DEPARTMENT OF INFORMATICS

TECHNICAL UNIVERSITY MUNICH

Master Thesis

On the Parametrized Complexity of Semitotal Domination on Graph Classes

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A Survey and a Linear Kernel For Planar Graphs

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I confirm that this master thesis is my own work and I have documented all sources and material used.

København, August 23, 2022 Lukas Retschmeier

Acknowledgments

Contents

Abstract

Abstract all the way

Chapter 1

Introduction

Parametrized Complexity emerging branch. Books about that

Semitotal domination introduced by

1.1 Content of the thesis

In this thesis we continue the systematic analysis of the SEMITOTAL DOMINATING SET problem by focusing on the parametrized complexity of the problem.

Although the problem already had a lot of attention regarding classical complexity (CITE), only few results are currently known for the parametrized variant.

As far as we have seen, even the w-hardness of the general case has not been explicitely been proofen in the literature.

In this thesis we continue the journey towards a systematic analysis by stating some hardness results for specific graph classes for the problem.

Our contributions Our main contributations consist of first showing the $w[2]$-hardness of SEMITOTAL DOMINATING SET for XXXX graphs.

As the DOMINATING SET problem and the TOTAL DOMINATING SET problem both admit a linear kernel for planar graphs, it is interesting to analyse wether this results also holds for the Semitotal Dominating Set problem which lays in between these two.

Having these kernels also for other variants like Edge Dominating Set, Efficient Dominating Set, Connected Dominating Set, Red-Blue Dominating Set lent us a great confidence that the result will also work for Semitotal Dominating Set on planar graphs.

Following the approach from ... which alraedy relies on the technique given in, we give some simple data reduction rules for Semitotal Dominating Set on planar graphs leading to a linear kernel. More precisely, we are going to proof the following central theorem of this thesis:

Theorem 1. *The* Semitotal Dominating Set *problem parametrized by solution size admits a linear kernel on planar graphs. There exists a polynomial-time algorithms that given a planar graph (G, k), either correctly reports that (G, k) is a NO-instance or returns an equivalent instance (G', k) such that XXX.*

Dominating Set problem and Total Dominating Set problem, both already

Chapter 2

Preliminaries

We start by recapping some basic notation in Graph Theory and Parametrized Complexity.

Continuing an intensive study of parametrized complexity of that problem.

2.1 Graph Theory

We quickly state the following definitions given by [Die10, p. xxx].

Definition 1 (Graph). *A graph is a pair $G = (V, E)$ of two sets where V denotes the vertices and $E \subseteq V \times C$ the edges of the graph. A vertex $v \in V$ is incident with an edge $e \in E$ if $v \in e$. Two vertices x, y are adjacent, or neighbours, if $\{x, y\} \in E$.*

Definition 2 (Special Graph Notations [Die10, p. 27]). *A simple Graph*

A directed Graph is a graph

A Multi Graph

A Planar Graph

Definition 3 (Adjacent Vertices).

Definition 4 (Closed and Open Neighborhoods of Vertices)**.** *+ Sets*

Definition 5 (Induced Subgraph)**.** *asd*

Special Graph Classes

We call the class of graphs without any special restrictions "General Graphs".

Definition 6 (r-partite Graphs)**.** *Let $r \geq 2$ be an integer. A Graph $G = (V, E)$ is called "r-partite" if V admits a parititon into r classes such that every edge has its ends in different classes: Vertices in the same partition class must not be adjacent.*

For the case $r = 2$ we say that the G is "bipartite"

Definition 7 (Chordal Graphs)**.**

Definition 8 (Split Graphs)**.**

2.2 Parametrized Complexity

2.2.1 Fixed Parameter Tractability

Fixed Parameter Intractability: The W Hierarchy

2.2.2 Kernelization

Chapter 3

On Parametrized Dominating Set

3.1 Semitotal Domination

Semitotal Dominating Set

Definition, dominating number

Complexity Status of Semitotal Dominating Set

3.2 *w[i]*-Intractibility

Now some w[i] hard classes.

3.2.1 Warm-Up: *W[2]*-hard on General Graphs

As any bipartite graphswith bipartition can be split further into r-partite graphsthis results also implies the $w[1]$-hardness of r-partite graphs

3.2.2 *W[2]*-hard on Bipartite Graphs

Definition 9 (Bipartite Graph, [BM08, p.5])**.** *A bipartite graphs is a Graph G whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y. Such a partition (X,Y) is called a bipartition of G.*

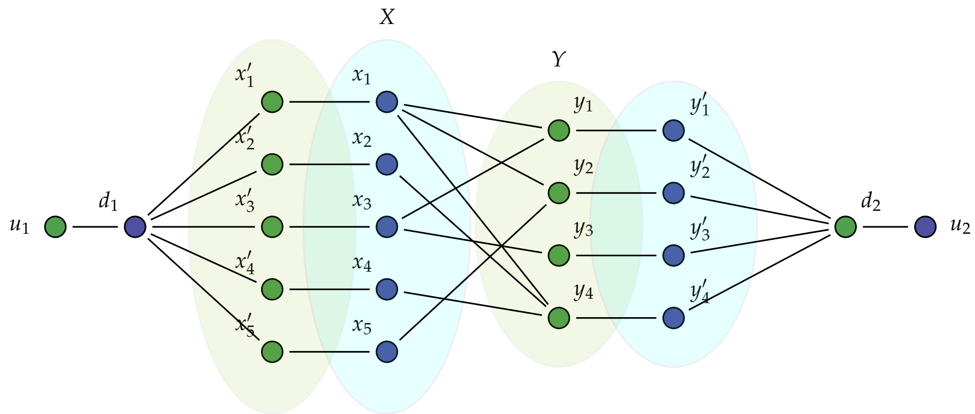

Figure 3.1: Constructing G′ from a bipartite Graph G by duplicating the vertices and adding a dominating tail

Theorem 2. *Semitotal Dominating Set is* $\omega[2]$ *hard for bipartite Graphs*

Proof. Given a bipartite Graph $G = (\{X \cup Y\}, E)$, we construct a bipartite Graph G′ in the following way:

1. For each vertex $x_i \in X$ we add a new vertex x'_i and an edge (x_i, x'_i) in between.

2. For each vertex $y_j \in Y$ we add a new vertex y'_j and an edge (y_j, y'_j) in between.

3. We add two P_1, namely (u_1, d_1) and (u_2, d_2), and connect them with all (d_1, x'_i) and (d_2, y'_j) respectively.

Observation: G′ is clearly bipartite as all y'_j and x'_i form again an Independent Set. Setting $X' = X \cup \{u_2\} \cup \bigcup y'_i$ and $Y' = Y \cup \{u_1\} \cup \bigcup x'_i$ form the partitions of bipartite G′.

Corollary 1. *G has a Dominating Set of size k iff G has a Semitotal Dominating Set of size* $k' = k + 2$

$\Rightarrow$: Asume there exists a Dominating Set D in G with size k. $DS = D \cup \{d_1, d_2\}$ is a Semitotal Dominating Set in G' with size $k' = k + 2$, because d_1 dominates u_1 and all x'_i; d_2 dominates u_2 and all y'_i. Hence, it is a Semitotal Dominating Set, because $\forall v \in (D \cap X) : d(v, d_1) = 2$ and $\forall v \in (D \cap Y) : d(v, d_2) = 2$

$\Leftarrow$: On the contrary, asume any Semitotal Dominating Set SD in G' with size k'. WLOG we can asume that $u_1, u_2 \notin DS$.

Our construction forces d_1, $d_2 \in DS$. Because all x'_i are only important in dominating x_i (y'_i for y_i resp.) as $d_1, d_2 \in DS$. If $x'_i \in DS$ we simply exchange it with x_i (for y'_i and y_i respectively) in our DS keeping the size of the dominating set. $D = DS \setminus \{d_1, d_2\}$ give us a Dominating Set in G with size $k = k' - 2$

As G′ can be constructed in $\mathcal{O}(n)$ and parameter k is only blown up by a constant, this reduction is a FPT reduction. As Dominating Set is $w[2]$ hard for bipartite Graphs[1] so is Semitotal Dominating Set. □

3.2.3 *W[2]*-hard on Chordal Graphs

3.2.4 *W[2]*-hard on Split Graphs

[1]Citation needed!

Chapter 4

A Linear Kernel for Planar Semitotal Domination

TODO Alber et. al, Total Domination.

4.1 The Main Idea and The Big Picture

4.2 Definitions

Definition 10. *Let $G = (V, E)$ be a graph and let $v \in V$. We denote by $N(v) = \{u \in V : \{u, v\} \in E\}$ the neighborhood of v. We split $N(v)$ into three subsets:*

$$N_1(v) = \{u \in N(v) : N(u) \setminus (N(v) \cup \{v\}) \neq \emptyset\} \tag{4.1}$$

$$N_2(v) = \{u \in N(v) \setminus N_1(v) : N(u) \cap N_1(v) \neq \emptyset\} \tag{4.2}$$

$$N_3(v) = N(v) \setminus (N_1(v) \cup N_2(v)) \tag{4.3}$$

In order to inhance future readability, we add some syntactical sugar. For $i, j \in [1, 3]$, we denote $N_{i,j}(v) = N_i(v) \cup N_j(v)$.

Definition 11. *Let $G = (V, E)$ be a graph and $v, w \in V$. We denote by $N(v, w) = N(v) \cup N(w)$ the neighborhood of the pair v, w. We split $N(v, w)$ into three subsets:*

$$N_1(v, w) = \{u \in N(v, w) \mid N(u) \setminus (N(v, w) \cup \{v, w\}) \neq \emptyset\} \tag{4.4}$$

$$N_2(v, w) = \{u \in N(v, w) \setminus N_1(v, w) \mid N(u) \cap N_1(v, w) \neq \emptyset\} \tag{4.5}$$

$$N_3(v, w) = N(v, w) \setminus (N_1(v, w) \cup N_2(v, w)) \tag{4.6}$$

Again, for $i, j \in [1, 3]$, we denote $N_{i,j}(v, w) = N_i(v, w) \cup N_j(v, w)$.

4.3 Deducing Reduction Rules

4.3.1 Reduction Rule I: Getting Rid of unneccessary $N_3(v)$ vertices

Rule 1. *Let $G = (V, E)$ be a graph and let $v \in V$. If $|N_3(v)| \geq 1$:*

- *remove $N_3(v)$ from G,*
- *add a vertex v' and an edge $\{v, v'\}$*

Lemma 3. *Let $G = (V, E)$ be a a graph and let $v \in V$. If G' is the graph obtained by applying Rule 1 on V, then G has SDS of size k if and only if G' has one.*

Proof. This will be the proof for this lemma X □

4.3.2 Reduction Rule II: Shrinking the Size of a Region

Extending the approach for a linear kernel for Dominating Set proposed by Alber et al. in [AFN04], Garnero and Stau transferred these results in [GS18] to the Total Dominating Set problem.

Their idea was to relax the reduction rules in such a way that the witness properties for total domination are being preserved.

Following their approach in one of the first verions of [GS14], we stating reduction rules that. Interestingly, the reduction rules given in the latest version of this paper was not transferable to Semitotal Dominating Set, but an older version giving slightly easier reduction rules could be adjusted to our problem.

which relies on the technique first introduced by Alber et al we try to reduce the neighborhood for two given vertices v and w

This observation gives motivation to define the following sets:

$$\mathcal{D} = \{\tilde{D} \subseteq N_{2,3}(v,w) \mid N_3(v,w) \subseteq \bigcup_{v \in \tilde{D}} N(v),\ |\tilde{D}| \leq 3\} \tag{4.7}$$

$$\mathcal{D}_v = \{\tilde{D} \subseteq N_{2,3}(v,w) \cup \{v\} \mid N_3(v,w) \subseteq \bigcup_{v \in \tilde{D}} N(v),\ |\tilde{D}| \leq 3,\ v \in \tilde{D}\} \tag{4.8}$$

$$\mathcal{D}_w = \{\tilde{D} \subseteq N_{2,3}(v,w) \cup \{w\} \mid N_3(v,w) \subseteq \bigcup_{v \in \tilde{D}} N(v),\ |\tilde{D}| \leq 3,\ w \in \tilde{D}\} \tag{4.9}$$

Note, that

Rule 2. *Let $G = (V, E)$ be a graph and two distinct $v, w \in V$. If $\mathcal{D} = \emptyset$ we apply the following:*

Case 1: *if $\mathcal{D}_v = \emptyset$ and $D_w = \emptyset$*

- *Remove $N_{2,3}(v,w)$*
- *Add vertices v' and w' and two edges $\{v, v'\}$ and $\{w, w'\}$*
- *If there was a common neighbor of v and w in $N_{2,3}(v,w)$ add another vertex y and two connecting edges $\{v, y\}$ and $\{y, w\}$*

Case 2: *if $\mathcal{D}_v \neq \emptyset$ and $D_w \neq \emptyset$*
Do nothing[1]

Case 3: *if $\mathcal{D}_v \neq \emptyset$ and $D_w = \emptyset$*

- *Remove $N_{2,3}(v) \Cap N_3(v,w)$*
- *Add $\{v, v'\}$*

Case 4: *if $\mathcal{D}_v = \emptyset$ and $D_w \neq \emptyset$*
*This case is symmetrical to **Case 3**.*

[1]Originally, reduce Simple Regions [STAU]

Before proofing the correctness of this reduction we will give some intuition behind these rules.

Lemma 4. *Let $G = (V, E)$ be a plane graph, $v, w \in V$ and $G' = (V', E')$ be the graph obtained after application of Rule 2 on the pair $\{v, w\}$. Then G has SDS of size k if and only if G' has SDS of size k.*

4.3.3 Reduction Rule III: Shrinking Simple Regions

Rule 3. *Let $G = (V, E)$ be a plane graph, $v, w\ inV$ and R be a simple region between v and w. If $|V(R) \setminus \{v, w\}| \geq 7$*

- *Remove $N_3 * (v, w)$*
- *Add two vertices h_1 and h_2 and four edges $\{v, h_1\}$, $\{v, H_2\}$, $\{w, h_1\}$ and $\{w, h_2\}$*

Lemma 5 (Correctness of Rule 3)**.** *Let $G = (V, E)$ be a plane graph, $v, w \in V$ and $G' = (V', E')$ be the graph obtained after application of Rule 3 on the pair $\{v, w\}$. Then G has SDS of size k if and only if G' has SDS of size k.*

The application of Rule 3 gives us a bound on the number of vertices inside a simple region.

Corollary 2. *Let $G = (V, E)$ be a graph, $v, w \in V$ and R a simple region between v and w. If* **??** *has been applied, the simply region has size at most XXXX.*

4.3.4 Computing Maximal Simple Regions between two vertices

For the sake of completeness, we state an algorithm how a maximal simple regionbe-tween two vertices $v, w \in V$ can be computed in time $\mathcal{O}(d(v) + d(w))$:

4.4 Bounding the Size of the Kernel

Lemma 6. *Given a plane Graph $G = (V, E)$ reduced under R2 and a region R(v, w), if $\mathcal{D}_v \neq$ (resp. $\mathcal{D}_w \neq \emptyset$), $N_3(v, w) \cap V(R)$ can be covered by:*

- 11 *simple regionif $\mathcal{D}_w \neq \emptyset$,*
- 14 *simple regionif $N_{2,3}(v) \cap N_3(v, w) = \emptyset$*

Proof. □

Lemma 7 (#Vertices inside a Region after Rules 1 and 2 and ??). *Let $G = (V, E)$ be a plane graph reduced under Rules 1 to 3. Furthermore, let D be a SDS of G and let $v, w \in D$. Then a region R between v and w has size at most xxx.*

Proof. □

Lemma 8 (#Vertices outside a Region)**.**

Proof. □

Lemma 9 (Number of Regions in a Maximum Region Decomposition)**.**

Proof. □

Lemma 10 (Running Time of Reduction Procedure)**.**

Proof. □

We now have all the weapons set up to proof Theorem 1:

Theorem 1. *The* SEMITOTAL DOMINATING SET *problem parametrized by solution size admits a linear kernel on planar graphs. There exists a polynomial-time algorithms that given a planar graph (G, k), either correctly reports that (G, k) is a NO-instance or returns an equivalent instance (G', k) such that XXX.*

Proof. XXXXXXsdfjöakfjölajdflöajsfölajsöflkjaslödjaöslf jöalkdjf aölsfjaöslfjöalsfjaölsdföalkföalksdjföaksdföalksdföalksföalksdfaksdf ajkf öaklsdf öalkdf öal lökjadf □

Chapter 5

Open Questions and Further Research

* Chordal Bipartite Grap hs a very interesting case. * Improve the Kernel Bound

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