menge ist und jeder Knoten $v \in D$ maximal einen Abstand von zwei zu einem anderen Zeugen $v' \in D$ besitzt.

Das MINIMUM SEMITOTAL DOMINATING SET Problem frägt nach einer halbtotalen stabilen Menge von minimaler Kardinalität. Sei $\gamma_{t2}(G)$ die minimale Kardinalität einer halbtotalen stabilen Menge. Diese ist zwischen der minimalen stabilen Menge $\gamma(G)$ and der minimalen total stabilen Menge $\gamma_t(G)$ eingezwängt. Gegeben $G = (V, E)$ und ein positives k, das SEMITOTAL DOMINATION ENTSCHEIDUNG s Problem frägt, ob $\gamma_{t2} \leq k$ ist.

Nachdem das Problem von Goddard, Henning und McPillan [37] eingeführt wurde, konnte **NP**-vollständigkeit für viele Graphklassen wie *allgemeine*, *split*, *plättbare*, *chordal bipartite* und *zirkuläre* Graphen bereits gezeigt werden [48, 53]. Andererseits, existieren polynomialzeit Algorithmen sowohl für *AT-freie*, *block* und *interval* Graphen, als auch für *Graphen mit beschränkter mim-width* und *Graphen mit beschränkter clique-width* [18, 31, 47, 48, 53].

Nach einer umfassenden Analyse zum Stand des Problems, beginnen wir eine systematische Analyse aus Sicht *parametrisierter Komplexität* und zeigen, dass SEMITOTAL DOMINATING SET W_2-hart für bipartite und split Graphen ist, wenn parameterisiert durch die Größe der Lösungsmenge. Basierend auf vorangegangener Arbeit [2, 34] war es uns möglich Reduktionsregeln anzugeben, die einen linearen Problemkern der Größe $359k$ für plättbare Graphen erzeugt. Dies vervollständigt existierende lineare Problemkerne ähnlicher Problem wie CONNECTED DOMINATING SET, PLANAR RED-BLUE DOMINATING SET, EFFICIENT DOMINATING SET, EDGE DOMINATING SET, INDEPENDENT DOMINATING SET oder DIRECTED DOMINATING SET.

Schlagworte: Stabile Menge; Halbtotale Stabile Menge; Parametrisierte Komplexität; Plättbare Graphen; Linearer Problemkern; Problemreduktion; Graph Theorie

CHAPTER 1

INTRODUCTION

We have seen [. . .], which is to say, all meaning comes from analogies.

Douglas Richard Hofstadter, *I am A Strange Loop*

Quack! Quack! They were careless for a second, and immediately the *dreaded geesiosi* clan took the opportunity to conquer the *Merganser Lake*, which belongs to your befriended ducks instantly! Now they are sitting on all the beautiful water lilies and refuse to give them back. The desperate ducks rely on your assistance! They have given you a map of the lake (see the left side of Figure 1.2) and marked all the water lilies in green. You instantly assured of helping and started to analyze their situation!

You see that the *geesiosi* members are terrified of the ducks' quacking, and you conclude that one single duck could free up an entire water lily by sitting there and driving away all the geese on neighboring plants! After thinking about this for a while, you realized that this might be the critical observation to regaining the lake! After some more deep contemplating, you came up with a good assignment of ducks to water lilies, where only a minimum number of ducks is required to liberate the whole territory again.

Happy with your first idea, you present it to the *Supreme Duck Decision Board*, but the *Chief Strategy Duck* shared her worries with you: "We have to hold the fort and protect the lake against another future rush of the *geesiosi*!", they said, "and it is a tedious task

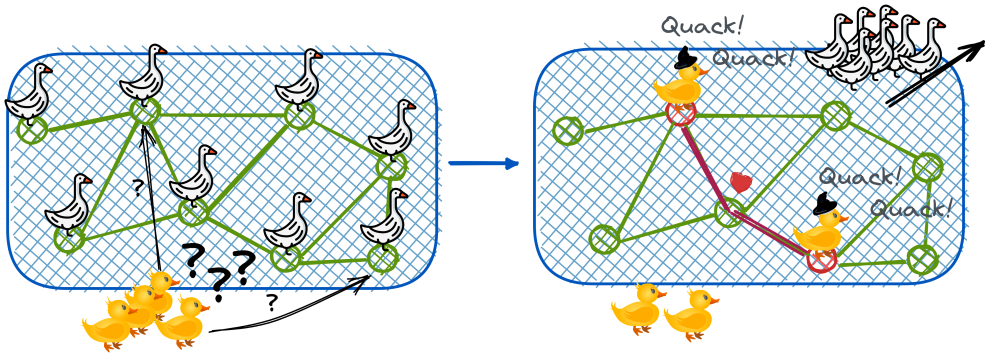


Figure 1.2: *Left: All water lilies are occupied by members of the geesiosi clan! The handwritten arrows have been your first solution proposal which was refused by the Supreme Duck Decision Board. Right: Your second and final solution: Two ducks are enough to make all geesiosis flee. Furthermore, they are only two water lilies apart (red line) and therefore have someone to quack together with!*

to sit alone on a water lily waiting the whole day! They would rather want to have another duck not too far away to have someone around to quack with together!"

After revising your solution, you came up with a new one where there is always another friend sitting at most two water lilies away. (see the right side of Figure 1.2). Now they should be close enough to dispel boredom, and your ducks were fully satisfied with your suggestion. Fantastic!

While you saw the chosen two ducks being sent out over the water's surface, you were still thinking about the problem. It looked so easy at first, but in the end, one had to try all the possible configurations (and of course, you did not tell the ducks that it was that simple because they think you are a wizard!). You wonder whether there is a way where you do not have to check all the possible configurations.

Back in your library, you learn from some ancient scrolls that this problem has already been formalized by a professor called Henning [48] as the Semitotal Dominating Set problem, which is a variant of the intensively studied Dominating Set problem. You read that both problems are **NP**-Complete [32, 48], and they are probably tough to solve in the general case efficiently, but there might still be hope if additional information is known. You look back into the map (Figure 1.2) and observe that none of the 'quacking'-relations do cross with each other, and you are getting curious if it can be used for something. . .

1.1 Content of the Thesis

Emerged during the last two decades, *parameterized complexity* is a modern branch of computer science that showed many practical implications. This thesis systematically analyzes the Semitotal Dominating Set problem through the lens of *parameterized complexity*.

- Chapter 2 will give the necessary definitions in the fields of *graph theory* and *parameterized complexity*.
- In Chapter 3, we will discuss the Semitotal Dominating Set problem and its relation to Dominating Set and Total Dominating Set in more detail. As they are closely related, we will gather the complexity status for various graph classes and compare them with each other in Section 3.2. We will then show $\mathbf{W_2}$-intractability for general, bipartite, split (and chordal) graphs.
- Chapter 4 is the mainstay of this thesis. We are going to construct a linear kernel for Planar Semitotal Dominating Set following an approach first suggested by Alber, Fellows and Niedermeier [2].
- In Chapter 5, we will give further ideas on how to improve the kernel and an outlook about interesting open problems for Semitotal Dominating Set in general.

Our contributions While many authors have already stated positive results - for example, there exist polynomial-time algorithms for *AT-free*, *block* and *interval* graphs as well as for *graphs of bounded mim-width* and *graphs of bounded clique-width* [18, 31, 47, 48, 53] - **NP**-completeness was shown for various graph classes like general graphs, *split*, *planar*, *chordal bipartite* and *circle* graphs [48, 53].

We will further investigate these **NP**-Complete cases by applying the framework of *parameterized complexity*. We could show W_2-intractability for *general*, *bipartite*, *chordal*, *split*, and *perfect graphs* using parameterized reductions from Dominating Set when parameterized by solution size.

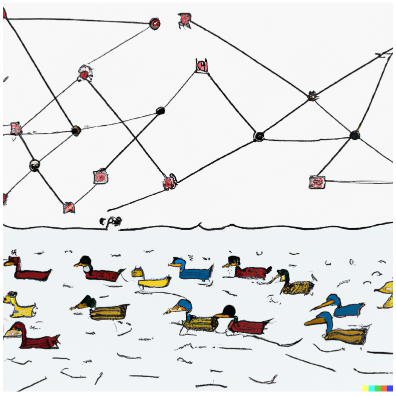
In a groundbreaking paper, Alber, Fellows, and Niedermeier [2] first gave a linear kernel for Planar Dominating Set. They discovered that a planar graph can be decomposed into a linear number of smaller regions. This motivated the introduction of local reduction rules that shrink the number of vertices in such a region to a constant size. Following up on this result, a plethora of other explicit linear kernels for domination problems on planar graphs were found [1, 35, 38, 62] and it made us believe we can also transfer them to Planar Semitotal Dominating Set. Our hunch turned out to be true, and by adjusting the reduction rules given by Garnero and Sau [34][1] for Planar Total Dominating Set, we were able to give an explicit kernel for Planar Semitotal Dominating Set of size $359k$. More precisely, we are going to prove the following theorem:

[1] We will rely on two different versions of this paper throughout the thesis. The *arXiv* versions are explicitly marked.

Theorem 1. *The* Semitotal Dominating Set *problem parameterized by solution size admits a linear kernel on planar graphs. There exists a polynomial-time algorithm that, given a planar graph (G, k), either correctly reports that (G, k) is a NO-instance or returns an equivalent instance (G', k) such that $|V(G')| \leq 359 \cdot k$.*

CHAPTER 2

TERMINOLOGY AND PRELIMINARIES



"All we have to decide is what to do with the time that is given to us."
J. R. R. Tolkien, *Gandalf* in *Lord of the Rings*

This chapter will introduce the core definitions used throughout this thesis. Most of the graph theory definitions are taken from [22]. For definitions in the area of *parameterized complexity*, the book by Cygan et al. [19] gives an excellent introduction. For standard mathematical notation, we refer the reader to any introductory textbook into discrete mathematics (e.g., [69]).

2.1 Graph Theory

If not explicitly stated otherwise, the following definitions were taken from the book *Graph Theory* written by Reinhard Diestel [23].

2.1.1 Basic Terminology

Definition 2.1.1 (Graph). *A simple graph is a pair $G = (V, E)$ of two sets where V denotes the vertices and $E \subseteq V \times V$ the edges of the graph. A vertex $v \in V$ is incident with an edge $e \in E$ if $v \in e$. Two vertices x, y are adjacent, or neighbours, if $\{x, y\} \in E$. By this definition, graph loops and multiple edges are excluded.*

A multigraph is a pair (V, E) of disjoint sets together with a map $E \to V \cup [V]^2$ assigning to every edge either one or two vertices, its ends. Multigraphs can have loops and multiple edges.

We usually denote the vertex set by $V(G)$ and its edge set by $E(G)$.

Unless stated otherwise, we usually consider only *simple graphs*, but the notion of *multigraphs* gets essential when we later talk about the *underlying multigraph* of a *D-region decomposition*.

Definition 2.1.2 (Subgraph and Induced Subgraph)**.** *Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. If $V' \subseteq V$ and $E' \subseteq E$ then G' is a subgraph of G. If G is a subgraph of G' and G' contains all the edges to G with both endpoints in $V(G')$, then G' is an induced subgraph of G, and we write $G' = G[V(G')]$.*

Definition 2.1.3 (Degrees)**.** *Let $G = (V, E)$ be a graph. The degree $d_G(v)$ (shortly $d(v)$ if G is clear from the context) of a vertex $v \in V$ is the number of neighbors of v. We call a vertex of degree 0 isolated, and one of degree 1 a pendant. If all the vertices of G have the same degree k, then g is k-regular.*

Definition 2.1.4 (Closed and Open Neighborhoods [4])**.** *Let $G = (V, E)$ be a (non-empty) graph. The set of all neighbors of v is the open neighborhood of v and denoted by $N(v)$; the set $N[v] = N(v) \cup \{v\}$ is the closed neighborhood f v in G. When G needs to be made explicit, those open and closed neighborhoods are denoted by $N_G(v)$ and $N_G[v]$.*

Definition 2.1.5 (Isomorphic Graphs)**.** *Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. We call G and G' isomorphic, if there exists a bijection $\phi : V \to V'$ with $\{x, y\} \in E \Leftrightarrow \phi(x)\phi(y) \in E'$ for all $x, y \in V$. Such a map ϕ is called isomorphism.*

If a graph G is isomorphic to another graph h, we denote $G \simeq H$.

Definition 2.1.6 (Paths and Cycles)**.** *A path is a non-empty graph $P = (V, E)$ of the form $V = \bigcup_{i \in [k]} \{x_i\}$ and $E = \bigcup_{i \in [k-1]} \{x_i x_{i+1}\}$ where all the x_i's are distinct. The vertices x_0 and x_k are linked by P and are called the ends of P. The length of a path is its number of edges, and the path on n vertices is denoted by P_n. We refer to a path P by a natural sequence of its vertices: $P = x_0 x_1 ... x_k$. Such a path P is a path between x_0 and x_k, or a x_0, x_k-path. If $P = x_0 ... x_k$ is a path and $k \geq 2$, the graph with vertex set $V(P)$ and edge set $E(P) \cup \{x_k x_0\}$ is a cycle. The cycle on n vertices is denoted as C_n. The distance $d_G(v, w)$ from a vertex v to a vertex w in a graph g is the length of the shortest path between v and w. If v and w are not linked by any path in G, we set $d_G(v, w) = \infty$. Again, if G is clear from the context, we omit the subscripted G and just write $d(v, w)$ instead.*

2.1.2 Graph Classes

A *graph class* is a set of graphs $\mathfrak{G}$ that is closed under isomorphism, that is, if $G \in \mathfrak{G}$ and a $H \simeq G$ then $H \in \mathfrak{G}$ as well.

Definition 2.1.7 (Graph Parameters)**.** *Let $G = (V, E)$ be a graph. An independent set of G is a set of pairwise non-adjacent vertices. A clique of G is a set of pairwise adjacent vertices. A vertex cover of G is a subset of vertices containing at least one endpoint of every edge. A dominating set is a subset D of vertices such that all vertices not contained in are adjacent to some vertex in D.*

Graph Class 1 (r-partite)**.** *Let $r \geq 2$ be an integer. A Graph $G = (V, E)$ is called r-partite if V admits a partition into r classes such that every edge has its ends in different classes: Vertices in the same partition class must not be adjacent. A 2-partite graph is called bipartite.*

An r-partite graph in which every two vertices from different partition classes are adjacent is called complete. For the complete bipartite graph on bipartitions $X \uplus Y$ of size m and n, we shortly write $K_{m,n}$.

Graph Class 2 (Complete)**.** *If all vertices of a graph $G = (V, E)$ are pairwise adjacent, we say that G is complete. A complete graph on n vertices is a K_n. A K_3 is called a triangle.*

Graph Class 3 (Chordal)**.** *For a graph $G = (V, E)$, an edge that joins two vertices of a cycle, but is not itself an edge of the cycle is a chord of that cycle.*

Furthermore, we say G is chordal (or triangulated) if each of its cycles of length at least four has a chord. In other words, it contains no induced cycle other than triangles.

Graph Class 4 (Split)**.** *A split graph is a graph $G = (V, E)$ whose vertices can be partitioned into a clique and an independent set.*

Graph Class 5 (Planar)**.** *A plane graph is a pair (V, E) of finite sets with the following properties:*

- *$V \subseteq \mathbb{R}^2$ (Vertices),*
- *Every edge is an arc between two vertices,*
- *different edges have different sets of endpoints, and*
- *The interior of an edge contains no vertex and no point of any other edge*

An embedding in the plane, or plane embedding, of an (abstract) graph G is an isomorphism between G and a plane graph H. A plane graph can is a concrete ***embedding*** *of the planar graph into the "plane" $\mathbb{R}^2$.*

2.2 Computational Complexity Theory

Computational complexity investigates how many computational resources are required to solve a specific problem. We are about to introduce two of the most important classes of problems in classical complexity theory:

The Classes [3]

If we denote **DTIME** as the set of decision problems that are solvable in $\mathcal{O}(n^k)$ time by a deterministic Turing Machine, we can define the class **P** as:

$$\mathrm{P} := \bigcup_{k \in \mathbb{N}} (\mathrm{DTIME}(n^k))$$

A language $L \subseteq \{0,1\}^*$ is in **NP** if there exists a polynomial $p : \mathbb{N} \to \mathbb{N}$ and a polynomial-time Turing Machine M such that for every $x \in \{0,1\}^*$,

$$x \in L \Leftrightarrow \exists u \in \{0,1\}^{p(|x|)} s.t.\ M(x,u) = 1$$

If $x \in L$ and $u \in \{0,1\}^{p(|x|)}$ satisfy $M(x,u) = 1$, then we call u a *certificate* for x.

P denotes the class of all problems that are *efficiently solvable*, whereas **NP** contains all problems whose solution can efficiently be verified. Note that **P** $\subseteq$ **NP**, but the opposite is unknown.

2.2.1 NP-Completeness

A significant discovery in the early 1970s was that some problems in **NP** are *at least as hard as* as any other problem in **NP** by reducing them among each other. This spans a whole "web of reductions" [3] and gives strong evidence that none of these problems can be solved efficiently. The first results in this new field had been published independently by Cook [17] and Levin [60] after Karp [51] had introduced this notion of problem reductions. The Cook-Levin-Theorem [17] proved that the Boolean Satisfiability (SAT) problem is **NP**-Complete any problem in NP can be reduced to SAT.

as hard as SAT, because a fast algorithm for P would immediately give a fast algorithm for SAT as well. one single algorithm for any of these problems would be enough to efficiently solve all of them. For a comprehensive introduction to classical complexity theory, the reader is referred to [3].

Definition 2.2.1 (Reductions, **NP**-hardness and **NP**-Completeness [3])**.** *We say that a language $A \subseteq \{0,1\}^*$ is polynomial-time Karp reducible to a language $B \subseteq \{0,1\}^*$ (denote $A \leq_p B$) if there is a poly-time computable function $f : \{0,1\}^* \to \{0,1\}^*$ such that for every $x \in \{0,1\}^*$, $x \in A$ if and only if $f(x) \in B$.*
*We say that a problem B is **NP**-hard if $A \leq_p B$ for every $A \in$ **NP** and B is **NP**-Complete if additionally $B \in NP$ holds.*

There are thousands of **NP**-Complete problems we do not expect to be solvable in polynomial time. The famous question of whether **P** = **NP** or not is still one of the biggest open questions in mathematics bountied with one million dollars by the *Clay*

Mathematical Institute [30]. Most of the domination problems like Dominating Set, Semitotal Dominating Set, Total Dominating Set are **NP**-Complete.

Coping with NP-Completeness Even though we do not expect **NP**-Complete problems to have a polynomial-time algorithm, there are some strategies to cope with them. We can either give up the exactness of a solution to possibly find fast *approximation algorithms* or abandon the search for a polynomial-time algorithm in favor of finding good *Exact Exponential (EEA) Algorithms* instead.

A third technique is using additional structural parameters of a specific problem instance and therefore **restricting the input to special cases**. This idea lead to the development of *parameterized complexity*.

2.2.2 Definitions in Parameterized Complexity

Introduced by Downey and Fellows [24], parameterized complexity extends the classical theory with a framework that allows a more finely-grained analysis of computationally hard problems. The idea is to measure a problem in terms of input size and an additional (structural) parameter k.

We like to find an algorithm that is only exponential in a function $f(k)$, but polynomial in the instance size. k denotes how difficult the problem is:

If k is small then the problem can still be considered tractable although the underlying **NP**-hard problem counts as intractable in general. Therefore k can be seen as a measure of the difficulty of a given instance. If not marked otherwise, all definitions are taken from [19].

Definition 2.2.2 (Parameterized Problem). *A parameterized problem is a $L \subseteq \Sigma^* \times \mathbb{N}$ (Σ finite fixed alphabet) for an instance $(x, k) \in \Sigma^* \times \mathbb{N}$, where k is called the parameter.*

The size of an instance of an instance (x, k) of a parameterized problem is $|(x, k)| = |x| + k$ where the parameter k is encoded in unary by convention.

2.2.3 Fixed-Parameter Tractability

We say that a problem is *fixed-parameter tractable (fpt)* if problem instances of size n can be solved in $f(k)n^{\mathcal{O}(1)}$ time for some function f independent of n. Like the class **P** can be seen as a notion of *tractability* in classical complexity theory, there is an equivalent in parameterized complexity, which we denote as Fixed-Parameter Tractable (**FPT**) and which we can define the following way:

The Class FPT

A parameterized problem $L \subseteq \Sigma^* \times \mathbb{N}$ is called *fixed-parameter tractable* if there exists an algorithm A (called a *fixed-parameter algorithm*), a computable function $f : \mathbb{N} \to \mathbb{N}$ and a constant c such that, given $(x, k) \in \Sigma^* \times \mathbb{N}$, the algorithm $\mathcal{A}$ correctly decides whether $(x, k) \in L$ in time bounded by $f(k) \cdot |(x, k)|^c$. The complexity class containing all fixed-parameter tractable problems is called **FPT**.

2.2.4 Kernelization

A kernelization algorithm is a natural and intuitive way to approach problems and can be seen as a preprocessing procedure that simplifies parts of an instance already before the actual solving algorithm is run. A visualization of this idea can be seen in Figure 2.2. One can introduce *reduction rules* that iteratively reduce the instance until we are left with a small kernel.

Definition 2.2.3 (Kernelization and Reduction Rules)**.** *A kernelization algorithm or kernel is an algorithm $\mathfrak{A}$ for a parameterized problem Q that given an instance (I, k) of Q runs in polynomial time and returns an equivalent instance (I', k') of Q. Moreover, we require that size$_{\mathfrak{A}}(k) \leq g(k)$ for some computable function $g : \mathbb{N} \to \mathbb{N}$.*

A reduction rule is a function $\phi : \Sigma^* \times \mathbb{N} \to \Sigma^* \times \mathbb{N}$ that maps an instance (x, k) to an equivalent instance (x', k') such that ϕ is computable in time polynomial in $|x|$ and k.

A reduction rule is sound (or safe) if $(I, k) \in Q \Leftrightarrow (I', k') \in Q$.

We can give a precise definition of the size of the kernel, after a preprocessing algorithm $\mathfrak{A}$ has been executed. *size*$_{\mathfrak{A}}$ denotes the largest size of any instance I after $\mathfrak{A}$ has been applied. We consider the size to be infinite if it cannot be bounded by a function in k.

Definition 2.2.4 (Output size of a Preprocessing Algorithm)**.** *The output size of a preprocessing algorithms $\mathfrak{A}$ is defined as*

$$\text{size}_{\mathfrak{A}}(k) = \sup\{|I'| + k' : (I', k') = \mathfrak{A}(I, k), I \in \Sigma^*\}$$

If we bound size$_{\mathfrak{A}}$ by a polynomial in k, we say that the problem admits a **polynomial kernel**. Analogous, if the size after the reduction is only linear k, we refer to it as a **linear** kernel.

The following Lemma 2.2.1 shows the relation between the complexity class **FPT** and a kernelization algorithm. If we find a kernelization algorithm $\mathfrak{A}$ for a (decidable) problem P, we immediately obtain an fpt algorithm by first running the $\mathfrak{A}$ on an instance I of P in polynomial time. Assuming that P can be solved by an algorithm $\mathfrak{M}$ running in time $g(n)$ we can use the fact that the kernel is bounded by a function $f(k)$ and apply

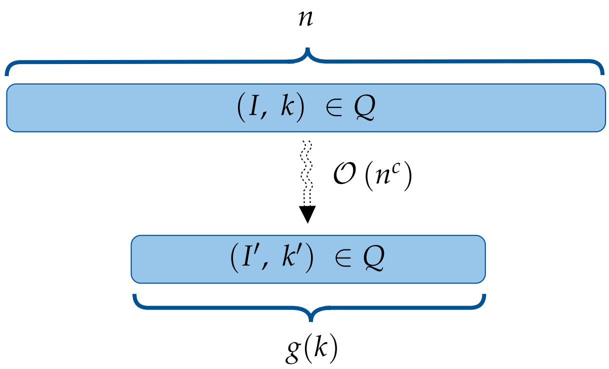


Figure 2.2: *Kernelization: Reducing an instance (I, k) of size n to a smaller instance (I', k') in polynomial time. The size of the kernel is a function $g(k)$ only dependent on k.*

$\mathfrak{M}$ on the kernel resulting in a total running time of the order $\mathcal{O}(g(f(k)) \cdot \text{poly}(n))$ which is fpt. Surprisingly, also the converse is true:

Lemma 2.2.1. *If a parameterized problem Q is* **FPT** *if and only if it admits a kernelization algorithm.*

In Chapter 4 we will use this and by explicitly constructing a kernel for Planar Semitotal Dominating Set, we show membership of the problem in **FPT**.

2.2.5 Reductions and Parameterized Intractability

It is natural to ask whether all (hard) problems are also fixed-parameter tractable. The answer is no and parameterized complexity has another tool in its toolbox that can be used to show that a problem is unlikely to be in **FPT**. The idea is to transfer the concepts of **NP**-hardness from Section 2.2.1 and reductions from the classical setting to the parameterized world. This raises the need for a new type of reduction that ensures that a reduced instance (I', k') is not only created in fpt time, but the new parameter k' depends only on the size of the parameter in the original instance.

There exists a whole hierarchy of classes $\mathbf{FPT} \subseteq \mathbf{W_1} \subseteq \mathbf{W_2} \subseteq \ldots \subseteq \mathbf{W_t} \subseteq \ldots$, which is known as the **W**-hierarchy. It is strongly believed that $\mathbf{FPT} \subsetneq \mathbf{W_t}$ and therefore, we do not expect the existence of an algorithm solving any $\mathbf{W_t}$-hard problem in fpt time.

Definition 2.2.5 (Parameterized Reduction)**.** *Let $A, B \subseteq \Sigma^* \times \mathbb{N}$ two parameterized problems. A parameter preserving reduction from A to B is an algorithm that, given an instance (x, k) of A, outputs an instance (x', k') of B such that:*

- *(x, k) is a yes instance of A* ***iff*** *(x', k') is a yes instance of B,*
- *$k' \leq g(k)$ for some computable function g, and*
- *runs in fpt-time $f(k) \cdot |x|^{\mathcal{O}(1)}$ for some computable function f.*

As shown in Lemmas 2.2.2 and 2.2.3 [19] this definition ensures that reductions are transitive and closed under fpt reductions.

Lemma 2.2.2 (Closed under fpt-reductions)**.** *If there is a parameterized reduction from A to B and $B \in \mathbf{FPT}$, then $A \in \mathbf{FPT}$, too.*

Lemma 2.2.3 (Transitivity)**.** *If there are parameterized reductions from A to B and from B to C, then there is a parameterized reduction from A to C.*

If there exists a parameterized reduction transforming a $\mathbf{W_t}$-hard problem A to another problem B, then B is $\mathbf{W_t}$-hard as well. We can define the classes $\mathbf{W_1}$ and $\mathbf{W_2}$, by giving two problems that are complete for these classes.

The Classes $\mathbf{W_1}$ [25] and $\mathbf{W_2}$ [19]

INDEPENDENT SET is $\mathbf{W_1}$-complete.

DOMINATING SET is $\mathbf{W_2}$-complete.

A problem P is in the class $\mathbf{W_1}$ (resp. $\mathbf{W_2}$) if there is a parameterized reduction from P to INDEPENDENT SET (DOMINATING SET).

We have omitted a more precise definition via the WEIGHTED BOOLEAN SATISFIABILIY problem as it is not important for our work. We refer the interested reader to [19, 28] for more details.

CHAPTER 3

ON PARAMETERIZED SEMITOTAL DOMINATION

"This set must be small, but also complete,
to conquer the graph with ease and fleet,
with vertices chosen so carefully,
the solution is found, so elegantly."

ChatGPT, Generated by OpenAI Language Model,
Knowledge Cutoff: 2021-09

In connection with various chessboard problems, the concept of domination can be traced back to the mid-1800s. For example, De Jaenosch attempted in 1862 to solve the minimum number of queens required to fully cover an $n \times n$-chessboard [49]. Because of the immense amount of publications related to domination that followed, Haynes, Hedetniemi, and Slater started a comprehensive survey of the literature in 1998 [42, 43]. 20 years later, by a series of three more books, Haynes, Henning and Hedetniemi complemented the survey with the latest developments [44, 45, 46].

We are now introducing the problems of Dominating Set, Semitotal Dominating Set and Total Dominating Set and dedicate the rest of the chapter to giving a current status about the complexity status of various graph classes.

3.1 Domination Problems

We are now going to define the three most important domination problems for this work: Dominating Set, Semitotal Dominating Set and Total Dominating Set. Recall that a dominating set of a graph $G = (V, E)$ is a subset $D \subseteq V$, such that every vertex $V \setminus D$ is adjacent to some vertex in D. We say that a vertex d is a *dominating vertex* or *dominator* if $d \in D$ and that d *dominates* all of its neighbors. In other words: every vertex $v \notin D$ needs to have at least one neighbor in D.

DOMINATING SET (DOM) [19, p. 586]

Input	Graph $G = (V, E)$, $k \in \mathbb{N}$
Question	Is there a set $X \subseteq V$ of size at most k such that $N[X] = V$?

The Dominating Set problem asks for a subset D of size at most k whose set of neighbors are adjacent to all the other remaining vertices in a graph. The following Fact 3.1.1 motivates the definition of the following variants of the problem.

Fact 3.1.1. *Let $G = (V, E)$ be a connected graph and D a ds with $|D| > 1$. For any $v_1 \in D$ there exists at least one v_2 with $d(v_1, v_2) \leq 3$.*

Proof. Assume a ds D and $v_1 \in D$ for which no other dominating vertex is in a distance less than three. Then exists v_2 with $d(v_1, v_2) > 3$. By connectivity there is a path $p = (v_1, p_1, p_2, ..., p_i, v_2)$ from v_1 to v_2 and no $p_i \in D$. For D to be a valid ds, $p_2, ...p_{i-1}$ have to be dominated by a neighbor as well and therefore, there must exist a $u \in N(p_2) \cap D$ with $d(v_1, u) \leq 2$ giving a contradiction. □

If we also demand the dominating vertices to be dominated, need the notion of Total Domination. The Total Dominating Set problem adds one additional constraint: Every vertex $v \in D$ in the dominating set must also be dominated by some vertex $v' \in D$ which we call the *witness* of v.

TOTAL DOMINATING SET (TDOM) [19, p. 596]

Input	Graph $G = (V, E)$, $k \in \mathbb{N}$
Question	Does there exists a set $X \subseteq V$ with $|X| \leq k$ vertices such that for every $u \in V(G)$ there exists $v \in X$ with $\{u, v\} \in E$?

Finally, Semitotal Domination was introduced by Goddard, Henning and McPillan [37] as a relaxation of Total Domination. Assume that we have an arbitrary dominating set D for some (connected) graph $G = (V, E)$ that has size at most two. It is easy to observe that for each $v \in D$ there must be at least one other dominating vertex $v' \in D$ that is at most three steps away because otherwise, this would not be a dominating set. On the other side, by definition of a total dominating set D, every dominating vertex $d \in D$ has another vertex in $d' \in N(d) \cap D$ that is also dominating. It is natural to ask what happens if we restrict this distance property to be at most two, which leads us straight to the idea of Semitotal Domination. For a semitotal dominating set D, we say that v witnesses v' if $v, v' \in D$ and $d(v, v') \leq 2$.

SEMITOTAL DOMINATING SET (SDOM) [37]

Input	Graph $G = (V, E)$, $k \in \mathbb{N}$
Question	Is there a subset $X \subseteq V$ with $\lvert X\rvert \leq k$ such that $N[X] = V$ and for all $d_1 \in X$ there exists another $d_2 \in X$ such that $d(d_1, d_2) \leq 2$?

Figure 3.2 demonstrates that the minimum ds, sds and tds can strictly differ in the size on a fixed graph $G = (V, E)$. a minimum ds (left) does not need any witnesses at all and we can choose d_1 and d_2 to dominate the graph. As $d(d_1, d_2) = 3$, we have to introduce additional dominating vertices for a minimum sds (middle) and tds (right). Their only purpose is, to bridge the gap between d_1 and d_2.

Definition 3.1.1 (Domination Numbers)**.** *The domination number in a graph G is the minimum cardinality of a dominating set (ds) of G, denoted as $\gamma(G)$. The total domination number is the minimum cardinality of a total dominating set (tds) of G, denoted by $\gamma_t(G)$. The semitotal domination number is the minimum cardinality of a semitotal dominating set (sds) of G, denoted by $\gamma_t(G)$.*

We say that a ds D is minimal if no proper subset $S' \subset S$ is a ds and that D is a minimum if it is the smallest ds.

Because any tds is also an sds and every sds is also a ds, $\gamma t2$ is squeezed between γ and γ_t. The following fact was first observed by Goddard and Henning [37]:

Fact 3.1.2. *For every graph G with no isolated vertex, $\gamma(G) \leq \gamma_{t2}(G) \leq \gamma_t(G)$.*

It turns out that for some graphs, all of these inequalities are strict. See Figure 3.2 for an example, where $\gamma < \gamma_{t2} < \gamma_t$.

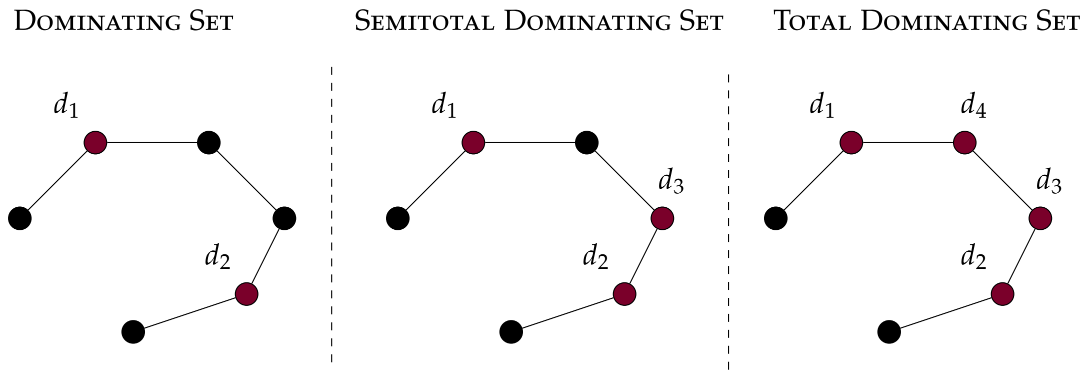


Figure 3.2: *An example for a minimum dominating set, semitotal dominating set and a total dominating set, where $\gamma(G) < \gamma_{2t}(G) < \gamma_t(G)$ are strict. In the first case, only two vertices suffice to dominate all others. In the second one, we need a witness between d_1 and d_2 that is at most distance two. In the last case, d_1 and d_2 both need a neighbor in the total dominating set.*

3.2 Complexity Status of Semitotal Dominating Set

Goddard et al. [37] first showed that sdom is **NP**-Complete in the general case and triggered the systematic analysis of the problem. Table 3.1 complements the complexity status for Dominating Set (dom), Semitotal Dominating Set (sdom) and Total Dominating Set (tdom) when restricted to specific graph classes which was first surveyed by Galby et al. [31] and adds an extra column showing the parameterized complexity. Furthermore, Figure 3.3 visualizes the big picture of graph inclusions.

A polynomial-time algorithm has been shown for graphs of bounded *mim*-width by Galby et al. [31]. Because decomposition trees of constant *mim*-width can be found for *permutation*, *convex*, *interval*, *(circular k-) trapezoid*, *circular permutation*, *Dilworth-k*, *k-polygon*, *circular arc*, *complements of d-degenerate*, *bipartite permutation* and *convex bipartite* graphs in polynomial time [5], this algorithm immediately implies the existence of a polynomial time algorithm for these graph classes as well. Additionally, Galby et. al resolved the complexity for *bipartite permutation* and *convex bipartite* graphs.

Furthermore, new polynomial time algorithms have been discovered for *strongly chordal* [70], *AT-free* [53] and *block* [47] graphs and a new linear time algorithm has been found for interval graphs [67] beating the previous $\mathcal{O}(n^2)$-algorithm given by Henning and Pandey [48]. On the negative side, Semitotal Dominating Set on *circle graphs* [53] graphs and *undirected path graphs* [48] was shown to be **NP**-Complete.

As γ_{2t} is squeezed between γ and γ_t, a comparison of the computational complexity of Semitotal Dominating Set and Total Dominating Set for specific graph classes on the dominating problems might already indicate the complexity of sdom. There is symmetry between these domination problems and the complexity of many graph classes mirrors each other. Nevertheless, it looks like SDOM has more in common with DOM than with TDOM. For example for *chordal bipartite* graphs, SDOM and DOM are **NP**-Complete while it is polynomial-time solvable for TDOM. Contrary, SDOM and DOM have a polynomial-time algorithm for *strongly chordal* graphs, but TDOM is **NP**-Complete. In all currently known cases, Semitotal Dominating Set follows the complexity of Dominating Set.

This thesis goes the first step in approaching the problem from the perspective of *parameterized complexity*. If not mentioned otherwise, the problem is always parameterized by the size of a solution and the results of this thesis are highlighted in **bold** text. Therefore, we have added the parameterized complexity for those graph classes as well and used it as an orientation. As we do not believe that SDOM is fundamentally different than DOM, our intuition is that also the parameterized complexity mimicks SDOM. We were able to prove $\mathbf{W_2}$-hardness for *bipartite* and *split* graphs by giving a graph structure preserving reducing from DOM and we give an explicit construction of a kernel for the *planar* case, which exists for DOM [2] and TDOM [33] as well.

Many gaps are still open for further research. A discussion about them can be found at the end of this thesis in Chapter 5.

Table 3.1: *Comparison between the complexities of* Dominating Set, Semitotal Dominating Set *and* Total Dominating Set *in the classical and parameterized setting. Open problems are marked with an "?".*

Graph Class	**Dominating Set**		**Semitotal Dominating Set**		**Total Dominating Set**	
	classical	Parameterized	classical	Parameterized	classical	Parameterized
bipartite	NPc [6]	W_2 [68]	NPc [48]	W_2 (this)	NPc [59]	W_2 (cite!)
line graph of bipartite	NPc [54]	? (?)	NPc [31]	? (?)	NPc [63]	? (?)
circle	NPc [52]	W_1 [11]	NPc [53]	? (?)	NPc [63]	W_1 [11]
chordal	NPc [10]	W_2 [68]	NPc [48]	W_2 (this)	NPc [65]	W_1 [15] by *split*
s-chordal , $s > 3$	NPc [61]	W_2 [61]	XXX	XX	NPc [61]	W_1 [61]
split	NPc [6]	W_2 [68]	NPc [48]	$\mathbf{W_2}$ **this**	NPc [65]	W_1 [15]
3-claw-free	NPc [20]	FPT [20]	Prob. Unk	Prob. Unk	NPc [63]	Unknown
t-claw-free, $t > 3$	NPc [20]	W_2 [20]	Prob. Unknown	Unknown	NPc [63]	Prob. Unknown
chordal bipartite	NPc [64]	? (?)	NPc [48]	?	P [21]	
planar	NPc (Sources!)	FPT [2]	NPc	**FPT (this)**	NPc	FPT [34]
undirected path	NPc [10]	FPT [27]	NPc [47]	?	NPc [58]	?
dually chordal	P [12]		? (partial res. [31])		P [57]	
strongly chordal	P [26]		P [70]		NPc [26]	
AT-free	P [56]		P [53]		P [56]	
tolerance	P [36]		?		?	
block	P [26]		P [47]		P [14]	
interval	P [16]		P [67]		P [7]	
bounded clique-width	P [18]		P [18]		P [18]	
bounded mim-width	P [5, 13]		P [31]		P [5, 13]	

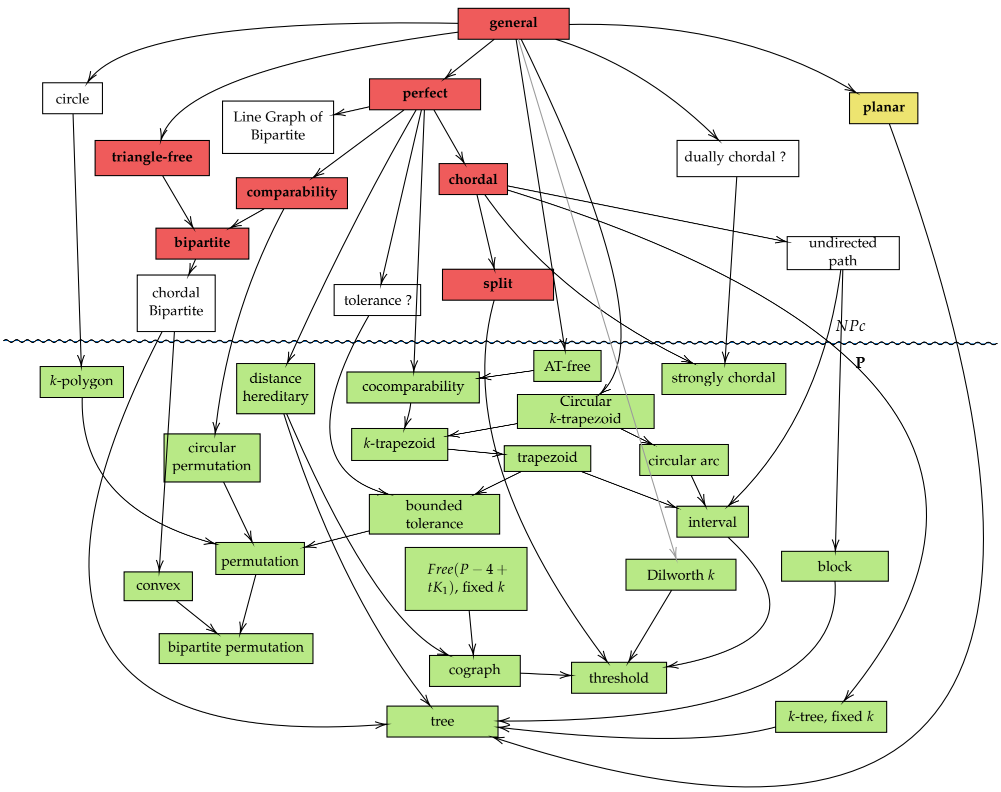


Figure 3.3: *Computational Complexity of* SEMITOTAL DOMINATING SET. *Graph classes that admit a polynomial time algorithm are marked in green, those with an fpt algorithm in yellow, those that are not fpt by solution size in red and those that are unknown are left white. An additional "?" marks those problems, where the classical complexity is also unknown.*

3.3 Fixed-Parameter Intractability

We will first show that Semitotal Dominating Set is W_2-hard on *bipartite graphs* by a parameterized reduction from Dominating Set on bipartite graphs which is known to be W_1.

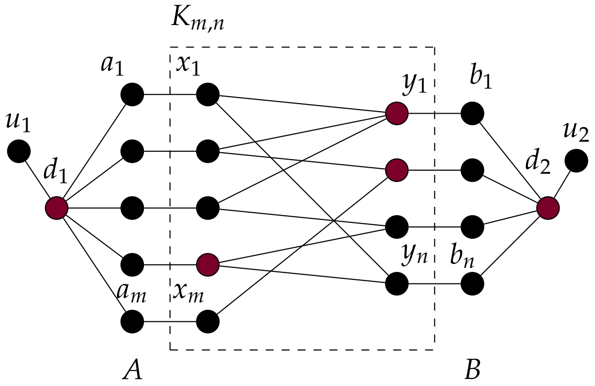


Figure 3.4: *Constructing a bipartite G' from the bipartite graph $K_{m,n}$ by duplicating all vertices and adding exactly two forced witnesses.*

Theorem 2. Semitotal Dominating Set *is W_2-hard when restricted to bipartite graphs.*

Proof. Given a bipartite $G = (X \cup Y, E)$, we construct a bipartite graph $G' = (\{X' \cup Y'\}, E')$:

1. For each $x_i \in X$, we add a new vertex $a_i \in A$ and an edge $\{x_i, a_i\}$ in between.
2. For each $y_j \in Y$, we add a new vertex $b_j \in B$ and an edge $\{y_j, b_j\}$ in between.
3. We add four vertices with edges $\{u_1, d_1\}$ and $\{u_2, d_2\}$, and connect them with all $\{d_1, a_i\}$ and $\{d_2, b_j\}$ ($i \in [m]$ and $j \in [n]$) respectively.

Observe that G' is bipartite because A and B form an independent set on G' that can be cross-wise attached to one of the previous vertex sets. Setting $X' = X \cup \{u_2, d_1\} \cup B$ and $Y' = X \cup \{u_1, d_2\} \cup A$ form the partitions of the new bipartite G'.

It is left to show that G has a ds D of size k if and only if G' has a sds D' of size $k' = k + 2$.

First, assume a ds D in G of size k. We know that $D' = D \cup \{d_1, d_2\}$ is an sds in G' of size $k' = k + 2$, because d_1 dominates u_1 and all $a_i \in A$; d_2 dominates u_2 and all $b_i \in B$. The rest is dominated by the same vertices as they were in G, but now all of them have either d_1 or d_2 as a witness. therefore by construction of G' we have $\forall v \in (D \cap X) : d(v, d_1) = 2$ and $\forall v \in (D \cap Y) : d(v, d_2) = 2$.

Contrary, assume an sds D' in G' with size k'. Wlog assume that $u_1, u_2 \notin D'$, because choosing d_1 and d_2 is always equally good and does break witness properties and therefore the construction forces $d_1, d_2 \in D'$. All $a_i \in A$ can only be used to dominate their neighboring x_i ($b_i \in B$ for y_i), because $d_1, d_2 \in D$ is the only second neighbor they have. If $a_i, b_i \in D'$ we replace it with x_i and y_i preserving the size D. As d_1 and d_2 suffice to provide a witness for every vertex in the graph and do not lose any other witnesses, this operation is sound. Hence, $D = D' \setminus \{d_1, d_2\}$ gives a ds in G of size $k = k' - 2$.

As G' can be constructed in linear time and the parameter k is only blown up by a constant, this reduction is an fpt reduction. Because Dominating Set is already W_2-hard on bipartite graphs [68, Th. 1], we imply that Semitotal Dominating Set is $w[2]$-hard as well. □

Next, we will prove intractability for *split* graphs which directly implies intractability for *chordal* graphs as well.

Theorem 3. Semitotal Dominating Set *is W_1-hard when restricted to split graphs.*

Proof. Given a split graph $G = \{(K \cup I), E\}$ transform it to a split graph G' by adding two vertices d and u and edges $\{d, u\} \cup \{d, k_i\}$ for all $k_i \in K$. **??** visualizes this reduction. Because d only increases the size of the clique K by one and u fits into the independent set I, G' is again a split graph. It is left to show that G has a ds of size k if and only if G' has a sds of size $k' = k + 1$.

First, assume any ds D of size k in G. Letting $D' = D \cup d$ gives an sds in G'. The contrary assume an sds D' of size k in G'. If a vertex $v \in D \cap I$, we simply flip it to a neighbor in K preserving the dominating set. $d \in D'$ is only used in G' for dominating u and can be ignored in G. Therefore letting $D = D' \setminus \{d\}$ gives a valid ds in G of size $k - 1$. Together with the fact that Dominating Set is W_2-hard on *split* graphs [68], the claim follows. □

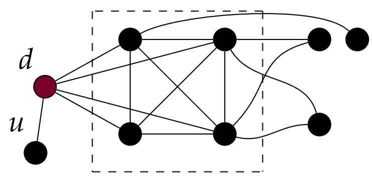


Figure 3.5: *Constructing another split graph G' by increasing the size of the complete graph by one with a new vertex d and adding one additional vertex u.*

3.3.1 *W[2]*-hard on s Chordal IDK

Although the previous result implies $w[2]$-hardness for chordal graphs, we found another reduction from Dominating Set on chordal graphs.

We will introduce the notion of an elimination ordering.

Definition 3.3.1 ([Rose1960]). *In a graph $G = (V, E)$ with n vertices, a vertex is called* ***simplicial*** *if and only if the subgraph of G induced by the vertex set $\{v\} \cup N(v)$ is a complete graph.*

G is said to have a ***perfect eliminiation ordering*** *if and only if there is an ordering $(v_1, ...v_n)$ of the vertices, such that each v_i is simplicial in the subgraph induced by the vertices $v_1, ..., v_i\{\}$*

The following lemma shows that

Lemma 3.3.1 ([Rose1960]). *A graph $G = (V, E)$ is chordal if and only if G has a perfect elimination ordering.*

Theorem 4. Semitotal Dominating Set *restricted to chordal graphs is $\omega[2]$-hard.*

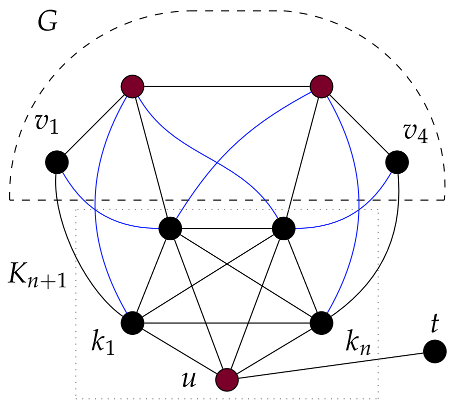


Figure 3.6: *Constructing a chordal G' from the chordal graph P_4 by adding a K_5, connecting its vertices pairwise to G. Adding the (blue) auxiliary vertices are necessary to preserve chordality.*

Proof. We will give a reduction from Dominating Set on chordal graphs. Given $G = (V, E)$ with vertex set $V = \{v_1, ...v_n\}$, we construct a chordal graph G' as described below:

1. Add one complete graph K_{n+1} consisting of the vertices $\{k_1, ..., k_n, u\}$ and an edge $\{v_i, k_i\}$ to each vertex $v_i \in V$ of G. One vertex of the complete subgraph is not connected to any $v \in V$. Denote it as u.

2. Add one additional vertex t and connect it with u vie the edge $\{u, t\}$.

3. For all vertices $v_i \in V$ in G, add a new edge $\{n, k_i\}$ for all neighbors $n \in N(v_i)$.

An example reduction on the graph P_4 is shown in Section 3.3.1.

Corollary 3.3.1. *$N(v_i) \in G$ forms a clique iff $N(v_i)$ forms a clique in G'*

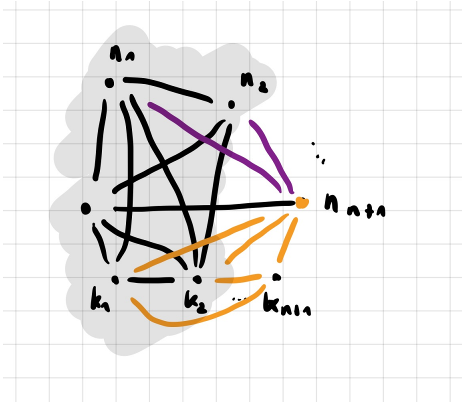


Figure 3.7: Induction Step

Proof. Asuming that $N(v_i)$ forms a clique in G, we show that it also forms a clique in G′ by induction over the number of neighbors $z = abs(N(v_i))$ in G.

- $z = 0$: Holds trivially as we do not have a neighbor in G and in G′ the connected k_i forms a P_1, hence a clique.
- $z = z + 1$:

  By IH, we already know that all neighbors $n_1, ..., n_z$ form a clique together with their vertices in k_i. As $k_{z+1}, v_{z+1} \in N(v_i)$ now also in G′, we show that $N(v_i)$ still forms clique in G′.

  Let k_i be the vertex that was connected with n_i during step 1. All we have to show is that v_{z+1} and k_{z+1} extend our previous clique, hence are fully connected with $N(v_i)$.

  v_{z+1} connects to $N(v_i)$ in G by assumption. By our construction, there exists an edge to $k_1, ..., k_z$, because we add an edge (n_{z+1}, k_i) if there is an edge from (n_{z+1}, n_i). (See fig 3.7)

  k_{z+1} form a complete subgraph with the other k_i and is connected to all n_i by construction because the edge (n_{z+1}, n_i) exists.

  Therefore, $N(v_i)$ will also form a clique in G'.

On the other side, if $N(v_i)$ forms a clique in G′, the vertices of $N(v_i)$ in G form an induced subgraph of G′, hence preserving the clique.

■

Corollary 3.3.2. *G is chordal iff G′ is chordal.*

Proof. ⇒: Assume G chordal. Then exists a total elimination order $o = (v_1, ..., v_n)$ in G where removing v_j sequentially returns cliques in $N(v_i)$. Define $o' = (v_1, ..., v_n, k_1, ..., k_n, u, t)$.

Applying corollary 3.3.1 states that $(v_1, ...v_n)$ always gives cliques in G and according to corollary 3.3.1 also in G'. As the rest is directly part of a clique in G' by definition with an additional vertex of degree 1, o' is a total elimination order for G', hence G' chordal. $\Leftarrow$: Holds as o' is always a total elimination order in G' and removing the complete subgraph K_{n+1} and u gives a total elimination order in G. ∎

Corollary 3.3.3. *G has a Dominating Set of size k iff G' has an sds of size $k+1$*

Proof. Assume a ds D of size k in G. $D \cup \{u\}$ is an sds in G' of size $k+1$, because u dominates t and for each $v \in DS : d(v, u) \leq 2$. Contrary, assume an sds SD in G'. To dominate t, $u \in SD$ must hold, hence already dominating the complete subgraph K_{n+1}. If a vertex $k_i \in SD$, we exchange it with v_i not losing the domination property. Taking $D = SD - \{u\}$ gives our desired ds of size k. ∎

As this reduction runs in FPT time and the parameter is only bounded by a function of k, this is an FPT reduction. As Dominating Set on Chordal Graphs is $w[2] - hard$, so is Semitotal Dominating Set on Chordal Graphs. □

CHAPTER 4

A LINEAR KERNEL FOR PLANAR SEMITOTAL DOMINATION

The best way to explain it is to do it.
Lewis Caroll, *Alice in Wonderland*

We are going to present a polynomial-time preprocessing procedure giving a linear kernel for Planar Semitotal Dominating Set parameterized by solution size. Based on the technique first introduced by Alber, Fellows and Niedermeier [2] in 2004, an abundance of similar results to other domination problems emerged which gave us the belief we can transfer these results to Semitotal Dominating Set. Table 4.1 gives an overview of the status of kernels for the planar case on various domination problems. All of these results introduce reduction rules bounding the number of vertices inside so-called "regions" which can be obtained by a special decomposition of the planar graph.

In the following years, this approach bore fruits in other planar problems as well: a 11/3 kernel for Connected Vertex Cover given in [55], $624k$ for Maximum Triangle Packing in [71], $40k$ for Induced Matching in [50], $13k$ for Feedback Vertex Set [9] and further linear kernels for Full-Degree Spanning Tree in [39] and Cycle Packing in [33].

In the impending years, many results generalized this approach to larger graph classes. Fomin and Thilikos [29] started by proving that the initial reduction rules [2] can also be extended to obtain a linear kernel on graphs with bounded genus g for Dominating Set. Gutner [40] advanced in 2008 by giving a linear kernel for

Problem	Best Known Kernel	Source
Planar Dominating Set	$67k$	[22][1]
Planar Total Dominating Set	$410k$	[34][2]
Planar Semitotal Dominating Set	$359k$	**This work**
Planar Edge Dominating Set	$14k$	[38, Th. 2]
Planar Efficient Dominating Set	$84k$	[38, Th. 4]
Planar Red-Blue Dominating Set	$43k$	[35]
Planar Connected Dominating Set	$130k$	[62]
Planar Directed Dominating Set	Linear	[1]

[1] There is a master's thesis by Halseth [41] claiming a bound of $43k$, but a conference or journal version was not found.

[2] Improved their own results from first $694k$ [34, arXiv v2]

Table 4.1: An overview about existing kernels for planar dominating problems

$K_{3,h}$-topological-minor-free graph classes and a polynomial kernel for K_h-topological-minor-free graph classes. In 2012 Philip, Raman and Sikdar [66] showed that $K_{i,j}$-free graph classes admit a polynomial kernel. In an attempt to expand these ideas to other problems as well, Bodlaender et al. [8] proved that all problems expressible in counting monadic second-order logic satisfying a coverability property admit a polynomial kernel on graphs of bounded genus g. These meta-results are interesting from a theoretical point of view, but the constants for the kernels obtained by these methods are too large to be of practical interest. The question of how an efficient kernel for the Planar Semitotal Dominating Set problem can be constructed remains. In the following, we will transfer the linear kernel for Planar Total Dominating Set described by Garnero and Sau [34, arXiv v2] to Planar Semitotal Dominating Set giving us an explicitly constructed kernel with "reasonable" small constants. Therefore, we were able to modify the original reduction rules to fit into Planar Semitotal Dominating Set. Our main challenge was to ensure that vertices that are important as witnesses are being preserved.

The Main Idea A given planar graph $G = (V, E)$ with a given vertex set $D \subseteq V$ can be decomposed into at most $3 \cdot |D| - 6$ so-called "regions" (see Definition 4.1.7). If D is a given semitotal dominating set of size $|D|$, the total number of regions in this decomposition depends linearly on the size of D. If we define *reduction rules* (see Rules 1 to 3) minimizing the number of vertices in and around a region, we can bound the size of a reduced graph. In our case, we give rules to reduce a region down to a constant number of vertices. We can create an equivalent instance of G whose number of remaining vertices depends linearly on the size of an enquiried solution. Such a reduction gives a *kernel* for Planar Semitotal Dominating Set.

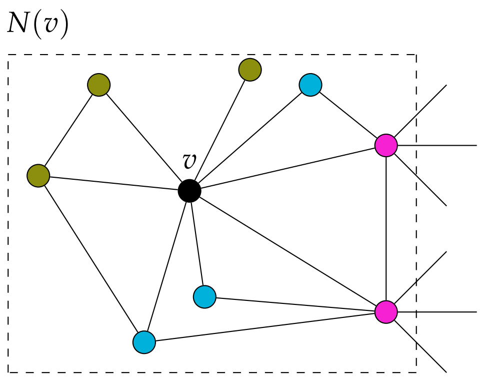


Figure 4.2: *The neighborhood of a single vertex v split to $N_1(v)$ (purple), $N_2(v)$ (blue), and $N_3(v)$ (green). $N_1(v)$'s are those having neighbors outside $N(v)$, $N_2(v)$'s are a buffer between $N_1(v)$ and $N_3(v)$, and $N_3(v)$-vertices are confined in $N(v)$.*

Interestingly, the reduction rules do not rely on the decomposition itself, but rather consider the neighborhood of every pair of vertices in the graph. The decomposition itself has just used a tool for analyzing the kernel size after the reduction.

4.1 Definitions

Before giving the exact reduction rules, we need some definitions exposing the nice properties we are going to exploit. These are the same as given by Garnero and Sau for Planar Total Dominating Set in [34, arXiv v2] and for Planar Red-Blue Dominating Set in [35] which in turn reused ideas introduced by Alber, Fellows and Niedermeier [2] for Planar Dominating Set.

The main idea is to partition the neighborhoods of both a single vertex and a pair of vertices, respectively into three distinct subsets which intuitively classify how much these vertices are confined and how closely they are related to the rest of the graph.

Definition 4.1.1. *Let $G = (V, E)$ be a graph and let $v \in V$. We denote by $N(v) = \{u \in V : \{u, v\} \in E\}$ the neighborhood of v. We split $N(v)$ into three subsets:*

$$N_1(v) = \{u \in N(v) : N(u) \setminus N[v] \neq \emptyset\} \tag{4.1}$$
$$N_2(v) = \{u \in N(v) \setminus N_1(v) : N(u) \cap N_1(v) \neq \emptyset\} \tag{4.2}$$
$$N_3(v) = N(v) \setminus (N_1(v) \cup N_2(v)) \tag{4.3}$$

For $i, j \in [1, 3]$, we denote $N_{i,j}(v) := N_i(v) \cup N_j(v)$. Furthermore, we call a vertex v' confined by a vertex v, if $N(v') \subseteq N[v]$.

$N_1(v)$ is all the neighbors of v which have at least one neighbor outside of $N(v)$ and therefore connects v with the rest of the graph. They are the only vertices with the power to dominate vertices outside the neighborhood of v

$N_2(v)$ contains all neighbors of v not from $N_1(v)$ for which at least one neighbor is in $N_1(v)$. These vertices do not have any function as a dominating vertex and are placed in between a vertex from $N_1(v)$ and those from $N_3(v) \cup \{v\}$. Furthermore, they are useless as witnesses, because either we can replace them with v (sharing the same neighborhood) or when being a witness for v, we replace it with a $z \in N_1(v)$.

$N_3(v)$ vertices are sealed off from the rest of the graph. They are useless as dominating vertices: For all $z \in N_3(v)$ it holds that $N(z) \subseteq N(v)$ by definition and thus, we would always prefer v as a dominating vertex instead of z. They can be important as a witness for v if $N_1(v) \cup N_2(v) = \emptyset$. This can only happen if v forms its connected component with only $N_3(v)$ vertices as neighbors. We will be using this observation in Rule 1 where we shrink $|N_3(v)| \leq 1$.

Next, we are going to extend this notation to a pair of vertices. Using this, Rule 2 will later try to reduce the neighborhood of two vertices, and similar to Definition 4.1.1, we observe nice properties. Again, the idea is to classify how strongly the joined neighborhood $N(v) \cup N(w)$ of two vertices is connected to the rest of the graph.

Definition 4.1.2. *Let $G = (V, E)$ be a graph and $v, w \in V$. We denote by $N(v, w) := N(v) \cup N(w)$ the joined neighborhood $N(v) \cup N(w)$ of the pair v, w and split $N(v, w)$ into three distinct subsets:*

$$N_1(v, w) = \{u \in N(v, w) \mid N(u) \setminus (N(v, w) \cup \{v, w\}) \neq \emptyset\} \tag{4.4}$$

$$N_2(v, w) = \{u \in N(v, w) \setminus N_1(v, w) \mid N(u) \cap N_1(v, w) \neq \emptyset\} \tag{4.5}$$

$$N_3(v, w) = N(v, w) \setminus (N_1(v, w) \cup N_2(v, w)) \tag{4.6}$$

Again, for $i, j \in [1, 3]$, we denote $N_{i,j}(v, w) = N_i(v, w) \cup N_j(v, w)$.

$N_1(v, w)$ contains those vertices having at least one neighbor outside $N[v] \cup N[w]$, $N_2(v, w)$-vertices are in between those from $N_3(v, w) \cup \{v, w\}$ and $N_1(v, w)$, and $N_3(v, w)$ contains vertices isolated from the rest of the graph. You can see an example in Figure 4.3.

A vertex $v \in N_i(v)$ is not necessarily also in $N_i(v, w)$! Observe the vertex z in Figure 4.3. Unlike the sets $N_1(v), N_2(v)$ and $N_3(v)$, in every of the distinct sets $N_i(v, w)$ ($i \in [3]$) can be vertices that belong to a Semitotal Dominating Set. In Figure 4.4 examples are given for these distinct cases.

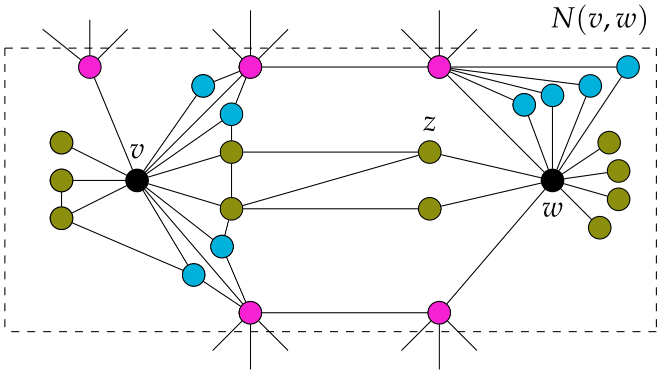


Figure 4.3: *The neighborhood of a pair of vertices. Vertices from $N_3(v, w)$ are colored green, $N_2(v, w)$'s blue and $N_1(v, w)$'s purple. Note that $z \in N_1(w)$, because there is an edge to a neighbor of v, but $z \notin N_1(v, w)$ (and rather $z \in N_3(v, w)$).*

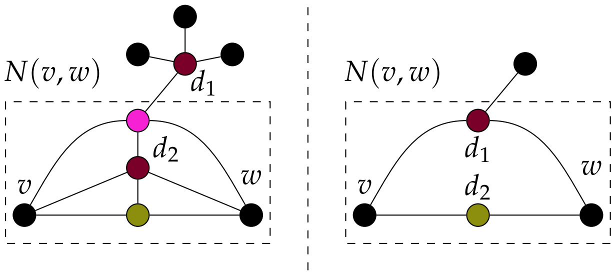


Figure 4.4: *Left:* $\{d_1, d_2\}$ with $d_2 \in N_2(v, w)$ form the only minimum semitotal dominating set. *Right:* $d_1 \in N_1(v, w)$ and $d_2 \in N_3(v, w)$ optimal.

4.1.1 Reduced Graph

Before stating the reduction rules, we want to clarify when we consider a graph $G = (V, E)$ to be *reduced*.

Definition 4.1.3 ([35]). *A graph $G = (V, E)$ is reduced under a set of rules if either none of them can be applied to G or the application of any of them creates a graph isomorphic to G.*

This definition differs from the definition usually given in literature where a graph G is *reduced* under a set of reduction rules if none of them can be applied to G anymore (compare e.g. [28]). Some of our reduction rules (Rule 1 or Rule 2) could be applied *ad infinitum* creating an endless loop that does not change G anymore. Our definition guarantees termination in that case. All of the given reduction rules are local and only need the neighborhood of at most two vertices and replace them partially with gadgets of constant size. Checking whether the application of one of the rules creates an isomorphic graph can be accomplished in constant time.

4.1.2 Regions in Planar Graphs

Alber, Fellows and Niedermeier [2] gave a novel approach to look at planar graphs. In their analysis, they stated a constructive algorithm that decomposes a planar graph into local "regions". Intuitively, assume that we have a fixed plane embedding of a planar graph $G = (V, E)$. If we pick two distinct vertices v and w from a given Semitotal Dominating Set $D \subseteq V$ that are at most of distance two apart, we can try to find two distinct paths from v to w that span up the boundaries of a face and enclose as many other vertices as possible.

The following definitions are based on those given by Garnero and Sau in [34, arXiv v2] and will lead toward a clean definition of a *region* and what we understand as a *D-region decomposition*. More detailed explanations and concrete examples can be found in their paper.

Definition 4.1.4. *Two simple paths P_1, P_2 in a plane graph G are confluent if at least one of the following statements holds:*

1. *they are vertex-disjoint*
2. *they are edge-disjoint and for every common vertex u, if v_i, w_i are the neighbors of u in p_i, for $i \in [1, 2]$, it holds that $[v_1, w_1, v_2, w_2]$*
3. *they are confluent after contracting common edges*

Definition 4.1.5. *Let $G = (V, E)$ be a plane graph and let $v, w \in V$ be two distinct vertices. A region $R(v, w)$ (also denoted as vw-region R) is a closed subset of the plane, such that:*

1. *the boundary of R is formed by two confluent simple vw-paths with length at most 3*

2. *every vertex in R belongs to $N(v, w)$, and*

3. *the complement of R in the plane is connected.*

We denote with ∂R the set of vertices on the boundary of R (including the poles) and by $V(R)$ the set of vertices laying (on the plane embedding) in R. Furthermore, we call $|V(R)|$ the size of the region.

The poles of R are the vertices v and w. The boundary paths are the two vw-paths that form ∂R

Definition 4.1.6. *Two regions R_1 and R_2 are non-crossing, if:*

1. *$(R_1 \setminus \partial R_1) \cap R_2 = (R_2 \setminus \partial R_2) \cap R_1 = \emptyset$, and*

2. *the boundary paths of R_1 are pairwise confluent with the ones in R_2*

We now have all the definitions ready to formally define a maximal *D-region decomposition* on planar graphs:

Definition 4.1.7. *Given a plane graph $G = (V, E)$ and $D \subseteq V$, a $\underline{D - region\ Decomposition}$ of G is a set $\mathfrak{R}$ of regions with poles in D such that:*

1. *for any vw-region $R \in \mathfrak{R}$, it holds that $D \cap V(R) = \{v, w\}$, and*

2. *all regions are pairwise non-crossing.*

We denote $V(\mathfrak{R}) = \bigcup\limits_{R \in \mathfrak{R}} V(R)$.

A D-region decomposition is <u>maximal</u> if there is no region $R \notin \mathfrak{R}$ such that $\mathfrak{R}' = \mathfrak{R} \cup \{R\}$ is a D-region decomposition with $V(\mathfrak{R}) \subsetneq V(\mathfrak{R}')$.

Figure 4.5 gives an example of how to decompose a graph into a maximal $D - region$ decomposition with a given semitotal dominating set D of size 3.

We are introducing a special subset of a region, namely *simple region* where every vertex is a common neighbor of v and w. They will appear in many unexpected astonishing places and are an important tool to operate on small parts of a plane graph. The upcoming Rule 3 will bound the size of these *simple regions*. Interestingly, in the first version o f the paper about the linear kernel for PLANAR TOTAL DOMINATING SET [34, arXiv v2], they were not given independently but covered by one of their reduction rules (Rule 2). As it turned out, the analysis is getting simpler if we treat them in a separate rule (In our case: Rule 3) and so did Garnero and Sau [34] in a revised version of their paper four years later.

Definition 4.1.8. *A simple vw-region is a vw-region such that:*

1. *its boundary paths have length at most 2, and*

2. *$V(R) \setminus \{v, w\} \subseteq N(v) \cap N(w)$.*

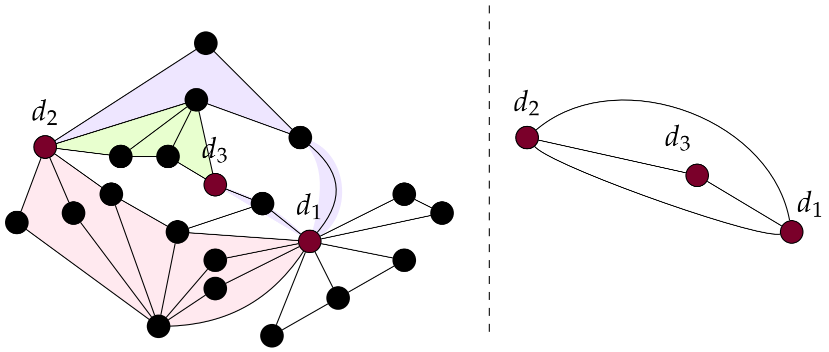


Figure 4.5: *Left: A maximal D-region decomposition $\mathfrak{R}$, where $D = \{d_1, d_2, d_3\}$ form a* Semitotal Dominating Set. *There are two regions between d_2 and d_1 (purple and pink), one region between d_1 and d_3 (purple) and one region between d_2 and d_3 (green). Observe that this D-region decomposition, some neighbors of d_1 are not covered by any vw-region for any $v, w \in D$. Our reduction rules are going to take care of them and bound this number of vertices to obtain the kernel. Right: The corresponding underlying multigraph $G_{\mathfrak{R}}$. Every edge denotes a region between d_i and d_j.*

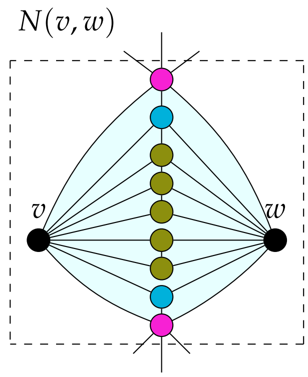


Figure 4.6: *A simple region with two vertices from $N_1(v, w)$ (purple) setting the boundary, two vertices from $N_2(v, w)$ (blue) and some vertices from $N_3(v, w)$ (green) in between.*

Figure 4.6 shows an example of a simple region containing 9 distinct vertices.

In the analysis, we will also use properties of the *underlying multigraph* of a *D-region decomposition* $\mathfrak{R}$. Refer to Figure 4.5 for an example.

Definition 4.1.9. *Let $G = (V, E)$ be a plane graph, let $D \subseteq V$ and let $\mathfrak{R}$ be a D-region decomposition of G. The underlying multigraph $G_{\mathfrak{R}} = (V_{\mathfrak{R}}, E_{\mathfrak{R}})$ of $\mathfrak{R}$ is such that $V_{\mathfrak{R}} = D$ and there is an edge $\{v, w\} \in E_{\mathfrak{R}}$ for each vw-region $R(v, w) \in \mathfrak{R}$*

4.2 The Big Picture

Figure 4.7 gives a high-level overview of how we are going to obtain the linear kernel for Planar Semitotal Dominating Set. We will first derive three different reduction rules (Rules 1 to 3 are green in the overview), prove that they preserve the solution size k and run in polynomial-time. Then we use the existence of a maximal *D-region decomposition* $\mathfrak{R}$ on planar graphs to bound the number of vertices that fly around a given region $R \in \mathfrak{R}$. This will lead us towards a bound on the number of vertices inside R. Furthermore, we observe that the number of vertices that are not enclosed in R, but lie outside the border is bounded, too. We will often encounter hidden simple regions which are reduced by Rule 3 and therefore of constant size by Corollary 4.3.1. As we know that the total number of regions R in the *D-region decomposition* is linear in k, we obtained a linear kernel for the Planar Semitotal Dominating Set as well.

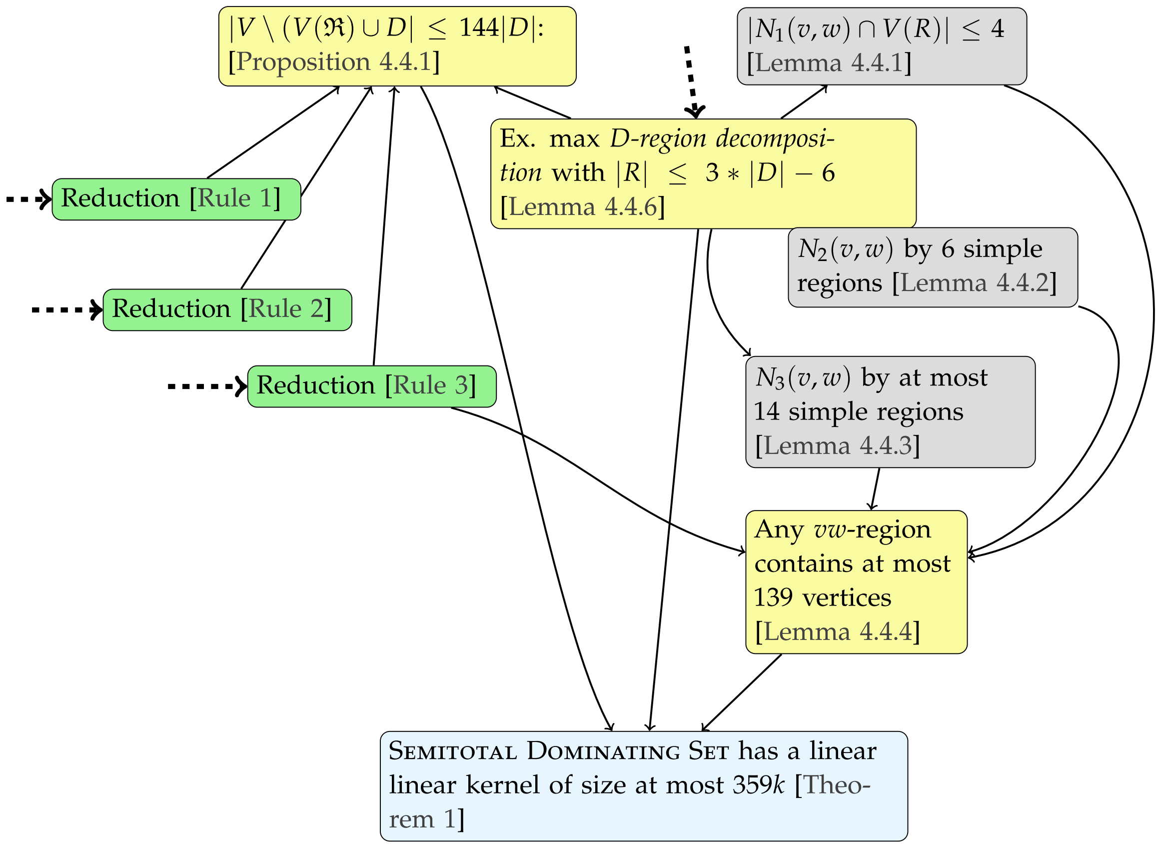


Figure 4.7: *The plan for obtaining a linear kernel for* Planar Semitotal Dominating Set. *Starting with the reduction rules (Green) we will derive the number of vertices inside and outside of a vw-region.*

4.3 The Reduction Rules

Following the ideas proposed by Garnero and Sau [34, arXiv v2], we state modified reduction rules that after exhaustive application will lead to a linear kernel. We note that especially for Rule 2, we relied on the second version of the paper electronically submitted to *arXiv* because in the following years they improved their kernel size at the cost of making them more dependent on properties of a total dominating set and the latest rules stated will not work for PLANAR SEMITOTAL DOMINATING SET. By a deeper look into the structure of simple regions, we were able to give a slightly more complex reduction Rule 3 that achieves the same bound as proven in [34]. The main challenge in our case was to preserve possible witness properties in the graph because a vertex inside a region can be important as a witness for vertices in another region. This was not an issue for a total dominating set, because there, these witnesses must be close and they do not have an effect on more distanced vertices.

4.3.1 Reduction Rule I: Shrinking $N_3(v)$

The idea of the first rule is the observation that a vertex $v' \in N_{2,3}(v)$ dominates v and possibly vertices from $N_2(v)$ and $N_3(v)$. As $N(v') \subseteq N(v)$ and the Fact 4.3.1 that a witness for v' is also a witness for v, we can use v instead of v' as a dominating vertex. Therefore, we can remove $N_{2,3}$ from the graph. Nevertheless, v' can be a witness for v itself and might be required in a solution. Our rule ensures that at least one $N_3(v)$-vertex is preserved. An example of this rule is shown in Figure 4.8.

Fact 4.3.1. *Let $G = (V, E)$, $v \in V$ and $v' \in N_{2,3}(v)$. Any witness $w \neq v$ for v' is also a witness for v.*

Proof. By assumption, v' is witnessed by a vertex $w \neq v$ with $d(v', w) \leq 2$. It follows directly from the definition of $v' \in N_{2,3}(v)$ that $N(v') \subseteq N[v]$ and hence v' is *confined* inside the neighborhood of v. Every path $P = (v', n, w)$ from v' to possible witness w within two steps must pass at least one vertex $n \in N[v]$ as all neighbors. This implies that there exists also a path from v to w with a length of at most two and v □

Rule 1. *Let $G = (V, E)$ be a graph and let $v \in V$. If $|N_3(v)| \geq 1$:*

- *remove $N_{2,3}(v)$ from G,*
- *add a vertex v' and an edge $\{v, v'\}$*

We can now prove the correctness of this rule.

Lemma 4.3.1. *Let $G = (V, E)$ be a graph and let $v \in V$. If G' is the graph obtained by applying Rule 1 on G, then G has a semitotal dominating set of size k if and only if G' has a semitotal dominating set of size k.*

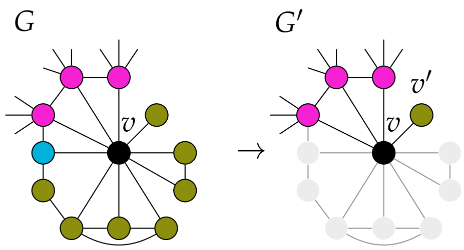


Figure 4.8: *Simplifying $N_{23}(v)$: As $N_3(v) \geq 1$, we remove $N_{23}(v)$ and add a new witness v'. $N_1(v)$ remains untouched.*

Proof. Assume D to be an sds set in G of size k. Because Rule 1 has been applied, we can assume $N_2(v) \neq \emptyset$ in G. To dominate $N_3(v)$, $|D \cap (N_{2,3}(v) \cup \{v\})| \neq \emptyset$. By Fact 4.3.1, we know that a witness for d (which is not v) is also a witness for v and therefore, we can replace any $D \cap N_{2,3}(v)$ by v in D. Henceforth, assume $v \in D$ and $N_{2,3}(v)$ is already dominated by v. If $|D \cap (N_{2,3}(v))| \geq 1$, set $D' = D \setminus N_{2,3}(v) \cup \{v'\}$, else $D' = D$. A $z \in D \cap N_{2,3}(v)$ could have been a witness for v and therefore we choose $v' \in D'$ preserving witnesses. In both cases, v' is by assumption dominated by v and $|D| \leq |D'|$.

Let D' be an sds in G'. We assume that $v \in v'$ because v' has to be dominated and v is always a better choice than v'. If $v' \in D'$ then we have to preserve a witness for v in G. We know $N_3(v) \neq \emptyset$ and therefore replace it with an arbitrary vertex $d \in N_3(v)$ in G. If a witness for v came from $N_1(v) \cup \{N(w) \setminus N(v) | w \in N_1(v)\}$, they have not been touched by the reduction. In summary, if $v' \in D'$, we set $D = D' \cup \{d\} \setminus \{v'\}$ for a $d \in N_3(v)$ and otherwise $D = D'$. In both cases, $N_{2,3}(v)$ is dominated by v and $|D| = |D'|$. □

Lemma 4.3.2. *A plane graph G of n vertices is reduced under Rule 1 in time $\mathcal{O}(n)$*

Proof. As Rule 1 stayed the same, the proof directly follows [2, Lemma 2]. □

Note that we need our definition of a reduced instance given in 4.1.3. If Rule 3 is being applied, it will still leave us with a vertex $z \in N_3(v)$ allowing this rule to be applied again.

4.3.2 Reduction Rule II: Shrinking the Size of a Region

The second rule is the heart of the whole reduction and minimizes the neighborhood of two distinct vertices. The rule follows Garnero and Sau's approach [34] for Planar Total Dominating Set. Especially Rule 2 given in [34, arXiv v2] was not transferable to Planar Semitotal Dominating Set, because it heavily relies on the property of a total dominating set that a witness w for v **must** be a direct neighbor of w. In the case of the more relaxed Semitotal Dominating Set, the witness is allowed to be further away.

It can be observed that in the worst case four vertices are needed to semitotally dominate $N(v,w)$ of two vertices $v,w \in V$: v, w and two witnesses for them. Exemplary, observe the graph consisting of two distinct $K_{1,m}$ with $m \in \mathbb{N}$ with centers v and w.

Before we give the concrete reduction rule, we need to define three sets. Intuitively, we first try to find a set $\tilde{D} \subseteq N_{2,3}(v,w)$ of size at most three dominating $N_3(v,w)$ without using v or w. If no such set exists, we allow v (resp. w) and try to find one again. If we now find such a set, we can conclude that v (w) must be part of a solution.

Definition 4.3.1. *Let $G = (V,E)$ be a graph and let $v,w \in V$. We now consider all the sets that can dominate $N_3(v,w)$:*

$$\mathcal{D} = \{\tilde{D} \subseteq N_{2,3}(v,w) \mid N_3(v,w) \subseteq \bigcup_{v\in\tilde{D}} N(v),\ |\tilde{D}| \leq 3\} \tag{4.7}$$

$$\mathcal{D}_v = \{\tilde{D} \subseteq N_{2,3}(v,w) \cup \{v\} \mid N_3(v,w) \subseteq \bigcup_{v\in\tilde{D}} N(v),\ |\tilde{D}| \leq 3,\ v \in \tilde{D}\} \tag{4.8}$$

$$\mathcal{D}_w = \{\tilde{D} \subseteq N_{2,3}(v,w) \cup \{w\} \mid N_3(v,w) \subseteq \bigcup_{v\in\tilde{D}} N(v),\ |\tilde{D}| \leq 3,\ w \in \tilde{D}\} \tag{4.9}$$

Furthermore, we shortly denote $\bigcup \mathcal{D}_v = \bigcup\limits_{D\in\mathcal{D}_v} D$ and $\bigcup \mathcal{D}_w = \bigcup\limits_{D\in\mathcal{D}_w} D$.

Assuming that v and w are closely connected with $d(v,w) \leq 2$, it might suffice to consider only sets of size at most three, because an intermediate vertex could witness v and w at the same time. In the later analysis, the *D-region decomposition* exactly creates regions around $N(v,w)$ requiring at least one path from v to w lengthed two. As the following rule is only used to locally investigate such regions, we could add the requirement of a distance of two to it and work with sets of size at most three. We believe that this could further improve the kernel.

We are now ready to state Rule 2. An exemplary application is shown in Figure 4.9.

Rule 2. *Let $G = (V,E)$ be a graph and v,w be two distinct vertices from V. If $\mathcal{D} = \emptyset$ (Definition 4.3.1) we apply the following:*

Case 1: *if $\mathcal{D}_v = \emptyset$ and $D_w = \emptyset$*

- *Remove $N_{2,3}(v,w)$*
- *Add vertices v' and w' and two edges $\{v,v'\}$ and $\{w,w'\}$*
- *If there was a common neighbor of v and w in $N_{2,3}(v,w)$, add another vertex y and two connecting edges $\{v,y\}$ and $\{y,w\}$*
- *If there was no common neighbor of v and w in $N_{2,3}(v,w)$, but at least one path of length three from v to w via only vertices from $N_{2,3}(v,w)$, add two vertices y and y' and connecting edges $\{v,y\}$, $\{y,y'\}$ and $\{y',w\}$*

Case 2: *if* $\mathcal{D}_v \neq \emptyset$ *and* $D_w = \emptyset$

- *Remove* $N_{2,3}(v)$
- *Add* $\{v, v'\}$

Case 3: *if* $\mathcal{D}_v = \emptyset$ *and* $D_w \neq \emptyset$
*This case is symmetrical to **Case (2)**.*

In **Case (1)**, we know by Fact 4.3.3 that v and w must be in D. Therefore, we introduce two forcing vertices v' and w' in G' and remove $N_{2,3}(v, w)$ as these vertices are dominated by v and w. Removing $N_{2,3}(v, w)$ entirely, could lose paths of length less than three and destroy solutions: If $d(v, w) \leq 2$ then v can directly witness w and if $d(v, w) = 3$, one vertex on this path could be a witness for both.

Again by Fact 4.3.3 we know for **Cases (2)** and **(3)** that $v \in D$ and similar to Rule 1 we can simplify the neighborhood $N_{2,3}(v)$. Fact 4.3.2 states, that these vertices are only useful for witnessing v, but do not go beyond what v already witnesses. Observe that removing $N_{2,3}$ cannot break any connectivity as all vertices in $N_{2,3}(v)$ are confined in v.

The case where $\mathcal{D}_v \neq \emptyset \wedge \mathcal{D}_w \neq \emptyset$ is not necessary for reasons clarified later.

Before proving Rule 2 we will deduce some *facts* which are implied by the definitions above. These facts justify the definition of the sets $\mathcal{D}$, $\mathcal{D}_v$ and $\mathcal{D}_w$.

Fact 4.3.2. *Let $G = (V, E)$ be a graph, let $v, w \in V$, and let G' be the graph obtained by the application of Rule 2 on v, w. If $\mathcal{D} = \emptyset$, then G has a solution of size at most k if and only if it has a solution of size at most k containing at least one of the two vertices $\{v, w\}$.*

Proof. Because $\mathcal{D} = \emptyset$, an sds of G must contain at least one of $\{v, w\}$ or at least four vertices from $N_{2,3}(v, w)$. In the second case, these four vertices can be replaced with v, w and two neighbors of v and w as witnesses. In all cases k becomes either smaller or stays unchanged. □

The second fact states that if $\mathcal{D}_v$ (resp. $\mathcal{D}_w$) is empty, too, we only need to consider solutions containing w (resp. v):

Fact 4.3.3. *Let $G = (V, E)$ be a graph, let $v, w \in V$, and let G' be the graph obtained by the application of Rule 2 on v, w. If $\mathcal{D} = \emptyset$ and $\mathcal{D}_w = \emptyset$ (resp. $\mathcal{D}_v = \emptyset$) then G' has a solution of size at most k if and only if it has a solution containing v of size at most k (resp. w).*

Proof. As $\mathcal{D}_v = \emptyset$, no set of the form $\{v\}$, $\{v, u\}$ or $\{v, u, u'\}$ with $u, u' \in N_{2,3}(v, w)$ can dominate $N_3(v, w)$. Since also $\mathcal{D} = \emptyset$ any sds of G has to contain v or at least four vertices by Fact 4.3.2. In the last case, we replace these four vertices with v, w and two neighbors respectively and we can conclude that v belongs to the solution. Again, k becomes either smaller or stays unchanged. □

We now have all the tools to prove the correctness of Rule 2.

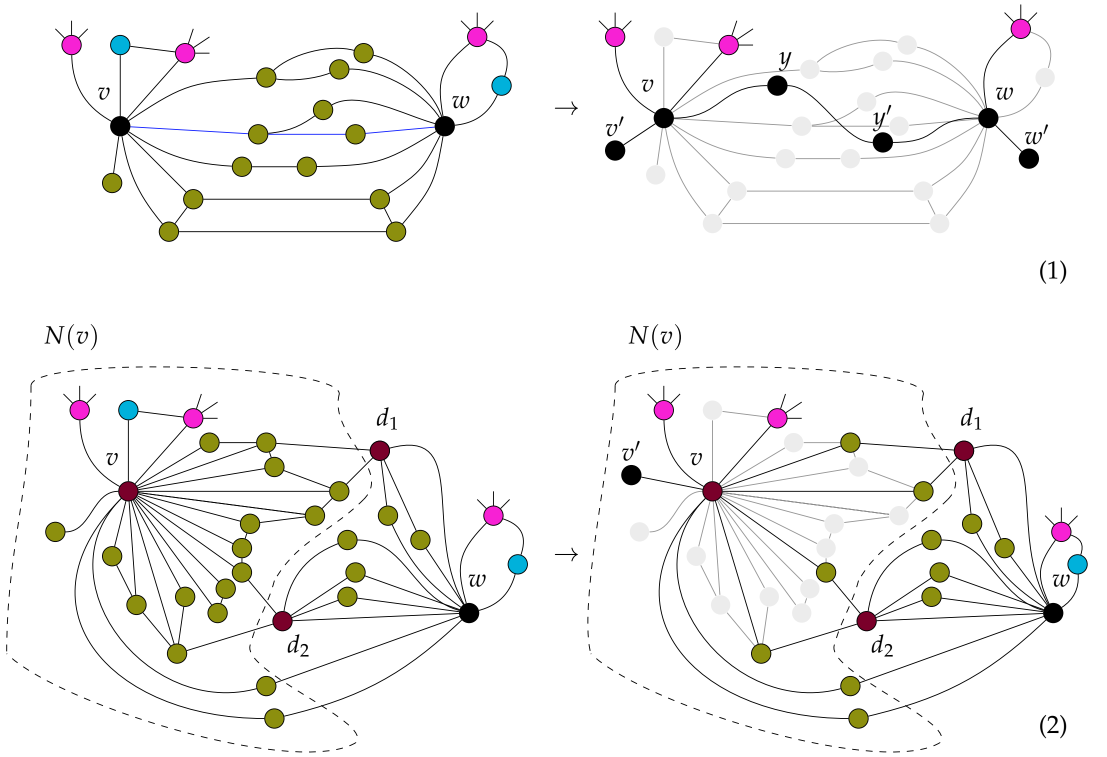


Figure 4.9: *An application of Rule 2: (a) $\mathcal{D}_v = \mathcal{D}_w = \emptyset$ and **Case (1)** applies. Both v and w must be in the sds and we can remove $N_{2,3}(v, w)$ and add $\{v', w'\}$. Furthermore, we need to preserve a path of length 3 from v to w by adding $\{y, y'\}$ as well. (b) **Case (2)** has been applied and the $N_{2,3}(v)$ removed. Observe that those vertices that cross the dotted line are vertices from $N_1(v)$ and not removed.**Case (3)** is symmetric.*

Lemma 4.3.3. *Let $G = (V, E)$ be a plane graph, $v, w \in V$ and $G' = (V', E')$ be the graph obtained after application of Rule 2 on the pair $\{v, w\}$. Then G has a semitotal dominating set of size at most k if and only if G' has a semitotal dominating set of size at most k.*

Proof. We will prove the claim by independently analyzing the different cases.

Assume an sds D in G and by assumption $\mathcal{D} = \emptyset$. We show that G' has an sds with $|D'| \leq |D|$.

1. Assume $\mathcal{D}_v = \emptyset \wedge \mathcal{D}_w = \emptyset$. By Fact 4.3.3, both $v, w \in D$. Therefore, v',w', and potentially y and y' are dominated by v or w in G'.

   We have three cases: Either v and w have their witnesses (e.g. via two distinct $N_1(v, w)$-vertices); they are of distance three and share one witness on a path from v to w (which could go through $N_{2,3}(v,w)$ and therefore will be kept by the vertices y and y) is required, or they can be of distance less than three, such that they already witness each other directly. Furthermore, a direct edge $\{v,w\}$ will not be reduced.

We will now build D' depending on which vertices from $D \cap N_{2,3}(v,w)$ have been removed.

- If the rule has not removed any $d \in D$, we simply set $D' = D$. If v was a witness for w (and vice versa), Rule 2 will preserve it by introducing the vertex y. Otherwise, these witnesses are preserved.
- If $d(v,w) > 3$, then v and w are not sharing any common witnesses. If the rule has removed a vertex from $D \cap N(v)$, we set $D' = D \setminus N_{2,3}(v,w) \cup \{v'\}$. If the rule has removed a vertex from $D \cap N(w)$, we set $D' = D \setminus N_{2,3}(v,w) \cup \{w'\}$. If the rule has removed a vertex from $(D \cap N(v))$ and a vertex from $(D \cap N(w))$, we set $D' = D \setminus N_{2,3}(v,w) \cup \{v', w'\}$.
- If $d(v,w) = 3$ and the vertices y and y' get introduced preserving one path from v to w, because there have been a path via $N_{2,3}(v,w)$-vertices containing a single witness for both v and w. If the rule removed a dominating vertex $D \cap N_{2,3}(v,w)$, we set $D' = D \setminus N_{2,3}(v,w) \cup \{y\}$. Note that we could also choose $y' \in D'$, because y's only function is to be a single witness for v and w and every other vertex it could be a witness for, will also be witnessed by $v, w \in D'$ (Fact 4.3.2).
- If $d(v,w) \leq 2$, then v and w directly witness each other and the reduction must preserve this relation, which is accomplished by introducing the single bridging vertex y. Even if the rule has removed a vertex $z \in D \cap N_{2,3}(v,w)$, we can ignore that, because Fact 4.3.2 states that v and w will witness the same vertices as z did. Hence, we set $D' = D \setminus N_{2,3}(v,w)$.

In all of the cases, it follows that D' is an sds of G' with $|D'| \leq |D|$

2. $\mathcal{D}_v \neq \emptyset \wedge \mathcal{D}_w = \emptyset$: As $\mathcal{D}_w = \emptyset$ and Fact 4.3.3, we know that $v \in D$ and v dominates $N_{2,3}(v)$. If a vertex $d \in D \cap N_{2,3}(v)$ was removed, we set $D' = D \setminus N_{2,3}(v) \cup \{v'\}$, else $D' = D$. Deleting dominating vertices $d \in D \cap N_{2,3}(v)$ does not destroy the witness properties of the graph, because by Fact 4.3.2 we know that everything d could witness, is also witnessed by v. If d was a witness for v, we have replaced it with v' in G'. Note that otherwise a vertex from $N_1(v) \cup \{p \in (N(z) \setminus N(v)) | z \in N_1(v)\}$ is a witness for v that is not touched by this reduction. Clearly, $|D'| \leq |D|$ holds.
3. $\mathcal{D}_v = \emptyset \wedge \mathcal{D}_w \neq \emptyset$: Symmetrical to previous case.

Let D' be a Semitotal Dominating Set in G' and $\mathcal{D} = \emptyset$. We show that G has an sds D with $|D| \leq |D'|$ by distinguishing the different cases again.

1. $\mathcal{D}_v = \emptyset \wedge \mathcal{D}_w = \emptyset$: In any case we know that $v, w \in D$ to dominate v' and w' and therefore also dominating $N_{2,3}(v,w)$ in G. To preserve the distance $d(v,w)$ the rule might have introduced additional vertices y and y'.
   - If only y was introduced we know that there was a common neighbor $n \in N(v) \cap N(w)$ of v and w. y allows v to witness w (and vice versa) and is not part of a solution itself. (assuming $y \notin D'$). Hence, we set $D = D'$.
   - If y and y' were added, a solution could use one of them to provide a single witness for v and w. There exists a path $p = (v, n_1, n_2, w)$ from v to w in G only using vertices from $N_{2,3}(v,w)$. As n_1 and n_2 both witness v and w, we put one of them in D if at least one of y or y are dominating vertices in G'. Hence, if $y \in D'$ or $y \in D'$, we set $D = D' \setminus \{y, y'\} \cup \{n_1\}$.
2. $\mathcal{D}_v \neq \emptyset \wedge \mathcal{D}_w = \emptyset$: Clearly, $v \in D'$ to dominate v'. If $v \in D'$, we set $D = D' \setminus \{v'\} \cup d$ for some vertex $d \in N_{2,3}(v,w)$ and otherwise $D = D'$. If v' was the witness of v, it is now replaced by d and D is an SDS with $|D| \leq |D'|$.
3. $\mathcal{D}_v = \emptyset \wedge \mathcal{D}_w \neq \emptyset$: Symmetrical to previous case.

In all cases, we have shown that $|D| \leq |D'|$ and D is an sds of G.

□

4.3.3 Reduction Rule III: Shrinking Simple Regions

Recall that a simple region is a region, where the entire neighborhood of the poles is shared.

By planarity, a *simple region* has at most two vertices from $N_1(v,w)$ (namely the border ∂), two vertices from $N_2(v,w)$ connected to the border and unlimited $N_3(v,w)$ vertices squeezed in the middle.

With a more sophisticated analysis, we were able to modify this rule such that the bound given in [34] remained valid although we might have to add some more vertices as originally described.

In contr

While first embedding a reduction procedure for the Planar Total Dominating Set directly into their rule two, Garnero and Sau [34] decided to decouple it into a separate rule.

Unfortunately, a total dominating set is easier to handle

Rule 3. *Let $G = (V, E)$ be a plane graph, $v, w \in V$ and R be a simple region between v and w. If $|V(R) \setminus \{v, w\}| \geq 5$ apply the following:*

Case 1: *If $G[R \setminus \partial R] \cong P_3$, then:*

- *remove $V(R \setminus \partial R)$*

- *add vertex y with edges $\{v, y\}$ and $\{y, w\}$*

Case 2: *If $G[R \setminus \partial R] \ncong P_3$, then*

- *remove $V(R \setminus \partial R)$*
- *add vertices y, y' and four edges $\{v, y\}$, $\{v, y'\}$, $\{y, w\}$ and $\{y', w\}$*

Recap that we denoted ∂R as the set of boundary vertices of the (simple) region R, which includes v and w and possibly up to two vertices on the border of R

Lemma 4.3.4 (Correctness of Rule 3)**.** *Let $G = (V, E)$ be a plane graph, $v, w \in V$ and $G' = (V', E')$ be the graph obtained after application of Rule 3 on the pair $\{v, w\}$. Then G has semitotal dominating set of size k if and only if G' has semitotal dominating set of size k.*

Proof. Consider an sds D in G. We show that G' also has an sds with $|D'| \leq |D|$. By assumption, we have $|V(R) \setminus \{v, w\}| \geq 5$, R is a simple region and therefor $d(v, w) \leq 2$.

First, we can safely assume that the set of border vertices ∂R contains exactly two vertices. If $|\partial R \setminus \{v, w\}| < 2$, the region could not have vertices strictly inside, because the boundary path of ∂R does not enclose any vertices. Hence, at least three vertices must lie strictly inside R.

On the other side, we only need to observe those cases, where both $v, w \notin D$. Again, if a vertex $v' \in V(R \setminus \partial R)$ together with v or w are in D, we can replace them with both v and w which preserves the dominating and witnessing properties and we can set $D' = D \setminus \{v'\} \cup \{v, w\}$. Note that even if v' was a sole witness for v, we know that $d(v, w) \leq 2$, and therefore v is a witness for w and vice versa. If no vertex in $V(R \setminus \partial R)$ is dominating, we just set $D' = D$.

Furthermore, we assume $|D \cap V(R \setminus \partial R)| \leq 1$ (at most one vertex strictly inside the region is dominating) because otherwise, we replace those two vertices by v and w which trivially dominate and witness the same vertices.

If $|\partial R \setminus \{v, w\} \cap D| \geq 1$ we can replace the vertex with v or w and set $D' = D \setminus V(R \setminus \partial R) \cup \{v\}$ (TODO Argument why this is ok)

In summary, we only need to consider in the following cases that no vertex from $\partial R \in D$, $|V(R \setminus \partial R)| \geq 3$ and at most one vertex strictly inside R is dominating.

First, assume $V(R \setminus \partial R) \cong P_3$. We denote this induced path as (p_1, p_2, p_3). If neither $p_1, p_2, p_3 \in D$, we set $D' = D$, trivially preserving a Semitotal Dominating Set. Otherwise, $p_2 \in D$ is forced because this is the only way to dominate $V(R \setminus \partial R)$ with exactly one vertex inside. We know that **Case (1)** has been applied and we set $D' = D \setminus \{p2\} \cup \{y\}$. which dominates $\{y, y'\}$ and witnesses $N(v) \cup N(w)$ which are the same vertices as p_2 did. This case is depicted in Figure 4.10.

Contrary, assume $V(R \setminus \partial R) \ncong P_3$. Again, let us denote this induced path as (p_1, p_2, p_3). We observe that without contradicting the planarity of G the induced subgraph of the vertices $G[R \setminus \partial R]$ must be a subgraph of a P_j for $j = |V(R \setminus \partial R)|$ and

$j \geq 3$. By assumption, we either have this as either a P_3 with at least one missing edge or a larger P_j. Therefore, it is impossible to dominate all these vertices with one single vertex p_i if we also assume that no vertex in ∂R is dominating. Hence, we set $D' = D$.

In all cases $|D| = |D'|$

$\Leftarrow$ Consider an sds D in G'. We show that G has an sds with $|D| \leq |D'|$. By assumption, we have $\mathcal{D} = \emptyset$. We analyze both cases separately.

First, assume $V(R \setminus \partial R) \cong P_3$. Here, **Case (1)** has been applied and $V(R \setminus R)$ replaced by one single vertex y. We denote the induced path in G as (p_1, p_2, p_3). If $y \in D'$, we set $D = D' \setminus \{y\} \cup \{p_2\}$. We know that p_2 dominates p_1,p_3 in G and witnesses the same vertices as in G'. Otherwise, we just set $D = D'$.

Contrary, assume $V(R \setminus \partial R) \ncong P_3$ and **Case (2)** has been applied and $V(R \setminus R)$ replaced by two vertices y and y'. Observe that neither y nor y' is useful in any Semitotal Dominating Set in G': If both $y, y' \in D'$, we simply replace them by v and w and if only $y \in D$ (resp. y'), then we need either $v \in D'$ or $w \in D'$ to dominate y' and again, we replace them by v and w. We can even ignore that y or y' is important as a witness, because as $d(v, w) \leq 2$, v, w and v always witness each other. Hence *at least one of* $\{v, w\}$ must be in an sds simulating an "OR-gadget". Therefore, also $V(R \partial R)$ is dominated by either v or w. All the witnesses stayed the same and hence, we simply set $D = D'$.

In all of the cases $|D| = |D'|$ □

The application of Rule 3 gives us a bound on the number of vertices inside a simple region.

Corollary 4.3.1. *Let $G = (V, E)$ be a graph, $v, w \in V$ and R a simple region between v and w. If Rule 3 has been applied, this simple region has a size of at most 4.*

Proof. If $|V(R) \setminus \{v, w\}| < 5$ then the rule would not have changed G and the size of the region would already be smaller than 5. Assuming $|V(R) \setminus \{v, w\}| \geq 5$ in both cases $V(R \setminus \partial R)$ gets removed and at most two new vertices added. As the boundary in a simple region contains at most two vertices distinct from v and w, the size of the simple region is bounded by at most four. □

4.3.4 Computing Maximal Simple Regions between two vertices

For the sake of completeness, we state an algorithm on how a maximal simple region-between two vertices $v, w \in V$ can be computed in time $\mathcal{O}(d(v) + d(w))$.

4.4 Bounding the Size of the Kernel

We now put all the pieces together to prove the main result: A kernel which siye is linearly bounded by the solution size k. For that purpose, we distinguish between

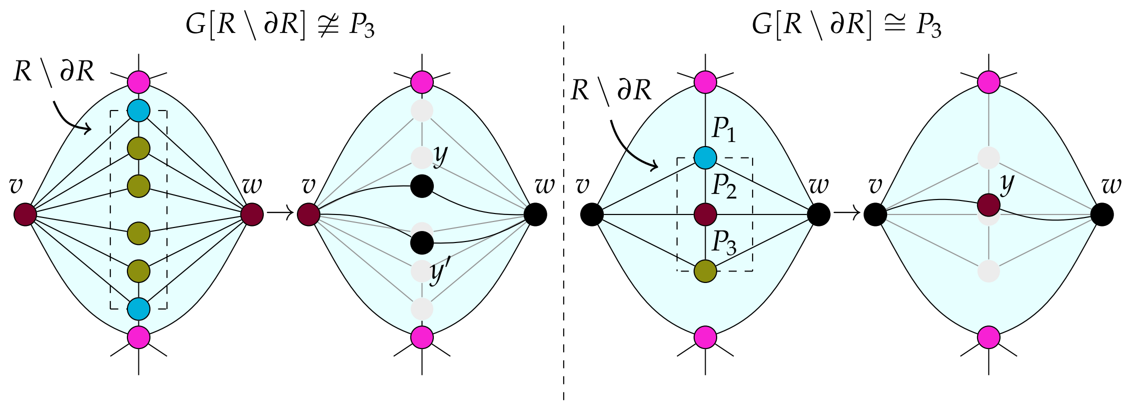


Figure 4.10: *Both cases of the application of Rule 3. Left: the vertices inside the region are not isomorphic to a P_3, which means that* **Case (2)** *will be applied and two new vertices being added. Right: They are isomorphic to a P_3 and we can replace the whole inner region with one single vertex by* **Case (1)**.

those vertices that are covered inside a region in a maximal *D-region decomposition* and those that are not. In both cases, our reduction rules bound the number of vertices to a constant size. As Lemma 4.4.6 states that for any given dominating set D, we can partition the whole graph into a linear number of regions in k, we know that we also only have linearly many vertices left in the whole graph. In particular, we show that given a Semitotal Dominating Set D of size k, there exists a maximal *D-region decomposition* $\mathfrak{R}$ such that:

1. $\mathfrak{R}$ has only at most $3|D| - 6$ regions
2. $V(\mathfrak{R})$ covers most vertices of V. There are at most $144 \cdot |D|$ vertices outside of any region.
3. each region of $\mathfrak{R}$ contains at most 139 vertices.

Combining these three statements will give us a linear kernel. Figure 4.7 depicts these three goals in *yellow*.

4.4.1 Bounding the Size of a Region

We start with a more fine-grained analysis of the impact of the different cases of Rule 2 on a vw-region. The main idea is to count the number of simple regions in the vw-region and then use the bound on the size of a simple region after Rule 3 was applied exhaustively. The bound was obtained in Corollary 4.3.1.

Lemma 4.4.1. *Given a plane Graph $G = (V, E)$ and a vw-region R, let D be a semitotal dominating set and let $\mathfrak{R}$ be a maximal D-region decomposition of G. For any vw-regions*

$R \in \mathfrak{R}$ it holds that $|N_1(v,w) \cap V(R)| \leq 4$ and these vertices lay exactly on the boundary ∂R of R.

Proof. The same argument as proposed by Alber, Fellows and Niedermeier [2], and Garnero and Sau[33, Proposition 2, Revision 2018] applies here: Let $P_1 = (v, u_1, u_2, w)$ and $P_2 = (v, u_3, u_4, w)$ be the two boundary paths enclosing the vw-region R. By the definition of a region, they have a length of at most 3. Because every vertex in R belongs to $N(v,w)$, but a vertex from $N_1(v,w)$ also has neighbors outside $N(v,w)$, it *must* lie on one of the boundary paths P_1, P_2. Therefore, R has at most four boundary vertices and $|N_1(v,w) \cap V(R)| \leq 4$.

Clearly, the worst case is achieved, if the two confluent paths P_1 and P_2 are vertex-disjoint. □

Lemma 4.4.2. *[34, See Fact 5, arXiv v2] Given a reduced plane graph $G = (V,E)$ and a vw-region R, $N_2(v,w) \cap V(R)$ can be covered by at most 6 simple regions.*

Proof. Let $P_1 = (v, u_1, u_2, w)$ and $P_2 = (v, u_3, u_4, w)$ be the two boundary paths of R. As in the previous Lemma 4.4.1, the worst case is achieved if they are vertex-disjoint. Otherwise, a smaller bound would be obtained.

By definition of $N_2(v,w)$, vertices from $N_2(v,w) \cap V(R)$ are common neighbors of v or w and one of $\{u_1, u_2, u_3, u_4\}$. By planarity, we can cover $N_2(v,w) \cap V(R)$ with at most 6 simple regions among 8 pairs of vertices (See fig. 4.11).

□

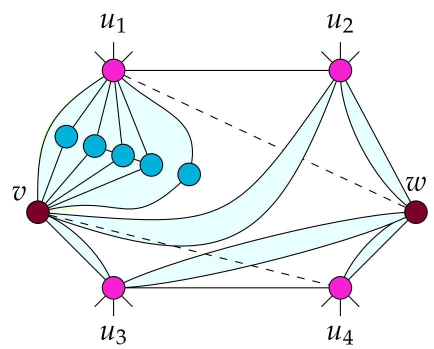


Figure 4.11: *Bounding the maximum number of simple regions inside a region $R(v,w)$. $N_2(v,w)$ is covered by 6 green (simple) regions. The two dashed edges indicate that they are among the 8 possible pairs of vertices, but a simple region between them would contradict the planarity.*

We continue by giving a constant bound on the number of simple regions that cover all $N_3(v,w)$ vertices in a given region.

Lemma 4.4.3. *Given a plane Graph $G = (V, E)$ reduced under Rule 2 and a region R(v, w), if $\mathcal{D}_v \neq \emptyset$ (resp. $\mathcal{D}_w \neq \emptyset$), $N_3(v, w) \cap V(R)$ can be covered by:*

1. 11 *simple regions if* $\mathcal{D}_w \neq \emptyset$,
2. 14 *simple regions if* $N_{2,3}(v) \cap N_3(v, w) = \emptyset$

Observe that in the first case, we can assume that no case of Rule 2 has been applied, but the claim is a direct consequence of the assumption $\mathcal{D}_v \neq \emptyset$ and $\mathcal{D}_w \neq \emptyset$. If **Case (2)** or **Case (3)** have been applied, $N_{2,3}(v, w)$ gets reduced and the second case can be applied. For the sake of completeness, we will restate (a slightly adjusted version of) the proof from Garnero and Sau [34, Fact 6, arXiv v2].

Note that this analysis provides a quick upper bound that might not be tight and executing it more sophisticated could yield a better bound because taking both $\mathcal{D}_v \neq \emptyset$ and $\mathcal{D}_w \neq \emptyset$ in concern, our regions might get even more restricted.

Proof. We partition $N_3(v, w)$ into the distinct $N_3(v, w) \setminus N(w)$, $N_3(v, w) \setminus N(v)$ and $N_3(v, w) \cap N(v) \cap N(w)$ and then analyze how many simple regions can there be in the worst case.

1. Because $\mathcal{D}_v \neq \emptyset$ there exists $D = \{v, u, u'\} \in \mathcal{D}_v$ (a smaller set will give a better bound). By definition we know that D dominates $N_3(v, w)$ and also $N_3(v, w) \setminus N(v)$. Therefore, all vertices in $N_3(v, w) \setminus N(v)$ must be neighbors of w and either u or u' and in the worst case at most three simple regions are required. By assumption, $\mathcal{D}_w \neq \emptyset$ as well, and therefore $N_3(v, w) \setminus N(w)$ is bounded by at most three simple regions, too. By planarity, we can cover the remaining common neighbors in $N_3(v, w) \cap N(v) \cap N(w)$ with at most 5 vertices and in total, we can cover $\mathbf{N_3(v, w) \cap R(v, w)}$ **by at most** $\mathbf{3 + 3 + 5 = 11}$ simple regions.
2. No property of a reduced instance is used, so the proof shown in [34] holds.

Cases 2 to 4 of Figure 4.12 visualize these simple regions around $N_3(v, w) \cap V(R)$ with simple regions in the relevant cases.[1]

□

Lemma 4.4.4 (#Vertices inside a Region after Rules 1 to 3). *Let $G = (V, E)$ be a plane graph reduced under Rules 1 to 3. Furthermore, let D be an sds of G and let $v, w \in D$. Any vw-region R contains at most 87 vertices distinct from its poles.*

Proof. By Lemmas 4.4.1 and 4.4.2 and Corollary 4.3.1 to bound the number of vertices inside a simple region, we know that $|N_1(v, w) \cap V(R)| \leq 4$. Furthermore, $|N_2(v, w) \cap V(R)| \leq 6 \cdot 4 = 24$, because after the reduction a simple region has at

[1] In revision 2018 of [34], Garnero and Sau removed this proof, because they changed Rule 2 and the overall proof was tuned.

most 4 vertices distinct from its poles and has at most 6 simpler regions covering all $N_2(v,w)$.

It is remaining to bound for $|N_3(v,w) \cap V(R)|$, but gladly we have Rule 2, which gracefully took care of them! We will now shortly do a distinction about the different cases of Rule 2. Figure 4.12 shows the worst-case amount of simple regions the individual cases can have.

Case 0: Rule 2 has **not** been applied in the following two cases: Either $\mathcal{D} \neq \emptyset$ or $\mathcal{D} = \emptyset \wedge \mathcal{D}_v \neq \emptyset \wedge \mathcal{D}_w \neq \emptyset$:

1. If $\mathcal{D} \neq \emptyset$, there exists a set $\tilde{D} = \{d_1, d_2, d_3\} \in \mathcal{D}$ of at most three vertices dominating $N_3(v,w)$. We observe that vertices from $|N_3(v,w) \cap V(R)|$ are common neighbors of either v or w (by the definition of a vw-region) and at least one vertex from $\tilde{D}$, because someone has to dominate them and we know that only the poles or vertices in $\tilde{D}$ come into question. Without violating planarity, we can span at most 6 distinct simple regions. Using the bound of simple regions (worst case shown in Corollary 4.3.1) and including $\left|\tilde{D}\right| = 3$, we can conclude $|N_3(v,w) \cap V(R)| \leq 6 \cdot 4 + 3 = 27$.
2. If $\mathcal{D} = \emptyset$, $\mathcal{D}_v \neq \emptyset$ and $\mathcal{D}_w \neq \emptyset$, we can apply Lemma 4.4.3 and although Rule 2 has not changed the graph G, we can cover R with at most 11 simple regions giving us $|N_3(v,w) \cap V(R)| \leq 11 \cdot 4 = 44$ vertices.

Case 1: If Rule 2 **Case (1)** has been applied,
$|N_2(v,w) \cap V(R)|$was entirely removed and $|N_3(v,w) \cap V(R)|$replaced by at most four new vertices v', w' and y and y'. Hence $|N_3(v,w) \cap V(R)| \leq 4$.

Case 2: If Rule 2 **Cases (2)** and **(3)** have been applied,
we know that $N_{2,3}(v) \cap N_3(v,w) \subseteq N_{2,3}(v)$ was removed and replaced by one single vertex. Applying Lemma 4.4.3, we can cover $N_3(v,w) \setminus \{v'\} \cap V(R)$ with at most 14 simple regions giving us $||N_3(v,w) \cap V(R)|| \leq 14 \cdot 4 + 1 = 57$.

All in all, as $V(R) = \{v,w\} \cup (N_1(v,w) \cup N_2(v,w) \cup N_3(v,w)) \cap V(R)$ we get

$$V(R) \leq 2 + 4 + 24 + \max(27, 44, 4, 57) = 87$$

□

4.4.2 Number of Vertices outside the Decomposition

We continue to bound the number of vertices that do not lay inside any region of a maximal *D-region decomposition* $\mathfrak{R}$, that is, we bound the size of $V \setminus V(\mathfrak{R})$. Rule 1 ensures that we only have a small amount of $N_3(v)$-pendants. We then try to cover the

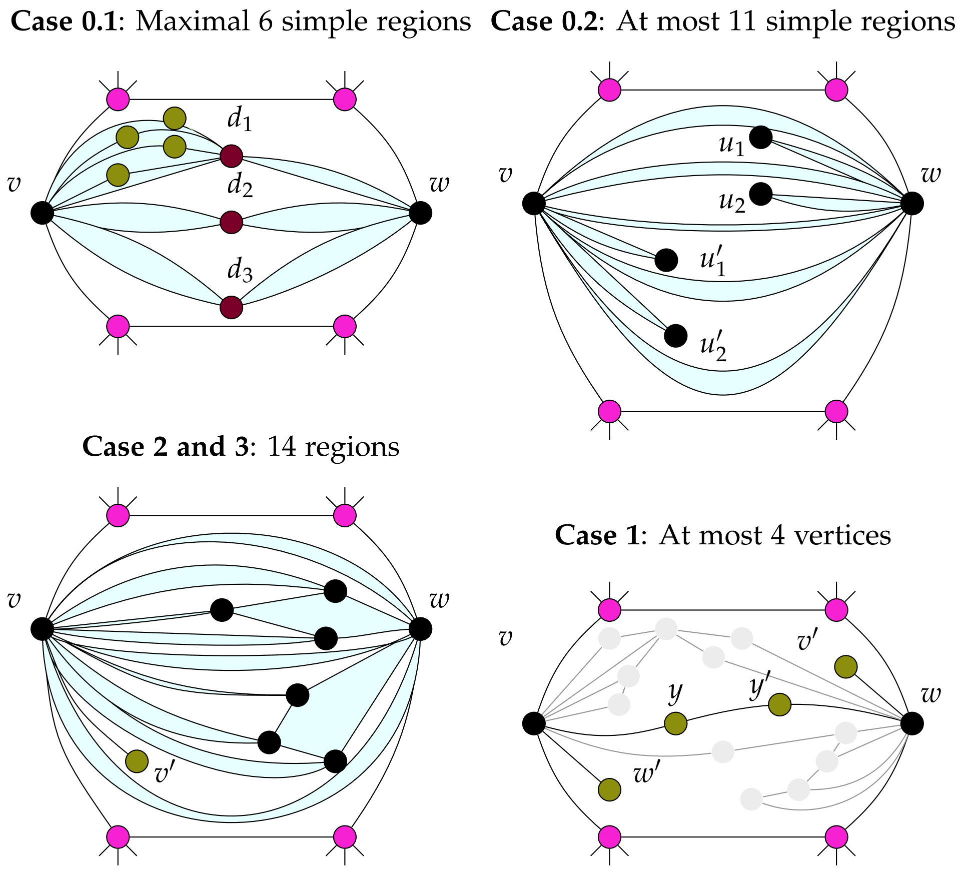


Figure 4.12: *Showing the worst case scenarios for the different cases in Lemma 4.4.4: **Case 0.1**: $\mathcal{D}$ is nonempty and we have three vertices that can dominate $N_{2,3}$ alone. They can span simple regions with the $N_3(v,w)$ vertices. **Case 1**: $N_{2,3}$ was removed and four vertices introduced. **Case 2 and 3**: At most 14 simple regions after $N_{2,3}$ has been replaced by a single v'. **Case 0.2**: $\mathcal{D}$, $\mathcal{D}_v$ and $\mathcal{D}_w$ are all empty, so the rule has not changed anything and we can cover $N_3(v,w) \cap V(R)$ with at most 11 simple regions.*

rest with as few simple regions as possible, because, by application of Rule 3, these simple regions are of constant size.
The following lemma states that no vertex from $N_1(v)$ will be outside of a maximal *D-region decomposition* which was already proven by Alber, Fellows and Niedermeier [2, Lemma 6].

Lemma 4.4.5. *Let $G = (V, E)$ be a plane graph and $\mathfrak{R}$ be a maximal D-region decomposition of a dominating set D. If $u \in N_1(v)$ for some vertex $v \in D$ then $u \in V(\mathfrak{R})$.*

In the following, we define $d_{G_\mathfrak{R}}(v) = |\{R(v,w) \in \mathfrak{R}, w \in D\}|$ to be the number of regions in $\mathfrak{R}$ having v as a pole.

Corollary 4.4.1. *Let $G = (V, E)$ be a graph and D be a set. For any maximal D-region decomposition $\mathfrak{R}$ on G it holds that $\sum_{v \in D} d_{G_{\mathfrak{R}}}(v) = 2 \cdot |\mathfrak{R}|$.*

Proof. The proof follows directly from the handshake lemma applied to the underlying multigraph $G_{\mathfrak{R}}$ where every edge between $v, w \in D$ represents a region between v and w in $\mathfrak{R}$. □

Proposition 4.4.1. *Let $G = (V, E)$ be a plane graph reduced under Rules 1 and 2 and let D be a semitotal dominating set of G. For a maximal D-region decomposition $\mathfrak{R}$, $|V \setminus (V(\mathfrak{R}) \cup D)| \leq 97|D|$*

With slight modifications, the proof given by Garnero and Sau [34, arXiv v2] will also apply for Semitotal Dominating Set. Although we assume G to be entirely reduced, the following proof only relies on Rules 1 and 3. The proof uses the observation that vertices from $N_2(v)$ that lie outside of a region must span simple regions between those from $\{v\} \cup N_1(v)$.

Proof. Again, we will follow the proof proposed by Alber, Fellows, Niedermeier [2, Proposition 2].

We use the size bound of a simple region we have proven in Corollary 4.3.1. In particular, we are going to show that $V \setminus V(\mathfrak{R}) \leq 48 \cdot |\mathfrak{R}| + 2 \cdot |D|$. Lemma 4.4.6 will then give the desired bound.

Let D be an sds, $\mathfrak{R}$ be a maximal *D-region decomposition* and $v \in D$. Since D dominates all vertices in the graph, we can consider V as $\bigcup_{v \in D} N(v)$ and thus, we only need to bound the sizes of $N_1(v) \setminus V(\mathfrak{R})$, $N_2(v) \setminus V(\mathfrak{R})$ and $N_3(v) \setminus V(\mathfrak{R})$ separately.

$\mathbf{N_3(v)}$: As we know that Rule 1 has been exhaustively applied, we trivially see that $|N_3(v)| \leq 1$ and hence,

$$\left| \bigcup_{v \in D} N_3(v) \setminus V(\mathfrak{R}) \right| \leq |D|$$

$\mathbf{N_2(v)}$: According to Garnero and Sau [34, Proposition 2], we know that $N_2(v) \setminus V(\mathfrak{R})$ in a reduced graph can be covered by at most $4d_{G_{\mathfrak{R}}}(v)$ simple regions between v and some vertices from $N_1(v)$ on the boundary of a region in $\mathfrak{R}$. As their argument Figure 4.13 gives some intuition, but intuitively, we can span two simple regions to each of the vertices from $N_1(v)$ on the two border vertices for each $R \in \mathfrak{R}$.

Because we assume G to be reduced, by Corollary 4.3.1 a simple region can have at least 4 vertices distinct from its poles and hence,

$$\left|\bigcup_{v\in D} N_2(v)\setminus V(\mathfrak{R})\right| \leq 4\sum_{v\in D} 4\cdot d_{G_{\mathfrak{R}}}(v)$$
$$= 16\cdot\sum_{v\in D} d_{G_{\mathfrak{R}}}(v) \tag{4.10}$$
$$\overset{\text{Cor. 4.4.2}}{\leq} 32|\mathfrak{R}|$$

$\mathbf{N_1(v)}$: Because every semitotal dominating set is also a dominating set, we can apply Lemma 4.4.5 and conclude that $N_1(v)\subseteq V(\mathfrak{R})$. Hence,

$$\left|\bigcup_{v\in D} N_1(v)\setminus V(\mathfrak{R})\right| = 0$$

Summing up these three upper bounds for each $v\in D$ we obtain the result using the equation from Lemma 4.4.6:

$$\begin{aligned}
|V\setminus V(\mathfrak{R})\cup D)| &\leq 32\cdot|\mathfrak{R}|+|D| && \text{(Lemma 4.4.6)}\\
&\leq 32\cdot(3|D|-6)+|D|\\
&\leq 96|D|+|D|\\
&= 97|D| && (4.11)
\end{aligned}$$

□

4.4.3 Bounding the Number of Regions

We are now utilizing the final tool in our toolbox. Alber, Fellows and Niedermeier [2, Proposition 1] gave an explicit greedy algorithm to construct a maximal *D-region decomposition* for a DOMINATING SET. The existence of this algorithm is the core for all following works because they always involve region decompositions. For example, Garnero and Sau used it for PLANAR RED-BLUE DOMINATING SET [35] and TOTAL DOMINATING SET [34]. This missing puzzle piece will now assemble everything we have set up so far giving us the linear kernelization we are looking for.

For the following lemma, Alber, Fellows and Niedermeier[2] required a reduced instance and their reduction rules for PLANAR DOMINATING SET differed from ours. Luckily, they do not rely on any specific properties following from a reduced graph and therefore, we can just use it for our kernelization algorithm as well.

This was already observed by Garnero and Sau [34] and a more formal proof along with the description of the algorithm was provided.

Because every semitotal dominating set is indeed also a dominating set, we can safely apply it to PLANAR SEMITOTAL DOMINATING SET as well.

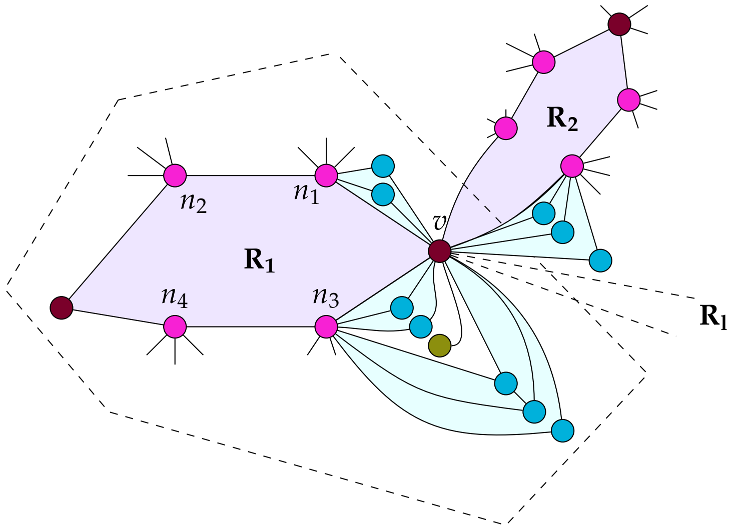


Figure 4.13: *Bounding the number of $N_2(v)$-vertices around a dominating vertex v given a maximal D-region decomposition $\mathfrak{R}$. v is a pole of $R_1, R_2, \ldots R_j$ and can span simple regions with the help of $N_2(v)$-vertices to at most two $N_1(v)$-vertices per R_i. Each region has at most four vertices in $N_1(v,w) \subseteq N_1(v)$ on the boundary of R_j, but only at most two can be used for a simple region: For Example trying to construct a simple region between v and n_2 would contradict the maximality of $\mathfrak{R}$. Furthermore, because rule Rule 1 has removed all but **one** vertex from $N_3(v)$, we intuitively can span two regions to each of the $N_1(v)$-vertices. Furthermore, the size of these simple regions is bounded after the application of Rule 3.*

Lemma 4.4.6 ([2, Proposition 1 and Lemma 5]). *Let G be a reduced plane graph and let D be a* Semitotal Dominating Set *with $|D| \geq 3$. There is a maximal D-region decomposition of G such that $|R| \leq 3 \cdot |D| - 6$.*

Lemma 4.4.7 (Running Time of Reduction Procedure).

Proof. □

By utilizing all the previous results, we are now finally ready to proof the Theorem 1:

Theorem 1. *The* Semitotal Dominating Set *problem parameterized by solution size admits a linear kernel on planar graphs. There exists a polynomial-time algorithm that, given a planar graph (G,k), either correctly reports that (G,k) is a NO-instance or returns an equivalent instance (G',k) such that $|V(G')| \leq 359 \cdot k$.*

Proof. Let $G = (V, E)$ be the plane input graph and $G' = (V', E')$ be the graph obtained by the exhaustive application of Rules 1 to 3. As none of our rules change the size of a possible solution $D' \subseteq V'$ in G', we know by Lemma 4.3.1, Lemma 4.3.3 and

Lemma 4.3.4 that G' has a Semitotal Dominating Set of size k if and only if G has an sds set of size k. Furthermore, by Lemma 4.4.7, the preprocessing procedure runs in polynomial time.

By taking the size of each region proven in Proposition 4.4.1, the total number of regions in a maximal *D-region decomposition* (Lemma 4.4.6) and the number of vertices that can lay outside of any region (Proposition 4.4.1), we obtain the following bound:

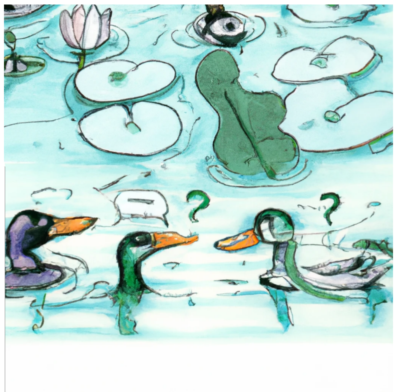
$$87 \cdot (3k - 6) + 97 \cdot k + k < 359 \cdot k \tag{4.12}$$

If $|V(G')| > 359 \cdot k$ G we replace G' by one single vertex v, which is trivially a *no*-instance, because v has no witness to form an sds.

□

CHAPTER 5

OPEN QUESTIONS AND FURTHER RESEARCH



"Peter does what he usually does when he doesn't know what to do next: he gives up."

Qualityland, *Marc-Uwe Kling*

This chapter discusses open problems for Semitotal Dominating Set for further research.

Improving Kernel With $359 \cdot k$, the size of our kernel is very high and the theoretical bounds are too large for practical applications. It would be interesting to improve them it. One idea would be the following:

Rule 2 uses the fact that $N(v, w)$ can be dominated by at most four vertices: v,w and two neighbors as a witness. Observe that if $d(v, w) \leq 3$, this witness might be shared in an sds, because choosing one single witness on the path from v to w is sufficient to semitotally dominate $N(v, w)$. Therefore $\mathcal{D}$, $\mathcal{D}_v$ and $\mathcal{D}_w$ could be redefined to contain sets of size at most two (instead of three) which will improve the reduction. Note that in our analysis, the poles of a region R in the *D-region decomposition* $\mathfrak{R}$ satisfy $d(v, w) \leq 3$ by definition. therefore, requiring $d(v, w) \leq 3$ would be ok and we can still assume that Rule 2 has been applied for any $R \in \mathfrak{R}$. $d(v, w)$ can be calculated in linear time and would not blow up our runtime.

Experimental Results According to experiments shown by Alber et al. [2], more than 79% of the vertices and 88% of the edges have been removed by the reduction rules from a sample set of random planar graphs with up to 4000 vertices. As our theoretical bounds are quite high, it would be interesting to see how much our reduction rules remove in practice.

Other Open Prolems The classical complexities for *dually chordal* and *tolerance* graphs have already been asked for by Galby et al. [31] and are still open.

Furthermore, it would be interesting to complement the parameterized complexities on the hard classes we have started in this work. open are those for *circle*, *chordal bipartite* and *undirected path* graphs. We think that SDOM on *chordal bipartite* graphs is at least W_1-hard when parameterized by solution size, but we have been unable to prove it. Furthermore, the reduction for *circle* graphs [53] to show **NP**-completeness depends on the input size, but DOM and TDOM have already been shown to be W_1-hard on *circle* graphs [11]. Maybe these reductions can be adjusted to show W_1-hardness for SDOM as well. Last but not least, there exists an fpt algorithm for DOM on *undirected path* graphs [27]. Does it transfer to SDOM as well?

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