DEPARTMENT OF INFORMATICS

TECHNICAL UNIVERSITY MUNICH

Master Thesis

On the Parametrized Complexity of Semitotal Domination on Graph Classes

Lukas Retschmeier

DEPARTMENT OF INFORMATICS

TECHNICAL UNIVERSITY MUNICH

Master Thesis

On the Parametrized Complexity of Semitotal Domination on Graph Classes

Author:	Lukas Retschmeier
Supervisor:	Prof. Debarghya Ghoshdastidar (TUM)
Advisor:	Prof. Paloma T. Lima (ITU)
Submission Date:	October 16, 2022

I confirm that this master thesis is my own work and I have documented all sources and material used.

København S Lukas Retschmeier
October 16, 2022

Acknowledgments

CONTENTS

ABSTRACT

Abstract all the way

CHAPTER 1

INTRODUCTION

Parametrized Complexity emerging branch. Books about that

Semitotal domination was introduced by

Idea: Lake with stones, and a family of ducks of fixed size wants to occupy the lake so that no other clan tries to take it over. Rules: * A duck can quack freeing up neighboring stones. * Ducks don't like to be alone and want to quack together. So for every duck their must be another duck that is not further than two stones away. Q: Can our ducklings occupy the whole lake while not feeling lonely?

1.1 Content of the thesis

In this thesis, we continue the systematic analysis of the Semitotal Dominating Set problem by focusing on the parametrized complexity of the problem.

Although the problem already had a lot of attention regarding classical complexity (CITE), only a few results are currently known for the parametrized variant.

As far as we have seen, even the w-hardness of the general case has not been explicitly been proven in the literature.

In this thesis, we continue the journey toward a systematic analysis by stating some hardness results for specific graph classes for the problem.

Our contributions Our main contributions consist of first showing the $w[2]$-hardness of Semitotal Dominating Set for XXXX graphs.

As the Dominating Set problem and the Total Dominating Set problem both admit a linear kernel for planar graphs, it is interesting to analyse wether this results also holds for the Semitotal Dominating Set problem which lays in between these two.

Having these kernels also for other variants like Edge Dominating Set, Efficient Dominating Set, Connected Dominating Set, Planar Red-Blue Dominating Set lent us great confidence that the result will also work for Semitotal Dominating Set on planar graphs.

Following the approach from ... which already relies on the technique given in, we give some simple data reduction rules for Semitotal Dominating Set on planar graphs leading to a linear kernel. More precisely, we are going to proof the following central theorem of this thesis:

With some modifications, we were able to transfer the approach given by Garnero and Sau in [12] to the Semitotal Dominating Set problem.

Theorem 1. *The* Semitotal Dominating Set *problem parametrized by solution size admits a linear kernel on planar graphs. There exists a polynomial-time algorithms that given a planar graph* (G, k)*, either correctly reports that* (G, k) *is a NO-instance or returns an equivalent instance* (G', k) *such that* $|V(G')| \leq 561 \cdot k$.

Dominating Set problem and Total Dominating Set problem, both already

CHAPTER 2

PRELIMINARIES

We start by recapping some basic notation in Graph Theory and Parametrized Complexity.

Continuing an intensive study of parametrized complexity of that problem.

2.1 Graph Theory

We quickly state the following definitions given by [8, p. xxx].

Definition 1 (Graph [8, p. 3])**.** *A graph is a pair $G = (V, E)$ of two sets where V denotes the vertices and $E \subseteq V \times V$ the edges of the graph. A vertex $v \in V$ is incident with an edge $e \in E$ if $v \in e$. Two vertices x, y are adjacent, or neighbours, if $\{x, y\} \in E$.*

Definition 2 (Vertex Degrees)**.** *The degree $d_G(v)$ (If G is clear, also $d(v)$) of a vertex v is the number of neighbors of v. We call a vertex of degree 0 as isoliated and one of degree 1 as a pendant.*

Definition 3 (isomorphic Graphs, [8, p. 3])**.** *Let $G = (V, E)$ and $G' = (V', E')$ be two graph. We call G and G' isomorphic, if there exists a bijection $\phi : V \to V'$ with $\{x, y\} \in E \Leftrightarrow \phi(x)\phi(y) \in E'$ for all $x, y \in V$. Such a map ϕ is called isomorphism.*

Definition 4 (Special Graph Notations [8, p. 27])**.** *A simple Graph*
A directed Graph is a graph
A Multi Graph
A Planar Graph

Definition 5 (Closed and Open Neighborhoods [3])**.** *Let $G = (V, E)$ be a (non-empty) graph. The set of all neighbors of v is the open neighborhood of v and denoted by $N(v)$; the set $N[v] = N(v) \uplus \{v\}$ is the closed neighborhood f v in G. When G needs to be made explicit, those open and closed neighborhoods are denoted by $N_G(v)$ and $N_G[v]$.*

Definition 6 (Induced Subgraph)**.** *asd*

Definition 7 (Isomorphic Graph)**.** *asd*

Special Graph Classes

We call the class of graphs without any special restrictions "General Graphs".

Definition 8 (r-partite Graphs)**.** *Let $r \geq 2$ be an integer. A Graph $G = (V, E)$ is called "r-partite" if V admits a parititon into r classes such that every edge has its ends in different classes: Vertices in the same partition class must not be adjacent.*

For the case $r = 2$ we say that the G is "bipartite"

Definition 9 (Chordal Graphs)**.**

Definition 10 (Split Graphs)**.**

2.2 Parametrized Complexity

We are now giving a short introduction into the world of parametrized complexity. * General Introduction

Ways to cope with NP-hard problem.

2.2.1 Fixed Parameter Tractability

Fixed Parameter Intractability: The W Hierarchy

2.2.2 Kernelization

Formal Definitions

Definition 11 (TODO CITE)**.** *A Kernelization Algorithm or kernel is an algorithm $\mathfrak{A}$ for a parametrized Problem Q, that given an instance (I, k) of Q works in polynomial timeand returns an equivalent instance (I', k') of Q. Moreover, we require that $size_{\mathfrak{A}}(k) \leq g(k)$ for some computable function $g : \mathcal{N} \to \mathcal{N}$*

CHAPTER 3

ON PARAMETRIZED SEMITOTAL DOMINATION

3.1 Semitotal Domination

3.1.1 Preliminaries

* Witness * domination

Let D be a dominating set of G and $w \in V(G) \setminus D$. For any neighbor $v \in D \cap N(w)$, we say that d_1 *dominates* w For two dominating vertices $d_1, d_2 in D$. If

SEMITOTAL DOMINATING SET

Definition, dominating number

Complexity Status of Semitotal Dominating Set

3.2 *w[i]*-Intractibility

Now some w[i] hard classes.

3.2.1 Warm-Up: *W[2]*-hard on General Graphs

As any bipartite graphswith bipartition can be split further into r-partite graphsthis results also implies the $w[1]$-hardness of r-partite graphs

3.2.2 *W[2]*-hard on Bipartite Graphs

Definition 12 (Bipartite Graph, [6, p.5]). *A bipartite graphs is a Graph G whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y. Such a partition (X,Y) is called a bipartition of G.*

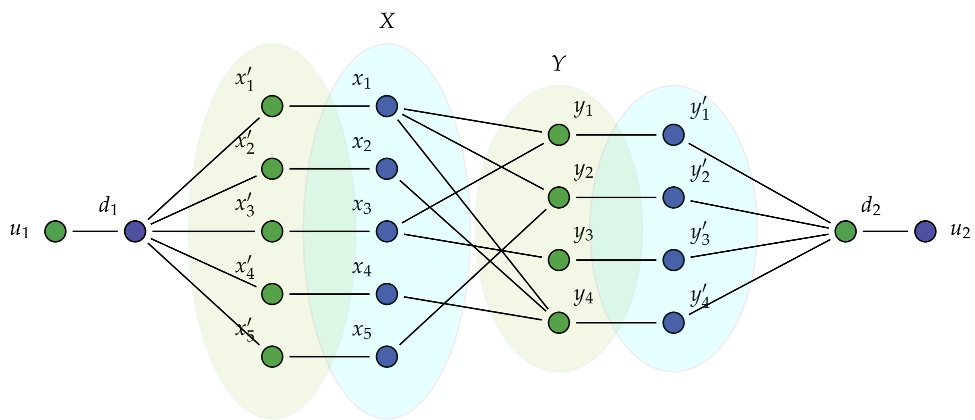

Figure 3.1: Constructing G′ from a bipartite Graph G by duplicating the vertices and adding a dominating tail

Theorem 2. *Semitotal Dominating Set is* $\omega[2]$ *hard for bipartite Graphs*

Proof. Given a bipartite Graph $G = (\{X \cup Y\}, E)$, we construct a bipartite Graph G′ in the following way:

1. For each vertex $x_i \in X$ we add a new vertex x'_i and an edge (x_i, x'_i) in between.
2. For each vertex $y_j \in Y$ we add a new vertex y'_j and an edge (y_j, y'_j) in between.
3. We add two P_1, namely (u_1, d_1) and (u_2, d_2), and connect them with all (d_1, x'_i) and (d_2, y'_j) respectively.

Observation: G′ is clearly bipartite as all y'_j and x'_i form again an Independent Set. Setting $X' = X \cup \{u_2\} \cup \bigcup y'_i$ and $Y' = Y \cup \{u_1\} \cup \bigcup x'_i$ form the partitions of bipartite G′.

Corollary 1. *G has a Dominating Set of size k iff G has a Semitotal Dominating Set of size* $k' = k + 2$

$\Rightarrow$: Asume there exists a Dominating Set D in G with size k. $DS = D \cup \{d_1, d_2\}$ is a Semitotal Dominating Set in G' with size $k' = k + 2$, because d_1 dominates u_1 and all x'_i; d_2 dominates u_2 and all y'_i. Hence, it is a Semitotal Dominating Set, because $\forall v \in (D \cap X) : d(v, d_1) = 2$ and $\forall v \in (D \cap Y) : d(v, d_2) = 2$

$\Leftarrow$: On the contrary, asume any Semitotal Dominating Set SD in G' with size k'. WLOG we can asume that $u_1, u_2 \notin DS$.

Our construction forces d_1, $d_2 \in DS$. Because all x'_i are only important in dominating x_i (y'_i for y_i resp.) as $d_1, d_2 \in DS$. If $x'_i \in DS$ we simply exchange it with x_i (for y'_i and y_i respectively) in our DS keeping the size of the dominating set. $D = DS \setminus \{d_1, d_2\}$ give us a Dominating Set in G with size $k = k' - 2$

As G′ can be constructed in $\mathcal{O}(n)$ and parameter k is only blown up by a constant, this reduction is a FPT reduction. As Dominating Set is $w[2]$ hard for bipartite Graphs[1] so is Semitotal Dominating Set. □

3.2.3 *W[2]*-hard on Chordal Graphs

Theorem 3. *Semitotal Dominating Set is $\omega[2]$ hard on Chordal Graphs*

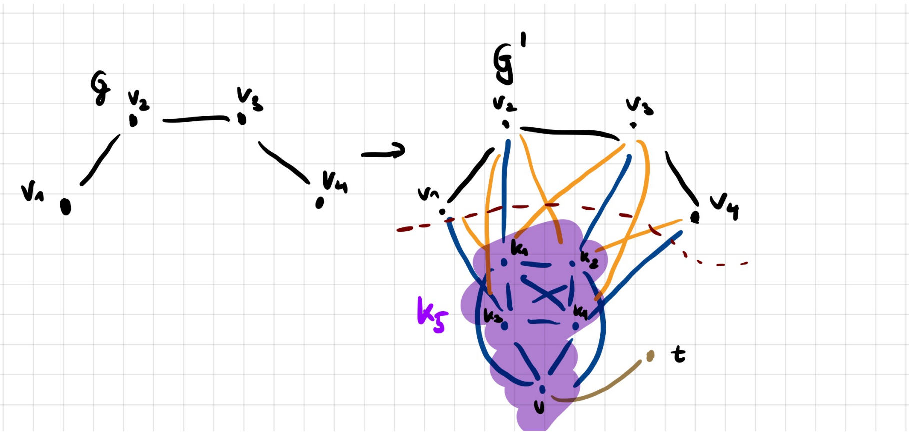

Figure 3.2: Constructing G' by adding a K_5 and the vertex t

Proof. Given a chordal graph $G = (V = \{v_1, ..., v_n\}, E)$, we construct a chordal graph G′ as described below (See also fig 3.2):

1. Add a K_{n+1} consisting of the vertices $\{k_1, ..., k_n, u\}$ and add an edge (v_i, k_i) to each vertex v_i of G. One vertex u in the clique will remain untouched.

2. Add one additional vertex t and connect it with u.

3. For all vertices v_i in G, add a new edge (n, k_i) for all $n \in N(v_i)$.

Corollary 2. *$N(v_i) \in G$ forms a clique iff $N(v_i)$ forms a clique in G'*

Proof. Asuming that $N(v_i)$ forms a clique in G, we show that it also forms a clique in G′ by induction over the number of neighbors $z = abs(N(v_i))$ in G.

[1] Citation needed!

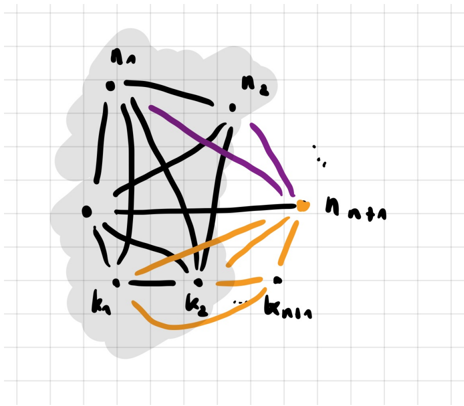

Figure 3.3: Induction Step

- $z = 0$: Holds trivially as we do not have a neighbor in G and in G′ the connected k_i forms a P_1, hence a clique.
- $z = z + 1$:

  By IH, we already know that all neigbors $n_1, ..., n_z$ form a clique together with their vertices in k_i. As $k_{z+1}, v_{z+1} \in N(v_i)$ now also in G′, we show that $N(v_i)$ still forms clique in G′.

  Let k_i be the vertex that was connected with n_i during step 1. All we have to show is that v_{z+1} and k_{z+1} extend our previous clique, hence are fully connected with $N(v_i)$.

  v_{z+1} connects to $N(v_i)$ in G by assumption. By our construction, there exists an edge to $k_1, ..., k_z$, because we add an edge (n_{z+1}, k_i) if there is an edge from (n_{z+1}, n_i). (See fig 3.3)

  k_{z+1} form a complete subgraph with the other k_i and is connected to all n_i by construction because the edge (n_{z+1}, n_i) exists.

  Therefore, $N(v_i)$ will also form a clique in G'.

On the other side, if $N(v_i)$ forms a clique in G′, the vertices of $N(v_i)$ in G just form an induced subgraph of G′, hence preserving the clique.

■

Corollary 3. *G is Chordal iff G′ is chordal.*

Proof. $\Rightarrow$: Asume G chordal. Then exists a total elemination order $o = (v_1, ..., v_n)$ in G where removing v_j sequentially returns cliques in $N(v_i)$. Define $o' = (v_1, ..., v_n, k_1, ..., k_n, u, t)$. Applying corollary 2 states that $(v_1, ...v_n)$ always gives cliques in G and according to corollary 2 also in G′. As the rest is directly part of a clique in G′ by definition with an additional vertex of degree 1, o′ is a total elemination order for G', hence G′ chordal.

$\Leftarrow$: Holds as o′ is always a total elemination order in G′ and removing the complete subgraph K_{n+1} and u gives a total elemination order in G. ■

Corollary 4. *G has a Dominating Set of size k iff G' has a dominating set of size $k+1$*

Proof. Asume a Dominating Set D of size k in G. $D \cup \{u\}$ is a Semitotal Dominating Set in G′ of size k + 1, because u dominates t and for each $v \in DS : d(v,u) \leq 2$.

Contrary, asume a Semitotal Dominating Set SD in G'. In order to dominate t, $u \in SD$ must hold, hence already dominating the complete subgraph K_{n+1}. If a vertex $k_i \in SD$, we exchange it with v_i still preserving a Dominating Set. Taking $D = SD - \{u\}$ gives our desired Dominating Set of size k. ■

As this reduction runs in FPT time and the parameter is only bounded by a function of k, this is a FPT reduction. As Dominating Set on Chordal Graphs is $w[2]-hard$, so is SDS on Chordal Graphs.

□

3.2.4 *W[2]*-hard on Split Graphs

CHAPTER 4

OPEN QUESTIONS AND FURTHER RESEARCH

* Chordal Bipartite Grap hs a very interesting case. * Improve the Kernel Bound

CHAPTER 5

A LINEAR KERNEL FOR PLANAR SEMITOTAL DOMINATION

The best way to explain it is to do it.
Lewis Caroll, *Alice in Wonderland*

We are going to present a polynomial-time preprocessing procedure that gives a linear kernel for Planar Semitotal Dominating Set parametrized by solution size. Based on the technique first introduced by Alber et al. ([2]) in 2004, an abundance of similar results to other domination problems emerged which gave us the belief that we can transfer these results to Semitotal Dominating Set. Table 5.1 gives an overview of the status of kernels for the planar case on various domination problems. All of these results introduce reduction rules bounding the number of vertices inside so-called "regions" which can be obtained by a special decomposition of the planar graph.

In the following years, this approach bore fruits in other planar problems as well like Connected Vertex Cover ($11/3k$ in [19]), Maximum Triangle Packing ($624k$ in [22]) Induced Matching ($40k$ in [18]) Full-Degree Spanning Tree ([15]) Feedback Vertex Set ($13k$ in [5]) and Cycle Packing ([11])

In the impending years, many results generalized this approach to larger graph classes. Fomin and Thilikos started by proofing that the initial reduction rules given by Alber et al. [2] can also be used to obtain a linear kernel on graphs with bounded genus g ([10]). Gutner advanced in 2008 by showing that the problem has a linear kernel on $K_{3,h}$-topological-minor-free graph classes and a polynomial kernel for K_h-topological-minor-free graph classes ([16]). In 2012 Philip, Raman and Sikdar showed that even $K_{i,j}$-free graph classes admit a polynomial kernel for Dominating Set ([21]). In an attempt to extend these ideas to other problems as well, Bodlaender et al. proved that all problems expressible in counting monadic second-order logic that satisfy a coverability property admit a polynomial kernel on graphs of bounded genus g ([4]).

Problem	**Best Known Kernel**	**Source**
PLANAR DOMINATING SET	$67k$	[7][1]
PLANAR TOTAL DOMINATING SET	$410k$	[12][2]
PLANAR SEMITOTAL DOMINATING SET	$561k$	**This work**
PLANAR EDGE DOMINATING SET	$14k$	[14, Th. 2]
PLANAR EFFICIENT DOMINATING SET	$84k$	[14, Th. 4]
PLANAR RED-BLUE DOMINATING SET	$43k$	[13]
PLANAR CONNECTED DOMINATING SET	$130k$	[20]
PLANAR DIRECTED DOMINATING SET	?	[1]

[1]There is also a master's thesis claiming a bound of 43k [17], but a conference or journal version was not found.

[2]Improved their own results from first $694k$ [12, Revision 2012]

Table 5.1: An overview about existing kernels for planar dominating set variants

These meta-results are interesting from a theoretical point of view, but the constants for the kernels obtained by these methods are too large to be of practical interest. The question of how an efficient kernel can be constructed remains. We will show that the kernel described by Garnero and Sau in their original version of [12, Revision 2014] for PLANAR TOTAL DOMINATING SET can also be "recycled" for PLANAR SEMITOTAL DOMINATING SET giving us an explicitly constructed kernel with "reasonable" small constants.

5.1 The Main Idea

The main idea is to use the fact that given a plane graph $G = (V, E)$ and given a vertex set $D \subseteq V$, G can be decomposed into at most $(3 \cdot |D| - 6)$ so-called "regions" (Definition 19). If D is now a given SEMITOTAL DOMINATING SET of size $|D|$, we know that the total number of regions depends linearly on the size of D. If we define *reduction rules* (Rules 1 to 3) that try to minimize the number of vertices in and around a region we can overall bound the size of a reduced graph G'. We give rules that ensure that each region is of constant size after they have been applied.

The reduction rules do not rely on the decomposition itself, but rather consider the neighborhood of every pair of vertices in the graph.

5.2 Definitions

In this section, we are giving key definitions for our reduction rules. They are inspired by those given by Garnero and Sau (PLANAR TOTAL DOMINATING SET in [12, Revision

2014] or PLANAR RED-BLUE DOMINATING SET in [13]) and used ideas given by Alber et al. in [2] for PLANAR DOMINATING SET.

We are splitting the neighborhood of a single vertex and a pair of vertices into three distinct subsets which intuitively give us a level of "confinement" of these vertices and how closely they are related to the rest of the graph ([12]).

Definition 13. *Let $G = (V, E)$ be a graph and let $v \in V$. We denote by $N(v) = \{u \in V : \{u, v\} \in E\}$ the neighborhood of v. We split $N(v)$ into three subsets:*

$$N_1(v) = \{u \in N(v) : N(u) \setminus N[v] \neq \emptyset\} \tag{5.1}$$
$$N_2(v) = \{u \in N(v) \setminus N_1(v) : N(u) \cap N_1(v) \neq \emptyset\} \tag{5.2}$$
$$N_3(v) = N(v) \setminus (N_1(v) \cup N_2(v)) \tag{5.3}$$

In order to inhance future readability, for $i, j \in [1, 3]$, we denote $N_{i,j}(v) := N_i(v) \cup N_j(v)$.

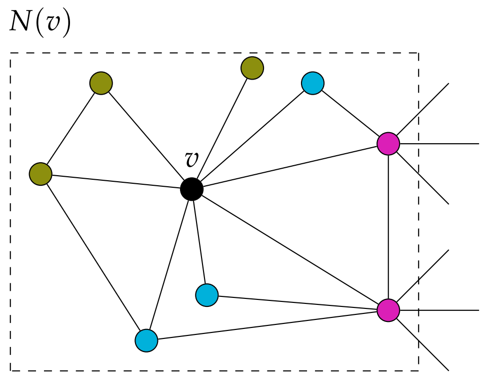

Figure 5.1: *The neighborhood of a single vertex v split to $N_1(v)$ (purple), $N_2(v)$ (blue), and $N_3(v)$ (green). $M_1(v)$ are those having neighbors outside $N(v)$, $N_2(v)$ are a buffer between $N_1(v)$ and $N_3(v)$, and $N_3(v)$-vertices are confined in $N(v)$*

Intuitively, these sets are classifying neighbors of v by how much they can interact with the rest of the graph and how much they are locally centered around v:

$N_1(v)$ are all neighbors of v which have at least one adjacent vertex that is outside of $N(v)$ and therefore connect v with the rest of the graph. They could possibly belong to a SEMITOTAL DOMINATING SET.

$N_2(v)$ are all neighbors of v that have at least one neighbor in $N_1(v)$. These vertices do not have any function as a dominating vertex and can be seen as a *buffer* bridging $N_1(v)$-vertices with those from $N_3(v) \cup \{v\}$. Furthermore, they are useless as witnesses, because either we can replace them by v (sharing the same neighborhood) or when being a witness for v, we replace it with one $z \in N_1(v)$.

$N_3(v)$ vertices are sealed off from the rest of the graph. They are useless as dominating vertices: For all $z \in N_3(v)$ it holds that $N(z) \subseteq N(v)$ by definition and thus, we would always prefer v as a dominating vertex instead of z. Nevertheless, they can be important as a witness for v in the case that $N_1(v) \cup N_2(v) = \emptyset$. We are using this observation in Rule 1 where we shrink $|N_3(v)| \leq 1$.

Next, we are going to extend this notation also to a pair of vertices. Using this, Rule 2 will later try to reduce the neighborhood of two vertices, and similar to Definition 13, we can observe nice properties. Again, the idea is to classify how strongly the shared neighborhood $N(v) \cup N(w)$ is connected with the rest of the graph.

Definition 14. *Let $G = (V, E)$ be a graph and $v, w \in V$. We denote by $N(v, w) := N(v) \cup N(w)$ the shared neighborhood of the pair v, w and split $N(v, w)$ into three distinct subsets:*

$$N_1(v,w) = \{u \in N(v,w) \mid N(u) \setminus (N(v,w) \cup \{v,w\}) \neq \emptyset\} \tag{5.4}$$

$$N_2(v,w) = \{u \in N(v,w) \setminus N_1(v,w) \mid N(u) \cap N_1(v,w) \neq \emptyset\} \tag{5.5}$$

$$N_3(v,w) = N(v,w) \setminus (N_1(v,w) \cup N_2(v,w)) \tag{5.6}$$

Again, for $i, j \in [1,3]$, we denote $N_{i,j}(v,w) = N_i(v,w) \cup N_j(v,w)$.

$N_1(v,w)$ contains those vertices connected with the rest of the graph, $N_2(v,w)$ are a *buffer* between $N_3(v,w) \cup \{v,w\}$ and $N_(v,w)$, and $N_3(v,w)$ are those vertices isolated from the rest of the graph.

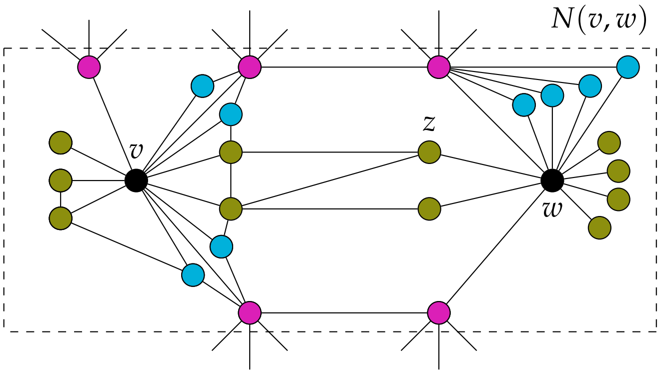

Figure 5.2: *The neighborhood of a pair of vertices. Furthermore, note that $z \in N_1(w)$, because there is an edge to v, but $z \notin N_1(v,w)$ and $z \in N_3(v,w)$*

Note that vertices in $N_i(v)$ do not necessarily also correspond to a vertex in $N_i(v,w)$. For example Fig. 5.2 gives an example, where z belongs to $N_1(v)$, but not to $N_1(v,w)$.

5.2.1 Reduced Graph

Before stating the reduction rules, we want to clarify when we consider a graph to be a *reduced*.

Definition 15 (Reduced Graph [12, p. 13] and [13])**.** *A Graph G is reduced under a set of rules if either none of them can be applied to G or the application of any of them creates a graph isomorphic to G.*

This definition differs from the definition usually given in literature where a graph G is *reduced* under a set of reduction rules if none of them can be applied to G anymore (compare e.g. [9]). Some of our reduction rules (Rule 1 or Rule 2) could be applied *ad infinitum* creating an endless loop that does not change G any more. Our definition guarantees termination in that case. All of the given reduction rules are local and only need the neighborhood of at most two vertices and replace them partially with gadgets of constant size. Now checking whether a graph after applying the rule has been isomorphically changed can be trivially accomplished in constant time.

In our case, we say G is reduced if all of the Rules 1 to 3 have exhaustively been applied.

5.2.2 Regions in Planar Graphs

Alber et al. ([2]) introduced a novel approach how to look at planar graphs. In their analysis, they gave a constructive algorithm that decomposes a planar graph into local areas which they call "regions". Vaguely said, a region is a set of vertices that are enclosed by a boundary path in a fixed planar embedding.

The following definitions are based on those given by Garnero and Sau ([12, Revision 2014]) and will lead toward a clean definition of a *region* and what we understand as a *D-region decomposition*.

Definition 16. *Two simple paths p_1, p_2 in a plane graph G are* confluent *if:*

1. *they are vertex-disjoint*
2. *they are edge-disjoint and for every common vertex u, if v_i, w_i are the neighbors of u in p_i, for $i \in [1, 2]$, it holds that $[v_1, w_1, v_2, w_2]$, or*
3. *they are confluent after contracting common edges*

Definition 17. *Let $G = (V, E)$ be a plane graph and let $v, w \in V$ be two distinct vertices. A* region $R(v, w)$ *(also denoted as vw-region) is a closed subset of the plane, such that:*

1. *the boundary of R is formed by two confluent simple vw-paths with length at most 3*
2. *every vertex in R belongs to $N(v, w)$, and*

3. *the complement of R in the plane is connected.*

We denote by ∂R the boundary of R and by $V(R)$ the set of vertices which lay (with the plane embedding) in R. Furthermore, we call $|V(R)|$ the size of the region.

The poles of R are the vertices v and w. The boundary paths are the two vw-paths that form ∂R

Definition 18. *Two regions R_1 and R_2 are non-crossing, if:*

1. $(R_1 \setminus \partial R_1) \cap R_2 = (R_2 \setminus \partial R_2) = \emptyset$*, and*
2. *the boundary paths of R_1 are pairwise confluent with the ones in R_2*

We now have all the definitions ready to formally a maximal *D-region decomposition* on planar graphs:

Definition 19. *Given a plane graph $G = (V, E)$ and $D \subseteq V$, a $\underline{D - region\ Decomposition}$ of G is a set $\mathfrak{R}$ of regions with poles in D such that:*

1. *for any vw-region $R \in \mathfrak{R}$, it holds that $D \cap V(R) = \{v, w\}$, and*
2. *all regions are pairwise non-crossing.*

We denote $V(\mathfrak{R}) = \bigcup\limits_{R \in \mathfrak{R}} V(R)$.

A D-region decomposition is maximal if there is no region $R \notin \mathfrak{R}$ such that $\mathfrak{R}' = \mathfrak{R} \cup \{R\}$ is a D-region decomposition with $V(\mathfrak{R}) \subsetneq V(\mathfrak{R}')$.

Fig. 5.3 gives an example of how to decompose a graph into a maximal $D - region$ decomposition with a given Semitotal Dominating Set D of size 3.

We are introducing a special subset of a region, namely *simple region* where every vertex is a common neighbor of v and w. They will appear in many unexpected astonishing places and are an important tool to operate on small parts of a plane graph. The upcoming Rule 3 will bound the size of these *simple regions*. Interestingly, in the first version of the paper about the linear kernel for Planar Total Dominating Set ([**Garnero2014**]), they were not given independently but covered by one of their reduction rules (Rule 2). As it turned out, the analysis is getting simpler if we treat them in a separate rule (In our case: Rule 3) and so did Garnero and Sau in a revised version of their paper four years later ([12]).

Definition 20. *A simple vw-region is a vw-region such that:*

1. *its boundary paths have length at most 2, and*
2. $V(R) \setminus \{v, w\} \subseteq N(v) \cap N(w)$.

Fig. 5.4 shows an example of a simple region containing 9 distinct vertices.

In the analysis, we will also use properties of the *underlying multigraph* of a *D-region decomposition* $\mathfrak{R}$. Refer to Fig. 5.3 for an example.

Definition 21. *Let $G = (V, E)$ be a plane graph, let $D \subseteq V$ and let $\mathfrak{R}$ be a D-region decomposition of G. The underlying multigraph $G_{\mathfrak{R}} = (V_{\mathfrak{R}}, E_{\mathfrak{R}})$ of $\mathfrak{R}$ is such that $V_{\mathfrak{R}} = D$ and there is an edge $\{v, w\} \in E_{\mathfrak{R}}$ for each vw-region $R(v, w) \in \mathfrak{R}$*

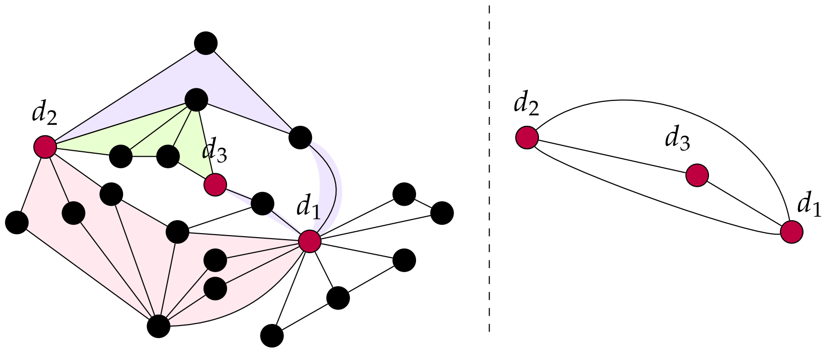

Figure 5.3: *Left: A maximal D-region decomposition $\mathfrak{R}$, where $D = \{d_1, d_2, d_3\}$ form a* Semitotal Dominating Set. *There are two regions between d_2 and d_1 (purple and pink), one region between d_1 and d_3 (green) and one region between d_2 and d_3 (purple). Observe that some neighbors of d_1 are not part of any vw-region. Our reduction rules are going to take care of of them and bound these number of vertices to obtain the kernel. Right: The corresponding underlying multigraph $G_{\mathfrak{R}}$. Every edge denotes a region between d_i and d_j*

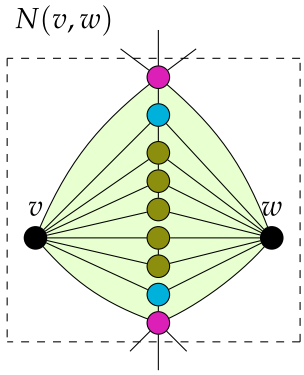

Figure 5.4: *A simple region with two vertices from $N_1(v, w)$ (purple) setting the boundary, two vertices from $N_2(v, w)$ (blue) and some vertices from $N_3(v, w)$ (green) in between.*

5.3 The Big Picture

Fig. 5.5 gives a high-level view of how we are going to obtain the linear kernel for Planar Semitotal Dominating Set. We will first five three different reduction rules (Rules 1 to 3 are green in the overview) and prove that they preserve the solution size k and run in polynomial-time. The idea is then to use a maximal *D-region decomposition* $\mathfrak{R}$ to bound the number of vertices that fly around a given region $R \in \mathfrak{R}$. This will lead us to a bound on the number of vertices inside R and those that are not belonging to any *vw*-region in $\mathfrak{R}$. We will often exploit Rule 3 and find hidden simple regions which we know are of constant size by Corollary 5.

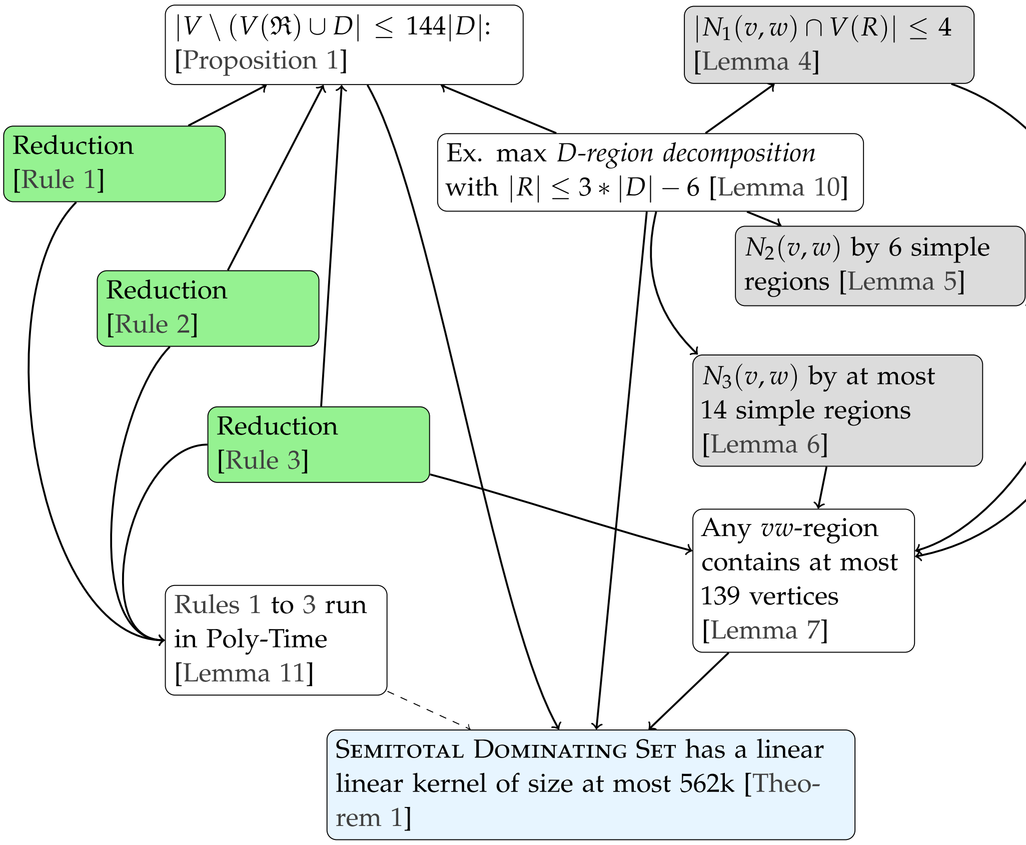

Figure 5.5: *The plan for obtaining a linear kernel for* Planar Semitotal Dominating Set. *Starting with the reduction rules we will try to derivate the number of vertices inside and outside of a vw-region.*

5.4 The Reduction Rules

Following the ideas proposed by Garnero and Sau in [12, Revision 2014], we state reduction rules that after exhaustive application will expose a linear kernel.

5.4.1 Reduction Rule I: Getting Rid of unneccessary $N_3(v)$ vertices

The idea behind our first rule is the observation that a vertex $v' \in N_3(v)$ dominates v and vertices from $N_2(v)$. As $N(v') \subseteq N(v)$ and the Fact 1 that a witness for v' is also witness for v, we can use v instead of z as a dominating vertex. Nevertheless, v' can be a witness for v itself and might be required in a solution. Our rule ensures that at least one $N_3(v)$-vertex is preserved.

Note that Rule 1 has only removed vertices that *must be dominated* by v in G. An example for this rule is shown in Fig. 5.6.

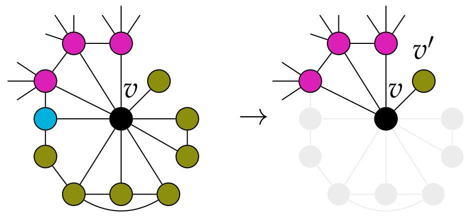

Figure 5.6: *Simplifying $N_{23}(v)$: As $N_3(v) \geq 1$, we remove $N_{23}(v)$ and add a new witness v'. $N_1(v)$ remains untouched.*

Rule 1. *Let $G = (V, E)$ be a graph and let $v \in V$. If $|N_3(v)| \geq 1$:*

- *remove $N_3(v)$ from G,*
- *add a vertex v' and an edge $\{v, v'\}$*

Fact 1. *Let $G = (V, E)$, $v \in V$ and $v' \in N_{2,3}(v)$. Any witness $w \neq v$ for v' is also a witness for v.*

Proof. By assumption v' is witnessed by a vertex $w \neq v$ with $d(v', w) \leq 2$. It follows directly from the definition of $v' \in N_{2,3}(v)$ that $N(v') \subseteq N(v)$ and hence v' is *confined* inside the neighborhood of v. Therefore, every possible witness that can be reached from v' within two steps can also be reached from v within two steps. □

We can now prove the correctness of this rule.

Lemma 1. *Let $G = (V, E)$ be a graph and let $v \in V$. If G' is the graph obtained by applying Rule 1 on G, then G has* Semitotal Dominating Set *of size k if and only if G' has one.*

Proof. $\Rightarrow$ Let D be a Semitotal Dominating Set in G of size k. Because Rule 1 has been applied, we can assume $N_3(v) \neq \emptyset$ in G.

In order to dominate $N_3(v)$ we either need $v \in D$ or at least one other vertex $d \in N_{2,3}(v) \cap D$. In the latter case, we can replace this vertex by v directly. By Fact 1, we know that a witness for d will also be a witness for v preserving a Semitotal Dominating Set. Hence, we now assume $v \in D$ and $N_{2,3}(v)$ is already dominated by v.

If Rule 1 has removed at least one dominating vertex, we set $D' = D \setminus N_{2,3}(v) \cup \{v'\}$ otherwise $D = D'$. In both cases, v' is dominated by v.

Because a vertex in $N_{2,3}(v) \cap D$ could have possible been a witness for v, we select $v' \in D'$ as a witness for v ensuring D' to be a Semitotal Dominating Set. In both cases $|D| \leq |D'|$.

$\Leftarrow$ Assume D' to be a Semitotal Dominating Set in G'. We know that $v \in D'$, because v' has to be dominated and it is always better to choose v.

If $v' \in D'$ is a witness for v in G', we have to preserve a witness in G as well. As we know that $N_3(v) \neq \emptyset$, we can replace it by an arbitrary vertex $d \in N_3(v)$ in G.

In the second case, the witness for v came either from a vertex $o \in N_1(v)$ or some neighbor $N(o) \setminus N(v)$ outside the neighborhood of v which has not been touched by this reduction rule.

In summary, if $v' \in D'$, we set $D = D' \cup \{d\} \setminus \{v'\}$ for a single $d \in N_3(v)$ and otherwise $D = D'$. In both cases, $N_{2,3}(v)$ is dominated by v and $|D| = |D'|$.

□

Note that we need our definition of a reduced instance given in 15. If Rule 3 is being applied, it will still leave us with a vertex $z \in N_3(v)$ allowing this rule to be applied again.

5.4.2 Reduction Rule II: Shrinking the Size of a Region

The second rule is the heart of the whole reduction and tries to minimize the neighborhood of two distinct vertices. The rule follows Garnero and Sau's approach in one of the early versions of **[Garnero2014]** for Planar Total Dominating Set. Interestingly, the reduction rules given in the latest version were not directly transferable to Planar Semitotal Dominating Set, because they heavily rely on properties only a Planar Total Dominating Set exposes and can not be generalized to the more relaxed Planar Semitotal Dominating Set.

It can be observed that in the worst case four vertices are needed to dominate $N_3(v, w)$ of two vertices $v, w \in V$: v, w and two witnesses for them. Exemplary, observe the graph consisting of two distinct $K_{1,m}$ with $m \in \mathbb{N}$ with centers v and w.

If $d(v, w) \leq 2$, it might even suffice to consider sets of size at most three, because then there would be an intermediate vertex that could witness v and w at the same time. We believe that the kernel can be improved by applying Rule 2 only on those distanced two vertices.

Before we give the concrete reduction rule, we need to define three sets. Intuitively, we first try to find a set $\tilde{D} \subseteq N_{2,3}(v, w)$ of size at most three dominating $N_3(v, w)$ without using v or w. If no such set exists, we allow v (resp. w) and try to find one again. If we now find such a set, we can conclude that v (w) must be part of a solution.

Definition 22. *Let $G = (V, E)$ be a graph and let $v, w \in V$. We now consider all the sets that can dominate $N_3(v, w)$:*

$$\mathcal{D} = \{\tilde{D} \subseteq N_{2,3}(v,w) \mid N_3(v,w) \subseteq \bigcup_{v \in \tilde{D}} N(v),\ |\tilde{D}| \leq 3\} \tag{5.7}$$

$$\mathcal{D}_v = \{\tilde{D} \subseteq N_{2,3}(v,w) \cup \{v\} \mid N_3(v,w) \subseteq \bigcup_{v \in \tilde{D}} N(v),\ |\tilde{D}| \leq 3,\ v \in \tilde{D}\} \tag{5.8}$$

$$\mathcal{D}_w = \{\tilde{D} \subseteq N_{2,3}(v,w) \cup \{w\} \mid N_3(v,w) \subseteq \bigcup_{v \in \tilde{D}} N(v),\ |\tilde{D}| \leq 3,\ w \in \tilde{D}\} \tag{5.9}$$

Furthermore, we shortly denote $\bigcup \mathcal{D}_v = \bigcup\limits_{D \in \mathcal{D}_v} D$ and $\bigcup \mathcal{D}_w = \bigcup\limits_{D \in \mathcal{D}_w} D$.

We are now ready to state Rule 2:

Rule 2. *Let $G = (V, E)$ be a graph and two distinct $v, w \in V$. If $\mathcal{D} = \emptyset$ we apply the following:*

Case 1: *if $\mathcal{D}_v = \emptyset$ and $D_w = \emptyset$*

- *Remove $N_{2,3}(v, w)$*
- *Add vertices v' and w' and two edges $\{v, v'\}$ and $\{w, w'\}$*
- *If there was a common neighbor of v and w in $N_{2,3}(v, w)$ add another vertex y and two connecting edges $\{v, y\}$ and $\{y, w\}$*
- *If there was no common neighbor of v and w in $N_{2,3}(v, w)$, but at least one path of length three from v to w via only vertices from $N_{2,3}(v, w)$, add two vertices y and y' and connecting edges $\{v, y\}$, $\{y, y'\}$ and $\{y', w\}$*

Case 2: *if $\mathcal{D}_v \neq \emptyset$ and $D_w = \emptyset$*

- *Remove $N_{2,3}(v)$*
- *Add $\{v, v'\}$*

Case 3: *if $\mathcal{D}_v = \emptyset$ and $D_w \neq \emptyset$*
*This case is symmetrical to **case (2)**.*

In **case (1)**, we know by Fact 3 that v and w must be in D. Therefore, we introduce two forcing vertices v' and w' in G' and remove $N_{2,3}(v, w)$ as these vertices are dominated by v and w. But if we remove $N_{2,3}(v, w)$ entirely, we could lose solutions: First, the case that v is a direct witness of w ($d(v, w) = 2$) and that there is one intermediate witness on a path of length three from v to w via vertices in $N_{2,3}(v, w)$, which could be a witness for both v and w at the same time. Note that if we would not distinguish between these two cases and just add one intermediate vertex, we would possibly generate wrong solutions, because v could always witness w.

Again by Fact 3 we know for **cases (1)** and **(2)** that $v \in D$ and similar to Rule 1 we can simplify the neighborhood $N_{2,3}(v)$. Fact 2 states, that these vertices are only useful for witnessing v, but do not go beyond what v already witnesses.

The cases where $\mathcal{D}_v \neq \emptyset$ and $\mathcal{D}_w \neq \emptyset$ are not required and in a later analysis only the existence of these two sets is required and the application of Rule 3 is critical.

Before proofing Rule 2 we will deduce some *facts* which are implied by the definitions above. These facts justify the definition of the sets $\mathcal{D}$, $\mathcal{D}_v$ and $\mathcal{D}_w$.

Fact 2. *Let $G = (V, E)$ be a graph, let $v, w \in V$, and let G' be the graph obtained by the application of Rule 2 on v, w. If $\mathcal{D} = \emptyset$, then G has a solution if and only if it has a solution containing at least one of the two vertices $\{v, w\}$.*

Proof. Because $\mathcal{D} = \emptyset$, any SDS of G has to contain v or w, or at least four vertices from $N_{2,3}$. In the second case, these four vertices can be replaced with v, w and two neighbors of v and w still forming a Semitotal Dominating Set. □

The second fact states that if $\mathcal{D}_v$ (resp. $\mathcal{D}_w$) is empty, too, we only need to consider solutions containing w (v):

Fact 3. *Let $G = (V, E)$ be a graph, let $v, w \in V$, and let G' be the graph obtained by the application of Rule 2 on v, w. If $\mathcal{D} = \emptyset$ and $\mathcal{D}_w = \emptyset$ (resp. $\mathcal{D}_v = \emptyset$) then G' has a solution if and only if it has a solution containing v (resp. w).*

Proof. As $\mathcal{D}_v = \emptyset$, no set of the form $\{v\}$, $\{v, u\}$ or $\{v, u, u'\}$ with $u, u' \in N_{2,3}(v, w)$ can dominate $N_3(v, w)$. Since also $\mathcal{D} = \emptyset$ any SDS of G has to contain v or at least four vertices by Fact 2. In the last case, we again replace these four vertices by v, w and two neighbors respectively and we can conclude that v belongs to the solution. □

Now we are ready to prove the correctness of Rule 2

Lemma 2. *Let $G = (V, E)$ be a plane graph, $v, w \in V$ and $G' = (V', E')$ be the graph obtained after application of Rule 2 on the pair $\{v, w\}$. Then G has SDS of size k if and only if G' has SDS of size k.*

Proof. We will prove the claim by analyzing the different cases of the rule independently.

- $\Rightarrow$ Consider a Semitotal Dominating Set D in G. We show that G' also has a SDS with $|D'| \leq |D|$. By assumption, we have $\mathcal{D} = \emptyset$.
  - a) $\mathcal{D}_v = \emptyset \wedge \mathcal{D}_w = \emptyset$: By applying Fact 3 twice, we know that both $v, w \in D$. Therefore, v',w', and potentially y and y' are dominated by v or w in G'.

    We now have three cases: Either v and w are so far away that both have their witnesses themselves; they are of distance three, such that potentially one witness on a path from v to w (which could go through $N_{2,3}(v,w)$ and will be removed) is required or they can be of distance less than three, such that they already witness each other directly.

We will now build D' depending on which vertices from $D \cap N_{2,3}(v,w)$ have removed.

- If the rule has not removed any $d \in D$, we simply set $D' = D$. If v was a witness for w (and vice versa), Rule 2 will preserve it by introducing the vertex y. Otherwise, these witnesses are preserved.
- If $d(v,w) > 3$, then v and w are not sharing any common witnesses. If the rule has removed a vertex from $D \cap N(v)$, we set $D' = D \setminus N_{2,3}(v,w) \cup \{v'\}$. If the rule has removed a vertex from $D \cap N(w)$, we set $D' = D \setminus N_{2,3}(v,w) \cup \{w'\}$. If the rule has removed a vertex from $(D \cap N(v))$ and a vertex from $(D \cap N(w))$, we set $D' = D \setminus N_{2,3}(v,w) \cup \{v', w'\}$.
- If $d(v,w) = 3$, then there could possibly be a path via $N_{2,3}(v,w)$ vertices containing a witness for both v and w. If the rule removed a vertex $D \cap N_{2,3}(v,w)$m, we set $D' = D \setminus N_{2,3}(v,w) \cup \{y\}$. Note that we could also choose $y' \in D'$, because y's only function is to be a single witness for v and w and everything it could witness too, will also be witnessed by $v, w \in D'$ (Fact 2).
- If $d(v,w) \leq 2$, then v directly witnesses w (and vice versa). Even if the rule has removed a vertex $z \in D \cap N_{2,3}(v,w)$, we can ignore that, because Fact 2 states that v and w will witness the same vertices as z did. Hence, we set $D' = D \setminus N_{2,3}(v,w)$.

In all of the cases, it follows that D' is a SDS of G' with $|D'| \leq |D|$

b) $\mathcal{D}_v \neq \emptyset \wedge \mathcal{D}_w = \emptyset$: As $\mathcal{D}_w = \emptyset$ and Fact 3, we know that $v \in D$ and v dominates $N_{2,3}(v)$. If a vertex $d \in D \cap N_{2,3}(v)$ was removed, we set $D' = D \setminus N_{2,3}(v) \cup \{v'\}$, else $D' = D$. Deleting dominating vertices $d \in D \cap N_{2,3}(v)$ does not destroy the witness properties of the graph, because by Fact 2 we know that everything d could witness, is also witnessed by v. If d was a witness for v, we have replaced it with v' in G'. Note that otherwise a vertex from $N_1(v) \cup \{p \in (N(z) \setminus N(v)) | z \in N_1(v)\}$ is a witness for v that is not touched by this reduction. Clearly, $|D'| \leq |D|$ holds.

c) $\mathcal{D}_v = \emptyset \wedge \mathcal{D}_w \neq \emptyset$: Symmetrical to previous case.

$\Leftarrow$ Let D' be a Semitotal Dominating Set in G' and $\mathcal{D} = \emptyset$. We show that G has a SDS D with $|D| \leq |D'|$ by distinguishing the different cases again.

a) $\mathcal{D}_v = \emptyset \wedge \mathcal{D}_w = \emptyset$: In any case we know that $v, w \in D$ to dominate v' and w' and therefore also dominating $N_{2,3}(v,w)$ in G. To preserve the distance $d(v,w)$ the rule might have introduced additional vertices y and y'.

- If only y was introduced we know that there was a common neighbor $n \in N(v) \cap N(w)$ of v and w. y allows v to witness w (and vice versa) and is not part of a solution itself. (assuming $y \notin D'$). Hence, we set $D = D'$.
- If y and y' were added, a solution could use one of them to provide a single witness for v and w. There exists a path $p = (v, n_1, n_2, w)$ from v to w in G only using vertices from $N_{2,3}(v,w)$. As n_1 and n_2 both witness v and w, we put one of them in D if at least one of y or y are dominating vertices in G'. Hence, if $y \in D'$ or $y \in D'$, we set $D = D' \setminus \{y, y'\} \cup \{n_1\}$.

b) $\mathcal{D}_v \neq \emptyset \wedge \mathcal{D}_w = \emptyset$: Clearly, $v \in D'$ to dominate v'. If $v \in D'$, we set $D = D' \setminus \{v'\} \cup d$ for some vertex $d \in N_{2,3}(v,w)$ and otherwise $D = D'$. If v' was the witness of v, it is now replaced by d and D is an SDS with $|D| \leq |D'|$.

c) $\mathcal{D}_v = \emptyset \wedge \mathcal{D}_w \neq \emptyset$: Symmetrical to previous case.

In all cases we have shown that $|D| \leq |D'|$ and D is a Semitotal Dominating Set of G.

□

5.4.3 Reduction Rule III: Shrinking Simple Regions

Rule 3. *Let $G = (V, E)$ be a plane graph, $v, w \in V$ and R be a simple region between v and w. If $|V(R) \setminus \{v, w\}| \geq 7$*

- *Remove $N_3(v, w)$*
- *Add two vertices h_1 and h_2 and four edges $\{v, h_1\}$, $\{v, h_2\}$, $\{w, h_1\}$ and $\{w, h_2\}$*

Lemma 3 (Correctness of Rule 3)**.** *Let $G = (V, E)$ be a plane graph, $v, w \in V$ and $G' = (V', E')$ be the graph obtained after application of Rule 3 on the pair $\{v, w\}$. Then G has SDS of size k if and only if G' has SDS of size k.*

The application of Rule 3 gives us a bound on the number of vertices inside a simple region.

Corollary 5. *Let $G = (V, E)$ be a graph, $v, w \in V$ and R a simple region between v and w. If Rule 3 has been applied, this simple region has size at most 6.*

Proof. Clearly, if $|V(R) \setminus \{v, w\}| < 7$ then the rule would not have changed G and the size of the region would already be bounded by 6. Asuming $|V(R) \setminus \{v, w\}| \geq 7$ we note that every simple region can have at most two distinct vertices from $N_1(v, w)$ and two distinct ones from $N_2(v, w)$ without breaking planarity. These vertices are not touched by the reduction. Adding the two vertices that are being added between v and w gives us the desired upper bound. □

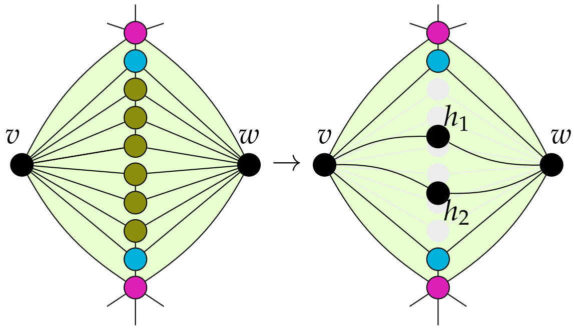

Figure 5.7: *TO BE DONE*

5.4.4 Computing Maximal Simple Regions between two vertices

For the sake of completeness, we state an algorithm how a maximal simple regionbetween two vertices $v, w \in V$ can be computed in time $\mathcal{O}(d(v) + d(w))$:

5.5 Bounding the Size of the Kernel

We are now putting all our pieces together to prove our main result: A linear bound on the kernel size. To do so, we distinguish between those vertices that are covered by a maximal *D-region decomposition* and those that are not. In both cases, our reduction rules bound the number of vertices to a constant size which means the kernel size does only depend on the number of regions of these decompositions. Fig. 5.3 states that for any solution D, we only have a linear number of regions that cover the whole graph. In particular, we show that given a SEMITOTAL DOMINATING SET D of size k, there exists a maximal *D-region decomposition* $\mathfrak{R}$ such that:

1. $\mathfrak{R}$ has only at most $3|D| - 6$ regions
2. $V(\mathfrak{R})$ covers most vertices of V. There are at most $144 \cdot |D|$ vertices outside of any region.
3. each region of $\mathfrak{R}$ contains at most XX vertices

Combining these three parts will give us a linear kernel.

5.5.1 Bounding the Size of a Region

We start with a more fine-grained analysis of the impact of the different cases of Rule 2 on a vw-region. The main idea is to count the number of simple regions in the vw-region and then use the bound on the size of a simple region after Rule 3 was applied exhaustively and which was obtained in Corollary 5.

Lemma 4. *Given a plane Graph $G = (V, E)$ and a vw-region R $|N_1(v, w) \cap V(R)| \leq 4$ and these vertices lay exactly on the boundary ∂R of R.*

Proof. □

Lemma 5. *[12, See Fact 5] Given a reduced plane graph $G = (V, E)$ and a region $R(v, w)$, $N_2(v, w) \cap V(R)$ can be covered by at most 6 simple regions.*

Proof. Let (v, u_1, u_2, w) and (v, u_3, u_4, w) be the two boundary paths of $R(v, w)$. (A shorter path would only lead to a smaller bound). By definition of $N_2(v, w)$, vertices from $N_2(v, w) \cap V(R)$ are common neighbors of v and w and $u_i, i \in [4]$. By planarity, we can cover $N_2(v, w) \cap V(R)$ with at most 6 simple regions. To see this, imagine the graph where edges denote all possible simple vw-regions (See fig. 5.8). There are at most 8 simple regions possible. but we have to remove at least two of them to maintain planarity.

Furthermore, assuming the graph to be reduced, any intermediate $N_3(v, w)$ which could separate multiple simple regions between v and u_i has been deleted by Rule 1 already. □

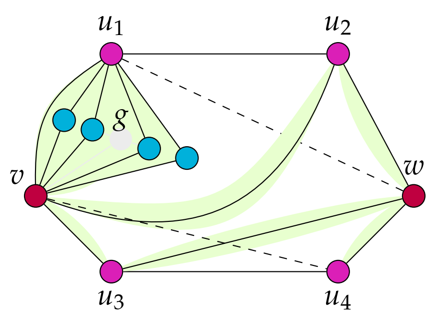

Figure 5.8: *Bounding the maximum number of simple regions inside a region $R(v, w)$. $N_2(v, w)$ is covered by 6 green (simple) regions. A dashed edge would also be an option but would contradict planarity. Note that the gray vertex g was reduced by Rule 1 allowing the formation of exactly one simple region between v and u_1*

We continue by giving a constant bound on the number of simple regions that cover all $N_3(v, w)$ vertices in a given region.

Lemma 6. *Given a plane Graph $G = (V, E)$ reduced under Rule 2 and a region R(v, w), if $\mathcal{D}_v \neq$ (resp. $\mathcal{D}_w \neq \emptyset$), $N_3(v, w) \cap V(R)$ can be covered by:*

1. 11 *simple regions if $\mathcal{D}_w \neq \emptyset$,*
2. 14 *simple regions if $N_{2,3}(v) \cap N_3(v, w) = \emptyset$*

Note, that the first case applies, when Case 2 & 3 of Rule 2 have been applied and the second one, when Case 4 of Rule 2 was applied.

Proof. We will just give some intuition because the proof of Garnero and Sau in **[Garnero2014]** does not use any special property exposed by the reduction rules. Figure (Add picture about figures) gives a visualization of the worst-case scenarios to cover $N_3(v,w) \cap V(R)$ with simple regions in the relevant cases.[1]

□

Lemma 7 (#Vertices inside a Region after Rules 1 to 3). *Let $G = (V,E)$ be a plane graph reduced under Rules 1 to 3. Furthermore, let D be an SDS of G and let $v, w \in D$. Any vw-region R contains at most 139 vertices distinct from its poles.*

Proof. By Lemmas 4 and 5 and Corollary 5 to bound the number of vertices inside a simple region, we know that $|N_1(v,w) \cap V(R)| \leq 4$ and $|N_2(v,w) \cap V(R)| \leq 6 \cdot 7 = 42$.

It is remaining to bound for $|N_3(v,w) \cap V(R)|$, but gladly we have Rule 2, which took care about them! Fig. 5.9 shows the worst case amount of simple regions the indidual cases can have.

Case 0: Rule 2 has **not** been applied in the following two cases: Either $\mathcal{D} \neq \emptyset$ or $(\mathcal{D} = \emptyset \land \mathcal{D}_v \neq \emptyset \land \mathcal{D}_w \neq \emptyset)$:

1. If $\mathcal{D} \neq \emptyset$, there exists a set $\tilde{D} = \{d_1, d_2, d_3\} \in \mathcal{D}$ of at most three vertices dominating $N_3(v,w)$. We observe that vertices from $|N_3(v,w) \cap V(R)|$ are common neighbors of either v or w (by the definition of a vw-region) and at least one vertex from $\tilde{D}$. Without violating planarity, we can span at most 6 simple regions. Using Corollary 5 and adding $|\tilde{D}| = 3$, we can conclude $|N_3(v,w) \cap V(R)| \leq 6 \cdot 6 + 3 = 39$.
2. If $\mathcal{D} = \emptyset$, $\mathcal{D}_v \neq \emptyset$ and $\mathcal{D}_w \neq \emptyset$, we can apply Lemma 6 and although Rule 2 has not changed the G, we can cover R with at most 11 simple regions giving us $|N_3(v,w) \cap V(R)| \leq 11 \cdot 6 = 66$ vertices.

Case 1: If Rule 2 **Case (1)** has been applied
In that case, $|N_2(v,w) \cap V(R)|$was entirely removed and at $|N_3(v,w) \cap V(R)|$replaced by at most three vertices (v', w' and y) added. Hence $|N_3(v,w) \cap V(R)| \leq 3$.

Case 2: If Rule 2 **Cases (2)** and **(3)** have been applied
We know that in this case $N_{2,3}(v) \cap N_3(v,w)$ was entirely removed and replaced by a single possible witness. Using Lemma 6, we can cover $(N_3(v,w) \setminus \{v'\} \cap V(R)$ with (at most) 14 simple regions giving us $||N_3(v,w) \cap V(R)|| \leq 14 \cdot 6 + 1 = 85$.

All in all, as $V(R) = \{v,w\} \cup (N_1(v,w) \cup N_2(v,w) \cup N_3(v,w)) \cap V(R)$ we get

$$V(R) \leq 2 + 4 + 42 + \max(39, 3, 66, 85) = 139$$

□

[1] Note: In a newer revision of their paper [12], Garnero and Sau removed this proof, because they changed Rule 2 and a more fine-grained analysis was made possible.

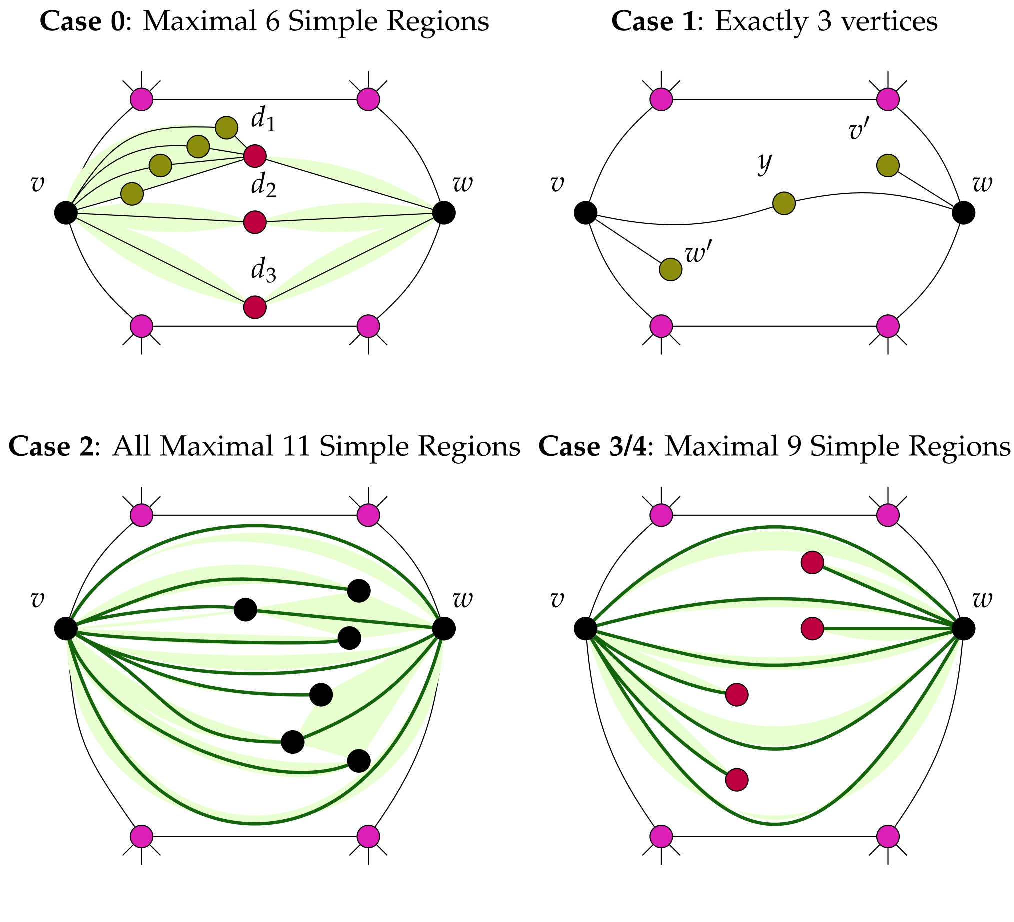

Figure 5.9: *TODO*

5.5.2 Number of Vertices outside the Decomposition

We continue to bound the number of vertices that do not lay inside any region of a maximal *D-region decomposition* $\mathfrak{R}$, that is, we bound the size of $V \setminus V(\mathfrak{R})$. Rule 1 ensures that we only have a small amount of $N_3(v)$-pendants. We then try to cover the rest with as few simple regions as possible, because, by application of Rule 3, these simple regions are of constant size.

Lemma 8. *[2] (Deprecated) Every vertex in $u \in V \setminus V(\mathfrak{R})$ is either in D or belongs to a set $N_2(v) \cup N_3(v)$.*

The following lemma states that no vertices from a set $N_1(v)$ will be outside of a maximal *D-region decomposition*.

Lemma 9. *[2, Lemma 6] Let $G = (V, E)$ be a plane graph and $\mathfrak{R}$ be a maximal D-region decomposition of a DS D. If $u \in N_1(v)$ for some vertex $v \in D$ then $u \in V(\mathfrak{R})$*

In the following, we define $d_{G_{\mathfrak{R}}}(v) = |\{R(v,w) \in \mathfrak{R}, w \in D\}|$ to be the number of regions in $\mathfrak{R}$ having v as a pole.

Corollary 6. *Let $G = (V, E)$ be a graph and D be a set. For any maximal D-region decomposition $\mathfrak{R}$ on G it holds that $\sum_{v \in D} d_{G_{\mathfrak{R}}}(v) = 2 \cdot |\mathfrak{R}|$.*

Proof. The proof follows directly from the handshake lemma applied to the underlying multigraph $G_{\mathfrak{R}}$. □

Proposition 1 (#Vertices outside a Region)**.** *Let $G = (V, E)$ be a plane graph reduced under Rules 1 and 2 and let D be a SDS of G. If G has a maximal D-region decomposition, then $|V \setminus (V(\mathfrak{R}) \cup D)| \leq 144|D|$*

With slight modifications, the proof given in [12, Revision 2014] will also work in our case. Note that although assuming the graph to be entirely reduced, the following proof only relies on Rules 1 and 3. The proof uses the observation that vertices from $N_2(v)$ span simple regions between those from $\{v\} \cup N_1(v)$.

Proof. Again, we will follow the proof proposed by Alber et al. [2, Proposition 2].

The proof does only rely on Rules 1 and 3 and we can use the number of vertices in a simple region we have proven in Corollary 5. In particular, we are going to proof that $V \setminus V(\mathfrak{R}) \leq 48 \cdot |\mathfrak{R}| + 2 \cdot |D|$. Directly placing in Lemma 10 will give the desired bound.

let $\mathfrak{R}$ be a maximal *D-region decomposition* and let $v \in D$. Since D dominates all vertices from V, we can consider V as $\bigcup_{v \in D} N(v)$ and thus, we only need to bound the sizes of $N_1(v) \setminus V(\mathfrak{R})$, $N_2(v) \setminus V(\mathfrak{R})$ and $N_3(v) \setminus V(\mathfrak{R})$ separately. In the following, let $v \in D$:

$\mathbf{N_3(v)}$: As we know that Rule 1 has been exhaustively applied, we trivially see that $|N_3(v)| \leq 1$ and hence,

$$\left| \bigcup_{v \in D} N_3(v) \setminus V(\mathfrak{R}) \right| \leq |D|$$

$\mathbf{N_2(v)}$: According to Garnero and Sau ([12, Proposition 2]), we know that $N_2(v) \setminus V(\mathfrak{R})$ can be covered by at most $4d_{G_{\mathfrak{R}}}(v)$ simple regions between v and some vertices from $N_1(v)$ on the boundary of a region in $\mathfrak{R}$. Figure 5.10 gives some intuition.

Because G is reduced by assumption, we know by Corollary 5 that a simple region can only have at least 6 vertices distinct from its poles and hence,

$$\begin{aligned} \left| \bigcup_{v \in D} N_2(v) \setminus V(\mathfrak{R}) \right| &\leq 6 \sum_{v \in D} 4 \cdot d_{G_{\mathfrak{R}}}(v) \\ &= 24 \cdot \sum_{v \in D} d_{G_{\mathfrak{R}}}(v) \\ &\overset{\text{Cor. 5.5.2}}{\leq} 48|\mathfrak{R}| \end{aligned} \tag{5.10}$$

$\mathbf{N_1(v)}$: By Lemma 9, we know that $N_1(v) \subseteq V(\mathfrak{R})$ and hence,

$$\left| \bigcup_{v \in D} N_1(v) \setminus V(\mathfrak{R}) \right| = 0$$

Summing up these three upper bounds for each $v \in D$ we obtain the result using the equation from Lemma 10:

$$\begin{aligned} |V \setminus V(\mathfrak{R}) \cup D)| &\leq 48 \cdot |\mathfrak{R}| + |D| && \text{(Lemma 10)} \\ &\leq 48 \cdot (3|D| - 6) + |D| \\ &\leq 144|D| + |D| \\ &= 145|D| && (5.11) \end{aligned}$$

□

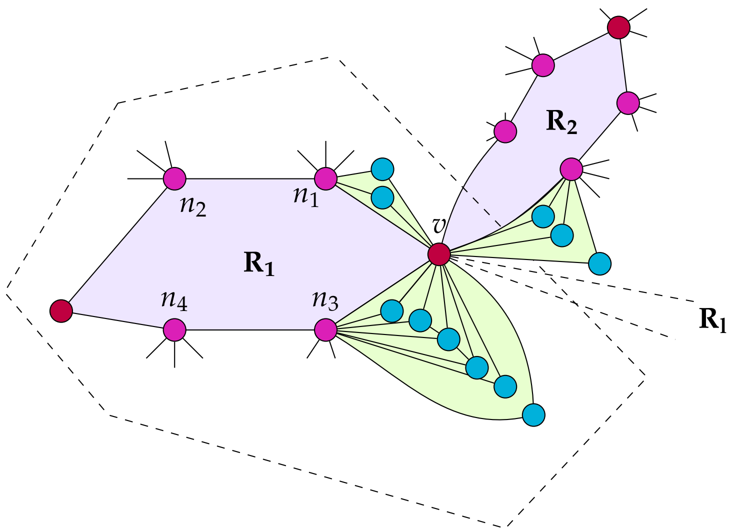

Figure 5.10: *Bounding the number of $N_2(v)$-vertices around a dominating vertex v given a maximal D-region decomposition $\mathfrak{R}$. v is a pole of $R_1, R_2, ...R_j$ and can span simple regions with the help of $N_2(v)$-vertices to at most two $N_1(v)$-vertices per R_i. Each region has at most four vertices in $N_1(v, w) \subseteq N_1(v)$ on the boundary of R_j, but only at most two can be used for a simple region: Observe that trying to build a simple region between v and n_2 in this example would contradict the maximality of $\mathfrak{R}$ Furthermore, the size of these simple regions is bounded after the application of Rule 3*

5.5.3 Bounding the Number of Regions

Alber et al. [2, Proposition 1] gave a greedy algorithm to construct a maximal *D-region decomposition* for a Dominating Set. Building upon these results, Garnero and Sau gave decomposition procedures for both Planar Red-Blue Dominating Set ([13]) and Total Dominating Set ([12]) relying on the same technique. This is the core of our linear kernelization because it states that given a Dominating Set D, we can decompose the graph into a *linear number* of regions.

The following lemma corresponds to [2, Proposition 1 and Lemma 5]. Although the authors gave different reduction rules and require a *reduced* instance as an assumption for the following lemma, they do not use any specific properties exposed by these rules. As any Semitotal Dominating Set is also a Dominating Set, we can safely apply it to our problem as well. For a more detailed and formal proof, one can also refer to [12, Proposition 1].

Lemma 10. *Let G be a reduced plane graph and let D be a* Semitotal Dominating Set *with* $|D| \geq 3$. *There is a maximal D-region decomposition of G such that* $|R| \leq 3 \cdot |D| - 6$

Proof. Follows directly from [2, Proposition 1 and Lemma 5] □

Lemma 11 (Running Time of Reduction Procedure)**.** *TODO Runsi in polynomial Time.*

Proof. □

By utilizing all the previous results, we are now finally ready to proof the Theorem 1:

Theorem 1. *The* Semitotal Dominating Set *problem parametrized by solution size admits a linear kernel on planar graphs. There exists a polynomial-time algorithms that given a planar graph* (G, k)*, either correctly reports that* (G, k) *is a NO-instance or returns an equivalent instance* (G', k) *such that* $|V(G')| \leq 561 \cdot k$.

Proof. Let $G = (V, E)$ be the plane input graph and $G' = (V', E')$ be the graph obtained by the exhautive application of the Rules 1 to 3. As none of our rules change the size of a possible solution D′ in G′, we know by Lemma 1, Lemma 2 and Lemma 3 that G′ has a Semitotal Dominating Set of size k if and only if G has a Semitotal Dominating Set of size k. In Lemma 11, we have proven that this preprocessing procedure runs in polynomial time.
Assume that G′ admits a solution D'.

By taking the size of each region proven in Proposition 1, the total number of regions in a maximal *D-region decomposition* (Lemma 10) and the number of vertices that can lay outside of any region (Proposition 1), we obtain the following bound:

$$139 \cdot (3k - 6) + 145 \cdot k + k \leq 561 \cdot k \tag{5.12}$$

If $|V(G')| > 561 \cdot k$ G is a NO-instance and we replace G′ by two single disconnected vertices (trivial NO-instance). Then the kernel is of the claimed size.

□

BIBLIOGRAPHY

[1] J. Alber, B. Dorn, and R. Niedermeier. "A General Data Reduction Scheme for Domination in Graphs." In: *SOFSEM 2006: Theory and Practice of Computer Science, 32nd Conference on Current Trends in Theory and Practice of Computer Science, Merin, Czech Republic, January 21-27, 2006, Proceedings*. Ed. by J. Wiedermann, G. Tel, J. Pokorný, M. Bieliková, and J. Stuller. Vol. 3831. Lecture Notes in Computer Science. Springer, 2006, pp. 137–147. DOI: 10.1007/11611257_{1}{1}.

[2] J. Alber, M. R. Fellows, and R. Niedermeier. "Polynomial-time data reduction for dominating set." In: (May 2004), pp. 363–384. DOI: 10.1145/990308.990309.

[3] R. Balakrishnan and K. Ranganathan. *A textbook of graph theory*. English. 2nd ed. Universitext. New York, NY: Springer, 2012. ISBN: 978-1-4614-4528-9; 978-1-4614-4529-6. DOI: 10.1007/978-1-4614-4529-6.

[4] H. L. Bodlaender, F. V. Fomin, D. Lokshtanov, E. Penninkx, S. Saurabh, and D. M. Thilikos. "(Meta) Kernelization." In: *J. ACM* 63.5 (2016), 44:1–44:69. DOI: 10.1145/2973749.

[5] M. Bonamy and Ł. Kowalik. "A 13k-kernel for planar feedback vertex set via region decomposition." en. In: *Theoretical Computer Science* 645 (Sept. 2016), pp. 25–40. ISSN: 0304-3975. DOI: 10.1016/j.tcs.2016.05.031.

[6] J. A. Bondy and U. S. R. Murty. *Graph Theory*. Graduate Texts in Mathematics. Springer, 2008. ISBN: 978-1-84628-970-5. DOI: 10.1007/978-1-84628-970-5.

[7] V. Diekert and B. Durand, eds. *STACS 2005, 22nd Annual Symposium on Theoretical Aspects of Computer Science, Stuttgart, Germany, February 24-26, 2005, Proceedings*. Vol. 3404. Lecture Notes in Computer Science. Springer, 2005. ISBN: 3-540-24998-2. DOI: 10.1007/b106485.

[8] R. Diestel. *Graph Theory*. Fourth. Vol. 173. Graduate Texts in Mathematics. Heidelberg; New York: Springer, 2010. ISBN: 9783642142789 3642142788 9783642142796 3642142796.

[9] F. V. Fomin, D. Lokshtanov, S. Saurabh, and M. Zehavi. *Kernelization. Theory of parameterized preprocessing*. English. Cambridge: Cambridge University Press, 2019. ISBN: 978-1-107-05776-0; 978-1-107-41515-7. DOI: 10.1017/9781107415157.

[10] F. V. Fomin and D. M. Thilikos. "Fast Parameterized Algorithms for Graphs on Surfaces: Linear Kernel and Exponential Speed-Up." en. In: *Automata, Languages and Programming*. Ed. by J. Díaz, J. Karhumäki, A. Lepistö, and D. Sannella. Lecture Notes in Computer Science. Berlin, Heidelberg: Springer, 2004, pp. 581–592. ISBN: 9783540278368. DOI: 10.1007/978-3-540-27836-8_50.

[11] V. Garnero, C. Paul, I. Sau, and D. M. Thilikos. "Explicit Linear Kernels for Packing Problems." en. In: *Algorithmica* 81.4 (Apr. 2019), pp. 1615–1656. ISSN: 1432-0541. DOI: 10.1007/s00453-018-0495-5.

[12] V. Garnero and I. Sau. "A Linear Kernel for Planar Total Dominating Set." In: *Discrete Mathematics & Theoretical Computer Science* Vol. 20 no. 1 (May 2018). DOI: 10.23638/DMTCS-20-1-14. eprint: 1211.0978.

[13] V. Garnero, I. Sau, and D. M. Thilikos. "A linear kernel for planar red-blue dominating set." In: *Discret. Appl. Math.* 217 (2017), pp. 536–547. DOI: 10.1016/j.dam.2016.09.045.

[14] J. Guo and R. Niedermeier. "Linear Problem Kernels for NP-Hard Problems on Planar Graphs." In: *Automata, Languages and Programming*. Ed. by L. Arge, C. Cachin, T. Jurdziński, and A. Tarlecki. Berlin, Heidelberg: Springer Berlin Heidelberg, 2007, pp. 375–386. ISBN: 978-3-540-73420-8.

[15] J. Guo, R. Niedermeier, and S. Wernicke. "Fixed-Parameter Tractability Results for Full-Degree Spanning Tree and Its Dual." en. In: *Parameterized and Exact Computation*. Ed. by H. L. Bodlaender and M. A. Langston. Lecture Notes in Computer Science. Berlin, Heidelberg: Springer, 2006, pp. 203–214. ISBN: 9783540391012. DOI: 10.1007/11847250_19.

[16] S. Gutner. "Polynomial Kernels and Faster Algorithms for the Dominating Set Problem on Graphs with an Excluded Minor." In: *Parameterized and Exact Computation: 4th International Workshop, IWPEC 2009, Copenhagen, Denmark, September 10-11, 2009, Revised Selected Papers*. Berlin, Heidelberg: Springer-Verlag, Dec. 2009, pp. 246–257. ISBN: 9783642112683.

[17] J. T. Halseth. "A 43k Kernel for Planar Dominating Set using Computer-Aided Reduction Rule Discovery." In: *University of Bergen, University Library* (Feb. 2016).

[18] I. Kanj, M. J. Pelsmajer, M. Schaefer, and G. Xia. "On the induced matching problem." en. In: *Journal of Computer and System Sciences* 77.6 (Nov. 2011), pp. 1058–1070. ISSN: 0022-0000. DOI: 10.1016/j.jcss.2010.09.001.

[19] Ł. Kowalik, M. Pilipczuk, and K. Suchan. "Towards optimal kernel for connected vertex cover in planar graphs." en. In: *Discrete Applied Mathematics* 161.7 (May 2013), pp. 1154–1161. ISSN: 0166-218X. DOI: 10.1016/j.dam.2012.12.001.

[20] W. Luo, J. Wang, Q. Feng, J. Guo, and J. Chen. "Improved linear problem kernel for planar connected dominating set." In: *Theor. Comput. Sci.* 511 (2013), pp. 2–12. DOI: 10.1016/j.tcs.2013.06.011.

[21] G. Philip, V. Raman, and S. Sikdar. "Polynomial kernels for dominating set in graphs of bounded degeneracy and beyond." In: *ACM Transactions on Algorithms* 9.1 (Dec. 2012), 11:1–11:23. ISSN: 1549-6325. DOI: 10.1145/2390176.2390187.

[22] J. Wang, Y. Yang, J. Guo, and J. Chen. "Linear Problem Kernels for Planar Graph Problems with Small Distance Property." en. In: *Mathematical Foundations of Computer Science 2011*. Ed. by F. Murlak and P. Sankowski. Lecture Notes in Computer Science. Berlin, Heidelberg: Springer, 2011, pp. 592–603. ISBN: 9783642229930. DOI: 10.1007/978-3-642-22993-0_53.

LIST OF FIGURES

LIST OF TABLES