



DEPARTMENT OF INFORMATICS

TECHNICAL UNIVERSITY MUNICH

Master Thesis

**On the Parametrized Complexity of
Semitotal Domination on Graph Classes**

Lukas Retschmeier





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**On the Parametrized Complexity of
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**Über die Parametrisierte Komplexität des
Problems der halbtotalen stabilen Menge
auf Graphklassen**

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Submission Date: October 31, 2022



I confirm that this master thesis is my own work and I have documented all sources and material used.

København S
October 31, 2022

Lukas Retschmeier

Acknowledgments

CONTENTS

ABSTRACT

Abstract

For a graph $G = (V, E)$, a set E is called a SEMITOTAL DOMINATING SET, if D is a dominating set and every vertex $v \in D$ is within distance two to another vertex $v' \in D$. The MINIMUM SEMITOTAL DOMINATING SET problem is to find a semi-total dominating set of minimum cardinality. The semitotal domination number $\gamma_{t2}(G)$ is the minimum cardinality of a semitotal dominating set and is squeezed between the domination number $\gamma(G)$ and the total domination number $\gamma_t(G)$.

After the problem was introduced by Goddard, Henning and McPillan in [Goddard2014], NP-completeness was shown for general graphs ([Henning2019]), split graphs ([Henning2019]), planar graphs ([Henning2019]), chordal bipartite graphs ([Henning2019]), circle graphs ([Kloks2021]) and subcubic line graphs of bipartite graphs ([Galby2020]). On the other side, there exist polynomial-time algorithms for AT-free graphs ([Kloks2021]), *graphs of bounded mim-width* ([Galby2020]), graphs of bounded clique-width ([Courcelle1990]), and interval graphs ([Henning2019]).

In this thesis, we start the systematic look through the lens of *parametrized complexity* by showing that SEMITOTAL DOMINATING SET is $\omega[2]$ -hard for bipartite graphs and chordal graphs. By applying the techniques proposed in [Alber2004] and [Garnero2018] for DOMINATING SET and TOTAL DOMINATING SET, we are going to construct a $561k$ kernel for SEMITOTAL DOMINATING SET in planar graphs. This result further complements known linear kernels for other domination problems like PLANAR CONNECTED DOMINATING SET, PLANAR RED-BLUE DOMINATING SET, PLANAR EFFICIENT DOMINATING SET, PLANAR EDGE DOMINATING SET, INDEPENDENT DOMINATING SET and PLANAR DIRECTED DOMINATING SET.

Keywords: Domination; Semitotal Domination; Parametrized Complexity; Planar Graphs; Linear Kernel

ZUSAMMENFASSUNG

Abstract

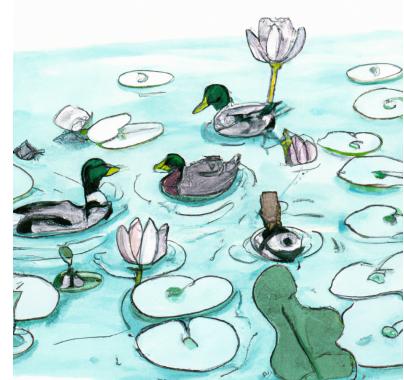
Hier kommt noch ein weiterer Abstract rein.

Schlagworte: Stabile Menge; Halbtotale Stabile Menge; Parametrisierte Komplexität; Plättbare Graphen; Linearer Problemkern

Abstract all the way

CHAPTER 1

INTRODUCTION



To do select another quote

Lewis Carroll, XXXX

Parametrized Complexity emerging branch. Books about that

Semitotal domination was introduced by

Quack! Quack! Idea: Lake with stones, and a family of ducks of fixed size want to occupy the lake so that no other clan tries to take it over. Rules: * A duck can quack freeing up neighboring stones. * Ducks don't like to be alone and want to quack together. So for every duck there must be another duck that is not further than two stones away. Q: Can our ducklings occupy the whole lake while not feeling lonely?

TODO A demo instance next to each other

Problem Definition

DOMINATING SET [Cygan2015]

Input:

Graph $G = (V, E)$ and an integer k

Question:

Is there a set $X \subseteq V$ of size at most k such that
 $N[X] = V$?

SEMITOTAL DOMINATING SET [Goddard2014]

Input: Graph $G = (V, E)$ and an integer k
Question: Is there a subset $X \subseteq V$ of size at most k such that $N[X] = V$ and for all $d_1 \in X$ there exists another $d_2 \in X$ such that $d(d_1, d_2) \leq 2$?

TOTAL DOMINATING SET [Cygan2015]

Input: Graph $G = (V, E)$ and an integer k
Question: Does there exist a set $X \subseteq V$ of at most k vertices of G such that for every $u \in V(G)$ there exists $v \in X$ with $\{u, v\} \in E$

1.1 Content of the thesis

In this thesis, we continue the systematic analysis of the SEMITOTAL DOMINATING SET problem by focusing on the parametrized complexity of the problem.

Although the problem already had a lot of attention regarding classical complexity (CITE), only a few results are currently known for the parametrized variant.

As far as we have seen, even the w -hardness of the general case has not been explicitly proven in the literature.

In this thesis, we continue the journey toward a systematic analysis by stating some hardness results for specific graph classes for the problem.

Our contributions Our main contributions consist of first showing the $w[2]$ -hardness of SEMITOTAL DOMINATING SET for XXXX graphs.

As the DOMINATING SET problem and the TOTAL DOMINATING SET problem both admit a linear kernel for planar graphs, it is interesting to analyze whether these results also hold for the SEMITOTAL DOMINATING SET problem which lies in between these two.

Having these kernels also for other variants like EDGE DOMINATING SET, EFFICIENT DOMINATING SET, CONNECTED DOMINATING SET, PLANAR RED-BLUE DOMINATING SET lent us great confidence that the result will also work for SEMITOTAL DOMINATING SET on planar graphs.

Following the approach from ... which already relies on the technique given in, we give some simple data reduction rules for SEMITOTAL DOMINATING SET on planar graphs leading to a linear kernel. More precisely, we are going to prove the following central theorem of this thesis:

1 Introduction

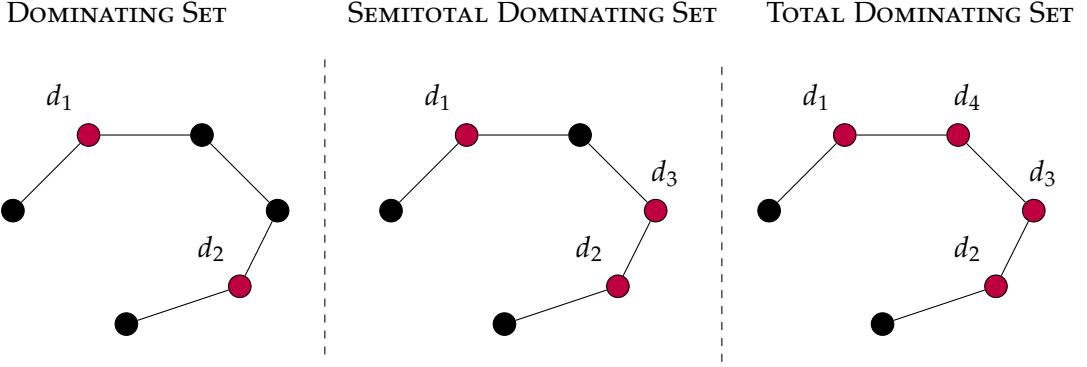


Figure 1.2: An example for a DOMINATING SET, SEMITOTAL DOMINATING SET and TOTAL DOMINATING SET, where $\gamma(G) < \gamma_{2t}(G) < \gamma_t(G)$ are strict. In the first case, only two vertices suffice to dominate all others. In the second one, we need a witness between d_1 and d_2 that is at most distance two. In the last case, d_1 and d_2 both need a neighbor in the TOTAL DOMINATING SET.

With some modifications, we were able to transfer the approach given by Garnero and Sau in [Garnero2018] to the SEMITOTAL DOMINATING SET problem.

Theorem 1. *The SEMITOTAL DOMINATING SET problem parametrized by solution size admits a linear kernel on planar graphs. There exists a polynomial-time algorithms that given a planar graph (G, k) , either correctly reports that (G, k) is a NO-instance or returns an equivalent instance (G', k) such that $|V(G')| \leq 561 \cdot k$.*

DOMINATING SET problem and TOTAL DOMINATING SET problem, both already

CHAPTER 2

PRELIMINARIES

We start by recapping some basic notation in Graph Theory and Parametrized Complexity.

Continuing an intensive study of parametrized complexity of that problem.

2.1 Graph Theory

We quickly state the following definitions given by [diestel10].

Definition 1 (Graph [diestel10]). A graph is a pair $G = (V, E)$ of two sets where V denotes the vertices and $E \subseteq V \times V$ the edges of the graph. A vertex $v \in V$ is incident with an edge $e \in E$ if $v \in e$. Two vertices x, y are adjacent, or neighbours, if $\{x, y\} \in E$.

Definition 2 (Vertex Degrees). The degree $d_G(v)$ (If G is clear, also $d(v)$) of a vertex v is the number of neighbors of v . We call a vertex of degree 0 as isolated and one of degree 1 as a pendant.

Definition 3 (isomorphic Graphs, [diestel10]). Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. We call G and G' isomorphic, if there exists a bijection $\phi : V \rightarrow V'$ with $\{x, y\} \in E \Leftrightarrow \phi(x)\phi(y) \in E'$ for all $x, y \in V$. Such a map ϕ is called isomorphism.

If a graph G is isomorphic to another graph H , we denote $G \simeq H$.

Definition 4 (Special Graph Notations [diestel10]). A simple Graph

A directed Graph is a graph

A Multi Graph

A Planar Graph

Definition 5 (Closed and Open Neighborhoods [Balakrishnan2012]). Let $G = (V, E)$ be a (non-empty) graph. The set of all neighbors of v is the open neighborhood of v and denoted by $N(v)$; the set $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of v in G . When G needs to be made explicit, those open and closed neighborhoods are denoted by $N_G(v)$ and $N_G[v]$.

Definition 6 (Induced Subgraph). asd

Definition 7 (Isomorphic Graph). asd

Graph Classes

We call the class of graphs without any special restrictions “General Graphs”.

Definition 8 (r-partite Graphs). Let $r \geq 2$ be an integer. A Graph $G = (V, E)$ is called “ r -partite” if V admits a partition into r classes such that every edge has its ends in different classes: Vertices in the same partition class must not be adjacent.

For the case $r = 2$ we say that the G is “bipartite”

Definition 9 (Chordal Graphs).

Definition 10 (Split Graphs).

Definition 11 (Path Graph P_i).

Definition 12 (Bipartite Graph, [Bondy2008]). A bipartite graphs is a Graph G whose vertex set can be partitioned into two subsets X and Y , so that each edge has one end in X and one end in Y . Such a partition (X, Y) is called a bipartition of G .

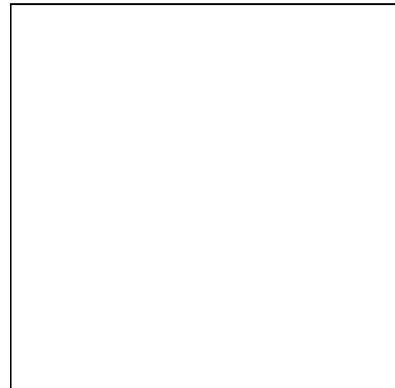
Definition 13 (Planar Graphs [diestel10]). A plane graph is a pair (V, E) of finite sets with the following properties:

- $V \subseteq \mathbb{R}^2$ (Vertices)
- Every edge is an arc between two vertices
- different edges have different sets of endpoints
- The interior of an edge contains no vertex and no point of any other edge

An embedding in the plane, or planar embedding, of an (abstract) graph G is an isomorphism between G and a plane graph H . A plane graph can be seen as a concrete **embedding** of the planar graph into the “plane” \mathbb{R}^2 .

CHAPTER 3

ON PARAMETRIZED SEMITOTAL DOMINATION



To do select another quote

Lewis Caroll, XXXX

For an introduction into classical complexity theory. Refer to the standard textbooks
and references. Rely on []

3.1 Parametrized Complexity

* Decision Problem

Definition 14 (Parametrized Problem[Cygan2015]). A parametrized problem is a $L \subseteq \Sigma^* \times \mathbb{N}$ (Σ finite fixed alphabet) for an instance $(x, k) \in \Sigma^* \times \mathbb{N}$, where k is called the parameter.

Definition 15 (Instance Size). The size of an instance of an instance (x, k) of a parametrized problem is $|(x, k)| = |x| + k$

We will now clarify the basic terminology within Parametrized Complexity. We are now giving a short introduction into the world of parametrized complexity. * General Introduction

Ways to cope with NP-hard problems.

3.1.1 Fixed Parameter Tractability

Definition 16 (The Class FPT [Cygan2015]). A parametrized problem $L \subseteq \Sigma^* \times \mathbb{N}$ is called *fixed-parameter tractable* if there exists an algorithm A (called a *fixed-parameter algorithm*), a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ and a constant c such that, given $(x, k) \in \Sigma^* \times \mathbb{N}$, the algorithm A correctly decides whether $(x, k) \in L$ in time bounded by $f(k) \cdot |(x, k)|^c$. The complexity class containing all fixed-parameter tractable problems is called *FPT*

3.1.2 Kernelization

Definition 17 (kernelization Algorithm[Cygan2015]). A *Kernelization Algorithm* or *kernel* is an algorithm \mathfrak{A} for a parametrized Problem Q , that given an instance (I, k) of Q works in polynomial time and returns an equivalent instance (I', k') of Q . Moreover, we require that $\text{size}_{\mathfrak{A}}(k) \leq g(k)$ for some computable function $g : \mathcal{N} \rightarrow \mathcal{N}$

If we bound the size of the kernel by a linear function $f(m) = \mathcal{O}(k)$, we say that the problem admits a **linear kernel**.

The main idea, preprocessing algorithm, shrink size as much as possible, sound reduction rules, small output instance

Definition 18 (Output size of a Preprocessing Procedure [Cygan2015]). The output size of a preprocessing algorithms \mathfrak{A} is defined as

$$\text{size}_{\mathfrak{A}}(k) = \sup \{|I'| + l' : (I', k') = \mathfrak{A}(I, k), I \in \Sigma^*\}$$

possibly infinite

Clearly, if there exists a kernelization algorithm for a problem L and an algorithm \mathfrak{A} with any runtime to decide L , the problem is in *FPT* because after the kernelization pre-processing has been applied, the size of the reduced instance is a function merely in k and independent of the input size n . In ?? we will explicitly construct a kernel for PLANAR SEMITOTAL DOMINATING SET and hence showing it to be in *FPT*.

Definition 19 (Reduction Rules [Cygan2015]). A *reduction rule* is a function $\phi : \Sigma^* \times \mathbb{N} \rightarrow \Sigma^* \times \mathbb{N}$ that maps an instance (x, k) to an equivalent instance (x', k') such that ϕ is computable in time polynomial in $|x|$ and k

Definition 20 (Equivalent Instance [Cygan2015]). This is a test

Definition 21. Soundness of a rule

A **reduction rule** is a function $\Sigma^* \times \mathbb{N}$ that maps an instance (x, k) to an equivalent instance (x', k') such that ϕ is computable in time polynomial in $|x|$ and k

3.1.3 Fixed Parameter Intractability: The w -Hierarchy

3.1.4 Compare to classical NP-Hardness theory

Parametrized Reductions

Definition 22 (Parametrized Reduction [Cygan2015]). Let $A, B \subseteq \Sigma^* \times \mathbb{N}$ two parametrized problems. A Parametrized Reduction from A to B is an algorithm that, given an instance (x, k) of A , outputs an instance (x', k') of B such that

- (x, k) is a yes instance of A iff (x', k') is a yes instance of B
- $k' \leq g(k)$ for some computable function g
- the running time is $f(k) \cdot |x|^{\mathcal{O}(1)}$ (FPT!)

The w -hierarchy

3.2 Semitotal Domination

Dominating Set

Semitotal Dominating Set

Total Dominating Set

we denote y as the dominating number. Clearly $y_t < y_s < y_d$. Notatio of an independent set

3.2.1 Preliminaries

* Witness * domination

Let D be a dominating set of G and $w \in V(G) \setminus D$. For any neighbor $v \in D \cap N(w)$, we say that d_1 dominates w . For two dominating vertices $d_1, d_2 \in D$. If

SEMITOTAL DOMINATING SET

Definition, dominating number

3.3 Complexity Status of Semitotal Dominating Set

3.4 $w[i]$ -Intractability

Now some $w[i]$ hard classes.

3.4.1 Warm-Up: $W[2]$ -hard on General Graphs

As any bipartite graphswith bipartition can be split further into r -partite graphsthis results also implies the $w[1]$ -hardness of r -partite graphs

3.4.2 W[2]-hard on Bipartite Graphs

We are showing that SEMITOTAL DOMINATING SET is $\omega[2]$ -hard on bipartite graphs by a parametrized reduction from DOMINATING SET on bipartite graphs which is known to be $\omega 2$ -hard ([Raman2008]).

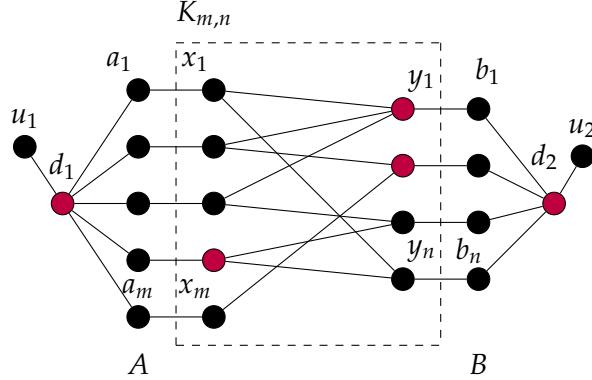


Figure 3.2: Constructing a bipartite G' from the bipartite graph $K_{m,n}$ by duplicating all vertices and adding exactly two forced witnesses.

Theorem 2. Semitotal Dominating Set is $\omega[2]$ hard for bipartite Graphs

Proof. Given a bipartite Graph $G = (\{X \cup Y\}, E)$, we construct a bipartite Graph $G' = (\{X' \cup Y'\}, E')$ in the following way:

1. For each vertex $x_i \in X$, we add a new vertex $a_i \in A$ and an edge $\{x_i, a_i\}$ in between.
2. For each vertex $y_j \in Y$, we add a new vertex $b_j \in B$ and an edge $\{y_j, b_j\}$ in between.
3. We add four vertices with edges $\{u_1, d_1\}$ and $\{u_2, d_2\}$, and connect them with all $\{d_1, a_i\}$ and $\{d_2, b_j\}$ ($i \in [m]$ and $j \in [n]$) respectively.

Observation: The constructed G' is bipartite because A and B form an independent set on G' that can be cross-wise attached to one of the previous vertex sets. Setting $X' = X \cup \{u_2, d_1\} \cup B$ and $Y' = X \cup \{u_1, d_2\} \cup A$ form the partitions of the new bipartite G' .

Corollary 1. G has a DOMINATING SET of size k iff G has a SEMITOTAL DOMINATING SET of size $k' = k + 2$

\Rightarrow Assume a DOMINATING SET in G of size k . We know that $D' = D \cup \{d_1, d_2\}$ is a SEMITOTAL DOMINATING SET in G' of size $k' = k + 2$, because d_1 dominates u_1

and all $a_i \in A$; d_2 dominates u_2 and all $b_i \in B$. The rest is dominated by the same vertices as they were in G , but now all of them have either d_1 or d_2 as a witness. More formally, we have by construction of G' that $\forall v \in (D \cap X) : d(v, d_1) = 2$ and $\forall v \in (D \cap Y) : d(v, d_2) = 2$.

- ⇐ On the contrary, assume any SEMITOTAL DOMINATING SET D' in G' with size k' . Without loss of generality, we can assume that $u_1, u_2 \notin D'$, because choosing d_1 and d_2 instead is always at least as good and does not violate any witnesses. Therefore, the construction forces $d_1, d_2 \in D'$.

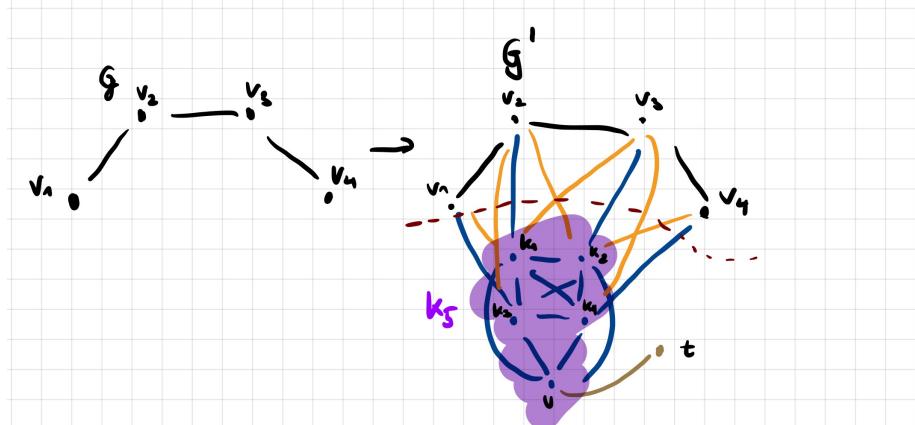
All $a_i \in A$ can only be useful to dominate their partnering x_i ($b_i \in B$ for y_i), because $d_1, d_2 \in D$ is the only second neighbor they have. If $a_i, b_i \in D'$ we replace it with x_i and y_i preserving the size D . As d_1 and d_2 suffice to provide a witness for every vertex in the graph and do not lose any other witnesses, this operation is sound.

In the end, $D = D' \setminus \{d_1, d_2\}$ gives us a DOMINATING SET in G with size $k = k' - 2$

As G' can be constructed in linear time and the parameter k is only blown up by a constant, this reduction is an FPT reduction. Because DOMINATING SET is already $w[2]$ -hard on bipartite graphs ([]), we imply that SEMITOTAL DOMINATING SET is $w[2]$ -hard as well. \square

3.4.3 $W[2]$ -hard on Chordal Graphs

Theorem 3. Semitotal Dominating Set is $w[2]$ hard on Chordal Graphs



3 On Parametrized Semitotal Domination

1. Add a K_{n+1} consisting of the vertices $\{k_1, \dots, k_n, u\}$ and add an edge (v_i, k_i) to each vertex v_i of G . One vertex u in the clique will remain untouched.
2. Add one additional vertex t and connect it with u .
3. For all vertices v_i in G , add a new edge (n, k_i) for all $n \in N(v_i)$.

Corollary 2. $N(v_i) \in G$ forms a clique iff $N(v_i)$ forms a clique in G'

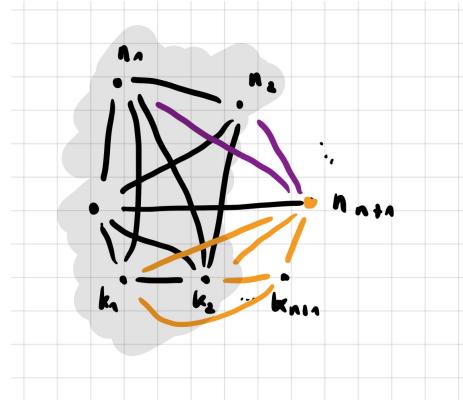


Figure 3.4: Induction Step

Proof. Assuming that $N(v_i)$ forms a clique in G , we show that it also forms a clique in G' by induction over the number of neighbors $z = \text{abs}(N(v_i))$ in G .

- $z = 0$: Holds trivially as we do not have a neighbor in G and in G' the connected k_i forms a P_1 , hence a clique.
- $z = z + 1$:

By IH, we already know that all neighbors n_1, \dots, n_z form a clique together with their vertices in k_i . As $k_{z+1}, v_{z+1} \in N(v_i)$ now also in G' , we show that $N(v_i)$ still forms clique in G' .

Let k_i be the vertex that was connected with n_i during step 1. All we have to show is that v_{z+1} and k_{z+1} extend our previous clique, hence are fully connected with $N(v_i)$.

v_{z+1} connects to $N(v_i)$ in G by assumption. By our construction, there exists an edge to k_1, \dots, k_z , because we add an edge (n_{z+1}, k_i) if there is an edge from (n_{z+1}, n_i) . (See fig ??)

k_{z+1} form a complete subgraph with the other k_i and is connected to all n_i by construction because the edge (n_{z+1}, n_i) exists.

Therefore, $N(v_i)$ will also form a clique in G' .

On the other side, if $N(v_i)$ forms a clique in G' , the vertices of $N(v_i)$ in G form an induced subgraph of G' , hence preserving the clique. ■

Corollary 3. G is Chordal iff G' is chordal.

Proof. \Rightarrow : Assume G chordal. Then exists a total elimination order $o = (v_1, \dots, v_n)$ in G where removing v_j sequentially returns cliques in $N(v_i)$. Define $o' = (v_1, \dots, v_n, k_1, \dots, k_n, u, t)$. Applying corollary ?? states that (v_1, \dots, v_n) always gives cliques in G and according to corollary ?? also in G' . As the rest is directly part of a clique in G' by definition with an additional vertex of degree 1, o' is a total elimination order for G' , hence G' chordal.
 \Leftarrow : Holds as o' is always a total elimination order in G' and removing the complete subgraph K_{n+1} and u gives a total elimination order in G . ■

Corollary 4. G has a Dominating Set of size k iff G' has a dominating set of size $k + 1$

Proof. Assume a Dominating Set D of size k in G . $D \cup \{u\}$ is a Semitotal Dominating Set in G' of size $k + 1$, because u dominates t and for each $v \in DS : d(v, u) \leq 2$.

Contrary, assume a Semitotal Dominating Set SD in G' . In order to dominate t , $u \in SD$ must hold, hence already dominating the complete subgraph K_{n+1} . If a vertex $k_i \in SD$, we exchange it with v_i still preserving a Dominating Set. Taking $D = SD - \{u\}$ gives our desired Dominating Set of size k . ■

As this reduction runs in FPT time and the parameter is only bounded by a function of k , this is a FPT reduction. As Dominating Set on Chordal Graphs is $w[2]$ – hard, so is SDS on Chordal Graphs. □

3.4.4 W[2]-hard on Split Graphs

TODO Getting started with that.

CHAPTER 4

A LINEAR KERNEL FOR PLANAR SEMITOTAL DOMINATION



The best way to explain it is to do it.

Lewis Caroll, *Alice in Wonderland*

We are going to present a polynomial-time preprocessing procedure giving a linear kernel for PLANAR SEMITOTAL DOMINATING SET parametrized by solution size. Based on the technique first introduced by Alber et al. ([Alber2004]) in 2004, an abundance of similar results to other domination problems emerged which gave us the belief we can transfer these results to SEMITOTAL DOMINATING SET. ?? gives an overview of the status of kernels for the planar case on various domination problems. All of these results introduce reduction rules bounding the number of vertices inside so-called “regions” which can be obtained by a special decomposition of the planar graph.

In the following years, this approach bore fruits in other planar problems as well like CONNECTED VERTEX COVER ($11/3k$ in [Kowalik2013]), MAXIMUM TRIANGLE PACKING ($624k$ in [Wang2011]) INDUCED MATCHING ($40k$ in [Kanj2011]) FULL-DEGREE SPANNING TREE ([Guo2006]) FEEDBACK VERTEX SET ($13k$ in [Bonamy2016]) and CYCLE PACKING ([Garnero2019]).

In the impending years, many results generalized this approach to larger graph classes. Fomin and Thilikos started by proofing that the initial reduction rules given by Alber et al. [Alber2004] can also be used to obtain a linear kernel on graphs with bounded genus g ([Fomin2004]). Gutner advanced in 2008 by giving a linear kernel on

Problem	Best Known Kernel	Source
PLANAR DOMINATING SET	$67k$	[Diekert2005] ¹
PLANAR TOTAL DOMINATING SET	$410k$	[Garnero2018] ²
PLANAR SEMITOTAL DOMINATING SET	$561k$	This work
PLANAR EDGE DOMINATING SET	$14k$	[Guo2007]
PLANAR EFFICIENT DOMINATING SET	$84k$	[Guo2007]
PLANAR RED-BLUE DOMINATING SET	$43k$	[Garnero2017]
PLANAR CONNECTED DOMINATING SET	$130k$	[Luo2013]
PLANAR DIRECTED DOMINATING SET	Linear	[Alber2006]

¹There is also a master's thesis claiming a bound of $43k$ [Halseth2016], but a conference or journal version was not found.

²Improved their own results from first $694k$ [Garnero2018]

Table 4.1: An overview about existing kernels for planar dominating set variants

$K_{3,h}$ -topological-minor-free graph classes and a polynomial kernel for K_h -topological-minor-free graph classes ([Gutner2009]). In 2012 Philip, Raman and Sikdar showed that $K_{i,j}$ -free graph classes admit a polynomial kernel for DOMINATING SET ([Philip2012]). In an attempt to extend these ideas to other problems as well, Bodlaender et al. proved that all problems expressible in counting monadic second-order logic satisfying a coverability property admit a polynomial kernel on graphs of bounded genus g ([Bodlaender2016]).

These meta-results are interesting from a theoretical point of view, but the constants for the kernels obtained by these methods are too large to be of practical interest. The question of how an efficient kernel can be constructed remains. We will transfer the kernel described by Garnero and Sau in their original version of [Garnero2018] for PLANAR TOTAL DOMINATING SET can also be “recycled” for PLANAR SEMITOTAL DOMINATING SET giving us an explicitly constructed kernel with “reasonable” small constants.

The Main Idea A given a plane graph $G = (V, E)$ with a given vertex set $D \subseteq V$ can be decomposed into at most $3 \cdot |D| - 6$ so-called “regions” (see ??). If D is a given SEMITOTAL DOMINATING SET of size $|D|$, the total number of regions in this decomposition depends linearly on the size of D . If we define *reduction rules* (??????) minimizing the number of vertices in and around a region, we can bound the size of a reduced graph. In our case, we give rules to reduce a region down to a constant number of vertices. We can create an equivalent instance of G whose number of remaining vertices lies linearly in the size of an enquired solution. Such a reduction gives a *Kernel* for PLANAR SEMITOTAL DOMINATING SET.

Interestingly, the reduction rules do not rely on the decomposition itself, but rather

consider the neighborhood of every pair of vertices in the graph. The decomposition itself has just used a tool for analyzing the kernel size after the reduction.

4.1 Definitions

Before giving the exact reduction rules, we need some definitions which expose some nice properties we are going to exploit. They are inspired by those given by Garnero and Sau (PLANAR TOTAL DOMINATING SET in [Garnero2018] or PLANAR RED-BLUE DOMINATING SET in [Garnero2017]) and reused ideas introduced by Alber et al. in [Alber2004] for PLANAR DOMINATING SET.

The idea is to split up the neighborhood of a single vertex and a pair of vertices into three distinct subsets which intuitively give us a level of “confinement” of these vertices and how closely they are related to the rest of the graph ([Garnero2018]).

Definition 23. Let $G = (V, E)$ be a graph and let $v \in V$. We denote by $N(v) = \{u \in V : \{u, v\} \in E\}$ the neighborhood of v . We split $N(v)$ into three subsets:

$$N_1(v) = \{u \in N(v) : N(u) \setminus N[v] \neq \emptyset\} \quad (4.1)$$

$$N_2(v) = \{u \in N(v) \setminus N_1(v) : N(u) \cap N_1(v) \neq \emptyset\} \quad (4.2)$$

$$N_3(v) = N(v) \setminus (N_1(v) \cup N_2(v)) \quad (4.3)$$

To enhance future readability, for $i, j \in [1, 3]$, we denote $N_{i,j}(v) := N_i(v) \cup N_j(v)$. Furthermore, we call a vertex v' confined by a vertex v , if $N(v') \subseteq N[v]$

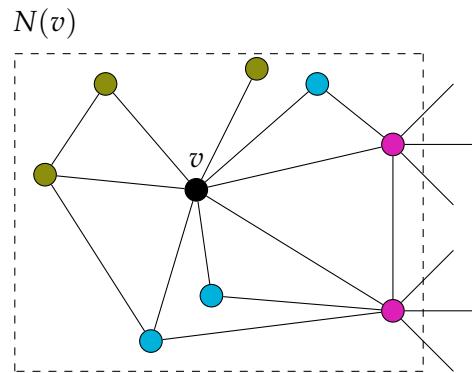


Figure 4.2: The neighborhood of a single vertex v split to $N_1(v)$ (purple), $N_2(v)$ (blue), and $N_3(v)$ (green). $N_1(v)$'s are those having neighbors outside $N(v)$, $N_2(v)$'s are a buffer between $N_1(v)$ and $N_3(v)$, and $N_3(v)$ -vertices are confined in $N(v)$.

- $N_1(v)$ is all the neighbors of v which have at least one adjacent vertex that is outside of $N(v)$ and therefore connects v with the rest of the graph. They can belong to a SEMITOTAL DOMINATING SET.
- $N_2(v)$ is all neighbors of v that have at least one neighbor in $N_1(v)$. These vertices do not have any function as a dominating vertex and can be seen as a *buffer* bridging $N_1(v)$ -vertices with those from $N_3(v) \cup \{v\}$. Furthermore, they are useless as witnesses, because either we can replace them with v (sharing the same neighborhood) or when being a witness for v , we replace it with a $z \in N_1(v)$.
- $N_3(v)$ vertices are sealed off from the rest of the graph. They are useless as dominating vertices: For all $z \in N_3(v)$ it holds that $N(z) \subseteq N(v)$ by definition and thus, we would always prefer v as a dominating vertex instead of z . Nevertheless, they can be important as a witness for v in the case where it is the only neighbor of v (this $N_1(v) \cup N_2(v) = \emptyset$). We are using this observation in ?? where we shrink $|N_3(v)| \leq 1$.

Next, we are going to extend this notation to a pair of vertices. Using this, ?? will later try to reduce the neighborhood of two vertices, and similar to ??, we observe nice properties. Again, the idea is to classify how strongly the shared neighborhood of two vertices $N(v) \cup N(w)$ is connected to the rest of the graph.

Definition 24. Let $G = (V, E)$ be a graph and $v, w \in V$. We denote by $N(v, w) := N(v) \cup N(w)$ the shared neighborhood of the pair v, w and split $N(v, w)$ into three distinct subsets:

$$N_1(v, w) = \{u \in N(v, w) \mid N(u) \setminus (N(v, w) \cup \{v, w\}) \neq \emptyset\} \quad (4.4)$$

$$N_2(v, w) = \{u \in N(v, w) \setminus N_1(v, w) \mid N(u) \cap N_1(v, w) \neq \emptyset\} \quad (4.5)$$

$$N_3(v, w) = N(v, w) \setminus (N_1(v, w) \cup N_2(v, w)) \quad (4.6)$$

Again, for $i, j \in [1, 3]$, we denote $N_{i,j}(v, w) = N_i(v, w) \cup N_j(v, w)$.

$N_1(v, w)$ contains those vertices connected with the rest of the graph by at least one vertex not in the neighborhood of v or w , $N_2(v, w)$ -vertices are a *buffer* between those from $N_3(v, w) \cup \{v, w\}$ and $N_1(v, w)$, and $N_3(v, w)$ contains vertices isolated from the rest of the graph. You can see an example in ??.

A vertex $v \in N_i(v)$ is not necessarily also in $N_i(v, w)$! Observe the vertex z in ?? Unlike $N_i(v)$ in every $N_i(v, w)$ ($i \in [3]$) can be in an optimal SEMITOTAL DOMINATING SET (compare ?? for an example in each of the sets).

4.1.1 Reduced Graph

Before stating the reduction rules, we want to clarify when we consider a graph to be a *reduced*.

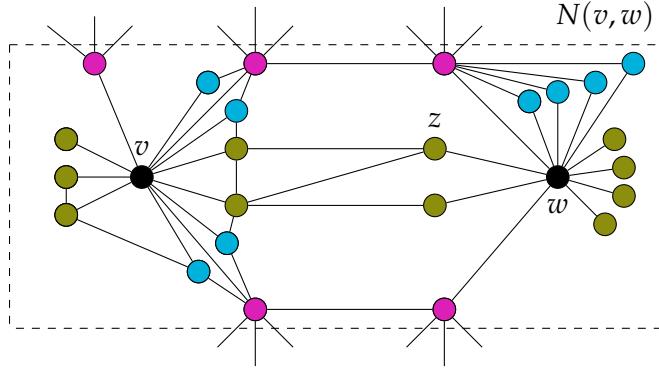


Figure 4.3: The neighborhood of a pair of vertices. Vertices from $N_3(v,w)$ are colored green, $N_2(v,w)$'s blue and $N_1(v,w)$'s purple. Note that $z \in N_1(w)$, because there is an edge to a neighbor of v , but $z \notin N_1(v,w)$ (and rather $z \in N_3(v,w)$).

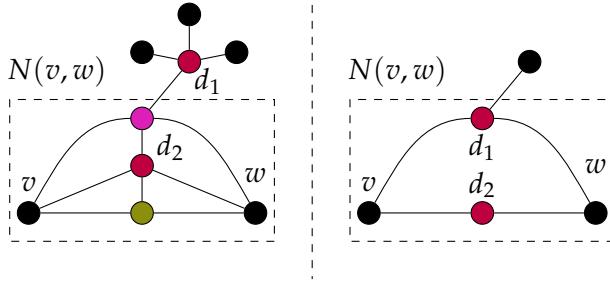


Figure 4.4: Left: $\{d_1, d_2\}$ with $d_2 \in N_2(v,w)$ form the only minimal SEMITOTAL DOMINATING SET. Right: $d_1 \in N_1(v,w)$ and $d_2 \in N_3(v,w)$ optimal.

Definition 25 (Reduced Graph [Garnero2018] and [Garnero2017]). A Graph G is reduced under a set of rules if either none of them can be applied to G or the application of any of them creates a graph isomorphic to G .

This definition differs from the definition usually given in literature where a graph G is *reduced* under a set of reduction rules if none of them can be applied to G anymore (compare e.g. [Fomin2019]). Some of our reduction rules (?? or ??) could be applied *ad infinitum* creating an endless loop that does not change G any more. Our definition guarantees termination in that case. All of the given reduction rules are local and only need the neighborhood of at most two vertices and replace them partially with gadgets of constant size. Now checking whether a graph after applying the rule has been isomorphically changed can be trivially accomplished in constant time.

In our case, we say G is reduced if all of the ????? have exhaustively been applied.

4.1.2 Regions in Planar Graphs

Alber et al. gave in [Alber2004] a novel approach to look at planar graphs. In their analysis, they stated a constructive algorithm that decomposes a planar graph into local “regions”. Vaguely said, a region is a set of vertices that are enclosed by a boundary path in a fixed planar embedding of the graph.

The following definitions are based on those given by Garnero and Sau in [Garnero2018] and will lead toward a clean definition of a *region* and what we understand as a *D-region decomposition*. More detailed explanations and concrete examples can be found in their paper.

Definition 26. Two simple paths p_1, p_2 in a plane graph G are confluent if:

1. they are vertex-disjoint
2. they are edge-disjoint and for every common vertex u , if v_i, w_i are the neighbors of u in p_i , for $i \in [1, 2]$, it holds that $[v_1, w_1, v_2, w_2]$, or
3. they are confluent after contracting common edges

Definition 27. Let $G = (V, E)$ be a plane graph and let $v, w \in V$ be two distinct vertices. A region $R(v, w)$ (synonymy denoted as *vw-region*) is a closed subset of the plane, such that:

1. the boundary of R is formed by two confluent simple vw -paths with length at most 3
2. every vertex in R belongs to $N(v, w)$, and
3. the complement of R in the plane is connected.

We denote with ∂R the boundary of R and by $V(R)$ the set of vertices laying (on the plane embedding) in R . Furthermore, we call $|V(R)|$ the size of the region.

The poles of R are the vertices v and w . The boundary paths are the two vw -paths that form ∂R .

Definition 28. Two regions R_1 and R_2 are non-crossing, if:

1. $(R_1 \setminus \partial R_1) \cap R_2 = (R_2 \setminus \partial R_2) = \emptyset$, and
2. the boundary paths of R_1 are pairwise confluent with the ones in R_2

We now have all the definitions ready to formally a maximal *D-region decomposition* on planar graphs:

Definition 29. Given a plane graph $G = (V, E)$ and $D \subseteq V$, a D – region Decomposition of G is a set \mathfrak{R} of regions with poles in D such that:

1. for any vw -region $R \in \mathfrak{R}$, it holds that $D \cap V(R) = \{v, w\}$, and

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2. all regions are pairwise non-crossing.

We denote $V(\mathfrak{R}) = \bigcup_{R \in \mathfrak{R}} V(R)$.

A D -region decomposition is maximal if there is no region $R \notin \mathfrak{R}$ such that $\mathfrak{R}' = \mathfrak{R} \cup \{R\}$ is a D -region decomposition with $V(\mathfrak{R}) \subsetneq V(\mathfrak{R}')$.

?? gives an example of how to decompose a graph into a maximal D -region decomposition with a given SEMITOTAL DOMINATING SET D of size 3.

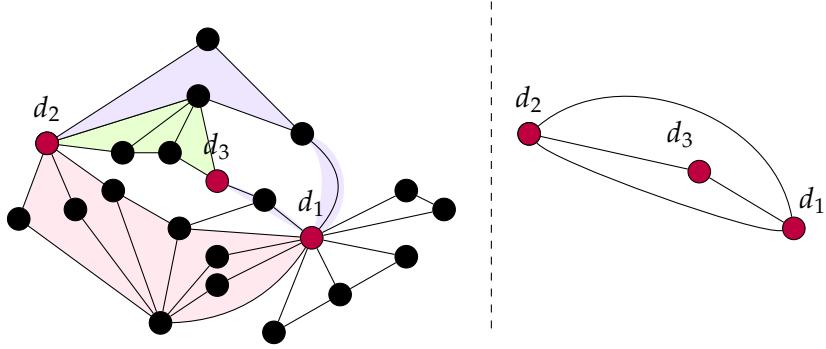


Figure 4.5: Left: A maximal D -region decomposition \mathfrak{R} , where $D = \{d_1, d_2, d_3\}$ form a SEMITOTAL DOMINATING SET. There are two regions between d_2 and d_1 (purple and pink), one region between d_1 and d_3 (green) and one region between d_2 and d_3 (purple). Observe that some neighbors of d_1 are not part of any vw -region. Our reduction rules are going to take care of them and bound these number of vertices to obtain the kernel. Right: The corresponding underlying multigraph $G_{\mathfrak{R}}$. Every edge denotes a region between d_i and d_j .

We are introducing a special subset of a region, namely *simple region* where every vertex is a common neighbor of v and w . They will appear in many unexpected astonishing places and are an important tool to operate on small parts of a plane graph. The upcoming ?? will bound the size of these *simple regions*. Interestingly, in the first version of the paper about the linear kernel for PLANAR TOTAL DOMINATING SET ([Garnero2018]), they were not given independently but covered by one of their reduction rules (Rule 2). As it turned out, the analysis is getting simpler if we treat them in a separate rule (In our case: ??) and so did Garnero and Sau in a revised version of their paper four years later ([Garnero2018]).

Definition 30. A simple vw -region is a vw -region such that:

1. its boundary paths have length at most 2, and

2. $V(R) \setminus \{v, w\} \subseteq N(v) \cap N(w)$.

?? shows an example of a simple region containing 9 distinct vertices.

In the analysis, we will also use properties of the *underlying multigraph* of a D -region decomposition \mathfrak{R} . Refer to ?? for an example.

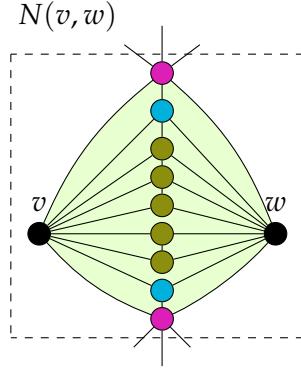


Figure 4.6: A simple region with two vertices from $N_1(v,w)$ (purple) setting the boundary, two vertices from $N_2(v,w)$ (blue) and some vertices from $N_3(v,w)$ (green) in between.

Definition 31. Let $G = (V, E)$ be a plane graph, let $D \subseteq V$ and let \mathfrak{R} be a D -region decomposition of G . The underlying multigraph $G_{\mathfrak{R}} = (V_{\mathfrak{R}}, E_{\mathfrak{R}})$ of \mathfrak{R} is such that $V_{\mathfrak{R}} = D$ and there is an edge $\{v, w\} \in E_{\mathfrak{R}}$ for each vw -region $R(v, w) \in \mathfrak{R}$

4.2 The Big Picture

?? gives a high-level view of how we are going to obtain the linear kernel for PLANAR SEMITOTAL DOMINATING SET. We will first derive three different reduction rules (?????? are green in the overview), prove that they preserve the solution size k and run in polynomial-time. Then we use the existence of a maximal D -region decomposition \mathfrak{R} on planar graphs to bound the number of vertices that fly around a given region $R \in \mathfrak{R}$. This will lead us towards a bound on the number of vertices inside R . Furthermore, we observe that the number of vertices that are not enclosed in R , but lie outside the border is bounded, too. We will often encounter hidden simple regions which are reduced by ?? and therefore of constant size by ?. As we know that the total number of regions R in the D -region decomposition is linear in k , we obtained a linear kernel for the PLANAR SEMITOTAL DOMINATING SET as well.

4.3 The Reduction Rules

Following the ideas proposed by Garnero and Sau in [Garnero2018], we state reduction rules that after exhaustive application will expose a linear kernel. We note that especially for ??, we relied on the first revision of the paper because in the following years their kernel size was improved for the cost of making them more specific to PLANAR TOTAL DOMINATING SET and we saw that the latest rules stated will not work for PLANAR SEMITOTAL DOMINATING SET. The main challenge in all of those rules is to maintain the witness properties of the graph because a vertex inside a region can be important as a

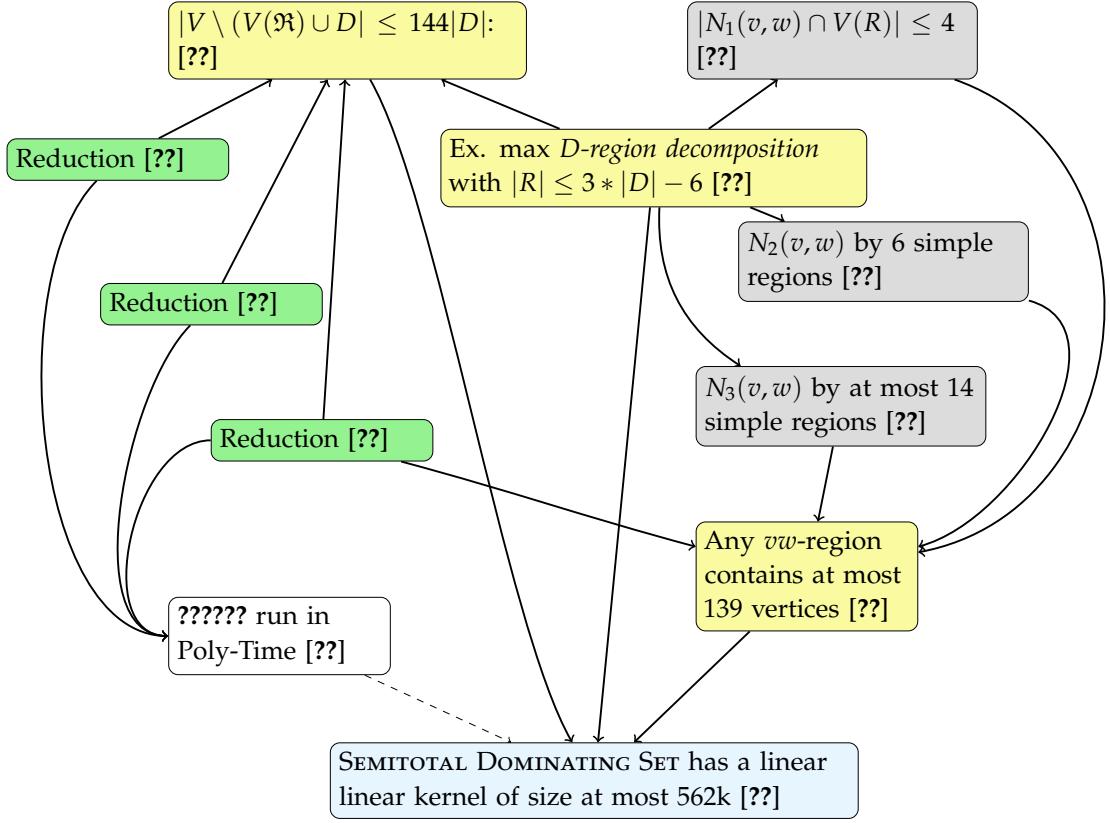


Figure 4.7: The plan for obtaining a linear kernel for PLANAR SEMITOTAL DOMINATING SET. Starting with the reduction rules (Green) we will derive the number of vertices inside and outside of a vw -region.

witness for vertices in another region and can not always simply be reduced. Therefore a more sophisticated analysis is required.

4.3.1 Reduction Rule I: Getting Rid of unnecessary $N_3(v)$ vertices

The idea of the first rule is the observation that a vertex $v' \in N_{2,3}(v)$ dominates v and possibly vertices from $N_2(v)$ and $N_3(v)$. As $N(v') \subseteq N(v)$ and the ?? that a witness for v' is also witness for v , we can use v instead of v' as a dominating vertex. Therefore, we can remove $N_{2,3}$ from the graph. Nevertheless, v' can be a witness for v itself and might be required in a solution. Our rule ensures that at least one $N_3(v)$ -vertex is preserved. An example for this rule is shown in ??.

Fact 1. Let $G = (V, E)$, $v \in V$ and $v' \in N_{2,3}(v)$. Any witness $w \neq v$ for v' is also a witness for v .

4.3 The Reduction Rules

Proof. By assumption v' is witnessed by a vertex $w \neq v$ with $d(v', w) \leq 2$. It follows directly from the definition of $v' \in N_{2,3}(v)$ that $N(v') \subseteq N[v]$ and hence v' is confined inside the neighborhood of v . Every path $P = (v', n, w)$ from v' to possible witness w within two steps must pass at least one vertex $n \in N[v]$ as all neighbors. This implies that there exists also a path from v to w with a length of at most two and v

□

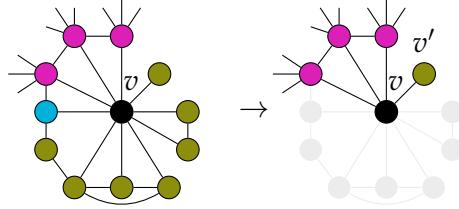


Figure 4.8: *Simplifying $N_{2,3}(v)$: As $N_3(v) \geq 1$, we remove $N_{2,3}(v)$ and add a new witness v' . $N_1(v)$ remains untouched.*

Rule 1. Let $G = (V, E)$ be a graph and let $v \in V$. If $|N_3(v)| \geq 1$:

- remove $N_{2,3}(v)$ from G ,
- add a vertex v' and an edge $\{v, v'\}$

We can now prove the correctness of this rule.

Lemma 1. Let $G = (V, E)$ be a graph and let $v \in V$. If G' is the graph obtained by applying ?? on G , then G has a SEMITOTAL DOMINATING SET of size k if and only if G' has a SEMITOTAL DOMINATING SET of size k .

Proof. \Rightarrow Let D be a SEMITOTAL DOMINATING SET in G of size k . Because ?? has been applied, we can assume $N_{2,3}(v) \neq \emptyset$ in G .

To dominate all vertices from $N_3(v)$ we either need $v \in D$ or at least one other vertex $d \in N_{2,3}(v) \cap D$. In the latter case, we can replace this vertex by v directly. By ??, we know that a witness for d (which is not v) is also a witness for v and therefore, the replacing preserves the SEMITOTAL DOMINATING SET. Henceforth, we assume $v \in D$ and $N_{2,3}(v)$ is already dominated by v .

If ?? has removed at least one dominating vertex from $N_{2,3}(v)$, we set $D' = D \setminus N_{2,3}(v) \cup \{v'\}$ otherwise $D = D'$. In the first case this vertex in $N_{2,3}(v) \cap D$ could have possibly been a witness for v , so we select $v' \in D'$ as a witness for v ensuring D' to be a SEMITOTAL DOMINATING SET. In both cases v' is by assumption dominated by v and $|D| \leq |D'|$.

4 A Linear Kernel for Planar Semitotal Domination

\Leftarrow Assume D' to be a SEMITOTAL DOMINATING SET in G' . We can assume that $v \in D'$, because v' has to be dominated and it is always better to choose v instead of v' .

If $v' \in D'$ is a witness for v in G' , we have to preserve a witness in G as well. As we know that $N_3(v) \neq \emptyset$, we can replace it by an arbitrary vertex $d \in N_3(v)$ in G .

In the second case, the witness for v came either from a vertex $o \in N_1(v)$ or some neighbor $N(o) \setminus N(v)$ outside the neighborhood of v which has not been touched by this reduction rule. In summary, if $v' \in D'$, we set $D = D' \cup \{d\} \setminus \{v'\}$ for any $d \in N_3(v)$ and otherwise $D = D'$. In both cases, $N_{2,3}(v)$ is dominated by v and $|D| = |D'|$.

□

Lemma 2. *A plane graph G of n vertices is reduced under ?? in time $\mathcal{O}(n)$*

Proof. As ?? stayed the same, the proof directly follows [Alber2004]. □

Note that we need our definition of a reduced instance given in ?. If ?? is being applied, it will still leave us with a vertex $z \in N_3(v)$ allowing this rule to be applied again.

4.3.2 Reduction Rule II: Shrinking the Size of a Region

The second rule is the heart of the whole reduction and it aims to minimize the neighborhood of two distinct vertices. The rule follows Garnero and Sau's approach in one of the early versions of [Garnero2018] for PLANAR TOTAL DOMINATING SET. The reduction rules given in the latest version ([Garnero2018]) were not directly transferable to PLANAR SEMITOTAL DOMINATING SET, because they heavily rely on the property of a TOTAL DOMINATING SET that a witness w for v **must** be a direct neighbor of w . In the case of the more relaxed SEMITOTAL DOMINATING SET, the witness is allowed to have more range.

It can be observed that in the worst case four vertices are needed to semi-totally dominate $N(v, w)$ of two vertices $v, w \in V$: v, w and two witnesses for them. Exemplary, observe the graph consisting of two distinct $K_{1,m}$ with $m \in \mathbb{N}$ with centers v and w .

Before we give the concrete reduction rule, we need to define three sets. Intuitively, we first try to find a set $\tilde{D} \subseteq N_{2,3}(v, w)$ of size at most three dominating $N_3(v, w)$ without using v or w . If no such set exists, we allow v (resp. w) and try to find one again. If we now find such a set, we can conclude that v (w) must be part of a solution.

Definition 32. *Let $G = (V, E)$ be a graph and let $v, w \in V$. We now consider all the sets that*

4.3 The Reduction Rules

can dominate $N_3(v, w)$:

$$\mathcal{D} = \{\tilde{D} \subseteq N_{2,3}(v, w) \mid N_3(v, w) \subseteq \bigcup_{v \in \tilde{D}} N(v), |\tilde{D}| \leq 3\} \quad (4.7)$$

$$\mathcal{D}_v = \{\tilde{D} \subseteq N_{2,3}(v, w) \cup \{v\} \mid N_3(v, w) \subseteq \bigcup_{v \in \tilde{D}} N(v), |\tilde{D}| \leq 3, v \in \tilde{D}\} \quad (4.8)$$

$$\mathcal{D}_w = \{\tilde{D} \subseteq N_{2,3}(v, w) \cup \{w\} \mid N_3(v, w) \subseteq \bigcup_{v \in \tilde{D}} N(v), |\tilde{D}| \leq 3, w \in \tilde{D}\} \quad (4.9)$$

Furthermore, we shortly denote $\bigcup \mathcal{D}_v = \bigcup_{D \in \mathcal{D}_v} D$ and $\bigcup \mathcal{D}_w = \bigcup_{D \in \mathcal{D}_w} D$.

Assuming that v and w are closely connected with $d(v, w) \leq 2$, it might suffice to consider only sets of size at most three, because an intermediate vertex could witness v and w at the same time. In the later analysis, the *D-region decomposition* exactly creates regions around $N(v, w)$ requiring at least one path from v to w lengthened two. As the following rule is only used to locally investigate such regions, we could add the requirement of a distance of two to it and work with sets of size at most three. We believe that this could further improve the kernel.

We are now ready to state ??:

Rule 2. Let $G = (V, E)$ be a graph and v, w be two distinct vertices from V . If $\mathcal{D} = \emptyset$ we apply the following:

Case 1: if $\mathcal{D}_v = \emptyset$ and $\mathcal{D}_w = \emptyset$

- Remove $N_{2,3}(v, w)$
- Add vertices v' and w' and two edges $\{v, v'\}$ and $\{w, w'\}$
- If there was a common neighbor of v and w in $N_{2,3}(v, w)$, add another vertex y and two connecting edges $\{v, y\}$ and $\{y, w\}$
- If there was no common neighbor of v and w in $N_{2,3}(v, w)$, but at least one path of length three from v to w via only vertices from $N_{2,3}(v, w)$, add two vertices y and y' and connecting edges $\{v, y\}$, $\{y, y'\}$ and $\{y', w\}$

Case 2: if $\mathcal{D}_v \neq \emptyset$ and $\mathcal{D}_w = \emptyset$

- Remove $N_{2,3}(v)$
- Add $\{v, v'\}$

Case 3: if $\mathcal{D}_v = \emptyset$ and $\mathcal{D}_w \neq \emptyset$

This case is symmetrical to ??.

4 A Linear Kernel for Planar Semitotal Domination

In ??, we know by ?? that v and w must be in D . Therefore, we introduce two forcing vertices v' and w' in G' and remove $N_{2,3}(v, w)$ as these vertices are dominated by v and w . But if we remove $N_{2,3}(v, w)$ entirely, we could lose solutions: First, the case that v is a direct witness of w ($d(v, w) = 2$) and that there is one intermediate witness on a path of length three from v to w via vertices in $N_{2,3}(v, w)$, which could be a witness for both v and w at the same time.

Note that if we would not distinguish between these two cases and had added one intermediate vertex in both of them, we would possibly have generated some wrong solutions, because v could always witness w .

Again by ?? we know for ??? that $v \in D$ and similar to ?? we can simplify the neighborhood $N_{2,3}(v)$. ?? states, that these vertices are only useful for witnessing v , but do not go beyond what v already witnesses. Observe that removing $N_{2,3}$ can not break any connectivity as all vertices in $N_{2,3}(v)$ are confined in v . The cases where $\mathcal{D}_v \neq \emptyset$ and $\mathcal{D}_w \neq \emptyset$ are not required and in a later analysis only the existence of these two sets is required and the application of ?? will be critical.

Before proofing ?? we will deduce some facts which are implied by the definitions above. These facts justify the definition of the sets \mathcal{D} , \mathcal{D}_v and \mathcal{D}_w .

Fact 2. Let $G = (V, E)$ be a graph, let $v, w \in V$, and let G' be the graph obtained by the application of ?? on v, w . If $\mathcal{D} = \emptyset$, then G has a solution if and only if it has a solution containing at least one of the two vertices $\{v, w\}$.

Proof. Because $\mathcal{D} = \emptyset$, any SDS of G has to contain v or w , or at least four vertices from $N_{2,3}(v, w)$. In the second case, these four vertices can be replaced with v, w and two neighbors of v and w still forming a SEMITOTAL DOMINATING SET. \square

The second fact states that if \mathcal{D}_v (resp. \mathcal{D}_w) is empty, too, we only need to consider solutions containing w (or v):

Fact 3. Let $G = (V, E)$ be a graph, let $v, w \in V$, and let G' be the graph obtained by the application of ?? on v, w . If $\mathcal{D} = \emptyset$ and $\mathcal{D}_w = \emptyset$ (resp. $\mathcal{D}_v = \emptyset$) then G' has a solution if and only if it has a solution containing v (resp. w).

Proof. As $\mathcal{D}_v = \emptyset$, no set of the form $\{v\}$, $\{v, u\}$ or $\{v, u, u'\}$ with $u, u' \in N_{2,3}(v, w)$ can dominate $N_3(v, w)$. Since also $\mathcal{D} = \emptyset$ any SDS of G has to contain v or at least four vertices by ?. In the last case, we again replace these four vertices with v, w and two neighbors respectively and we can conclude that v belongs to the solution. \square

Now we are ready to prove the correctness of ??

Lemma 3. Let $G = (V, E)$ be a plane graph, $v, w \in V$ and $G' = (V', E')$ be the graph obtained after application of ?? on the pair $\{v, w\}$. Then G has SDS of size k if and only if G' has SDS of size k .

Proof. We will prove the claim by analyzing the different cases of the rule independently.

4.3 The Reduction Rules

\Rightarrow Consider a SEMITOTAL DOMINATING SET D in G . We show that G' also has a SDS with $|D'| \leq |D|$. By assumption, we have $\mathcal{D} = \emptyset$.

- a) $\mathcal{D}_v = \emptyset \wedge \mathcal{D}_w = \emptyset$: By applying ?? twice, we know that both $v, w \in D$. Therefore, v', w' , and potentially y and y' are dominated by v or w in G' .

We now have three cases: Either v and w are so far away that both have their witnesses themselves; they are of distance three, such that potentially one witness on a path from v to w (which could go through $N_{2,3}(v, w)$ and will be removed) is required or they can be of distance less than three, such that they already witness each other directly.

We will now build D' depending on which vertices from $D \cap N_{2,3}(v, w)$ have removed.

- If the rule has not removed any $d \in D$, we simply set $D' = D$. If v was a witness for w (and vice versa), ?? will preserve it by introducing the vertex y . Otherwise, these witnesses are preserved.
- If $d(v, w) > 3$, then v and w are not sharing any common witnesses. If the rule has removed a vertex from $D \cap N(v)$, we set $D' = D \setminus N_{2,3}(v, w) \cup \{v'\}$. If the rule has removed a vertex from $D \cap N(w)$, we set $D' = D \setminus N_{2,3}(v, w) \cup \{w'\}$. If the rule has removed a vertex from $(D \cap N(v))$ and a vertex from $(D \cap N(w))$, we set $D' = D \setminus N_{2,3}(v, w) \cup \{v', w'\}$.
- If $d(v, w) = 3$, then there could possibly be a path via $N_{2,3}(v, w)$ vertices containing a witness for both v and w . If the rule removed a vertex $D \cap N_{2,3}(v, w)m$, we set $D' = D \setminus N_{2,3}(v, w) \cup \{y\}$. Note that we could also choose $y' \in D'$, because y 's only function is to be a single witness for v and w and everything it could witness too, will also be witnessed by $v, w \in D'$ (??).
- If $d(v, w) \leq 2$, then v directly witnesses w (and vice versa). even if the rule has removed a vertex $z \in D \cap N_{2,3}(v, w)$, we can ignore that, because ?? states that v and w will witness the same vertices as z did. Hence, we set $D' = D \setminus N_{2,3}(v, w)$.

In all of the cases, it follows that D' is a SDS of G' with $|D'| \leq |D|$

- b) $\mathcal{D}_v \neq \emptyset \wedge \mathcal{D}_w = \emptyset$: As $\mathcal{D}_w = \emptyset$ and ??, we know that $v \in D$ and v dominates $N_{2,3}(v)$. If a vertex $d \in D \cap N_{2,3}(v)$ was removed, we set $D' = D \setminus N_{2,3}(v) \cup \{v'\}$, else $D' = D$. Deleting dominating vertices $d \in D \cap N_{2,3}(v)$ does not destroy the witness properties of the graph, because by ?? we know that everything d could witness, is also witnessed by v . If d was a witness for v , we have replaced it with v' in G' . Note that otherwise a vertex from $N_1(v) \cup \{p \in (N(z) \setminus N(v)) | z \in N_1(v)\}$ is a witness for v that is not touched by this reduction. Clearly, $|D'| \leq |D|$ holds.

c) $\mathcal{D}_v = \emptyset \wedge \mathcal{D}_w \neq \emptyset$: Symmetrical to previous case.

\Leftarrow Let D' be a SEMITOTAL DOMINATING SET in G' and $\mathcal{D} = \emptyset$. We show that G has a SDS D with $|D| \leq |D'|$ by distinguishing the different cases again.

a) $\mathcal{D}_v = \emptyset \wedge \mathcal{D}_w = \emptyset$: In any case we know that $v, w \in D$ to dominate v' and w' and therefore also dominating $N_{2,3}(v, w)$ in G . To preserve the distance $d(v, w)$ the rule might have introduced additional vertices y and y' .

- If only y was introduced we know that there was a common neighbor $n \in N(v) \cap N(w)$ of v and w . y allows v to witness w (and vice versa) and is not part of a solution itself. (assuming $y \notin D'$). Hence, we set $D = D'$.

- If y and y' were added, a solution could use one of them to provide a single witness for v and w . There exists a path $p = (v, n_1, n_2, w)$ from v to w in G only using vertices from $N_{2,3}(v, w)$. As n_1 and n_2 both witness v and w , we put one of them in D if at least one of y or y' are dominating vertices in G' . Hence, if $y \in D'$ or $y' \in D'$, we set $D = D' \setminus \{y, y'\} \cup \{n_1\}$.

b) $\mathcal{D}_v \neq \emptyset \wedge \mathcal{D}_w = \emptyset$: Clearly, $v \in D'$ to dominate v' . If $v \in D'$, we set $D = D' \setminus \{v'\} \cup d$ for some vertex $d \in N_{2,3}(v, w)$ and otherwise $D = D'$. If v' was the witness of v , it is now replaced by d and D is an SDS with $|D| \leq |D'|$.

c) $\mathcal{D}_v = \emptyset \wedge \mathcal{D}_w \neq \emptyset$: Symmetrical to previous case.

In all cases we have shown that $|D| \leq |D'|$ and D is a SEMITOTAL DOMINATING SET of G .

□

4.3.3 Reduction Rule III: Shrinking Simple Regions

If two vertices v and w share enough neighbors with each other, we can sometimes conclude at least one of them to be in a solution.

By planarity, a *simple region* has at most two vertices from $N_1(v, w)$ (namely the border ∂), two vertices from $N_2(v, w)$ connected to the border and unlimited $N_3(v, w)$ vertices squeezed in the middle.

With a more sophisticated analysis, we were able to modify this rule such that the bound given in [Garnero2018] remained valid although we might have to add some more vertices as originally described.

Rule 3. Let $G = (V, E)$ be a plane graph, $v, w \in V$ and R be a simple region between v and w . If $|V(R) \setminus \{v, w\}| \geq 5$ apply the following:

Case 1: If $V(R \setminus \partial R) \cong P_3$ then

- Remove $V(R \setminus \partial R)$
- add vertex y with edges $\{v, y\}$ and $\{y, w\}$

Case 2: If $V(R \setminus \partial) \not\cong P_3$ then

- Remove $V(R \setminus \partial R)$
- add vertices y, y' and four edges $\{v, y\}, \{v, y'\}, \{y, w\}$ and $\{y', w\}$

Lemma 4 (Correctness of ??). Let $G = (V, E)$ be a plane graph, $v, w \in V$ and $G' = (V', E')$ be the graph obtained after application of ?? on the pair $\{v, w\}$. Then G has SDS of size k if and only if G' has SDS of size k .

Proof. \Rightarrow Consider a SEMITOTAL DOMINATING SET D in G . We show that G' also has a SDS with $|D'| \leq |D|$. By assumption, we have $\mathcal{D} = \emptyset$.

\Rightarrow Consider a SEMITOTAL DOMINATING SET D in G' . We show that G has a SDS with $|D| \leq |D'|$. By assumption, we have $\mathcal{D} = \emptyset$. □

The application of ?? gives us a bound on the number of vertices inside a simple region.

Corollary 5. Let $G = (V, E)$ be a graph, $v, w \in V$ and R a simple region between v and w . If ?? has been applied, this simple region has a size of at most 6.

Proof. If $|V(R) \setminus \{v, w\}| < 7$ then the rule would not have changed G and the size of the region would already be bounded by 6. Assuming $|V(R) \setminus \{v, w\}| \geq 7$ we note that every simple region can have at most two distinct vertices from $N_1(v, w)$ and two distinct ones from $N_2(v, w)$ without breaking planarity. These vertices are not touched by the reduction. Adding the two vertices that are being added between v and w gives us the desired upper bound. □

4.3.4 Computing Maximal Simple Regions between two vertices

For the sake of completeness, we state an algorithm on how a maximal simple region-between two vertices $v, w \in V$ can be computed in time $\mathcal{O}(d(v) + d(w))$:

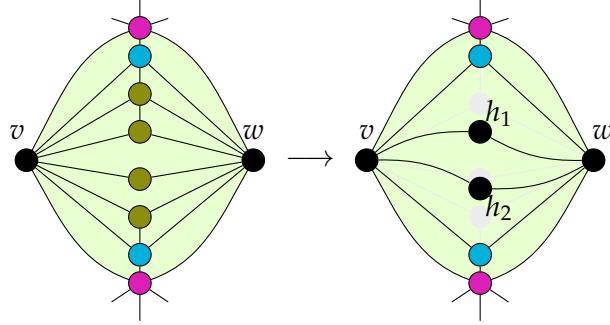


Figure 4.9: TO BE DONE

4.4 Bounding the Size of the Kernel

We now put all the pieces together to prove the main result: A kernel bounded linear in the solution size k . For that purpose, we distinguish between those vertices that are covered inside a region in a maximal *D-region decomposition* and those that are not. In both cases, our reduction rules bound the number of vertices to a constant size which means the kernel size does only depend on the number of regions of these decompositions. As ?? states that for any given dominating set D , we can partition the whole graph into a linear number of regions. In particular, we show that given a SEMITOTAL DOMINATING SET D of size k , there exists a maximal *D-region decomposition* \mathfrak{R} such that:

1. \mathfrak{R} has only at most $3|D| - 6$ regions
2. $V(\mathfrak{R})$ covers most vertices of V . There are at most $144 \cdot |D|$ vertices outside of any region.
3. each region of \mathfrak{R} contains at most 139 vertices

Combining these three statements will give us a linear kernel. ?? depicts them *yellow*.

4.4.1 Bounding the Size of a Region

We start with a more fine-grained analysis of the impact of the different cases of ?? on a vw -region. The main idea is to count the number of simple regions in the vw -region and then use the bound on the size of a simple region after ?? was applied exhaustively. The bound was obtained in ??.

Lemma 5. *Given a plane Graph $G = (V, E)$ and a vw -region R , let D be a SEMITOTAL DOMINATING SET and let \mathfrak{R} be a maximal D -region decomposition of G . For any vw -regions $R \in \mathfrak{R}$, $|N_1(v, w) \cap V(R)| \leq 4$ and these vertices lay exactly on the boundary ∂R of R .*

Proof. The proof directly follows from [Garnero2019], because every TOTAL DOMINATING SET is also SEMITOTAL DOMINATING SET. Actually, this claim also holds for

4.4 Bounding the Size of the Kernel

DOMINATING SET in Intuitively, this statement says that every $N(v, w)$

□

Lemma 6. [Garnero2018] Given a reduced plane graph $G = (V, E)$ and a region $R(v, w)$, $N_2(v, w) \cap V(R)$ can be covered by at most 6 simple regions.

Proof. Let (v, u_1, u_2, w) and (v, u_3, u_4, w) be the two boundary paths of $R(v, w)$. (A shorter path would only lead to a smaller bound). By definition of $N_2(v, w)$, vertices from $N_2(v, w) \cap V(R)$ are common neighbors of v and w and $u_i, i \in [4]$. By planarity, we can cover $N_2(v, w) \cap V(R)$ with at most 6 simple regions. To see this, imagine the graph where edges denote all possible simple vw -regions (See fig. ??). There are at most 8 simple regions possible. but we have to remove at least two of them to maintain planarity.

Furthermore, assuming the graph to be reduced, any intermediate $N_3(v, w)$ which could separate multiple simple regions between v and u_i has been deleted by ?? already.

□

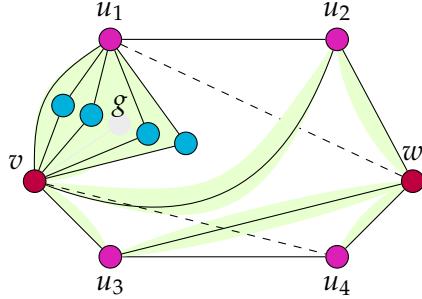


Figure 4.10: Bounding the maximum number of simple regions inside a region $R(v, w)$.

$N_2(v, w)$ is covered by 6 green (simple) regions. A dashed edge would also be an option but would contradict planarity. Note that the gray vertex g was reduced by ?? allowing the formation of exactly one simple region between v and u_1

We continue by giving a constant bound on the number of simple regions that cover all $N_3(v, w)$ vertices in a given region.

Lemma 7. Given a plane Graph $G = (V, E)$ reduced under ?? and a region $R(v, w)$, if $\mathcal{D}_v \neq \emptyset$ (resp. $\mathcal{D}_w \neq \emptyset$), $N_3(v, w) \cap V(R)$ can be covered by:

1. 11 simple regions if $\mathcal{D}_w \neq \emptyset$,
2. 14 simple regions if $N_{2,3}(v) \cap N_3(v, w) = \emptyset$

Note, that the first case applies, when Case 2 & 3 of ?? have been applied and the second one, when Case 4 of ?? was applied.

Proof. We will just give some intuition because the proof of Garnero and Sau in [Garnero2018] does not use any special property exposed by the reduction rules. Figure (Add picture about figures) gives a visualization of the worst-case scenarios to cover $N_3(v, w) \cap V(R)$ with simple regions in the relevant cases.¹

□

Lemma 8 (#Vertices inside a Region after ??????). *Let $G = (V, E)$ be a plane graph reduced under ??????. Furthermore, let D be an SDS of G and let $v, w \in D$. Any vw -region R contains at most 139 vertices distinct from its poles.*

Proof. By ??? and ?? to bound the number of vertices inside a simple region, we know that $|N_1(v, w) \cap V(R)| \leq 4$ and $|N_2(v, w) \cap V(R)| \leq 6 \cdot 7 = 42$.

It is remaining to bound for $|N_3(v, w) \cap V(R)|$, but gladly we have ??, which took care of them! ?? shows the worst-case amount of simple regions the individual cases can have.

Case 0: ?? has **not** been applied in the following two cases: Either $\mathcal{D} \neq \emptyset$ or $(\mathcal{D} = \emptyset \wedge \mathcal{D}_v \neq \emptyset \wedge \mathcal{D}_w \neq \emptyset)$:

1. If $\mathcal{D} \neq \emptyset$, there exists a set $\tilde{\mathcal{D}} = \{d_1, d_2, d_3\} \in \mathcal{D}$ of at most three vertices dominating $N_3(v, w)$. We observe that vertices from $|N_3(v, w) \cap V(R)|$ are common neighbors of either v or w (by the definition of a vw -region) and at least one vertex from $\tilde{\mathcal{D}}$. Without violating planarity, we can span at most 6 simple regions. Using ?? and adding $|\tilde{\mathcal{D}}| = 3$, we can conclude $|N_3(v, w) \cap V(R)| \leq 6 \cdot 6 + 3 = 39$.
2. If $\mathcal{D} = \emptyset$, $\mathcal{D}_v \neq \emptyset$ and $\mathcal{D}_w \neq \emptyset$, we can apply ?? and although ?? has not changed the G , we can cover R with at most 11 simple regions giving us $|N_3(v, w) \cap V(R)| \leq 11 \cdot 6 = 66$ vertices.

Case 1: If ?? ?? has been applied

In that case, $|N_2(v, w) \cap V(R)|$ was entirely removed and at $|N_3(v, w) \cap V(R)|$ replaced by at most three vertices (v' , w' and y) added. Hence $|N_3(v, w) \cap V(R)| \leq 3$.

Case 2: If ?? ???? have been applied

We know that in this case $N_{2,3}(v) \cap N_3(v, w)$ was entirely removed and replaced by a single possible witness. Using ??, we can cover $(N_3(v, w) \setminus \{v'\}) \cap V(R)$ with (at most) 14 simple regions giving us $||N_3(v, w) \cap V(R)|| \leq 14 \cdot 6 + 1 = 85$.

All in all, as $V(R) = \{v, w\} \cup (N_1(v, w) \cup N_2(v, w) \cup N_3(v, w)) \cap V(R)$ we get

$$V(R) \leq 2 + 4 + 42 + \max(39, 3, 66, 85) = 139$$

□

¹Note: In a newer revision of their paper [Garnero2018], Garnero and Sau removed this proof, because they changed ?? and a more fine-grained analysis was made possible.

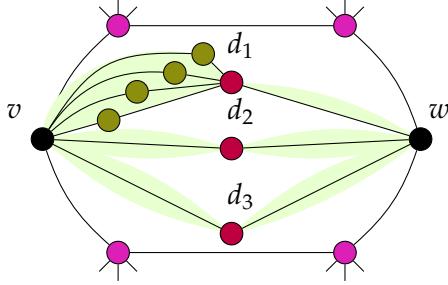
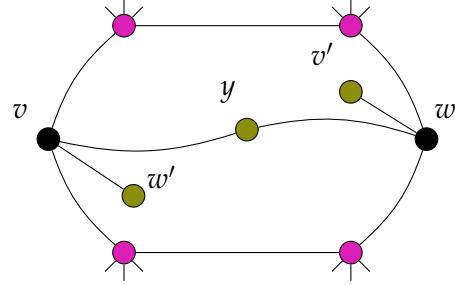
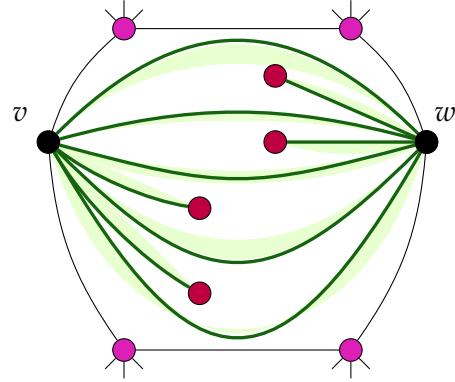
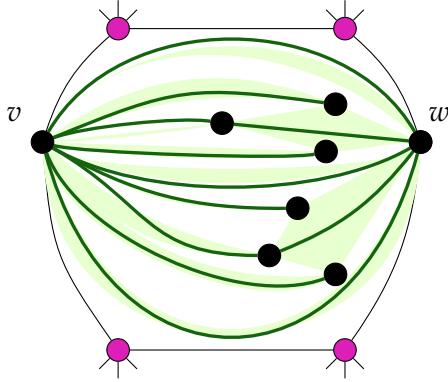
Case 0: Maximal 6 Simple Regions

Case 1: Exactly 3 vertices

Case 2: All Maximal 11 Simple Regions **Case 3/4:** Maximal 9 Simple Regions


Figure 4.11: TODO

4.4.2 Number of Vertices outside the Decomposition

We continue to bound the number of vertices that do not lay inside any region of a maximal D -region decomposition \mathfrak{R} , that is, we bound the size of $V \setminus V(\mathfrak{R})$. ?? ensures that we only have a small amount of $N_3(v)$ -pendants. We then try to cover the rest with as few simple regions as possible, because, by application of ??, these simple regions are of constant size.

Lemma 9. [Alber2004] (Deprecated) Every vertex in $u \in V \setminus V(\mathfrak{R})$ is either in D or belongs to a set $N_2(v) \cup N_3(v)$.

The following lemma states that no vertices from a set $N_1(v)$ will be outside of a maximal D -region decomposition.

Lemma 10. [Alber2004] Let $G = (V, E)$ be a plane graph and \mathfrak{R} be a maximal D -region decomposition of a DS D . If $u \in N_1(v)$ for some vertex $v \in D$ then $u \in V(\mathfrak{R})$

4 A Linear Kernel for Planar Semitotal Domination

In the following, we define $d_{G_{\mathfrak{R}}}(v) = |\{R(v, w) \in \mathfrak{R}, w \in D\}|$ to be the number of regions in \mathfrak{R} having v as a pole.

Corollary 6. *Let $G = (V, E)$ be a graph and D be a set. For any maximal D -region decomposition \mathfrak{R} on G it holds that $\sum_{v \in D} d_{G_{\mathfrak{R}}}(v) = 2 \cdot |\mathfrak{R}|$.*

Proof. The proof follows directly from the handshake lemma applied to the underlying multigraph $G_{\mathfrak{R}}$. \square

Proposition 1 (#Vertices outside a Region). *Let $G = (V, E)$ be a plane graph reduced under [????](#) and let D be a SDS of G . If G has a maximal D -region decomposition, then $|V \setminus (V(\mathfrak{R}) \cup D)| \leq 144|D|$*

With slight modifications, the proof given in [[Garnero2018](#)] will also work in our case. Note that although assuming the graph to be entirely reduced, the following proof only relies on [????](#). The proof uses the observation that vertices from $N_2(v)$ span simple regions between those from $\{v\} \cup N_1(v)$.

Proof. Again, we will follow the proof proposed by Alber et al. [[Alber2004](#)].

The proof does only rely on [????](#) and we can use the number of vertices in a simple region we have proven in [??](#). In particular, we are going to proof that $V \setminus V(\mathfrak{R}) \leq 48 \cdot |\mathfrak{R}| + 2 \cdot |D|$. Directly placing in [??](#) will give the desired bound.

let \mathfrak{R} be a maximal D -region decomposition and let $v \in D$. Since D dominates all vertices from V , we can consider V as $\bigcup_{v \in D} N(v)$ and thus, we only need to bound the sizes of $N_1(v) \setminus V(\mathfrak{R})$, $N_2(v) \setminus V(\mathfrak{R})$ and $N_3(v) \setminus V(\mathfrak{R})$ separately. In the following, let $v \in D$:

N₃(v): As we know that [??](#) has been exhaustively applied, we trivially see that $|N_3(v)| \leq 1$ and hence,

$$\left| \bigcup_{v \in D} N_3(v) \setminus V(\mathfrak{R}) \right| \leq |D|$$

N₂(v): According to Garnero and Sau ([\[Garnero2018\]](#)), we know that $N_2(v) \setminus V(\mathfrak{R})$ can be covered by at most $4d_{G_{\mathfrak{R}}}(v)$ simple regions between v and some vertices from $N_1(v)$ on the boundary of a region in \mathfrak{R} . Figure [??](#) gives some intuition.

Because G is reduced by assumption, we know by [??](#) that a simple region can only have at least 6 vertices distinct from its poles and hence,

$$\begin{aligned} \left| \bigcup_{v \in D} N_2(v) \setminus V(\mathfrak{R}) \right| &\leq 6 \sum_{v \in D} 4 \cdot d_{G_{\mathfrak{R}}}(v) \\ &= 24 \cdot \sum_{v \in D} d_{G_{\mathfrak{R}}}(v) \\ &\stackrel{\text{Cor. ??}}{\leq} 48|\mathfrak{R}| \end{aligned} \tag{4.10}$$

4.4 Bounding the Size of the Kernel

N₁(v): By ??, we know that $N_1(v) \subseteq V(\mathfrak{R})$ and hence,

$$\left| \bigcup_{v \in D} N_1(v) \setminus V(\mathfrak{R}) \right| = 0$$

Summing up these three upper bounds for each $v \in D$ we obtain the result using the equation from ??:

$$\begin{aligned} |V \setminus V(\mathfrak{R}) \cup D| &\leq 48 \cdot |\mathfrak{R}| + |D| && \text{(Lemma ??)} \\ &\leq 48 \cdot (3|D| - 6) + |D| \\ &\leq 144|D| + |D| \\ &= 145|D| \end{aligned} \tag{4.11}$$

□

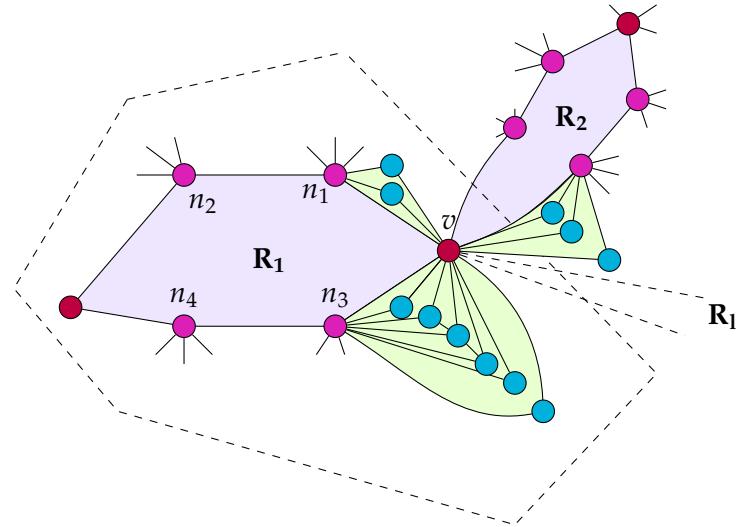


Figure 4.12: Bounding the number of $N_2(v)$ -vertices around a dominating vertex v given a maximal D -region decomposition \mathfrak{R} . v is a pole of R_1, R_2, \dots, R_j and can span simple regions with the help of $N_2(v)$ -vertices to at most two $N_1(v)$ -vertices per R_i . Each region has at most four vertices in $N_1(v, w) \subseteq N_1(v)$ on the boundary of R_j , but only at most two can be used for a simple region: Observe that trying to build a simple region between v and n_2 in this example would contradict the maximality of \mathfrak{R} . Furthermore, the size of these simple regions is bounded after the application of ??

4.4.3 Bounding the Number of Regions

Alber et al. [Alber2004] gave a greedy algorithm to construct a maximal *D-region decomposition* for a DOMINATING SET. Building upon these results, Garnero and Sau gave decomposition procedures for both PLANAR RED-BLUE DOMINATING SET ([Garnero2017]) and TOTAL DOMINATING SET ([Garnero2018]) relying on the same technique. This is the core of our linear kernelization because it states that given a DOMINATING SET D , we can decompose the graph into a *linear number* of regions.

The following lemma corresponds to [Alber2004]. Although the authors gave different reduction rules and require a *reduced* instance as an assumption for the following lemma, they do not use any specific properties exposed by these rules. As any SEMITOTAL DOMINATING SET is also a DOMINATING SET, we can safely apply it to our problem as well. For a more detailed and formal proof, one can also refer to [Garnero2018].

Lemma 11. *Let G be a reduced plane graph and let D be a SEMITOTAL DOMINATING SET with $|D| \geq 3$. There is a maximal D -region decomposition of G such that $|R| \leq 3 \cdot |D| - 6$*

Proof. Follows directly from [Alber2004] □

Lemma 12 (Running Time of Reduction Procedure).

Proof. □

By utilizing all the previous results, we are now finally ready to proof the ??:

Theorem ??. *The SEMITOTAL DOMINATING SET problem parametrized by solution size admits a linear kernel on planar graphs. There exists a polynomial-time algorithms that given a planar graph (G, k) , either correctly reports that (G, k) is a NO-instance or returns an equivalent instance (G', k) such that $|V(G')| \leq 561 \cdot k$.*

Proof. Let $G = (V, E)$ be the plane input graph and $G' = (V', E')$ be the graph obtained by the exhaustive application of the ??????. As none of our rules change the size of a possible solution D' in G' , we know by ??, ?? and ?? that G' has a SEMITOTAL DOMINATING SET of size k if and only if G has a SEMITOTAL DOMINATING SET of size k . In ??, we have proven that this preprocessing procedure runs in polynomial time. Assume that G' admits a solution D' .

By taking the size of each region proven in ??, the total number of regions in a maximal *D-region decomposition* (??) and the number of vertices that can lay outside of any region (??), we obtain the following bound:

$$139 \cdot (3k - 6) + 145 \cdot k + k \leq 561 \cdot k \quad (4.12)$$

If $|V(G')| > 561 \cdot k$ G is a NO-instance and we replace G' by two single disconnected vertices (trivial NO-instance). Then the kernel is of the claimed size. □

CHAPTER 5

OPEN QUESTIONS AND FURTHER RESEARCH



To do select another quote

Lewis Caroll, XXXX

* Chordal Bipartite Graph has a very interesting case. * Improve the Kernel Bound

5 Open Questions and Further Research

LIST OF FIGURES

LIST OF TABLES