DEPARTMENT OF INFORMATICS

TECHNICAL UNIVERSITY MUNICH

Master Thesis

Collection of Proofs

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On Parametrized Semitotal Dominating Set

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I confirm that this master thesis is my own work and I have documented all sources and material used.

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Acknowledgments

Abstract

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1 Proofs

(Required Definitions)

1. Chordal Graphs
2. Graph Theory: Open and Closed Neighborhood

Theorem 1. *A Graph is chordal if and only if there exists a Total Domination Order*

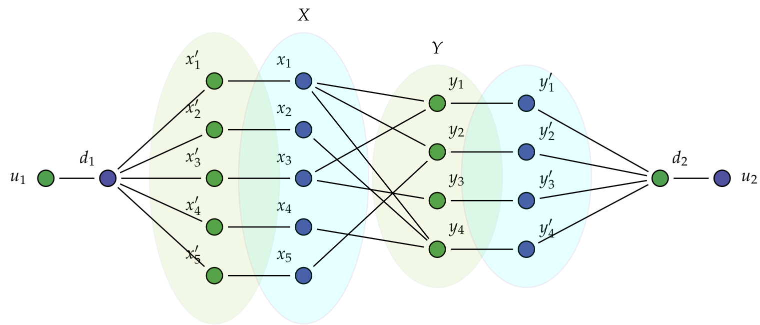

Figure 1.1: Constructing G′ from a bipartite Graph G by duplicating the vertices and adding a dominating tail

Theorem 2. *Semitotal Dominating Set is* $\omega[2]$ *hard for bipartite Graphs*

Proof. Given a bipartite Graph $G = (\{X \cup Y\}, E)$ where X and Y are Independent Sets, we construct a bipartite Graph G′ in the following way:

1. For each vertex $x_i \in X$ we add a new vertex x'_i and an edge (x_i, x'_i) in between.
2. For each vertex $y_j \in Y$ we add a new vertex y'_j and an edge (y_j, y'_j) in between.
3. We add two P_1 from (u_1, d_1) and (u_2, d_2) and connect them with all (d_1, x'_i) and (d_2, y'_j) respectively.

Observation: G′ is clearly bipartite as all y'_j and x'_i form again an Independent Set. Setting $X' = X \cup \{u_2\} \cup \bigcup y'_i$ and $Y' = Y \cup \{u_1\} \cup \bigcup x'_i$ form the partitions of bipartite G′.

Corollary 1. *G has a Dominating Set of size k iff G has a Semitotal Dominating Set of size* $k' = k + 2$

⇒: Asume there exists a Dominating Set D in G with size k. $DS = D \cup \{d_1, d_2\}$ is a Semitotal Dominating Set in G' with size $k' = k + 2$, because d_1 dominates u_1 and all x'_i; d_2 dominates u_2 and all y'_i. Hence, it is a Semitotal Dominating Set, because $\forall v \in (D \cap X) : d(v, d_1) = 2$ and $\forall v \in (D \cap Y) : d(v, d_2) = 2$

⇐: On the contrary, asume any Semitotal Dominating Set SD in G' with size k'. WLOG we can asume that $u_1, u_2 \notin DS$.

Our construction forces d_1, $d_2 \in DS$. Because all x'_i are only important in dominating x_i (y'_i for y_i resp.) as $d_1, d_2 \in DS$. If $x'_i \in DS$ we simply exchange it with x_i (for y'_i and y_i respectively) in our S keeping the size of the dominating set. $D = DS \setminus \{d_1, d_2\}$ give us a Dominating Set in G with size $k = k' - 2$

As G′ can be constructed in $\mathcal{O}(n)$ and parameter k is only blown up by a constant, this reduction is a FPT reduction. As Dominating Set is $w[2]$ hard for bipartite Graphs (CITE) so is Semitotal Dominating Set. □

Theorem 3. *Semitotal Dominating Set is $\omega[2]$ hard on Chordal Graphs*

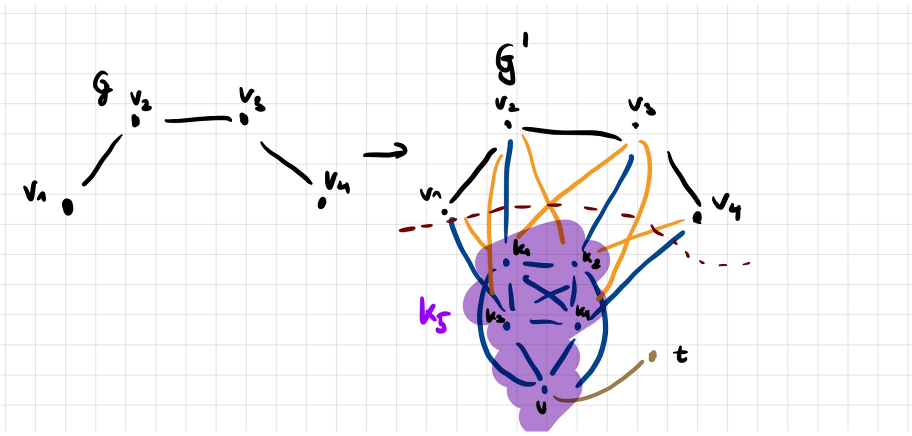

Figure 1.2: Constructing G' by adding a K_5 and the vertex t

Proof. Given a chordal graph $G = (V = \{v_1, ..., v_n\}, E)$, we construct a chordal graph G′ as described below (See also fig 1.2):

1. Add a K_{n+1} consisting of the vertices $\{k_1, ..., k_n, u\}$ and add an edge (v_i, k_i) to each vertex v_i of G. One vertex u in the clique will remain untouched.

2. Add one additional vertex t and connect it with u.

3. For all vertices v_i in G, add a new edge (n, k_i) for all $n \in N(v_i)$.

Corollary 2. *$N(v_i) \in G$ form a clique iff $N(v_i)$ forms a clique in G'*

Proof. Asuming that $N(v_i)$ forms a clique in G, we show that it also forms a clique in G′ by induction over the number of neighbors $z = abs(N(v_i))$ in G.

- $z = 0$: Holds trivially as we do not have a neighbor in G and in G′ the connected k_i forms a P_1, hence a clique.

- $z = z + 1$:

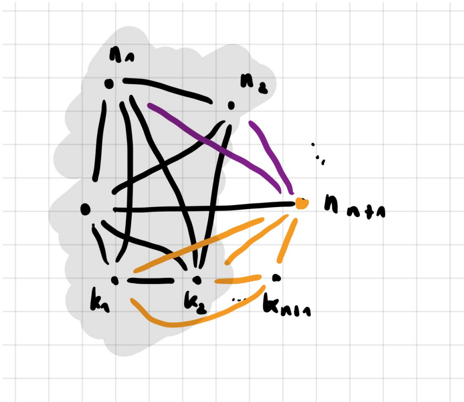

Figure 1.3: Induction Step

By IH, we already know that all neigbors $n_1, ..., n_z$ form a clique together with their vertices in k_i. As $k_{z+1}, v_{z+1} \in N(v_i)$ now also in G′, we show that $N(v_i)$ still forms clique in G′.

Let k_i be the vertex that was connected with n_i during step 1. All we have to show is that v_{z+1} and k_{z+1} extend our previous clique, hence are fully connected with $N(v_i)$.

v_{z+1} connects to $N(v_i)$ in G by assumption. By our construction, there exists an edge to $k_1, ..., k_z$, because we add an edge (n_{z+1}, k_i) if there is an edge from (n_{z+1}, n_i). (See fig 1.3)

k_{z+1} form a complete subgraph with the other k_i and is connected to all n_i by construction because the edge (n_{z+1}, n_i) exists.

Therefore, $N(v_i)$ will also form a clique in G'.

On the other side, if $N(v_i)$ forms a clique in G′, the vertices of $N(v_i)$ in G just form an induced subgraph of G′, hence preserving the clique.

■

Corollary 3. *G is Chordal iff G′ is chordal.*

Proof. $\Rightarrow$: Asume G chordal. Then exists a total elemination order $o = (v_1, ..., v_n)$ in G where removing v_j sequentially returns cliques in $N(v_i)$.

Define $o' = (v_1, ..., v_n, k_1, ..., k_n, u, t)$. Applying corollary 2 states that $(v_1, ... v_n)$ is a partial elemination order and as the rest is part of a clique with an additional vertex of degree 1, o′ is a total elemination order $o = (v_1, ..., b_n)$ for G'.

$\Leftarrow$: Holds as o′ is always a total elemination order in G′ and removing the complete subgraph K_{n+1} and u gives a total elemination order in G. ■

Corollary 4. *G has a Dominating Set of size k iff G' has a dominating set of size $k+1$*

Proof. Asume a Dominating Set D of size k in G. $D \cup \{u\}$ is a Semitotal Dominating Set in G′ of size k + 1, because u dominates t and for each $v \in DS : d(v,u) \leq 2$.

Contrary, asume a Semitotal Dominating Set SD in G'. In order to dominate t, $u \in SD$ must hold, hence already dominating the complete subgraph K_{n+1}. If a vertex $k_i \in SD$, we exchange it with v_i still preserving a Dominating Set. Taking $D = SD - \{u\}$ gives our desired Dominating Set of size k. ■

As this reduction runs in FPT time and the parameter is only bounded by a function of k, this is a FPT reduction. As Dominating Set on Chordal Graphs is $w[2] - hard$, so is SDS on Chordal Graphs.

□

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