DEPARTMENT OF INFORMATICS

TECHNICAL UNIVERSITY MUNICH

Master Thesis

On the Parametrized Complexity of Semitotal Domination on Graph Classes

Lukas Retschmeier

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Über die Parametrisierte Komplexität des Problems der halbtotalen stabilen Menge auf Graphklassen

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I confirm that this master thesis is my own work and I have documented all sources and material used.

København S
November 14, 2022

Lukas Retschmeier

Acknowledgments

CONTENTS

ABSTRACT

Abstract

For a graph $G = (V, E)$, a set D is called a Semitotal Dominating Set, if D is a dominating set and every vertex $v \in D$ is within distance two to another witness $v' \in D$. The Minimum Semitotal Dominating Set problem is to find a semi-total dominating set of minimum cardinality. The semitotal domination number $\gamma_{t2}(G)$ is the minimum cardinality of a semitotal dominating set and is squeezed between the domination number $\gamma(G)$ and the total domination number $\gamma_t(G)$. Given a graph $G = (V, E)$ and a positive integer k, the Semitotal Domination Decision problem asks if G has a Semitotal Dominating Set of size at most k.

After the problem was introduced by Goddard, Henning and McPillan in [18], NP-completeness was shown for general graphs [23], *split graphs* [23], *planar graphs* [23], *chordal bipartite graphs* [23], *circle graphs* [25] and *subcubic line graphs of bipartite graphs* [14]. On the other side, there exist polynomial-time algorithms for *AT-free graphs* [25], *graphs of bounded mim-width* [14], *graphs of bounded clique-width* [8], and *interval graphs* [23].

In this thesis, we start the systematic look through the lens of *parametrized complexity* by showing that Semitotal Dominating Set is $\omega[2]$-hard for bipartite graphs and split graphs. By applying the techniques proposed in [2] and [16] for Dominating Set and Total Dominating Set, we are going to construct a $359k$ kernel for Semitotal Dominating Set in planar graphs. This result further complements known linear kernels for other domination problems like Planar Connected Dominating Set, Planar Red-Blue Dominating Set, Planar Efficient Dominating Set, Planar Edge Dominating Set, Independent Dominating Set and Planar Directed Dominating Set.

Keywords: Domination; Semitotal Domination; Parametrized Complexity; Planar Graphs; Linear Kernel

ZUSAMMENFASSUNG

Abstract

Hier kommt noch ein weiterer Abstract rein.

Schlagworte: Stabile Menge; Halbtotale Stabile Menge; Parametrisierte Komplexität; Plättbare Graphen; Linearer Problemkern

Abstract all the way

CHAPTER 1

INTRODUCTION

Todo select another quote

Lewis Caroll, *XXXX*

Parametrized Complexity emerging branch. Books about that

Semitotal domination was introduced by

Quack! Quack! Idea: Lake with stones, and a family of ducks of fixed size want to occupy the lake so that no other clan tries to take it over. Rules: * A duck can quack freeing up neighboring stones. * Ducks don't like to be alone and want to quack together. So for every duck their must be another duck that is not further than two stones away. Q: Can our ducklings occupy the whole lake while not feeling lonely?

TODO A demo instance next to each other

Problem Definition

DOMINATING SET [9, p. 586]

Input:	Graph $G = (V, E)$ and an integer k
Question:	Is there a set $X \subseteq V$ of size at most k such that $N[X] = V$?

SEMITOTAL DOMINATING SET [18]

Input:	Graph $G = (V, E)$ and an integer k
Question:	Is there a subset $X \subseteq V$ of size at most k such that $N[X] = V$ and for all $d_1 \in X$ there exists another $d_2 \in X$ such that $d(d_1, d_2) \leq 2$?

TOTAL DOMINATING SET [9, p. 596]

Input:	Graph $G = (V, E)$ and an integer k
Question:	Does there exists a set $X \subseteq V$ of at most k vertices of G such that for every $u \in V(G)$ there exists $v \in X$ with $\{u, v\} \in E$

1.1 Content of the thesis

In this thesis, we continue the systematic analysis of the Semitotal Dominating Set problem by focusing on the parametrized complexity of the problem.

Although the problem already had a lot of attention regarding classical complexity (CITE), only a few results are currently known for the parametrized variant.

As far as we have seen, even the w-hardness of the general case has not been explicitly proven in the literature.

In this thesis, we continue the journey toward a systematic analysis by stating some hardness results for specific graph classes for the problem.

Our contributions Our main contributions consist of first showing the $w[2]$-hardness of Semitotal Dominating Set for XXXX graphs.
As the Dominating Set problem and the Total Dominating Set problem both admit a linear kernel for planar graphs, it is interesting to analyze whether these results also hold for the Semitotal Dominating Set problem which lies in between these two.

Having these kernels also for other variants like Edge Dominating Set, Efficient Dominating Set, Connected Dominating Set, Planar Red-Blue Dominating Set lent us great confidence that the result will also work for Semitotal Dominating Set on planar graphs.

Following the approach from ... which already relies on the technique given in, we give some simple data reduction rules for Semitotal Dominating Set on planar graphs leading to a linear kernel. More precisely, we are going to prove the following central theorem of this thesis:

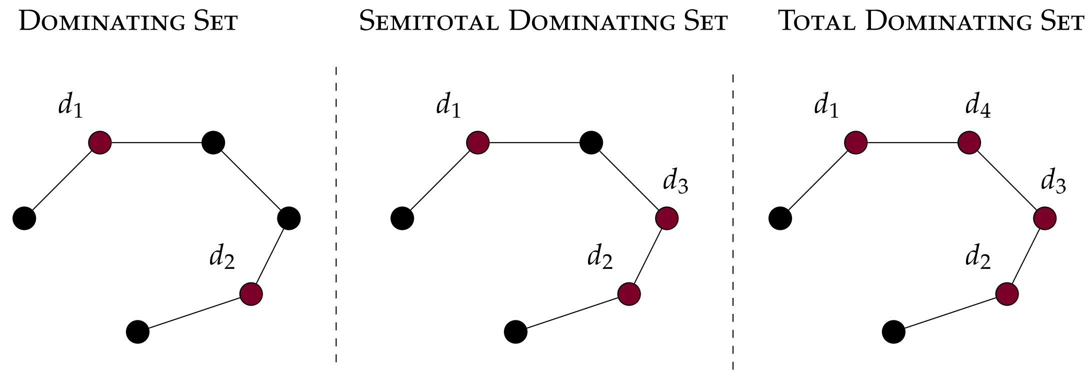


Figure 1.2: *An example for a* Dominating Set, Semitotal Dominating Set *and* Total Dominating Set, *where $\gamma(G) < \gamma_{2t}(G) < \gamma_t(G)$ are strict. In the first case, only two vertices suffice to dominate all others. In the second one, we need a witness between d_1 and d_2 that is at most distance two. In the last case, d_1 and d_2 both need a neighbor in the* Total Dominating Set.

With some modifications, we were able to transfer the approach given by Garnero and Sau in [16] to the Semitotal Dominating Set problem.

Theorem 1. *The* Semitotal Dominating Set *problem parametrized by solution size admits a linear kernel on planar graphs. There exists a polynomial-time algorithms that given a planar graph (G, k), either correctly reports that (G, k) is a NO-instance or returns an equivalent instance (G', k) such that $|V(G')| \leq 359 \cdot k$.*

Dominating Set problem and Total Dominating Set problem, both already

CHAPTER 2

PRELIMINARIES

We start by recapping some basic notation in Graph Theory and Parametrized Complexity.

Continuing an intensive study of parametrized complexity of that problem.

2.1 Graph Theory

We quickly state the following definitions given by [11, p. xxx].

Definition 2.1.1 (Graph [11, p. 3])**.** *A graph is a pair $G = (V, E)$ of two sets where V denotes the vertices and $E \subseteq V \times V$ the edges of the graph. A vertex $v \in V$ is incident with an edge $e \in E$ if $v \in e$. Two vertices x, y are adjacent, or neighbours, if $\{x, y\} \in E$.*

Definition 2.1.2 (Vertex Degrees)**.** *The degree $d_G(v)$ (If G is clear, also $d(v)$) of a vertex v is the number of neighbors of v. We call a vertex of degree 0 as isolated and one of degree 1 as a pendant.*

Definition 2.1.3 (isomorphic Graphs, [11, p. 3])**.** *Let $G = (V, E)$ and $G' = (V', E')$ be two graph. We call G and G' isomorphic, if there exists a bijection $\phi : V \to V'$ with $\{x, y\} \in E \Leftrightarrow \phi(x)\phi(y) \in E'$ for all $x, y \in V$. Such a map ϕ is called isomorphism.*

If a graph G is isomorphic to another graph h, we denote $G \simeq H$.

Definition 2.1.4 (Special Graph Notations [11, p. 27])**.** *A simple Graph*

A directed Graph is a graph

A Multi Graph

A Planar Graph

Definition 2.1.5 (Closed and Open Neighborhoods [3])**.** *Let $G = (V, E)$ be a (non-empty) graph. The set of all neighbors of v is the open neighborhood of v and denoted by $N(v)$; the set $N[v] = N(v) \uplus \{v\}$ is the closed neighborhood f v in G. When G needs to be made explicit, those open and closed neighborhoods are denoted by $N_G(v)$ and $N_G[v]$.*

Definition 2.1.6 (Induced Subgraph)**.** *asd*

Definition 2.1.7 (Isomorphic Graph)**.** *asd*

Graph Classes

We call the class of graphs without any special restrictions "General Graphs".

Definition 2.1.8 (r-partite Graphs)**.** *Let $r \geq 2$ be an integer. A Graph $G = (V, E)$ is called "r-partite" if V admits a partition into r classes such that every edge has its ends in different classes: Vertices in the same partition class must not be adjacent.*

For the case $r = 2$ we say that the G is "bipartite"

Definition 2.1.9 (Chordal Graphs)**.**

Definition 2.1.10 (Split Graphs)**.**

Definition 2.1.11 (Path Graph P_i)**.**

Definition 2.1.12 (Bipartite Graph, [7, p.5])**.** *A bipartite graphs is a Graph G whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y. Such a partition (X,Y) is called a bipartition of G.*

Definition 2.1.13 (Planar Graphs [11, Chapter 4])**.** *A plane graph is a pair (V, E) of finite sets with the following properties:*

- $V \subseteq \mathbb{R}^2$ *(Vertices)*
- *Every edge is an arc between two vertices*
- *different edges have different sets of endpoints*
- *The interior of an edge contains no vertex and no point of any other edge*

An embedding in the plane, or planar embedding, of an (abstract) graph G is an isomorphism between G and a plane graph H. A plane graph can be seen as a concrete ***embedding*** *of the planar graph into the "plane" $\mathbb{R}^2$.*

CHAPTER 3

ON PARAMETRIZED SEMITOTAL DOMINATION

Todo select another quote

Lewis Caroll, *XXXX*

For an introduction into classical complexity theory. Refer to the standard textbooks aaran und cpo. Rely an []

3.1 Parametrized Complexity

* Decision Problem

Definition 3.1.1 (Parametrized Problem[9, Def 1.1])**.** *A parametrized problem is a* $L \subseteq \Sigma^* \times \mathbb{N}$ *(*Σ *finite fixed alphabet) for an instance* $(x, k) \in \Sigma^* \times \mathbb{N}$*, where* k *is called the parameter.*

Definition 3.1.2 (Instance Size)**.** *The* ***size of an instance*** *of an instance* (x, k) *of a parametrized problem is* $|(x, k)| = |x| + k$

We will now clarify the basic terminology withing Parametrized Complexity. We are now giving a short introduction into the world of parametrized complexity. * General Introduction

Ways to cope with NP-hard problem.

3.1.1 Fixed Parameter Tractability

Definition 3.1.3 (The Class FPT [9, Def 1.2])**.** *A parametrized problem $L \subseteq \Sigma^* \times \mathbb{N}$ is called fixed-parameter tractable if there exists an algorithm A (called a fixed-parameter algorithm), a computable function $f : \mathbb{N} \to \mathbb{N}$ and a constant c such that, given $(x,k) \in \Sigma^* \times \mathbb{N}$, the algorithm $\mathcal{A}$ correctly decides whether $(x,k) \in L$ in time bounded by $f(k) \cdot |(x,k)|^c$. The complexity class containing all fixed-parameter tractable problems is called FPT*

3.1.2 Kernelization

Definition 3.1.4 (kernelization Algoritm[9, Def 2.1])**.** *A Kernelization Algorithm or kernel is an algorithm $\mathfrak{A}$ for a parametrized Problem Q, that given an instance (I,k) of Q works in polynomial time and returns an equivalent instance (I',k') of Q. Moreover, we require that $size_{\mathfrak{A}}(k) \leq g(k)$ for some computable function $g : \mathbb{N} \to \mathbb{N}$*

If we bound the size of the kernel by linear function $f(m) = \mathcal{O}(k)$, we say that the problem admits a **linear kernel**.

The main idea, preprocessing algorithm, shrink size as much as possible, sound reduction rules,small output instance

Definition 3.1.5 (Output size of a Preprocessing Procedure [9, p. 18])**.** *The output size of a preprocessing algorithms $\mathfrak{A}$ is defined as*

$$\mathrm{size}_{\mathfrak{A}}(k) = \sup\{\left|I'\right| + l' : (I',k') = \mathfrak{A}(I,k), I \in \Sigma^*\}$$

possibly infinite

Clearly, if there exists a kernelization algorithm for a problem L and an algorithm $\mathfrak{A}$ with any runtime to decide L, the problem is in FPT because after the kernelization pre-processing has been applied, the size of the reduced instance is a function merely in k and independent of the input size n. In Chapter 4 we will explicitly construct a kernel for Planar Semitotal Dominating Set and hence showing it to be in FPT.

Definition 3.1.6 (Reduction Rules [9, p. 18])**.** *A **reduction rule** is a function $\phi : \Sigma^* \times \mathbb{N} \to \Sigma^* \times \mathbb{N}$ that maps an instance (x,k) to an equivalent instance (x',k') such that phi is computable in time polynomial in $|x|$ and k*

Definition 3.1.7 (Equivalent Instance [9, p. 18])**.** *This is a test*

Definition 3.1.8. *Soundness of a rule*

A **reduction rule** is a function $\Sigma * \times \mathbb{N}$ that maps an instance (x,k) to an equivalent instance (x',k') such that xx is computable in time polynomial in $|x|$ and k

3.1.3 Fixed Parameter Intractability: The w-Hierarchy

3.1.4 Compare to classical NP-Hardness theory

Parametrized Reductions

Definition 3.1.9 (Parametrized Reduction [9, Def 13.1])**.** *Let $A, B \subseteq \Sigma^* \times \mathbb{N}$ two parametrized problems. A Parametrized Reduction from A to B is an algorithm that, given an instance (x, k) of A, outputs an instance (x', k') of B such that*

- *(x, k) is a yes instance of A **iff** (x', k') is a yes instance of B*
- *$k' \leq g(k)$ for some computable function g*
- *the running time is $f(k) \cdot |x|^{\mathcal{O}(1)}$ (FPT!)*

The w-hierarchy

3.2 Semitotal Domination

Dominating Set
Semitotal Dominating Set
Total Dominating Set
we denote y as the dominating number. Clearly $y_t < y_s < y_d$. Notatio of an independent set

3.2.1 Preliminaries

* Witness * u pendant ofrom a vertex c if $N(u) = \{w\}$ * domination
Let D be a dominating set of G and $w \in V(G) \setminus D$. For any neighbor $v \in D \cap N(w)$, we say that d_1 *dominates* w For two dominating vertices $d_1, d_2 inD$. If
Semitotal Dominating Set
Definition, dominating number

3.3 Complexity Status of Semitotal Dominating Set

3.4 *w[i]*-Intractibility

Now some w[i] hard classes.

3.4.1 Warm-Up: *W[2]*-hard on General Graphs

As any bipartite graphswith bipartition can be split further into r-partite graphsthis results also implies the $w[1]$-hardness of r-partite graphs

3.4.2 *W[2]*-hard on Bipartite Graphs

We are showing that Semitotal Dominating Set is $\omega[2]$-hard on bipartite graphs by a parametrized reduction from Dominating Set on bipartite graphs which is known to be ω2-hard ([29, Theorem 1]).

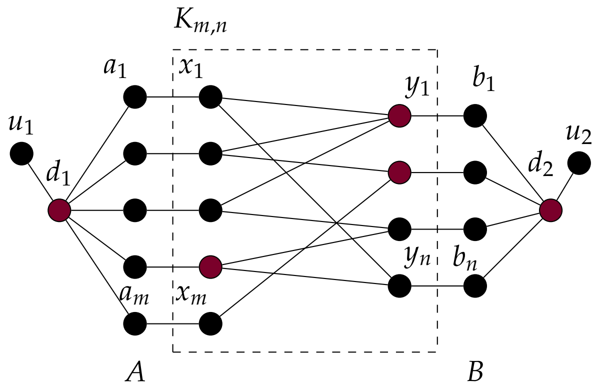


Figure 3.2: *Constructing a bipartite G' from the bipartite graph $K_{m,n}$ by duplicating all vertices and adding exactly two forced witnesses.*

Theorem 2. *Semitotal Dominating Set is $\omega[2]$ hard for bipartite Graphs*

Proof. Given a bipartite Graph $G = (\{X \cup Y\}, E)$, we construct a bipartite Graph $G' = (\{X' \cup Y'\}, E')$ in the following way:

1. For each vertex $x_i \in X$, we add a new vertex $a_i \in A$ and an edge $\{x_i, a_i\}$ in between.
2. For each vertex $y_j \in Y$, we add a new vertex $b_j \in B$ and an edge $\{y_j, b_j\}$ in between.
3. We add fource vertices with with edges $\{u_1, d_1\}$ and $\{u_2, d_2\}$, and connect them with all $\{d_1, a_i\}$ and $\{d_2, b_j\}$ ($i \in [m]$ and $j \in [n]$) respectively.

Observation: The constructed G' is bipartite because A and B form an independent set on G′ that can be cross-wise attached to one of the previous vertex sets. Setting $X' = X \cup \{u_2, d_1\} \cup B$ and $Y' = X \cup \{u_1, d_2\} \cup A$ form the partitions of the new bipartite G′.

Corollary 3.4.1. *G has a* Dominating Set *of size k iff G has a* Semitotal Dominating Set *of size $k' = k + 2$*

- ⇒ Assume a Dominating Set in G of size k. We know that $D' = D \cup \{d_1, d_2\}$ is a Semitotal Dominating Set in G' of size $k' = k + 2$, because d_1 dominates u_1

and all $a_i \in A$; d_2 dominates u_2 and all $b_i \in B$. The rest is dominated by the same vertices as they were in G, but now all of them have either d_1 or d_2 as a witness. More formally, we have by construction of G' that $\forall v \in (D \cap X) : d(v, d_1) = 2$ and $\forall v \in (D \cap Y) : d(v, d_2) = 2$.

$\Leftarrow$ On the contrary, assume any Semitotal Dominating Set D' in G' with size k'. Without loss of generality, we can assume that $u_1, u_2 \notin D'$, because choosing d_1 and d_2 instead is always as least as good and does not violate any witnesses. Therefore, the construction forces d_1, $d_2 \in D'$.

All $a_i \in A$ can only be useful to dominate their partnering x_i ($b_i \in B$ for y_i), because $d_1, d_2 \in D$ is the only second neighbor they have. If $a_i, b_i \in D'$ we replace it with x_i and y_i preserving the size D. As d_1 and d_2 suffice to provide a witness for every vertex in the graph and do not lose any other witnesses, this operation is sound.

In the end, $D = D' \setminus \{d_1, d_2\}$ gives us a Dominating Set in G with size $k = k' - 2$

As G′ can be constructed in linear time and the parameter k is only blown up by a constant, this reduction is an FPT reduction. Because Dominating Set is already $w[2]$-hard on bipartite graphs ([)], we imply that Semitotal Dominating Set is $w[2]$-hard as well. □

3.4.3 *W[2]*-hard on Split Graphs

TODO Getting started with that.

3.4.4 *W[2]*-hard on Chordal Graphs

Although the previous result implies $w[2]$-hardness for chordal graphs, we found another reduction from Dominating Set on chordal graphs.

Definition 3.4.1 ([4, IV. Triangulated Graphs])**.** *A graph G is called chordal (or in the older literature triangulated) graphs if for every cycle $c = [p_1, ..., p_n, p_1]$ of length $l > 3$ there is an edge of G joining two non-consecutive vertices of c. Such vertices are called chords of the cycle*

We will introduce the notion of an elimination ordering.

Definition 3.4.2 ([Rose1960])**.** *In a graph $G = (V, E)$ with n vertices, a vertex is called* ***simplicial*** *if and only if the subgraph of G induced by the vertex set $\{v\} \cup N(v)$ is a complete graph.*

G is said to have a ***perfect eliminiation ordering*** *if and only if there is an ordering $(v_1, ...v_n)$ of the vertices, such that each v_i is simplicial in the subgraph induced by the vertices $v_1, ..., v_i\{\}$*

The following lemma shows that

Lemma 3.4.1 ([Rose1960]). *A graph $G = (V, E)$ is chordal if and only if G has a perfect elimination ordering.*

Theorem 3. SEMITOTAL DOMINATING SET *restricted to chordal graphs is $\omega[2]$-hard.*

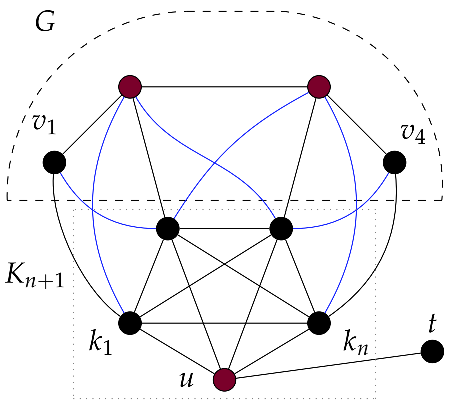


Figure 3.3: *Constructing a chordal G' from the chordal graph P_4 by adding a K_5, connecting its vertices pairwise to G. Adding the (blue) auxiliary vertices are necessary to preserve chordality.*

Proof. We will give a reduction from DOMINATING SET on chordal graphs. Given $G = (V, E)$ with vertex set $V = \{v_1, ...v_n\}$, we construct a chordal graph G' as described below:

1. Add one complete graph K_{n+1} consisting of the vertices $\{k_1, ..., k_n, u\}$ and an edge $\{v_i, k_i\}$ to each vertex $v_i \in V$ of G. One vertex of the complete subgraph is not connected to any $v \in V$. Denote it as u.

2. Add one additional vertex t and connect it with u vie the edge $\{u, t\}$.

3. For all vertices $v_i \in V$ in G, add a new edge $\{n, k_i\}$ for all neighbors $n \in N(v_i)$.

An example reduction on the graph P_4 is shown in Section 3.4.4.

Corollary 3.4.2. *$N(v_i) \in G$ forms a clique iff $N(v_i)$ forms a clique in G'*

Proof. Asuming that $N(v_i)$ forms a clique in G, we show that it also forms a clique in G′ by induction over the number of neighbors $z = abs(N(v_i))$ in G.

- $z = 0$: Holds trivially as we do not have a neighbor in G and in G′ the connected k_i forms a P_1, hence a clique.

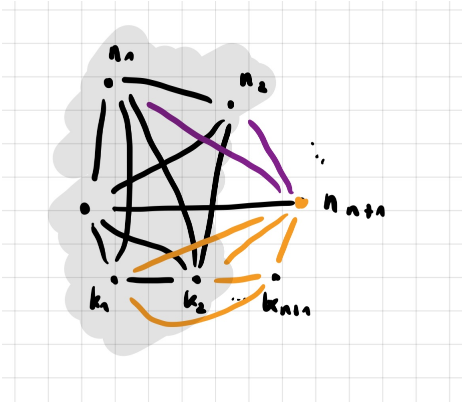


Figure 3.4: Induction Step

- $z = z + 1$:

  By IH, we already know that all neigbors $n_1, ..., n_z$ form a clique together with their vertices in k_i. As $k_{z+1}, v_{z+1} \in N(v_i)$ now also in G′, we show that $N(v_i)$ still forms clique in G′.

  Let k_i be the vertex that was connected with n_i during step 1. All we have to show is that v_{z+1} and k_{z+1} extend our previous clique, hence are fully connected with $N(v_i)$.

  v_{z+1} connects to $N(v_i)$ in G by assumption. By our construction, there exists an edge to $k_1, ..., k_z$, because we add an edge (n_{z+1}, k_i) if there is an edge from (n_{z+1}, n_i). (See fig 3.4)

  k_{z+1} form a complete subgraph with the other k_i and is connected to all n_i by construction because the edge (n_{z+1}, n_i) exists.

  Therefore, $N(v_i)$ will also form a clique in G'.

On the other side, if $N(v_i)$ forms a clique in G′, the vertices of $N(v_i)$ in G form an induced subgraph of G′, hence preserving the clique.

■

Corollary 3.4.3. *G is Chordal iff G′ is chordal.*

Proof. ⇒: Asume G chordal. Then exists a total elemination order $o = (v_1, ..., v_n)$ in G where removing v_j sequentially returns cliques in $N(v_i)$. Define $o' = (v_1, ..., v_n, k_1, ..., k_n, u, t)$. Applying corollary 3.4.2 states that $(v_1, ... v_n)$ always gives cliques in G and according to corollary 3.4.2 also in G′. As the rest is directly part of a clique in G′ by definition with an additional vertex of degree 1, o′ is a total elemination order for G', hence G′ chordal. ⇐: Holds as o′ is always a total elemination order in G′ and removing the complete subgraph K_{n+1} and u gives a total elemination order in G. ■

Corollary 3.4.4. *G has a Dominating Set of size k iff G' has a dominating set of size $k+1$*

Proof. Asume a Dominating Set D of size k in G. $D \cup \{u\}$ is a Semitotal Dominating Set in G′ of size k + 1, because u dominates t and for each $v \in DS : d(v,u) \leq 2$.

Contrary, asume a Semitotal Dominating Set SD in G'. In order to dominate t, $u \in SD$ must hold, hence already dominating the complete subgraph K_{n+1}. If a vertex $k_i \in SD$, we exchange it with v_i still preserving a Dominating Set. Taking $D = SD - \{u\}$ gives our desired Dominating Set of size k. ■

As this reduction runs in FPT time and the parameter is only bounded by a function of k, this is a FPT reduction. As Dominating Set on Chordal Graphs is $w[2] - hard$, so is SDS on Chordal Graphs.

□

CHAPTER 4

A LINEAR KERNEL FOR PLANAR SEMITOTAL DOMINATION

The best way to explain it is to do it.

Lewis Caroll, *Alice in Wonderland*

We are going to present a polynomial-time preprocessing procedure giving a linear kernel for Planar Semitotal Dominating Set parametrized by solution size. Based on the technique first introduced by Alber, Fellows and Niedermeier [2] in 2004, an abundance of similar results to other domination problems emerged which gave us the belief we can transfer these results to Semitotal Dominating Set. Table 4.1 gives an overview of the status of kernels for the planar case on various domination problems. All of these results introduce reduction rules bounding the number of vertices inside so-called "regions" which can be obtained by a special decomposition of the planar graph.

In the following years, this approach bore fruits in other planar problems as well: a 11/3 kernel for Connected Vertex Cover given in [26], $624k$ for Maximum Triangle Packing in [30], $40k$ for Induced Matching in [24], $13k$ for Feedback Vertex Set [6] and further linear kernels for Full-Degree Spanning Tree in [20] and Cycle Packing in [15].

In the impending years, many results generalized this approach to larger graph classes. Fomin and Thilikos [13] started by proofing that the initial reduction rules [2] can also be extended to obtain a linear kernel on graphs with bounded genus g for Dominating Set. Gutner [21] advanced in 2008 by giving a linear kernel for

Problem	Best Known Kernel	Source
Planar Dominating Set	$67k$	[10][1]
Planar Total Dominating Set	$410k$	[16][2]
Planar Semitotal Dominating Set	$359k$	**This work**
Planar Edge Dominating Set	$14k$	[19, Th. 2]
Planar Efficient Dominating Set	$84k$	[19, Th. 4]
Planar Red-Blue Dominating Set	$43k$	[17]
Planar Connected Dominating Set	$130k$	[27]
Planar Directed Dominating Set	Linear	[1]

[1]There is a master's thesis by Halseth [22] claiming a bound of $43k$, but a conference or journal version was not found.

[2]Improved their own results from first $694k$ [16, Revision 2012]

Table 4.1: An overview about existing kernels for planar dominating set variants

$K_{3,h}$-topological-minor-free graph classes and a polynomial kernel for K_h-topological-minor-free graph classes. In 2012 Philip, Raman and Sikdar [28] showed that $K_{i,j}$-free graph classes admit a polynomial kernel. In an attempt to expand these ideas to other problems as well, Bodlaender et al. [5] proved that all problems expressible in counting monadic second-order logic satisfying a coverability property admit a polynomial kernel on graphs of bounded genus g. These meta-results are interesting from a theoretical point of view, but the constants for the kernels obtained by these methods are too large to be of practical interest. The question of how an efficient kernel for the Planar Semitotal Dominating Set problem can be constructed remains. In the following, we will transfer the linear kernel for Planar Total Dominating Set described by Garnero and Sau [16, Revision 2014] to Planar Semitotal Dominating Set giving us an explicitly constructed kernel with "reasonable" small constants. Therefore, we were able to modify the original reduction rules to fit into Planar Semitotal Dominating Set. Our main challenge was to ensure that vertices that are important as witnesses are being preserved.

The Main Idea A given a planar graph $G = (V, E)$ with a given vertex set $D \subseteq V$ can be decomposed into at most $3 \cdot |D| - 6$ so-called "regions" (see definition 4.1.7). If D is a given Semitotal Dominating Set of size $|D|$, the total number of regions in this decomposition depends linearly on the size of D. If we define *reduction rules* (see rules 1 to 3) minimizing the number of vertices in and around a region, we can bound the size of a reduced graph. In our case, we give rules to reduce a region down to a constant number of vertices. We can create an equivalent instance of G whose number of remaining vertices depends linearly on the size of an enquiried solution. Such a reduction gives a *kernel* for Planar Semitotal Dominating Set.

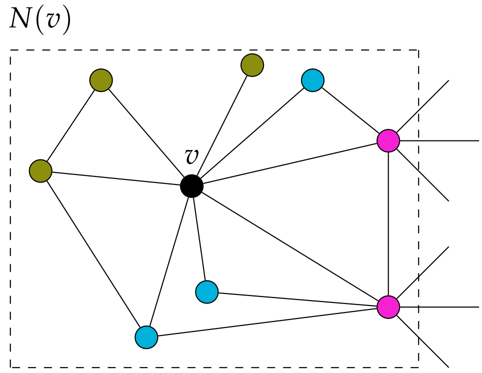


Figure 4.2: *The neighborhood of a single vertex v split to $N_1(v)$ (purple), $N_2(v)$ (blue), and $N_3(v)$ (green). $N_1(v)$'s are those having neighbors outside $N(v)$, $N_2(v)$'s are a buffer between $N_1(v)$ and $N_3(v)$, and $N_3(v)$-vertices are confined in $N(v)$.*

Interestingly, the reduction rules do not rely on the decomposition itself, but rather consider the neighborhood of every pair of vertices in the graph. The decomposition itself has just used a tool for analyzing the kernel size after the reduction.

4.1 Definitions

Before giving the exact reduction rules, we need some definitions exposing the nice properties we are going to exploit. These are the same as given by Garnero and Sau for Planar Total Dominating Set in [16, Revision 2014] and for Planar Red-Blue Dominating Set in [17]) which in turn reused ideas introduced by Alber, Fellows and Niedermeier [2] for Planar Dominating Set.

The main idea is to partition the neighborhoods of both a single vertex and a pair of vertices, respectively into three distinct subsets which intuitively classify how much these vertices are confined and how closely they are related to the rest of the graph

Definition 4.1.1. *Let $G = (V, E)$ be a graph and let $v \in V$. We denote by $N(v) = \{u \in V : \{u, v\} \in E\}$ the neighborhood of v. We split $N(v)$ into three subsets:*

$$N_1(v) = \{u \in N(v) : N(u) \setminus N[v] \neq \emptyset\} \tag{4.1}$$

$$N_2(v) = \{u \in N(v) \setminus N_1(v) : N(u) \cap N_1(v) \neq \emptyset\} \tag{4.2}$$

$$N_3(v) = N(v) \setminus (N_1(v) \cup N_2(v)) \tag{4.3}$$

For $i, j \in [1, 3]$, we denote $N_{i,j}(v) := N_i(v) \cup N_j(v)$. Furthermore, we call a vertex v' confined by a vertex v, if $N(v') \subseteq N[v]$

$\mathbf{N_1(v)}$ is all the neighbors of v which have at least one neighbor outside of $N(v)$ and therefore connects v with the rest of the graph. They are the only vertices with the power to dominate vertices outside the neighborhood of v

$\mathbf{N_2(v)}$ contains all neighbors of v not from $N_1(v)$ for which at least one neighbor is in $N_1(v)$. These vertices do not have any function as a dominating vertex and are placed in between a vertex from $N_1(v)$ and those from $N_3(v) \cup \{v\}$. Furthermore, they are useless as witnesses, because either we can replace them with v (sharing the same neighborhood) or when being a witness for v, we replace it with a $z \in N_1(v)$.

$\mathbf{N_3(v)}$ vertices are sealed off from the rest of the graph. They are useless as dominating vertices: For all $z \in N_3(v)$ it holds that $N(z) \subseteq N(v)$ by definition and thus, we would always prefer v as a dominating vertex instead of z. They can be important as a witness for v if $N_1(v) \cup N_2(v) = \emptyset$. This can only happen if v forms its connected component with only $N_3(v)$ vertices as neighbors. We will be using this observation in rule 1 where we shrink $|N_3(v)| \leq 1$.

Next, we are going to extend this notation to a pair of vertices. Using this, rule 2 will later try to reduce the neighborhood of two vertices, and similar to definition 4.1.1, we observe nice properties. Again, the idea is to classify how strongly the joined neighborhood $N(v) \cup N(w)$ of two vertices is connected to the rest of the graph.

Definition 4.1.2. *Let $G = (V, E)$ be a graph and $v, w \in V$. We denote by $N(v, w) := N(v) \cup N(w)$ the joined neighborhood $N(v) \cup N(w)$ of the pair v, w and split $N(v, w)$ into three distinct subsets:*

$$N_1(v,w) = \{u \in N(v,w) \mid N(u) \setminus (N(v,w) \cup \{v,w\}) \neq \emptyset\} \tag{4.4}$$

$$N_2(v,w) = \{u \in N(v,w) \setminus N_1(v,w) \mid N(u) \cap N_1(v,w) \neq \emptyset\} \tag{4.5}$$

$$N_3(v,w) = N(v,w) \setminus (N_1(v,w) \cup N_2(v,w)) \tag{4.6}$$

Again, for $i, j \in [1,3]$, we denote $N_{i,j}(v,w) = N_i(v,w) \cup N_j(v,w)$.

$N_1(v,w)$ contains those vertices having at least one neighbor outside $N[v] \cup N[w]$, $N_2(v,w)$-vertices are in between those from $N_3(v,w) \cup \{v,w\}$ and $N_1(v,w)$, and $N_3(v,w)$ contains vertices isolated from the rest of the graph. You can see an example in Fig. 4.3.

A vertex $v \in N_i(v)$ is not necessarily also in $N_i(v,w)$! Observe the vertex z in Fig. 4.3. Unlike the sets $N_1(v), N_2(v)$ and $N_3(v)$, in every of the distinct sets $N_i(v,w)$ ($i \in [3]$) can be vertices that belong to a Semitotal Dominating Set. In Fig. 4.4 examples are given for these distinct cases.

4.1.1 Reduced Graph

Before stating the reduction rules, we want to clarify when we consider a graph $G = (V, E)$ to be *reduced*.

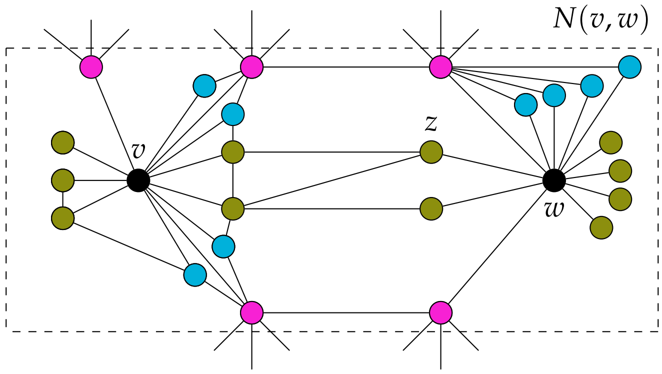


Figure 4.3: *The neighborhood of a pair of vertices. Vertices from $N_3(v,w)$ are colored green, $N_2(v,w)$'s blue and $N_1(v,w)$'s purple. Note that $z \in N_1(w)$, because there is an edge to a neighbor of v, but $z \notin N_1(v,w)$ (and rather $z \in N_3(v,w)$).*

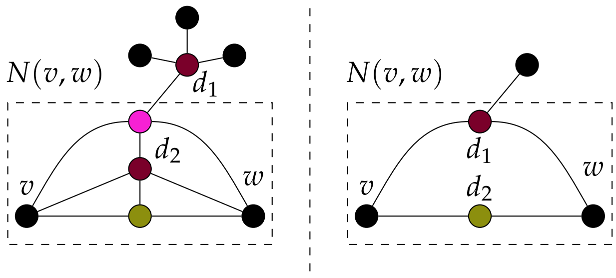


Figure 4.4: *Left:* $\{d_1, d_2\}$ with $d_2 \in N_2(v,w)$ form the only minimal Semitotal Dominating Set. *Right:* $d_1 \in N_1(v,w)$ and $d_2 \in N_3(v,w)$ optimal.

Definition 4.1.3 ([17]). *A graph $G = (V, E)$ is reduced under a set of rules if either none of them can be applied to G or the application of any of them creates a graph isomorphic to G.*

This definition differs from the definition usually given in literature where a graph G is *reduced* under a set of reduction rules if none of them can be applied to G anymore (compare e.g. [12]). Some of our reduction rules (rule 1 or rule 2) could be applied *ad infinitum* creating an endless loop that does not change G anymore. Our definition guarantees termination in that case. All of the given reduction rules are local and only need the neighborhood of at most two vertices and replace them partially with gadgets of constant size. Checking whether the application of one of the rules creates an isomorphic graph can be accomplished in constant time.

4.1.2 Regions in Planar Graphs

Alber, Fellows and Niedermeier [2] gave a novel approach to look at planar graphs. In their analysis, they stated a constructive algorithm that decomposes a planar graph into local "regions". Intuitively, assume that we have a fixed plane embedding of a planar graph $G = (V, E)$. If we pick two distinct vertices v and w from a given SEMITOTAL DOMINATING SET $D \subseteq V$ that are at most of distance two apart, we can try to find two distinct paths from v to w that span up the boundaries of a face and enclose as many other vertices as possible.

The following definitions are based on those given by Garnero and Sau in [16, Revision 2014] and will lead toward a clean definition of a *region* and what we understand as a *D-region decomposition*. More detailed explanations and concrete examples can be found in their paper.

Definition 4.1.4. *Two simple paths P_1, P_2 in a plane graph G are confluent if at least one of the following statements holds:*

1. *they are vertex-disjoint*
2. *they are edge-disjoint and for every common vertex u, if v_i, w_i are the neighbors of u in p_i, for $i \in [1, 2]$, it holds that $[v_1, w_1, v_2, w_2]$*
3. *they are confluent after contracting common edges*

Definition 4.1.5. *Let $G = (V, E)$ be a plane graph and let $v, w \in V$ be two distinct vertices. A region $R(v, w)$ (also denoted as vw-region R) is a closed subset of the plane, such that:*

1. *the boundary of R is formed by two confluent simple vw-paths with length at most 3*
2. *every vertex in R belongs to $N(v, w)$, and*
3. *the complement of R in the plane is connected.*

We denote with ∂R the set of vertices on the boundary of R (including the poles) and by $V(R)$ the set of vertices laying (on the plane embedding) in R. Furthermore, we call $|V(R)|$ the size of the region.

The poles of R are the vertices v and w. The boundary paths are the two vw-paths that form ∂R

Definition 4.1.6. *Two regions R_1 and R_2 are non-crossing, if:*

1. *$(R_1 \setminus \partial R_1) \cap R_2 = (R_2 \setminus \partial R_2) \cap R_1 = \emptyset$, and*
2. *the boundary paths of R_1 are pairwise confluent with the ones in R_2*

We now have all the definitions ready to formally define a maximal *D-region decomposition* on planar graphs:

Definition 4.1.7. *Given a plane graph $G = (V, E)$ and $D \subseteq V$, a $\underline{D - region\ Decomposition}$ of G is a set $\mathfrak{R}$ of regions with poles in D such that:*

1. *for any vw-region $R \in \mathfrak{R}$, it holds that $D \cap V(R) = \{v, w\}$, and*
2. *all regions are pairwise non-crossing.*

We denote $V(\mathfrak{R}) = \bigcup_{R \in \mathfrak{R}} V(R)$.
A D-region decomposition is maximal if there is no region $R \notin \mathfrak{R}$ such that $\mathfrak{R}' = \mathfrak{R} \cup \{R\}$ is a D-region decomposition with $V(\mathfrak{R}) \subsetneq V(\mathfrak{R}')$.

Fig. 4.5 gives an example of how to decompose a graph into a maximal $D - region$ decomposition with a given Semitotal Dominating Set D of size 3.

We are introducing a special subset of a region, namely *simple region* where every vertex is a common neighbor of v and w. They will appear in many unexpected astonishing places and are an important tool to operate on small parts of a plane graph. The upcoming rule 3 will bound the size of these *simple regions*. Interestingly, in the first version of the paper about the linear kernel for Planar Total Dominating Set ([16, Revision 2014]), they were not given independently but covered by one of their reduction rules (Rule 2). As it turned out, the analysis is getting simpler if we treat them in a separate rule (In our case: rule 3) and so did Garnero and Sau in a revised version of their paper four years later ([16]).

Definition 4.1.8. *A simple vw-region is a vw-region such that:*

1. *its boundary paths have length at most 2, and*
2. *$V(R) \setminus \{v, w\} \subseteq N(v) \cap N(w)$.*

Fig. 4.6 shows an example of a simple region containing 9 distinct vertices.

In the analysis, we will also use properties of the *underlying multigraph* of a *D-region decomposition* $\mathfrak{R}$. Refer to Fig. 4.5 for an example.

Definition 4.1.9. *Let $G = (V, E)$ be a plane graph, let $D \subseteq V$ and let $\mathfrak{R}$ be a D-region decomposition of G. The underlying multigraph $G_{\mathfrak{R}} = (V_{\mathfrak{R}}, E_{\mathfrak{R}})$ of $\mathfrak{R}$ is such that $V_{\mathfrak{R}} = D$ and there is an edge $\{v, w\} \in E_{\mathfrak{R}}$ for each vw-region $R(v, w) \in \mathfrak{R}$*

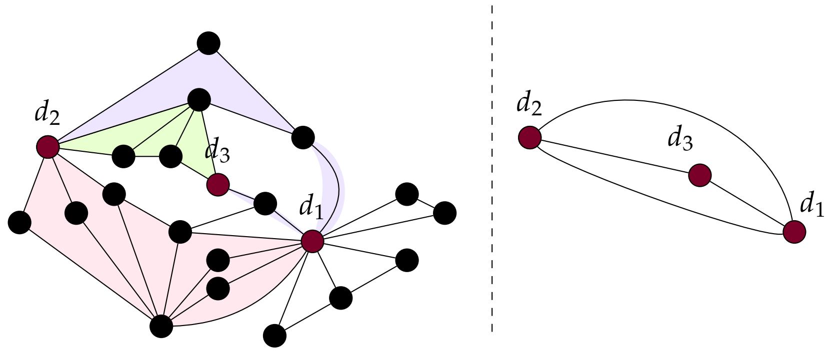


Figure 4.5: *Left: A maximal D-region decomposition $\mathfrak{R}$, where $D = \{d_1, d_2, d_3\}$ form a* Semitotal Dominating Set. *There are two regions between d_2 and d_1 (purple and pink), one region between d_1 and d_3 (purple) and one region between d_2 and d_3 (green). Observe that this D-region decomposition, some neighbors of d_1 are not covered by any vw-region for any $v, w \in D$. Our reduction rules are going to take care of them and bound this number of vertices to obtain the kernel. Right: The corresponding underlying multigraph $G_{\mathfrak{R}}$. Every edge denotes a region between d_i and d_j.*

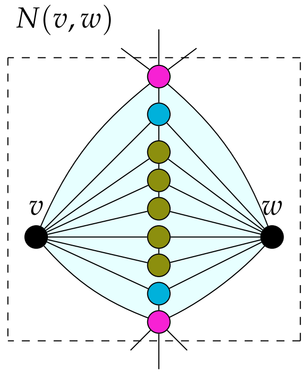


Figure 4.6: *A simple region with two vertices from $N_1(v,w)$ (purple) setting the boundary, two vertices from $N_2(v,w)$ (blue) and some vertices from $N_3(v,w)$ (green) in between.*

4.2 The Big Picture

Figure 4.7 gives a high-level overview of how we are going to obtain the linear kernel for Planar Semitotal Dominating Set. We will first derive three different reduction rules (rules 1 to 3 are green in the overview), prove that they preserve the solution size k and run in polynomial-time. Then we use the existence of a maximal *D-region decomposition* $\mathfrak{R}$ on planar graphs to bound the number of vertices that fly around a given region $R \in \mathfrak{R}$. This will lead us towards a bound on the number of vertices inside

R. Furthermore, we observe that the number of vertices that are not enclosed in R, but lie outside the border is bounded, too. We will often encounter hidden simple regions which are reduced by rule 3 and therefore of constant size by corollary 4.3.1. As we know that the total number of regions R in the *D-region decomposition* is linear in k, we obtained a linear kernel for the Planar Semitotal Dominating Set as well.

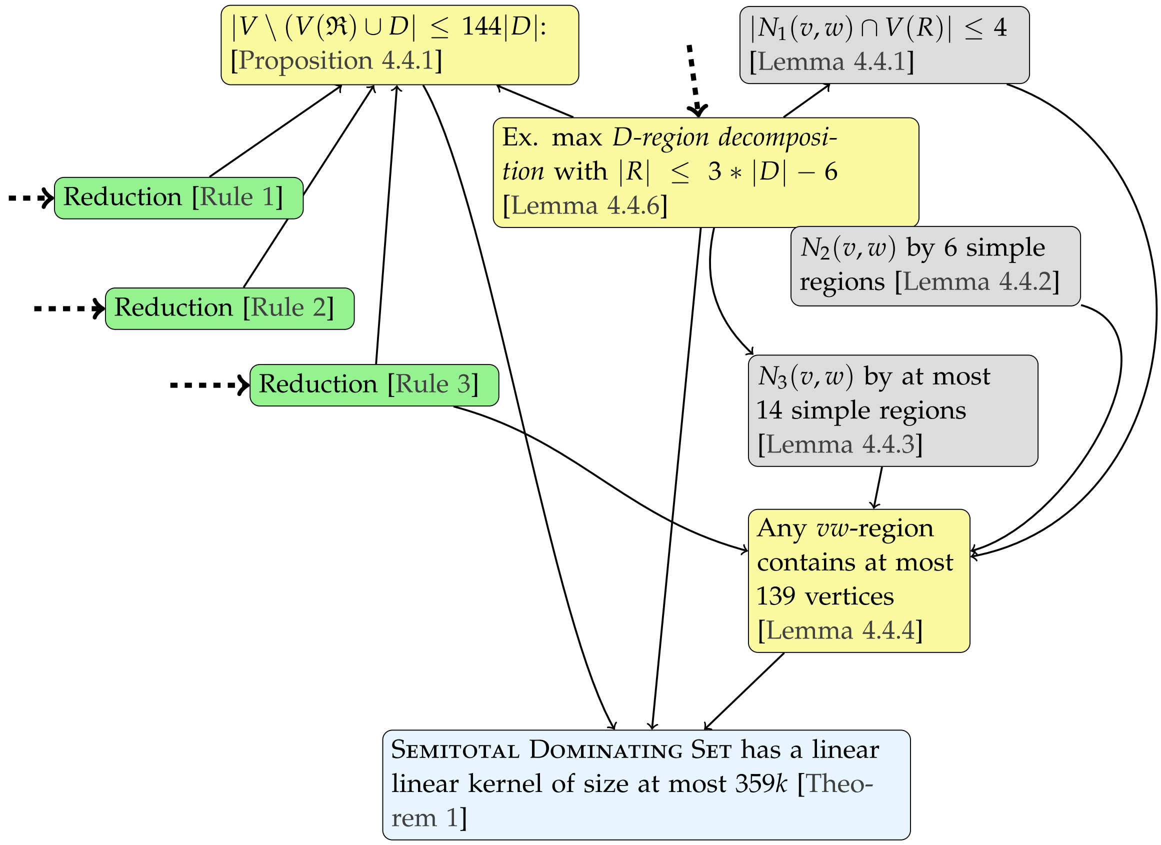


Figure 4.7: *The plan for obtaining a linear kernel for* Planar Semitotal Dominating Set. *Starting with the reduction rules (Green) we will derive the number of vertices inside and outside of a vw-region.*

4.3 The Reduction Rules

Following the ideas proposed by Garnero and Sau [16, Revision 2014], we state modified reduction rules that after exhaustive application will lead to a linear kernel. We note that especially for rule 2, we relied on the first version of the paper electronically published at *arXiv*, because in the following years they improved their kernel size at the cost of making them more specific to Total Dominating Set and the latest rules stated will not work for Planar Semitotal Dominating Set. By a deeper look into the structure of simple regions, we were able to give a slightly more complex reduction rule 3 that achieves the same bound as proven in [16, Revision 2018]. The main challenge in our case was to preserve possible witness properties in the graph because a vertex inside a region can be important as a witness for vertices in another region. This was not a problem for Total Dominating Set, because there, these witnesses must be close and they do not have an effect on more distanced vertices.

4.3.1 Reduction Rule I: Shrinking $N_3(v)$

The idea of the first rule is the observation that a vertex $v' \in N_{2,3}(v)$ dominates v and possibly vertices from $N_2(v)$ and $N_3(v)$. As $N(v') \subseteq N(v)$ and the fact 4.3.1 that a witness for v' is also a witness for v, we can use v instead of v' as a dominating vertex. Therefore, we can remove $N_{2,3}$ from the graph. Nevertheless, v' can be a witness for v itself and might be required in a solution. Our rule ensures that at least one $N_3(v)$-vertex is preserved. An example for this rule is shown in Fig. 4.8.

Fact 4.3.1. *Let $G = (V, E)$, $v \in V$ and $v' \in N_{2,3}(v)$. Any witness $w \neq v$ for v' is also a witness for v.*

Proof. By assumption, v' is witnessed by a vertex $w \neq v$ with $d(v', w) \leq 2$. It follows directly from the definition of $v' \in N_{2,3}(v)$ that $N(v') \subseteq N[v]$ and hence v' is *confined* inside the neighborhood of v. Every path $P = (v', n, w)$ from v' to possible witness w within two steps must pass at least one vertex $n \in N[v]$ as all neighbors. This implies that there exists also a path from v to w with a length of at most two and v □

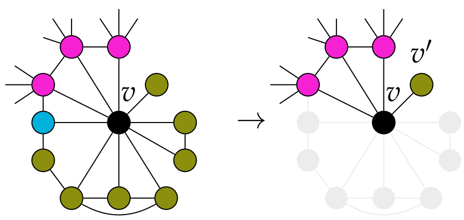


Figure 4.8: *Simplifying $N_{23}(v)$: As $N_3(v) \geq 1$, we remove $N_{23}(v)$ and add a new witness v'. $N_1(v)$ remains untouched.*

Rule 1. *Let $G = (V, E)$ be a graph and let $v \in V$. If $|N_3(v)| \geq 1$:*

- *remove $N_{2,3}(v)$ from G,*
- *add a vertex v' and an edge $\{v, v'\}$*

We can now prove the correctness of this rule.

Lemma 4.3.1. *Let $G = (V, E)$ be a graph and let $v \in V$. If G' is the graph obtained by applying rule 1 on G, then G has a* Semitotal Dominating Set *of size k if and only if G' has a* Semitotal Dominating Set *of size k.*

Proof. $\Rightarrow$ Let D be a Semitotal Dominating Set in G of size k. Because rule 1 has been applied, we can assume $N_{2,3}(v) \neq \emptyset$ in G.

To dominate all vertices from $N_3(v)$ we either need $v \in D$ or at least one other vertex $d \in N_{2,3}(v) \cap D$. In the latter case, we can replace this vertex by v directly. By fact 4.3.1, we know that a witness for d (which is not v) is also a witness for v and therefore, the replacing preserves the Semitotal Dominating Set. Henceforth, we assume $v \in D$ and $N_{2,3}(v)$ is already dominated by v.

If rule 1 has removed at least one dominating vertex from $N_{2,3}(v)$, we set $D' = D \setminus N_{2,3}(v) \cup \{v'\}$ otherwise $D = D'$. In the first case this vertex in $N_{2,3}(v) \cap D$ could have possible been a witness for v, so we select $v' \in D'$ as a witness for v ensuring D' to be a Semitotal Dominating Set. In both cases v' is by assumption dominated by v and $|D| \leq |D'|$.

$\Leftarrow$ Assume D' to be a Semitotal Dominating Set in G'. We can assume that $v \in D'$, because v' has to be dominated and it is always better to choose v instead of v.

If $v' \in D'$ is a witness for v in G', we have to preserve a witness in G as well. As we know that $N_3(v) \neq \emptyset$, we can replace it by an arbitrary vertex $d \in N_3(v)$ in G.

In the second case, the witness for v came either from a vertex $o \in N_1(v)$ or some neighbor $N(o) \setminus N(v)$ outside the neighborhood of v which has not been touched by this reduction rule. In summary, if $v' \in D'$, we set $D = D' \cup \{d\} \setminus \{v'\}$ for any $d \in N_3(v)$ and otherwise $D = D'$. In both cases, $N_{2,3}(v)$ is dominated by v and $|D| = |D'|$.

□

Lemma 4.3.2. *A plane graph G of n vertices is reduced under rule 1 in time $\mathcal{O}(n)$*

Proof. As rule 1 stayed the same, the proof directly follows [2, Lemma 2]. □

Note that we need our definition of a reduced instance given in 4.1.3. If rule 3 is being applied, it will still leave us with a vertex $z \in N_3(v)$ allowing this rule to be applied again.

4.3.2 Reduction Rule II: Shrinking the Size of a Region

The second rule is the heart of the whole reduction and it aims to minimize the neighborhood of two distinct vertices. The rule follows Garnero and Sau's approach [16] for Planar Total Dominating Set. Especially rule two given in the latest version of [16, Revision 2018] were not transferable to Planar Semitotal Dominating Set, because it heavily relies on the property of a Total Dominating Set that a witness w for v **must** be a direct neighbor of w. In the case of the more relaxed Semitotal Dominating Set, the witness is allowed to be farther away.

It can be observed that in the worst case four vertices are needed to semi-totally dominate $N(v, w)$ of two vertices $v, w \in V$: v, w and two witnesses for them. Exemplary, observe the graph consisting of two distinct $K_{1,m}$ with $m \in \mathbb{N}$ with centers v and w.

Before we give the concrete reduction rule, we need to define three sets. Intuitively, we first try to find a set $\tilde{D} \subseteq N_{2,3}(v, w)$ of size at most three dominating $N_3(v, w)$ without using v or w. If no such set exists, we allow v (resp. w) and try to find one again. If we now find such a set, we can conclude that v (w) must be part of a solution.

Definition 4.3.1. *Let $G = (V, E)$ be a graph and let $v, w \in V$. We now consider all the sets that can dominate $N_3(v, w)$:*

$$\mathcal{D} = \{\tilde{D} \subseteq N_{2,3}(v, w) \mid N_3(v, w) \subseteq \bigcup_{v \in \tilde{D}} N(v),\ |\tilde{D}| \leq 3\} \tag{4.7}$$

$$\mathcal{D}_v = \{\tilde{D} \subseteq N_{2,3}(v, w) \cup \{v\} \mid N_3(v, w) \subseteq \bigcup_{v \in \tilde{D}} N(v),\ |\tilde{D}| \leq 3,\ v \in \tilde{D}\} \tag{4.8}$$

$$\mathcal{D}_w = \{\tilde{D} \subseteq N_{2,3}(v, w) \cup \{w\} \mid N_3(v, w) \subseteq \bigcup_{v \in \tilde{D}} N(v),\ |\tilde{D}| \leq 3,\ w \in \tilde{D}\} \tag{4.9}$$

Furthermore, we shortly denote $\bigcup \mathcal{D}_v = \bigcup_{D \in \mathcal{D}_v} D$ and $\bigcup \mathcal{D}_w = \bigcup_{D \in \mathcal{D}_w} D$.

Assuming that v and w are closely connected with $d(v, w) \leq 2$, it might suffice to consider only sets of size at most three, because an intermediate vertex could witness v and w at the same time. In the later analysis, the *D-region decomposition* exactly creates regions around $N(v, w)$ requiring at least one path from v to w lengthed two. As the following rule is only used to locally investigate such regions, we could add the requirement of a distance of two to it and work with sets of size at most three. We believe that this could further improve the kernel.

We are now ready to state rule 2:

Rule 2. *Let $G = (V, E)$ be a graph and v, be two distinct vertices from V. If $\mathcal{D} = \emptyset$ we apply the following:*

Case 1: *if $\mathcal{D}_v = \emptyset$ and $D_w = \emptyset$*

- *Remove $N_{2,3}(v, w)$*
- *Add vertices v' and w' and two edges $\{v, v'\}$ and $\{w, w'\}$*
- *If there was a common neighbor of v and w in $N_{2,3}(v, w)$, add another vertex y and two connecting edges $\{v, y\}$ and $\{y, w\}$*

- *If there was no common neighbor of v and w in $N_{2,3}(v,w)$, but at least one path of length three from v to w via only vertices from $N_{2,3}(v,w)$, add two vertices y and y' and connecting edges $\{v,y\}$, $\{y,y'\}$ and $\{y',w\}$*

Case 2: *if $\mathcal{D}_v \neq \emptyset$ and $D_w = \emptyset$*

- *Remove $N_{2,3}(v)$*
- *Add $\{v,v'\}$*

Case 3: *if $\mathcal{D}_v = \emptyset$ and $D_w \neq \emptyset$*
*This case is symmetrical to **case (2)**.*

In **case (1)**, we know by fact 4.3.3 that v and w must be in D. Therefore, we introduce two forcing vertices v' and w' in G' and remove $N_{2,3}(v,w)$ as these vertices are dominated by v and w. But if we remove $N_{2,3}(v,w)$ entirely, we could lose solutions: First, the case that v is a direct witness of w ($d(v,w) = 2$) and that there is one intermediate witness on a path of length three from v to w via vertices in $N_{2,3}(v,w)$, which could be a witness for both v and w at the same time.

Note that if we would not distinguish between these two cases and had added one intermediate vertex in both of them, we would possibly have generated some wrong solutions, because v could always witness w.

Again by fact 4.3.3 we know for **cases (2)** and **(3)** that $v \in D$ and similar to rule 1 we can simplify the neighborhood $N_{2,3}(v)$. fact 4.3.2 states, that these vertices are only useful for witnessing v, but do not go beyond what v already witnesses. Observe that removing $N_{2,3}$ can not break any connectivity as all vertices in $N_{2,3}(v)$ are confined in v. The cases where $\mathcal{D}_v \neq \emptyset$ and $\mathcal{D}_w \neq \emptyset$ are not required and in a later analysis only the existence of these two sets is required and the application of rule 3 will be critical.

Before proofing rule 2 we will deduce some *facts* which are implied by the definitions above. These facts justify the definition of the sets $\mathcal{D}$, $\mathcal{D}_v$ and $\mathcal{D}_w$.

Fact 4.3.2. *Let $G = (V,E)$ be a graph, let $v,w \in V$, and let G' be the graph obtained by the application of rule 2 on v,w. If $\mathcal{D} = \emptyset$, then G has a solution if and only if it has a solution containing at least one of the two vertices $\{v,w\}$.*

Proof. Because $\mathcal{D} = \emptyset$, any SDS of G has to contain v or w, or at least four vertices from $N_{2,3}(v,w)$. In the second case, these four vertices can be replaced with v, w and two neighbors of v and w still forming a Semitotal Dominating Set. □

The second fact states that if $\mathcal{D}_v$ (resp. $\mathcal{D}_w$) is empty, too, we only need to consider solutions containing w (or v):

Fact 4.3.3. *Let $G = (V,E)$ be a graph, let $v,w \in V$, and let G' be the graph obtained by the application of rule 2 on v,w. If $\mathcal{D} = \emptyset$ and $\mathcal{D}_w = \emptyset$ (resp. $\mathcal{D}_v = \emptyset$) then G' has a solution if and only if it has a solution containing v (resp. w).*

Proof. As $\mathcal{D}_v = \emptyset$, no set of the form $\{v\}$, $\{v, u\}$ or $\{v, u, u'\}$ with $u, u' \in N_{2,3}(v, w)$ can dominate $N_3(v, w)$. Since also $\mathcal{D} = \emptyset$ any SDS of G has to contain v or at least four vertices by fact 4.3.2. In the last case, we again replace these four vertices with v, w and two neighbors respectively and we can conclude that v belongs to the solution. □

Now we are ready to prove the correctness of rule 2

Lemma 4.3.3. *Let $G = (V, E)$ be a plane graph, $v, w \in V$ and $G' = (V', E')$ be the graph obtained after application of rule 2 on the pair $\{v, w\}$. Then G has* Semitotal Dominating Set *of size k if and only if G' has* Semitotal Dominating Set *of size k.*

Proof. We will prove the claim by analyzing the different cases of the rule independently.

⇒ Consider a Semitotal Dominating Set D in G. We show that G' also has a SDS with $|D'| \leq |D|$. By assumption, we have $\mathcal{D} = \emptyset$.

a) $\mathcal{D}_v = \emptyset \wedge \mathcal{D}_w = \emptyset$: By applying fact 4.3.3 twice, we know that both $v, w \in D$. Therefore, v',w', and potentially y and y' are dominated by v or w in G'.

We now have three cases: Either v and w have their own witnesses (e.g. via two distinct $N_1(v, w)$-vertices); they are of distance three and share one witness on a path from v to w (which could go through $N_{2,3}(v, w)$ and therefore will be kept by the vertices y and y) is required, or they can be of distance less than three, such that they already witness each other directly. Furthermore, a direct edge $\{v, w\}$ will not be reduced.

We will now build D' depending on which vertices from $D \cap N_{2,3}(v, w)$ have been removed.

- If the rule has not removed any $d \in D$, we simply set $D' = D$. If v was a witness for w (and vice versa), rule 2 will preserve it by introducing the vertex y. Otherwise, these witnesses are preserved.
- If $d(v, w) > 3$, then v and w are not sharing any common witnesses. If the rule has removed a vertex from $D \cap N(v)$, we set $D' = D \setminus N_{2,3}(v, w) \cup \{v'\}$. If the rule has removed a vertex from $D \cap N(w)$, we set $D' = D \setminus N_{2,3}(v, w) \cup \{w'\}$. If the rule has removed a vertex from $(D \cap N(v))$ and a vertex from $(D \cap N(w))$, we set $D' = D \setminus N_{2,3}(v, w) \cup \{v', w'\}$.
- If $d(v, w) = 3$ and the vertices y and y' get introduced preserving one path from v to w, because there have been a path via $N_{2,3}(v, w)$-vertices containing a single witness for both v and w. If the rule removed a dominating vertex $D \cap N_{2,3}(v, w)$, we set $D' = D \setminus N_{2,3}(v, w) \cup \{y\}$. Note that we could also choose $y' \in D'$, because y's only function is to be a single witness for v and w and every other vertex it could be a witness for, will also be witnessed by $v, w \in D'$ (fact 4.3.2).

- If $d(v, w) \leq 2$, then v and w directly witness each other and the reduction must preserve this relation, which is accomplished by introducing the single bridging vertex y. Even if the rule has removed a vertex $z \in D \cap N_{2,3}(v, w)$, we can ignore that, because fact 4.3.2 states that v and w will witness the same vertices as z did. Hence, we set $D' = D \setminus N_{2,3}(v, w)$.

In all of the cases, it follows that D' is a SDS of G' with $|D'| \leq |D|$

b) $\mathcal{D}_v \neq \emptyset \wedge \mathcal{D}_w = \emptyset$: As $\mathcal{D}_w = \emptyset$ and fact 4.3.3, we know that $v \in D$ and v dominates $N_{2,3}(v)$. If a vertex $d \in D \cap N_{2,3}(v)$ was removed, we set $D' = D \setminus N_{2,3}(v) \cup \{v'\}$, else $D' = D$. Deleting dominating vertices $d \in D \cap N_{2,3}(v)$ does not destroy the witness properties of the graph, because by fact 4.3.2 we know that everything d could witness, is also witnessed by v. If d was a witness for v, we have replaced it with v' in G'. Note that otherwise a vertex from $N_1(v) \cup \{p \in (N(z) \setminus N(v)) | z \in N_1(v)\}$ is a witness for v that is not touched by this reduction. Clearly, $|D'| \leq |D|$ holds.

c) $\mathcal{D}_v = \emptyset \wedge \mathcal{D}_w \neq \emptyset$: Symmetrical to previous case.

$\Leftarrow$ Let D' be a Semitotal Dominating Set in G' and $\mathcal{D} = \emptyset$. We show that G has a SDS D with $|D| \leq |D'|$ by distinguishing the different cases again.

a) $\mathcal{D}_v = \emptyset \wedge \mathcal{D}_w = \emptyset$: In any case we know that $v, w \in D$ to dominate v' and w' and therefore also dominating $N_{2,3}(v, w)$ in G. To preserve the distance $d(v, w)$ the rule might have introduced additional vertices y and y'.

- If only y was introduced we know that there was a common neighbor $n \in N(v) \cap N(w)$ of v and w. y allows v to witness w (and vice versa) and is not part of a solution itself. (assuming $y \notin D'$). Hence, we set $D = D'$.

- If y and y' were added, a solution could use one of them to provide a single witness for v and w. There exists a path $p = (v, n_1, n_2, w)$ from v to w in G only using vertices from $N_{2,3}(v, w)$. As n_1 and n_2 both witness v and w, we put one of them in D if at least one of y or y are dominating vertices in G'. Hence, if $y \in D'$ or $y \in D'$, we set $D = D' \setminus \{y, y'\} \cup \{n_1\}$.

b) $\mathcal{D}_v \neq \emptyset \wedge \mathcal{D}_w = \emptyset$: Clearly, $v \in D'$ to dominate v'. If $v \in D'$, we set $D = D' \setminus \{v'\} \cup d$ for some vertex $d \in N_{2,3}(v, w)$ and otherwise $D = D'$. If v' was the witness of v, it is now replaced by d and D is an SDS with $|D| \leq |D'|$.

c) $\mathcal{D}_v = \emptyset \wedge \mathcal{D}_w \neq \emptyset$: Symmetrical to previous case.

In all cases, we have shown that $|D| \leq |D'|$ and D is a Semitotal Dominating Set of G.

□

4.3.3 Reduction Rule III: Shrinking Simple Regions

If two vertices v and w share enough neighbors with each other, we can sometimes conclude at least one of them to be in a solution.

By planarity, a *simple region* has at most two vertices from $N_1(v,w)$ (namely the border ∂), two vertices from $N_2(v,w)$ connected to the border and unlimited $N_3(v,w)$ vertices squeezed in the middle.

With a more sophisticated analysis, we were able to modify this rule such that the bound given in [16] remained valid although we might have to add some more vertices as orginally described.

Rule 3. *Let $G = (V,E)$ be a plane graph, $v, w \in V$ and R be a simple region between v and w. If $|V(R) \setminus \{v,w\}| \geq 5$ apply the following:*

Case 1: *If $G[R \setminus \partial R] \cong P_3$ then:*

- *remove $V(R \setminus \partial R)$*
- *add vertex y with edges $\{v,y\}$ and $\{y,w\}$*

Case 2: *If $G[R \setminus \partial R] \ncong P_3$ then*

- *remove $V(R \setminus \partial R)$*
- *add vertices y, y' and four edges $\{v,y\}$, $\{v,y'\}$, $\{y,w\}$ and $\{y',w\}$*

Recap that we denoted ∂R as the set of boundary vertices of the (simple) region R, which includes v and w and possibly up to two vertices on the border of R

Lemma 4.3.4 (Correctness of rule 3)**.** *Let $G = (V,E)$ be a plane graph, $v, w \in V$ and $G' = (V',E')$ be the graph obtained after application of rule 3 on the pair $\{v,w\}$. Then G has SDS of size k if and only if G' has SDS of size k.*

Proof. $\Rightarrow$ Consider a Semitotal Dominating Set D in G. We show that G' also has an Semitotal Dominating Set with $|D'| \leq |D|$. By assumption, we have $\mathcal{D} = \emptyset$ and $|V(R) \setminus \{v,w\}| \geq 5$.

First, we can safely assume that the set of border vertices ∂R contains exactly two vertices. If $|\partial R \setminus \{v,w\}| < 2$, the region could not have vertices strictly inside, because the boundary path of ∂R does not enclose any vertices. Hence, at least three vertices must lie strictly inside R.

On the other side, we only need to observe those cases, where both $v, w \notin D$. Again, if a vertex $v' \in V(R \setminus \partial R)$ together with v or w are in D, we can replace them with both v and w which preserves the dominating and witnessing properties and we can set $D' = D \setminus \{v'\} \cup \{v,w\}$ Note that even if v' was a sole witness for v, we know that $d(v,w) \leq 2$ (R being a simple region), and therefore v is a witness for w and vice versa. If no vertex in $V(R \setminus \partial R)$ is dominating, we just set $D' = D$.

Furthermore, we assume $|D \cap V(R \setminus \partial R)| \leq 1$ (at most one vertex strictly inside the region is dominating) because otherwise, we replace those two vertices by v and w which trivially dominate and witness the same vertices.

If $|\partial R \setminus \{v,w\} \cap D| \geq 1$ we can replace the vertex with v or w and set $D' = D \setminus V(R \setminus \partial R) \cup \{v\}$ (TODO Argument why this is ok)

Summing all up, we only need to consider in the following cases that no vertex from $\partial R \in D$ and at most one vertex strictly inside R is dominating.

First assume $V(R \setminus \partial R) \cong P_3$. We denote this induced path as (p_1, p_2, p_3). If neither $p_1, p_2, p_3 \in D$, we set $D' = D$, trivially preserving a Semitotal Dominating Set. Otherwise, $p_2 \in D$ is forced because this is the only way to dominate $V(R \setminus \partial R)$ with exactly one vertex inside. We know that **case (1)** has been applied and we set $D' = D \setminus \{p2\} \cup \{y\}$. which dominates $\{y, y'\}$ and witnesses $N(v) \cup N(w)$ which are the same vertices as p_2 did. in xxx this case is depicted.

Contrary, assume $V(R \setminus \partial R) \ncong P_3$. Again, let us denote this induced path as (p_1, p_2, p_3). In that case, we need

$\Rightarrow$ Consider a Semitotal Dominating Set D in G'. We show that G has an SDS with $|D| \leq |D'|$. By assumption, we have $\mathcal{D} = \emptyset$.

□

The application of rule 3 gives us a bound on the number of vertices inside a simple region.

Corollary 4.3.1. *Let $G = (V, E)$ be a graph, $v, w \in V$ and R a simple region between v and w. If rule 3 has been applied, this simple region has a size of at most 4.*

Proof. If $|V(R) \setminus \{v,w\}| < 5$ then the rule would not have changed G and the size of the region would already be smaller than 5. Assuming $|V(R) \setminus \{v,w\}| \geq 5$ in both cases $V(R \setminus \partial R)$ gets removed and at most two new vertices added. As the boundary in a simple region contains at most two vertices distinct from v and w, the size of the simple region is bounded by at most four. □

4.3.4 Computing Maximal Simple Regions between two vertices

For the sake of completeness, we state an algorithm on how a maximal simple region-between two vertices $v, w \in V$ can be computed in time $\mathcal{O}(d(v) + d(w))$.

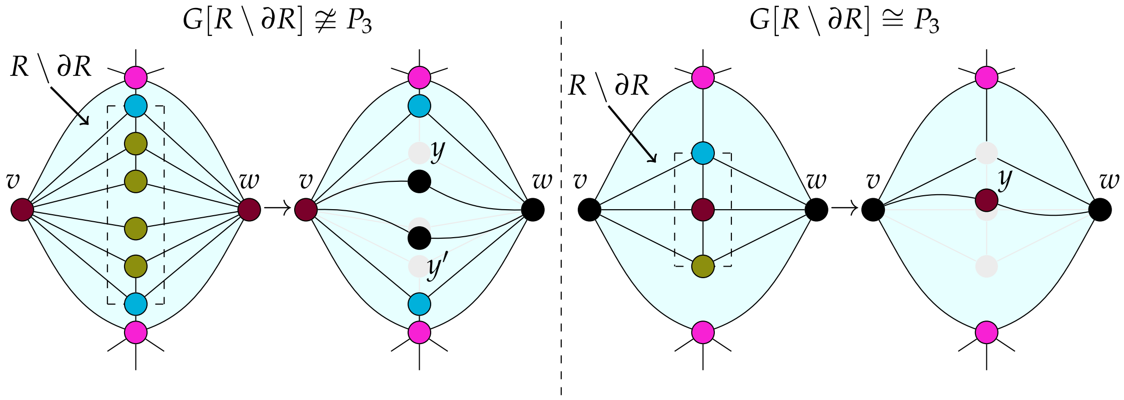


Figure 4.9: *Both cases of the application of rule 3. Left: the vertices inside the region are not isomorphic to a P_3, which means that* ***case (2)*** *will be applied and two new vertices being added. Right: They are isomorphic to a P_3 and we can replace the whole inner region with one single vertex by* ***case (1)****.*

4.4 Bounding the Size of the Kernel

We now put all the pieces together to prove the main result: A kernel which siye is linearly bounded by the solution size k. For that purpose, we distinguish between those vertices that are covered inside a region in a maximal *D-region decomposition* and those that are not. In both cases, our reduction rules bound the number of vertices to a constant size. As lemma 4.4.6 states that for any given dominating set D, we can partition the whole graph into a linear number of regions in k, we know that we also only have linearly many vertices left in the whole graph. In particular, we show that given a Semitotal Dominating Set D of size k, there exists a maximal *D-region decomposition* $\mathfrak{R}$ such that:

1. $\mathfrak{R}$ has only at most $3|D| - 6$ regions
2. $V(\mathfrak{R})$ covers most vertices of V. There are at most $144 \cdot |D|$ vertices outside of any region.
3. each region of $\mathfrak{R}$ contains at most 139 vertices.

Combining these three statements will give us a linear kernel. Figure 4.7 depicts these three goals in *yellow*.

4.4.1 Bounding the Size of a Region

We start with a more fine-grained analysis of the impact of the different cases of rule 2 on a vw-region. The main idea is to count the number of simple regions in the vw-region and then use the bound on the size of a simple region after rule 3 was applied exhaustively. The bound was obtained in corollary 4.3.1.

Lemma 4.4.1. *Given a plane Graph $G = (V, E)$ and a vw-region R, let D be a* Semitotal Dominating Set *and let $\mathfrak{R}$ be a maximal D-region decomposition of G. For any vw-regions $R \in \mathfrak{R}$ it holds that $|N_1(v, w) \cap V(R)| \leq 4$ and these vertices lay exactly on the boundary ∂R of R.*

Proof. The same argument as proposed by Alber, Fellows and Niedermeier [2], and Garnero and Sau[15, Proposition 2, Revision 2018] applies here: Let $P_1 = (v, u_1, u_2, w)$ and $P_2 = (v, u_3, u_4, w)$ be the two boundary paths enclosing the vw-region R. By the definition of a region, they have a length of at most 3. Because every vertex in R belongs to $N(v, w)$, but a vertex from $N_1(v, w)$ also has neighbors outside $N(v, w)$, it *must* lie on one of the boundary paths P_1, P_2. Therefore, R has at most four boundary vertices and $|N_1(v, w) \cap V(R)| \leq 4$.

Clearly, the worst case is achieved, if the two confluent paths P_1 and P_2 are vertex-disjoint. □

Lemma 4.4.2. *[16, See Fact 5] Given a reduced plane graph $G = (V, E)$ and a vw-region R, $N_2(v, w) \cap V(R)$ can be covered by at most 6 simple regions.*

Proof. Let $P_1 = (v, u_1, u_2, w)$ and $P_2 = (v, u_3, u_4, w)$ be the two boundary paths of R. As in the previous lemma 4.4.1, the worst case is achieved if they are vertex-disjoint. Otherwise, a smaller bound would be obtained.

By definition of $N_2(v, w)$, vertices from $N_2(v, w) \cap V(R)$ are common neighbors of v or w and one of $\{u_1, u_2, u_3, u_4\}$. By planarity, we can cover $N_2(v, w) \cap V(R)$ with at most 6 simple regions among 8 pairs of vertices (See fig. 4.10).

□

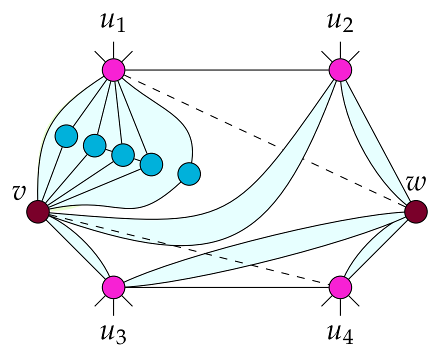


Figure 4.10: *Bounding the maximum number of simple regions inside a region $R(v, w)$. $N_2(v, w)$ is covered by 6 green (simple) regions. The two dashed edges indicate that they are among the 8 possible pairs of vertices, but a simple region between them would contradict the planarity.*

We continue by giving a constant bound on the number of simple regions that cover all $N_3(v,w)$ vertices in a given region.

Lemma 4.4.3. *Given a plane Graph $G = (V, E)$ reduced under rule 2 and a region R(v, w), if $\mathcal{D}_v \neq \emptyset$ (resp. $\mathcal{D}_w \neq \emptyset$), $N_3(v,w) \cap V(R)$ can be covered by:*

1. 11 *simple regions if $\mathcal{D}_w \neq \emptyset$,*
2. 14 *simple regions if $N_{2,3}(v) \cap N_3(v,w) = \emptyset$*

Observe that in the first case, we can assume that no case of rule 2 has been applied, but the claim is a direct consequence of the assumption $\mathcal{D}_v \neq \emptyset$ and $\mathcal{D}_w \neq \emptyset$. If **case (2)** or **case (3)** have been applied, $N_{2,3}(v,w)$ gets reduced and the second case can be applied. For the sake of completeness, we will restate (a slightly adjusted version of) the proof from Garnero and Sau [16, Revision 2014, Fact 6].

Note that this analysis provides a quick upper bound that might not be tight and executing it more sophisticated could yield a better bound because taking both $\mathcal{D}_v \neq \emptyset$ and $\mathcal{D}_w \neq \emptyset$ in concern, our regions might get even more restricted.

Proof. We partition $N_3(v,w)$ into the distinct $N_3(v,w) \setminus N(w)$, $N_3(v,w) \setminus N(v)$ and $N_3(v,w) \cap N(v) \cap N(w)$ and then analyze how many simple regions can there be in the worst case.

1. Because $\mathcal{D}_v \neq \emptyset$ there exists $D = \{v, u, u'\} \in \mathcal{D}_v$ (a smaller set will give a better bound). By definition we know that D dominates $N_3(v,w)$ and also $N_3(v,w) \setminus N(v)$. Therefore, all vertices in $N_3(v,w) \setminus N(v)$ must be neighbors of w and either u or u' and in the worst case at most three simple regions are required. By assumption, $\mathcal{D}_w \neq \emptyset$ as well, and therefore $N_3(v,w) \setminus N(w)$ is bounded by at most three simple regions, too. By planarity, we can cover the remaining common neighbors in $N_3(v,w) \cap N(v) \cap N(w)$ with at most 5 vertices and in total, we can cover $\mathbf{N_3(v,w) \cap R(v,w)}$ **by at most $\mathbf{3 + 3 + 5 = 11}$** simple regions.
2. No property of a reduced instance is used, so the proof shown in [16, Revision 2018] holds.

Cases 2 to 4 of Fig. 4.11 visualize these simple regions around $N_3(v,w) \cap V(R)$ with simple regions in the relevant cases.[1]

□

Lemma 4.4.4 (#Vertices inside a Region after rules 1 to 3). *Let $G = (V, E)$ be a plane graph reduced under rules 1 to 3. Furthermore, let D be an SDS of G and let $v, w \in D$. Any vw-region R contains at most 87 vertices distinct from its poles.*

[1] In revision 2018 of [16], Garnero and Sau removed this proof, because they changed rule 2 and the overall proof was tuned.

Proof. By lemmas 4.4.1 and 4.4.2 and corollary 4.3.1 to bound the number of vertices inside a simple region, we know that $|N_1(v,w) \cap V(R)| \leq 4$. Furthermore, $|N_2(v,w) \cap V(R)| \leq 6 \cdot 4 = 24$, because after the reduction a simple region has at most 4 vertices distinct from its poles and has at most 6 simpler regions covering all $N_2(v,w)$.

It is remaining to bound for $|N_3(v,w) \cap V(R)|$, but gladly we have rule 2, which gracefully took care of them! We will now shortly do a distinction about the different cases of rule 2. Figure 4.11 shows the worst-case amount of simple regions the individual cases can have.

Case 0: rule 2 has **not** been applied in the following two cases: Either $\mathcal{D} \neq \emptyset$ or $\mathcal{D} = \emptyset \wedge \mathcal{D}_v \neq \emptyset \wedge \mathcal{D}_w \neq \emptyset$:

1. If $\mathcal{D} \neq \emptyset$, there exists a set $\tilde{D} = \{d_1, d_2, d_3\} \in \mathcal{D}$ of at most three vertices dominating $N_3(v,w)$. We observe that vertices from $|N_3(v,w) \cap V(R)|$ are common neighbors of either v or w (by the definition of a vw-region) and at least one vertex from $\tilde{D}$, because someone has to dominate them and we know that only the poles or vertices in $\tilde{D}$ come into question. Without violating planarity, we can span at most 6 distinct simple regions. Using the bound of simple regions (worst case shown in corollary 4.3.1) and including $\left|\tilde{D}\right| = 3$, we can conclude $|N_3(v,w) \cap V(R)| \leq 6 \cdot 4 + 3 = 27$.

2. If $\mathcal{D} = \emptyset$, $\mathcal{D}_v \neq \emptyset$ and $\mathcal{D}_w \neq \emptyset$, we can apply lemma 4.4.3 and although rule 2 has not changed the graph G, we can cover R with at most 11 simple regions giving us $|N_3(v,w) \cap V(R)| \leq 11 \cdot 4 = 44$ vertices.

Case 1: If rule 2 **Case (1)** has been applied,
$|N_2(v,w) \cap V(R)|$was entirely removed and $|N_3(v,w) \cap V(R)|$replaced by at most four new vertices v', w' and y and y'. Hence $|N_3(v,w) \cap V(R)| \leq 4$.

Case 2: If rule 2 **Cases (2)** and **(3)** have been applied,
we know that $N_{2,3}(v) \cap N_3(v,w) \subseteq N_{2,3}(v)$ was removed and replaced by one single vertex. Applying lemma 4.4.3, we can cover $N_3(v,w) \setminus \{v'\} \cap V(R)$ with at most 14 simple regions giving us $||N_3(v,w) \cap V(R)|| \leq 14 \cdot 4 + 1 = 57$.

All in all, as $V(R) = \{v,w\} \cup (N_1(v,w) \cup N_2(v,w) \cup N_3(v,w)) \cap V(R)$ we get

$$V(R) \leq 2 + 4 + 24 + \max(27, 44, 4, 57) = 87$$

□

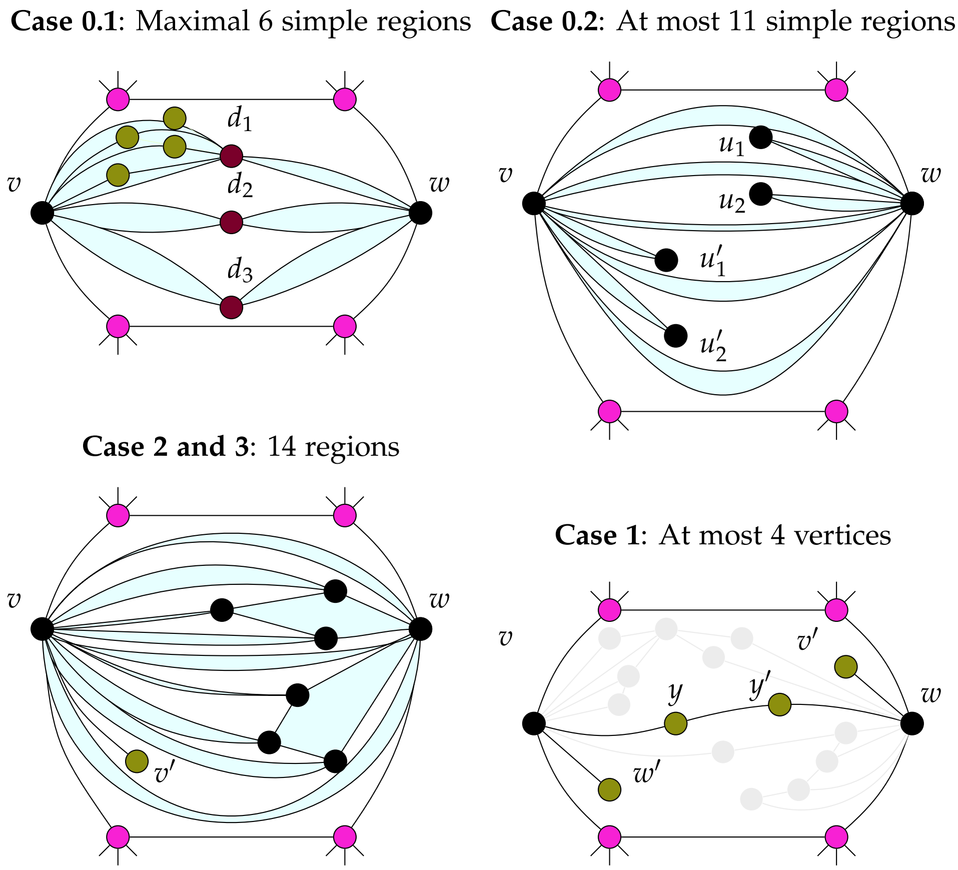


Figure 4.11: *Showing the worst case scenarios for the different cases in lemma 4.4.4:* ***Case 0.1****: $\mathcal{D}$ is nonempty and we have three vertices that can dominate $N_{2,3}$ alone. They can span simple regions with the $N_3(v,w)$ vertices.* ***Case 1****: $N_{2,3}$ was removed and four vertices introduced.* ***Case 2 and 3****: At most 14 simple regions after $N_{2,3}$ has been replaced by a single v'.* ***Case 0.2****: $\mathcal{D}$, $\mathcal{D}_v$ and $\mathcal{D}_w$ are all empty, so the rule has not changed anything and we can cover $N_3(v,w) \cap V(R)$ with at most 11 simple regions.*

4.4.2 Number of Vertices outside the Decomposition

We continue to bound the number of vertices that do not lay inside any region of a maximal *D-region decomposition* $\mathfrak{R}$, that is, we bound the size of $V \setminus V(\mathfrak{R})$. rule 1 ensures that we only have a small amount of $N_3(v)$-pendants. We then try to cover the rest with as few simple regions as possible, because, by application of rule 3, these simple regions are of constant size.
The following lemma states that no vertex from $N_1(v)$ will be outside of a maximal *D-region decomposition*.

Lemma 4.4.5. *[2, Lemma 6] Let $G = (V, E)$ be a plane graph and $\mathfrak{R}$ be a maximal D-region decomposition of a* Dominating Set *D. If $u \in N_1(v)$ for some vertex $v \in D$ then $u \in V(\mathfrak{R})$.*

In the following, we define $d_{G_\mathfrak{R}}(v) = |\{R(v,w) \in \mathfrak{R}, w \in D\}|$ to be the number of regions in $\mathfrak{R}$ having v as a pole.

Corollary 4.4.1. *Let $G = (V, E)$ be a graph and D be a set. For any maximal D-region decomposition $\mathfrak{R}$ on G it holds that $\sum_{v \in D} d_{G_\mathfrak{R}}(v) = 2 \cdot |\mathfrak{R}|$.*

Proof. The proof follows directly from the handshake lemma applied to the underlying multigraph $G_\mathfrak{R}$where every edge between $v, w \in D$ represents a region between v and w in $\mathfrak{R}$. □

Proposition 4.4.1. *Let $G = (V, E)$ be a plane graph reduced under rules 1 and 2 and let D be a* Semitotal Dominating Set *of G. For a maximal D-region decomposition $\mathfrak{R}$, $|V \setminus (V(\mathfrak{R}) \cup D)| \leq 97|D|$*

With slight modifications, the proof given in [16, Revision 2014] will also apply for Semitotal Dominating Set. Although we assume G to be entirely reduced, the following proof only relies on rules 1 and 3. The proof uses the observation that vertices from $N_2(v)$ that lie outside of a region must span simple regions between those from $\{v\} \cup N_1(v)$.

Proof. Again, we will follow the proof proposed by Alber, Fellows, Niedermeier [2, Proposition 2].

We use the size bound of a simple region we have proven in corollary 4.3.1. In particular, we are going to show that $V \setminus V(\mathfrak{R}) \leq 48 \cdot |\mathfrak{R}| + 2 \cdot |D|$. Lemma 4.4.6 will then give the desired bound.

Let D be a Semitotal Dominating Set, $\mathfrak{R}$ be a maximal *D-region decomposition* and $v \in D$. Since D dominates all vertices in the graph, we can consider V as $\bigcup_{v \in D} N(v)$ and thus, we only need to bound the sizes of $N_1(v) \setminus V(\mathfrak{R})$, $N_2(v) \setminus V(\mathfrak{R})$ and $N_3(v) \setminus V(\mathfrak{R})$ separately.

$\mathbf{N_3(v)}$: As we know that rule 1 has been exhaustively applied, we trivially see that $|N_3(v)| \leq 1$ and hence,

$$\left| \bigcup_{v \in D} N_3(v) \setminus V(\mathfrak{R}) \right| \leq |D|$$

$\mathbf{N_2(v)}$: According to Garnero and Sau [16, Proposition 2], we know that $N_2(v) \setminus V(\mathfrak{R})$ can be covered by at most $4d_{G_\mathfrak{R}}(v)$ simple regions between v and some vertices from $N_1(v)$ on the boundary of a region in $\mathfrak{R}$. Figure 4.12 gives some intuition, but intuitively, we can span two simple regions to each of the vertices from $N_1(v)$ on the two border vertices for each $R \in \mathfrak{R}$.

Because we assume G to be reduced, by corollary 4.3.1 a simple region can have at least 4 vertices distinct from its poles and hence,

$$\left| \bigcup_{v \in D} N_2(v) \setminus V(\mathfrak{R}) \right| \leq 4 \sum_{v \in D} 4 \cdot d_{G_{\mathfrak{R}}}(v)$$
$$= 16 \cdot \sum_{v \in D} d_{G_{\mathfrak{R}}}(v) \tag{4.10}$$
$$\overset{\text{Cor. 4.4.2}}{\leq} 32|\mathfrak{R}|$$

$\mathbf{N_1(v)}$: Because every Semitotal Dominating Set is also a Dominating Set, we can apply lemma 4.4.5 and conclude that $N_1(v) \subseteq V(\mathfrak{R})$. Hence,

$$\left| \bigcup_{v \in D} N_1(v) \setminus V(\mathfrak{R}) \right| = 0$$

Summing up these three upper bounds for each $v \in D$ we obtain the result using the equation from lemma 4.4.6:

$$\begin{aligned} |V \setminus V(\mathfrak{R}) \cup D)| &\leq 32 \cdot |\mathfrak{R}| + |D| && \text{(Lemma 4.4.6)} \\ &\leq 32 \cdot (3|D| - 6) + |D| \\ &\leq 96|D| + |D| \\ &= 97|D| && (4.11) \end{aligned}$$

□

4.4.3 Bounding the Number of Regions

Alber, Fellows and Niedermeier [2, Proposition 1] gave a greedy algorithm to construct a maximal *D-region decomposition* for a Dominating Set. Building upon these results, Garnero and Sau gave stated explicit decomposition algorithms for Planar Red-Blue Dominating Set [17] and Total Dominating Set [16] relying on the same technique. This is the core of our linear kernelization because it states that given a Dominating Set D, we can decompose the graph into a *linear number* of regions.

Although Alber, Fellows and Niedermeier[2] gave different reduction rules and require a *reduced* instance as an assumption for the following lemma, they do not use any specific properties exposed by these rules, which was already observed by Garnero and Stau [16, Revision xx]. As every Semitotal Dominating Set is also a Dominating Set, we can safely apply it to our problem as well. For a more detailed and formal proof, one can also refer to [16, Proposition 1].

Lemma 4.4.6 ([2, Proposition 1 and Lemma 5])**.** *Let G be a reduced plane graph and let D be a* Semitotal Dominating Set *with* $|D| \geq 3$. *There is a maximal D-region decomposition of G such that* $|R| \leq 3 \cdot |D| - 6$.

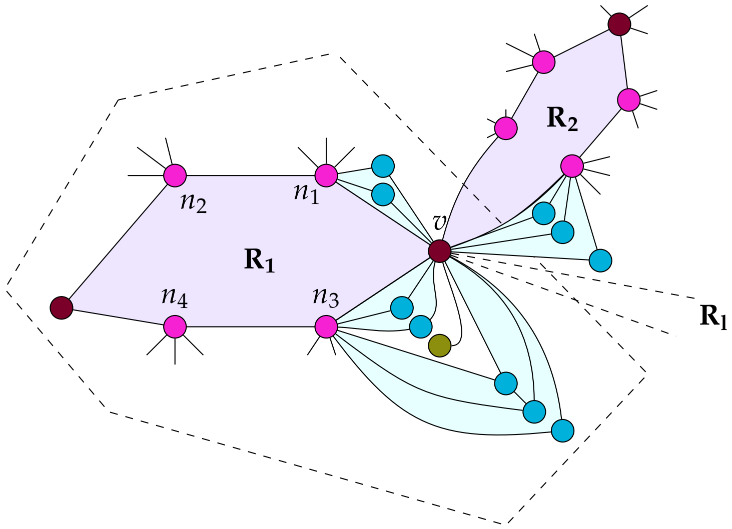


Figure 4.12: *Bounding the number of $N_2(v)$-vertices around a dominating vertex v given a maximal D-region decomposition $\mathfrak{R}$. v is a pole of $R_1, R_2, ...R_j$ and can span simple regions with the help of $N_2(v)$-vertices to at most two $N_1(v)$-vertices per R_i. Each region has at most four vertices in $N_1(v, w) \subseteq N_1(v)$ on the boundary of R_j, but only at most two can be used for a simple region: For Example trying to construct a simple region between v and n_2 would contradict the maximality of $\mathfrak{R}$. Furthermore, because rule rule 1 has removed all but **one** vertex from $N_3(v)$, we intuitively can span two regions to each of the $N_1(v)$-vertices. Furthermore, the size of these simple regions is bounded after the application of rule 3.*

Lemma 4.4.7 (Running Time of Reduction Procedure)**.**

Proof. □

By utilizing all the previous results, we are now finally ready to proof the theorem 1:

Theorem 1. *The* Semitotal Dominating Set *problem parametrized by solution size admits a linear kernel on planar graphs. There exists a polynomial-time algorithms that given a planar graph (G, k), either correctly reports that (G, k) is a NO-instance or returns an equivalent instance (G', k) such that $|V(G')| \leq 359 \cdot k$.*

Proof. Let $G = (V, E)$ be the plane input graph and $G' = (V', E')$ be the graph obtained by the exhaustive application of rules 1 to 3. As none of our rules change the size of a possible solution $D' \subseteq V'$ in G', we know by lemma 4.3.1, lemma 4.3.3 and lemma 4.3.4 that G' has a Semitotal Dominating Set of size k if and only if G has a Semitotal Dominating Set of size k. Furthermore, by lemma 4.4.7, the preprocessing procedure runs in polynomial time.

By taking the size of each region proven in Proposition 4.4.1, the total number of regions in a maximal *D-region decomposition* (lemma 4.4.6) and the number of vertices that can lay outside of any region (Proposition 4.4.1), we obtain the following bound:

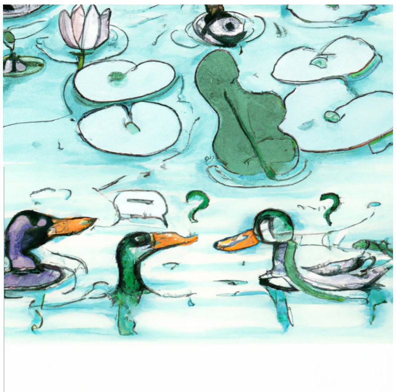
$$87 \cdot (3k - 6) + 97 \cdot k + k < 359 \cdot k \tag{4.12}$$

If $|V(G')| > 359 \cdot k$ G we replace G' by one single vertex v, which is trivially a *no*-instance, because v has no witness to form a Semitotal Dominating Set.

□

CHAPTER 5

OPEN QUESTIONS AND FURTHER RESEARCH



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Lewis Caroll, *XXXX*

* Chordal Bipartite Grap hs a very interesting case. * Improve the Kernel Bound

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