Graph Theory

Lukas Bach - lbach@outlook.de - lukasbach.com

List of Contents

lbach@outlook.de
lukasbach.com

1 Formales

21.02. Exam

2 Second Lecture

Given $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$, $dist_{Ham}(a, b) = |\{k: a_k \neq b_k, 1 \leq k \leq n\}|$ describes the Hamming-distance between a and b. Example: $dist_{Ham}\big((0,1,0), (0,0,1)\big) = 2$.

Proposition 3: If $\delta(G) \geq 2$ for a graph G, then G has a cycle of length at least $\delta + 1$, $\delta = \delta(G)$.

Proof: Let $P = v_0, v_1, \dots, v_n$ be a longest path in G. Then $N(v_0) \subseteq V(P)$. (N as neighbourhood of v_0 and P as vertex set of the path) Let $k = \max\{i: v_i \in N(v_0)\}$. We have $k \geq \delta$. Then $v_0 v_1, \dots, v_k v_0$ is a cycle of length of at least $s + 1$.

Definition: A walk in a graph G is an alternating sequence of vertices and edges: $v_0 e_0 v_1 e_1 \dots e_{n-1} v_n$ mit $e_i = v_i v_{i+1} \forall i = 0, \dots, n-1$.

Example: $W = v_0 e_0 v_1 e_1 v_2$

Edges may be included redundantly. This definition also holds for multigraphs. For non-multigraphs suffices a sequence of vertices.

In a walk $W = v_0 e_0 \dots v_n$, v_0 and v_n are endpoints.

If $v_0 = v_n$, the walk is considered closed.

Proposition 4: Given a graph G, $u, v \in V(G)$ with $u \neq v$, if there is a walk in G with endpoints u and v, then $\exists u - v - path$.

Proof: Let W be a u-v-walk with the smallest length. If W corresponds to a path, i.e. it has no repeated vertices: done.

Otherwise, there is a repeated vertex, i.e. $W = v_0 e_0 v_1 \dots e_{i-1} v_i e_i \dots e_{j-1} v_j e_j \dots v_n$ with $v_i = v_j$. Then $W' = v_0 e_0 \dots e_{i-1} v_i e_j \dots v_n$ is a shorter walk, which is a contradiction.

Proposition 5: If a graph has a closed walk of odd length ("odd walk"), then it has an odd cycle.

Proof: Let W be a closed odd walk of smallest length. If W corresponds to a cycle: done.

Otherwise is $W = v_0 e_0 v_1 \dots e_{i-1} v_i e_i \dots e_{j-1} v_j e_j \dots v_{n-1} e_{n-1} v_n$ where $v_i = v_j$. Then $W' = v_0 e_0 \dots v_i e_j \dots v_{n-1} e_{n-1} v_n$ and $W'' = v_i e_i \dots e_{j-1} v_j$. Then $length(W) = length(W') + length(W'')$ and W', W''-closed walks. Since length of W is odd ➔ either W' or W'' is an odd walk of length less than length of W, which is an contradiction.

Definition: Let $dist(u, v)$ be the distance between u and v. The diameter of a graph is $diam(G) = \max\{dist(u, v): u, v \in V(G)\}$.

Proposition 6: A graph is bipartite ⇔ it has no odd cycles.

Proof:

- "⇐": Let G be bipartite with parts A and B. Then any cycle has a form $a_1 b_1 a_2 b_2 \dots a_k b_k a_1$ where $a_i \in A$ and $b_i \in B$ for $i = 1, \dots, k$; Thus any cycle is even.
- "⇒": Assume G has no odd cycles. Assume tat G is connected. Let $v \in V(G)$. Let $A = \{u: u \in V(G); dist(u, v) \text{ } is \text{ } even\}$ and $B = \{u: u \in V(G); dist(u, v) \text{ } is \text{ } odd\}$. Let $u_1, u_2 \in B, U - 1 \neq u_2, u_1 u_2 \in E(G)$. Let P_i be a shortest path $(v - u_i - path)$, P_i has odd length, $i = 1,2$. Then P_1, P_2 and $u_1 u_2$ form and odd closed walk. By proposition 5, G has an odd cycle, which is a contradiction. Thus there are no edges with both endpoints in B.
  Let $u_1 u_2 \in A$, similar argument shows $u_1 u_2 \notin E(G)$.

Definition: Eulerian Tours are closed walks containing every edge of a given graph exactly once. Graphs having eulerian tours are called eulerian.

Example:

Proposition: A connected graph is eulerian if and only if each vertex has even degree.

Proof:

- "⇒": Assume G is eulerian. Then there is even number of edges incident to any vertex.
- "⇐": Assume all vertices of G have even degree. Let G' be a walk in G that contains the largest number of edges so that no edge in W is repeated. (Any walk will do).
  We shall show that W is eulerian.
  - Claim 1: W is closed.
    Assume not, i.e. $W = v_0 e_0 \dots v_n, v_0 \neq v_n$. Then v_0, v_n have odd degrees. But since the degree of v_0 is even, $\exists edge \text{ } e = v_0 y, e \notin W$.
    Then $W' = y e v_0 e_0 \dots v_n$ is a longer walk with non-repeated edges, which is a contradiction with the claim.

Assume that W does not contain all edges of G. Let E' be the set of edges of G not in W.

- Case 1: $\exists e \in E'$ that is incident to a vertex in W.

  Then let $e = v_i y$. Consider $W' = y v_i e_i \dots v_n e_{0 v_1} \dots e_{i-1} v_j$. Then W' has more edges than W, which is a contradiction.
- Case 2: $\forall e \in E'$, e is not incident to W. Then G is disconnected, which is a contradiction.

LEMMA: Each tree on at least 2 vertices has a leaf.

PROOF: Apply proposition 3.

3 Third Lecture

Lemma 7: $\forall tree\ T\ on \geq 2\ vertices$, T has a leaf.

3.1 Operations on Graphs

$$G_1 = (V_1, E_1), G_2 = (V_2, E_2)$$

$$G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2), G1\ cut\ G_2 = (V_1 cut\ V_2, E_1 cut E_2)$$

$$G = (V, E), v \in V, G - v = G - \{v\} = (V - \{v\}, E - \{vu, uv \in E\})$$

$$e \in E, G - e = (V, E - \{e\})$$

LEMMA 8: A tree on n vertices has $n - 1$ edges.

Proof: Induction on n.

- Basis: $n = 1$, 1 vertex, 0 edges.
- Step: Assume the statement holds for all trees on n vertices. Let T be a tree with $|V(T)| = n + 1, n + 1 \geq 2$. By Lemma 7, T has a leaf, v. Let $T' = T - v$. Since T is acyclic $\Rightarrow$ T' is acyclic. Since T is connected, $\forall\, u, w \in V(T)\ \exists u - w - path$ in T. If $u, v \neq v$, this path does not pass through v. Thus this is a path in T'. Thus T' is connected. By induction $|E(T')| = |V(T')| - 1$.
  Therefore $|E(T)| = |E(T')| - 1 = |V(T')| - 1 + 1 = |V(T')| = n$. qed

LEMMA 9: Every connected graph contains a spanning tree.

Note: If H is a subgraph of G, H is spanning if $V(H) = V(G)$.

Proof: Let G be a connected graph, let T be a spanning acyclic subgraph with maximum number of edges. (exists because of empty subgraph). Want to show that T is a tree. For that, we need to verify that T is connected.

Assume T is disconnected.

- Case 1: $\exists e \in E(G), e = xy$, x and y are distinct connected components of T. Then $T \cup e$ is acyclic, contractionary to maximality of T.
- Case 2: $\neg \exists e \in E(G)$, e has endpoints in distinct connected components of T. Contractionary to connectivity of G.

Qed

LEMMA 11: The vertices of a connected graph G can be ordered $v_1, v_2, .., v_n$ so that $G[\{v_1, v_2, \dots, v_i\}]$ is connected for i=1,…,n.

Proof: Induction on $|V(\text{G})|$.

- Basis $|V(G)| = 1$.
- Step: Assume the statement holds for any connected graph on $n - 1$ vertices.
  Let G be connected with $|V(G)| = n$.
  By Lemma 9, G has a spanning tree T, with a leaf v. Let $v_n = v$. Then $T - v_n$ is still connected, thus $G - v_n$ is also connected. By induction, $V(G_{v_n}) = \{v_1, \dots, v_{n-1}\}$ so that $(G - v_n)[\{v_1, \dots, v_i\}]$ is connected for$i = 1, \dots, n - 1$.

Qed

Definition: A graph G is k-degenerate for $\in \mathbb{N}$, if each induced subgraph of G has minimum degree of at most K. (TODO minimum!?)

Example: Any tree is 1-degenerate. K_n is $(n-1)$-degenerate.

Proposition 1.7 (Tree equivalence theorem): The following statements are equivalent:

1. G is a tree, i.e. is connected and acyclic.
2. G is connected and $\forall e \in E(G)\ \ G - e$ is disconnected (minimally connected)
3. G is acyclic and $\forall x, y \in V(G), xy \notin E(G)$, $G \cup xy$ has a cycle. (maximally acyclic).
4. G is connected and 1-degenerate.
5. G is connected and $|E(G)| = |V(G)| - 1$.
6. G is acyclic and $|E(G)| = |V(G)| - 1$.
7. $\forall u, v \in V(G)\ \exists unique\ u - v - path.$

Proof: $(1) \Leftrightarrow (3)$.

$(1) \Leftrightarrow (3)$. Let G be a connected acyclic graph. Let $x, y \in V(G), x \notin E(G)$. G is connected $\Rightarrow \exists x - y - path\ p$. Then $p \cup e$ is a cycle.

$(3) \Leftrightarrow (1)$. G is acyclic and $\forall x, y \in V(G), xy \notin E(G)$. $G \cup xy$ has a cycle. We need to check that G is connected. If not, pick x,y from distinct connected components of G. Then adding edge xy does not create a cycle. Which is a contradiction.

lbach@outlook.de
lukasbach.com

3.2 Matchings

Definition: A **MATCHING** is a 1-regular graph.

Definition: A **FOREST** is an acyclic graph.

Theorem 2.2 (Hall's matching theorem, Marriage Theorem): In a bipartite graph with parts A and B, there is a matching containing all vertices of A ⇔ the Hall's condition is satisfied.

Proof:

- ⇒: Obvious. Indeed, if M-matching saturating A, $N(S) \supseteq \{b: ba \in E(M), a \in S\}$
- ⇐: Assume that Hall's condition holds. Induction on A.
  - Basis: $|A| = 1$. Obviously holds.
  - Step: Assume the result holds for $|A| = k$, prove for $|A| = k + 1$.
    Case 1: $\forall S \subseteq A, S \neq A, |N(S)| > |S|$.
    (Abbildung)
    Let $x \in A, yx \in E(G)$. Let $G' = G - \{x, y\}$, i.e. a graph obtaining from G by deleting vertices x and y and adjacent edges. $|N_{G'}(S)| \geq |N_G(S)| - 1 \geq |S|$. By Induction, G' has a matching M' saturating $A - \{x\}$. Thus $M' \cup \{xy\}$-saturates A in G.
    Case 2: $\exists S \subseteq A, S \neq A, |N(S)| = |S|$.
    (Abbildung)
    Let A' be such a set, i.e. $|N(A')| = |A'|$. We shall apply induction to $(A', N(A'))$ and to $(A - A', B - N(A'))$. The Hall's condition holds for $G[A' \cup N(A')]$, thus $\exists M' - match$ saturating A'.
    Assume that Hall's condition fails in $G'' \coloneqq G[A - A') \cup B - N(A')]$. This means $\exists S \subseteq A - A'$ so that $|N_{G''}(S)| < |S|$. Consider $S' = S \cup A'$.
    $$|N_G(S')| = |N(A') \cup N_{G''}(S)| = |A'| + |N_{G''}(S)| < |A'| + |S| = |A' \cup S| = |S'|$$
    Which is a contradiction. Thus Hall's condition holds for G'', $\exists M'' - matching$ saturating $A - A'$ in G''. Then $M' \cup M''$-matching saturates A in G.
    Qed.

DEFINITION: (HALL'S CONDITION): A bipartite graph with parts A and B satisfies Hall's condition with respect to A if $\forall S \subseteq A: |N(S)| \geq |S|$, where $N(S) = \{b: b \in B, ba \in E \text{ for some } a \in S\}$.

4 Problem class 1

TODO Anne aufschrieb

Claim 3: If $d = 1$: $g(Q_d) = \infty$ and if $d \geq 2$: $g(Q_d) = 4$.

Proof: Q_1 is acyclic, and the girth of any acyclic is defined to be $\infty \Rightarrow g(Q_1) = \infty$.

Fix $d \geq 2$. I claim that Q_d has no triangle (i.e. a cycle of length 3). Suppose it does. For $v \in \{0,1\}^d$, let

$$|v| = \quad 1's \text{ in } v$$

. Suppose wlog (without loss of generality) that $|x|$ is odd. What can you say about $|y|$?: It must be even. Then also, $|z|$ must be odd. But then, $|x|$ is even, which is a contradiction.

(More generally, there are no odd cycles in Q_d.

Are there 4 cycles? Yes, the cycle $(00 \dots 0, 010 \dots 0, 1100 \dots 0, 1000 \dots 0, 00 \dots 0)$.

Task 2: For any tree T, T has at least $\Delta(T)$ leaves.

Solution:

By induction on the order of T. If T has at most 2 vertices, the assumption is true.

Suppose $n > 2$ is given and the result holds for all trees of order <n. Let T be a tree with $|T| = n$. We know from the lecture that T contains a leaf, say v, and that $T' = T - v$ is a tree.

Let u denote v's unique neighbor in T. We have that $\Delta(T') \in \{\Delta(T), \Delta(T) - 1\}$.

By induction hypothesis, T' has at least $\Delta(T')$ leaves.

Case 1: $\Delta(T') = \Delta(T)$. The leaves of T' or the leaves of T (Except possibly the vertex u). We get $\geq \Delta(T') - 1 + 1$ (-1 because we possibly delete u, +1 because v is a leaf in T) $= \Delta(T') = \Delta(T)$ many leafs.

Case 2: $\Delta(T') = \Delta(T) - 1$. This can only happen if $d_T(u) = \Delta(T)$ and u is the only such vertex.

Suppose $\Delta(T') \geq 2$. We know that u cannot be a leaf in T'. We get $\geq \Delta(T') + 1$ (+1 because v is a leaf in T) $= \Delta(T) - 1 + 1 = \Delta(T)$ leafs. Otherwise $\Delta(T') = 1$. From the base case, there are $\Delta(T') + 1$ leafs.

We get $\geq \Delta(T') + 1 - 1 + 1$ (-1 from u, +1 from adding v back)$= \Delta(T) - 1 + 1 = \Delta(T)$.

In all cases T has $\geq \Delta(T)$ leafs.

Task 3: Prove that for any graph G, either G or $\bar{G}$ is connected.

Solution:

Suppose that G is not connected. G has connected components $C_1, C_2, \dots, C_t, t \geq 2 \text{ each } C_i \neq \emptyset$.

Pick some vertices $u, v \in V(G)$.

- If they belong to distinct components, they are connected in $\bar{G}$ ($uv \in E(\bar{G})$.
- If they belong to the same component, there is a vertex in another component w with edges $uw, wv \in E(\bar{G})$.

Task 4: Prove that any graph has a vertex partition such that for any vertex $\geq \frac{1}{2}$ of its neighbors belong to the other set.

lbach@outlook.de
lukasbach.com

Idea: Find partition that maximizes the number of edges between the vertex sets, if that partition does not fulfill the requirement, proof that there is a better partition.

Solution:

Let G be a our graph and choose a partition X, Y of $V(G)$ so that it maximizes $e(X,Y) = |\{xy\} \in E(G): x \in X, y \in Y\}|$.

Claim: This partition satisfies the desired property. Suppose not, there is a vector $x \in X$ such that $d_Y(x) = |N(x) \cap Y| < \frac{1}{2} d_G(x)$.

Consider the partition of $V(G)$ given by $X \backslash \{x\}, Y \cup \{x\}$. This means there are $d_Y(x)$ cross edges lost and $D_G(x) - d_Y(x)$ cross edges gained.

Net gain in the new partition Y is $\left(d_G(x) - d_Y(x)\right) - d(Y\ (x) = d_G(x) - 2d_Y(x) > 0$, i.e. $e(X\backslash\{x\}, Y \cup \{x\}) > e(X,Y)$ which is a contradiction, because we have chosen X,Y to maximize that equation. Qed.

For practice: A tournament is an orientation of the complete graph (We choose directions on all edges). Show that any tournament contains a directed path through all vertices.

4.1 Hypergraphs

A hypergraph H is a pair (X, E) where $E \subset 2^X$ (2^X as powerset of X).

A hypergraph is r-uniform if $|e| = r\ \forall e \in E$.

Note: A graph is a 2-uniform hypergraph.

We sometimes encounter problems in this more general setting, but these problems have natural graphs associated to them.

$H(X, E)$ as hypergraph. Consider the incidence graph of H: Have vertice sets E and X. For $e_i \in E$ and $x \in X$, $e_i \sim x \Leftrightarrow x \in e_i$.

Example: Let G be a graph with $\delta(G) \geq 2$. Show that there is a connected graph G' with the same degree sequence as G.

Solution: Apply induction on the number of connected components.

If there is one component, G is already connected, so done.

Let $n > 1$ be given and suppose the result is true for all graphs with <n components.

Let G be given with n components, $\delta(G) \geq 2$. We know that since $\delta(G) \geq 2$, that $\delta(G[C_1]), \delta(G[C_2]) \geq 2$.

(Any graph G has a cycle of length $\delta(G) + 1$ (if $\delta(G) \geq 2$) by considering a longest path in G)

Fix cycles C, C' in C_1, C_2 respectively.

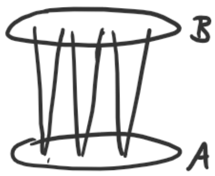
~~Let e an edge in C_1 and e' be an edge in C_2.~~

There is an edge e in C, e' in C' such that $G[C_1] - 1, G[C_2] - e'$ are both still connected. (Why? Consider spanning tree in each component).

From a new graph G' by removing e and e' and connecting C_1 and C_2 on the now open vertices (add edges $e_x e'_x, e_y e'_y$). The degree sequence is preserved. Number of components drops by 1 so we can apply induction. Qed.

5 Lecture 25.10.

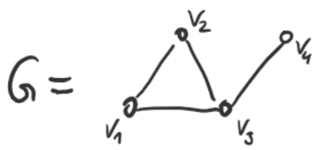
Hall's Condition:



$\forall S \subseteq A; |N(S)| \geq |S|$.

Definition: A vertex cover in a graph G is a set of vertices intercepting every edge of G.

Example:



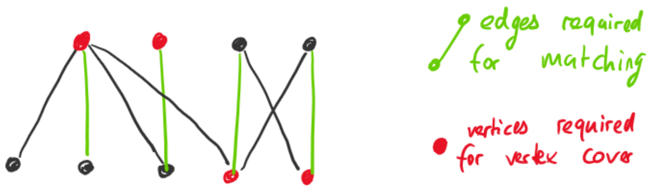
$\{v_1, v_3\}$ - vertex cover
$\{v_1, v_2, v_3\}$ - vertex cover

Let $c(G)$ be the size of a smallest (minimum) vertex cover. Example: $c(G) = 2$.

Let $m(G)$ be the size of a maximum vertex matching[1].

König's Theorem (1931): If G is a connected bipartite graph, then $C(G) = m(G)$.

Example:



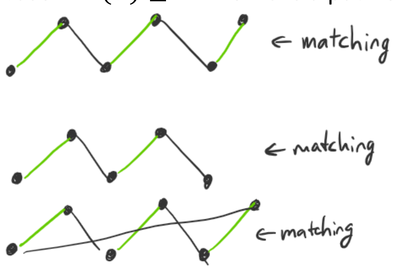
Proof (by Romeo Rizzi 1999): We shall show $c(G) \leq m(G)$ and $c(G) \geq m(G)$.

- $c(G) \geq m(G)$: True since $\forall e \in E(M)$, M-max matching e contains a vertex of a vertex cover.
- $c(G) \leq m(G)$:

[1] maximum = largest size, not maximal, maximal = could not be enlarged.

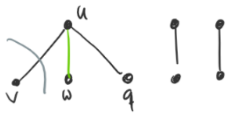
lbach@outlook.de
lukasbach.com

- Case 1: $\Delta(G) \leq 2$. Then G is a path or an even cycle.



$$m(P_{2n}) = n = c(P_{2n})$$
$$m(P_{2n-1}) = n = c(P_{2n-1})$$
$$m(C_{2n}) = n = c(C_{2n})$$

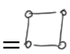
- Case 2: $\Delta(G) \geq 3$. Let G be a minimal counterexample in its number of edges. Let $\deg(u) \geq 3, v \in N(u)$.
  Case 2.1: $m(G - v) < m(G)$.



By minimality of G, $m(G - v) = c(G - v)$. Thus, $\exists X - vertex - cover$ of $G - v$, $|X| = c(G - v) = m(G - v) < m(G)$. Then $X \cup \{v\}$ will be a vertex cover of G of size $\leq m(G)$.
Case 2.2: $m(G - v) = m(G)$. ∃matching M of G of size $m(G), v \notin V(M)$. Let $uw \in E(M), q \in N(u)\{v, w\}$. Let $f = uq$. ∃ a vertex cover W of $G - f$ of size $|M| = m(G)$ (by minimality of G). Then W contains only vertices of M. Thus $v \notin W$, since $v \notin V(M)$. W must contain u. Thus W covers f as well, thus it covers G. Thus $c(G) \leq m(G)$.
Qed.

Definition: q(G):=number of odd components of G, i.e. components with odd number of vertices.

Example: $q(G_1) = 2, q(G_2) = 1, q(G_3) = 0$.

$G_1 =$ , $G_2 =$ , $G_3 =$ 

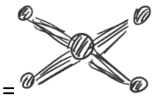
TUTTE'S MATCHING THEOREM: A graph G has a perfect (spanning) matching ⇔ $q(G - S) \leq |S| \ \forall S \subseteq V(G)$.

Proof:

- ⇒: Let G have a M-perfect match. Consider $S \subseteq V(G)$.

∀ odd component Q of $G - S$, ∃ an edge of M "from" Q "to" S. Thus $|S| \geq q(G - S)$.
- ⇐: Assume $q(G - s) \leq |S| \ \forall S \subseteq V$, but G has no perfect matching.
  Claim 1: $|V(G)|$ is even.

Indeed, take $S := \emptyset$, $q(G - \emptyset) \leq 0$, i.e. G has no odd components. Claim 1 qed.
Let $G'$with $G \subseteq G'$ so that G' has no perfect match but adding any edge to G' creates such.

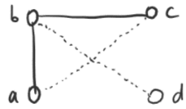
We shall show that G'=  (center component where each vertex is fully connected with all vertices of all other components).

Let $S := \{v \in G' : \deg_{G'}(v) = |V(G')| - 1\}$, i.e. set of vertices of "full" degree.

Claim 2: Each component of $G' - S$ is complete. If not, ∃

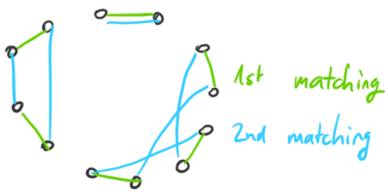
component in $G' - S$ with [(distance a→b exactly 2)] subgraph.

Since $b \notin S$, $\deg(b) < |V(G')| - 1$, i.e. $\exists d \in V: db \notin$



$E(G')$. I.e.. By maximality of G', $G' \cup \{ac\}$ has a perfect matching M_1, $G' \cup \{bd\}$ has a perfect matching M_2.

Consider $M_1 \cup M_2$:



It has components, even cycles and edges.
Case 1: ac, bd belong to different cycles of $M_1 \cup M_2$.
Let $ac \in E(C)$, let $M = M_2[V(C)] \cup M_1[V - V(C)]$. M is a perfect matching of G', which is a contradiction.
Case 2: $ac, bd \in E(C)$, C is a cycle of $M_1 \cup M_2$. $C \cup \{ac, bd\}$ has a perfect matching $\tilde{M}$, then $M = \tilde{M} \cup M_1[V - V(C)]$ is a perfect match of G', which is a contradiction.
Assume $q(G' - S) \leq |S|$. Since $|V(G')|$ is even and G' has special structure, G' has a perfect matching, this is a contradiction.
Thus $Q(G' - S) > |S|$. Therefore $q(G - S) \geq q(G' - S) > |S| \Rightarrow q(G - S) > |S|$, which is a contradiction.
$q(G - S) \geq q(G' - S)$ holds because: Components of $G - S$subgraphs of component of $G' - S$.

Summary for Tutte's theorem: G has a perfect matching ⇔ $\forall S \subseteq V(G), q(G - S) \leq |S|$.

If G-regular bipartite graph with parts $A, B \Rightarrow |A| = |B|$.

Proof: $E(G) = k \cdot |A|$ if G-k-regular. $E(G) = k \cdot |B|$. Qed.

lbach@outlook.de
lukasbach.com

6 Lecture 28.10.

TODO MARKIERUNGEN

Hall's theorem: For a bipartite $G = (A \cup B, E)$ it holds that: $\exists$matching saturating $A \Leftrightarrow \forall S \subseteq A: |N(S)| \geq |S|$.

(Reminder) Tutte's theorem: For a graph $G = (A \cup B, E)$, it holds that: $\forall S \subseteq V(G): g(G - S) \leq |S|$, where g denotes the number of contained odd components.

Idea of classical proof of Königs theorem: Consider a maximum matching M. An alternating path starts in $A \backslash V(M)$ and alternates between edges of M and $E(G) \backslash E(M)$.

Construct a vertex cover U as follows: $\forall a, b \in E(M), a \in A, b \in B$, pick b to be in U if $\exists$ an alternating path ending in b. Otherwise pick a.

Abbx1

Corollary 1 of Hall's theorem: If G is a bipartite graph with parts A and B, $\forall S \subseteq A, |N(S)| \geq |S| - q, q \in \mathbb{N}$.

Then G has a matching of size at least $|A| - q$.

Proof: Let G' be as follows:

Abbx2

Let X be a set of vertices, $|X| = q, X \cap A = \emptyset, X \cap B = \emptyset. V(G') = V(G) \cup X, E(G') = E(G) \cup \{ax: a \in A, x \in X\}$.

$$|N_{G'}(S)| \geq |N_G(S)| + q \geq (|S| - q) + q = |S|$$

$\Rightarrow$ by Hall's theorem, G' has a matching of size $|A|$. In this matching at most q edges are incident to X, the rest is in G. Thus G has a matching of size $\geq |A| - q$. Qed.

6.1 Colorings in graphs

A vertex coloring is a map $c: V(G) \to \{1, 2, \dots, k\}$. A coloring is proper if $c(u) \neq c(v)\ \forall uv \in E(G)$. The chromatic number $\chi(G)$ is the minimal number of colors in a proper coloring of G.

Example: Abbx3

Example: Bipartite graphs G always have $\chi(G) = 2$, by coloring each partition with one color.

The function $c: E(G) \to \{1, 2, \dots, k\}$ denotes an edge coloring of G. It is proper if $c(e) \neq c(e')$ for adjacent edges e, e'. The chromatic index or edge chromatic number $\chi'(G)$ is the minimum number of colors in a proper edge coloring of G.

Example: Abbx4

Example: $\chi'(K_{3,3}) = 3$.

A color class is a set of objects of the same color. In a proper edge coloring, each color class is a matching. In a proper vertex-coloring, each color class is an independent set (induces empty graph).

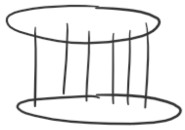
Abbx5

Note: If $\chi(G) = k$, then G is k-partite, i.e. a subgraph of a complete k-partite graph.

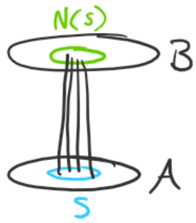
Corollary 2 (of Hall's Theorem): If G is a k-regular bipartite graph, then $\chi'(G) = k$.

Proof: Induction on k.

- $k = 1$: For graph G which looks like:



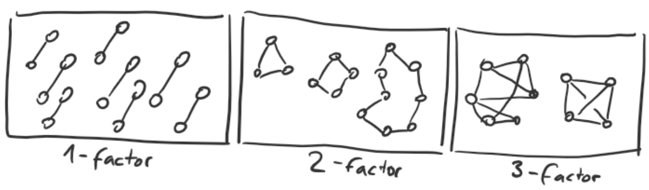
  G is matching, $\chi'(G) = 1$.
- $k \to k + 1$: G is $k + 1$-regular $\Rightarrow$ it has parts A, B with $|A| = |B|$.
  Claim: G has a perfect matching.
  Apply Hall's Theorem (we want $|N(S)| \geq |S|\ \forall S \subseteq A$).



  Fix $S \subseteq N(S)$. $e = |S| \cdot (k + 1)$ ("count from A"), $e \leq |N(S)|(k + 1) \Rightarrow |S|(k + 1) \leq |N(S)|(k + 1) \Rightarrow |S| \leq |N(S)|$.
  G has an perfect matching M. Assign all edges and M to the same color, apply induction to $G - E(M)$ (k-regular graph).

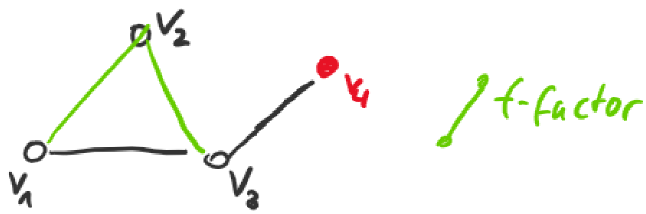
Qed.

DEFINITION: A perfect matching is a 1-factor. A k-factor is a k-regular spanning subgraph.



If $f: V(G) \mapsto \{0,1, \dots\}$, we say that $H \subseteq G$ is an f-factor if $d_H(v) = f(v)$. $f(v_1) = 1, f(v_2) = 2, f(v_3) = 1, f(v_4) = 0$

Example:

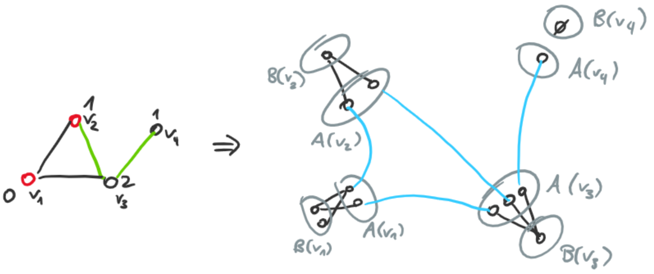


lbach@outlook.de
lukasbach.com

Claim: Graph G, $f: V(G) \mapsto \{0,1,\dots\}, f(v) \leq \deg_G(v)$. There is a graph G' such that G' has a pefect matching if and only if G has an f-factor.

Construction of G':$V(G')$-pairwise-vertex disjoint union of sets $A(v), B(v), v \in V(G)$. $|A(v)| = \deg_G(v)$, $|B(v)| = \deg_G(v) - f(v)$. $A(v), B(v)$'s induce no edges, there are all edges between $A(v)$ and $B(v)$ $\forall v$, $\bigcup_{v \in V} A(v)$ induces a matching such that $\exists$ unique edge between $A(u)$ & $A(v) \Leftrightarrow uv \in E(G)$.

Example:



f	v_1	v_2	v_3	v_4
	0	1	2	1

G has f-factor $\Leftrightarrow$ G' has 1-factor.

Proof: See image.

Definition H-factors: H as given graph, G as graph, $|V(G)|$ is divisible by $|V(H)|$. We say that a spanning subgraph of G is an H-factor if all its components are isomorphic to H.

Example: TODO Abb6

Hajnal & Szemeredi Theorem (1970): If n is divisible by k and $\delta(G) \geq \left(1 - \frac{1}{k}\right) n \Rightarrow$ G has a K_k-factor, $n = |V(G)|$.

7 Problem class 30.10.

…

8 Lecture 04.11.2019

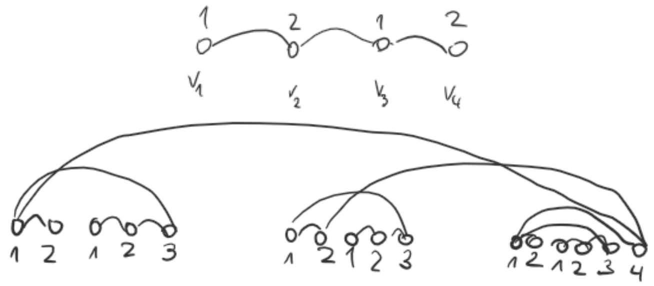
Last time:

- Perfect matchings
- Tuttes theorem
- k-factors, f-factors
- H-factors
- $\chi(G) \coloneqq min$ colors on $V(G)$ s.t. adjacent vertices $g \in T$ have different colors.

8.1 Facts on Colorings

Given a graph G and an ordering of its vertices $v_1, \dots, v_n$, we say that a vertex coloring c is greedy if it uses colors from $\{1, \dots\}$, colors $v_1, v_2, \dots$ in order, uses smallest available color on v_i, i.e. the smallest color is not on $N(v_i) \cap \{v_1, \dots, v_{i-1}\}$.

Examples:



Claim: For any graph G: $\chi(G) \leq \Delta(G) + 1$ with $\Delta(G)$ denoting the maximum degree of G.

Proof: Use greedy coloring.

Examples: $\chi(K_n) = n = (n-1) + 1, \chi(C_{2k+1} = 3 = 2 + 1$.

8.2 H-factors

Theorem Hajnal-Szemeredi (1970): If k divides $|V(G)|$ and $\delta(G) \geq \left(1 - \frac{1}{k}\right) |V(G)| \Rightarrow G$ has a K_k-factor.

Theorem Kühn-Osthns (2009): For graphs H, G, $|V(H)|$ divides $|V(G)|$ and $\delta(G) \geq \left(1 - \frac{1}{\chi^*(H)}\right) n + C$ for $C = C(H)$, then G has an H-factor, and $\chi^*(H) \in \{\chi(H), \chi_{cz}(H)\}$, $\chi_{cz}(H) = \frac{\chi(H)-1}{|V(H)| - \delta(H)} \cdot |V(H)|$, where $\sigma(H)$ is the size of a smallest color class in a proper coloring of H.

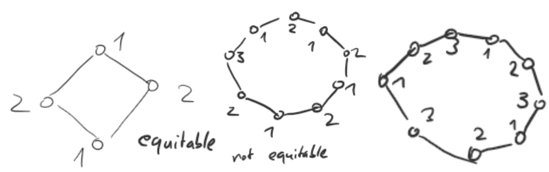
$$\chi(H)[-1]?? \subseteq \chi_{cz}(H) \subseteq \chi(H)$$

If $|V(G)|$ is divisible by $2k+1$ and $\delta(G) \geq \left(1 - \frac{1}{2 + \frac{1}{k}}\right) n + c \Rightarrow$ G has an C_{2k+1}-factor.

Corollary: $|V(H)|$ divides $|V(G)|$ and $\delta(G) \geq \left(1 - \frac{1}{\chi(H)}\right) |V(G)| + c \Rightarrow \exists$H-factor in G.

Definition: A proper coloring of a graph is equitable if color classes differ in size by at most 1.

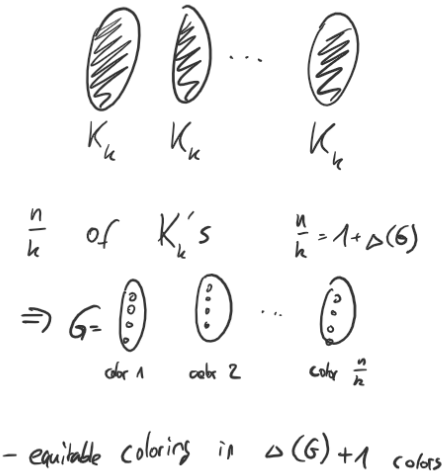
Example:



Corollary of Hajnal-Szemeredis Theorem: If $|V(G)|$ is divisible by $\Delta(G) + 1$, then there is an equitable coloring of G in $(\Delta(G) + 1)$ colors.

Proof: Let G with $(\Delta + 1)$ divisible by $|V(G)| = n$. Then $\delta(\bar{G}) \geq n - 1 - \Delta(G) = n\left(1 - \frac{1}{k}\right)$ with $\bar{G}$ being the complement of G. $(1 + \Delta)k = n$.

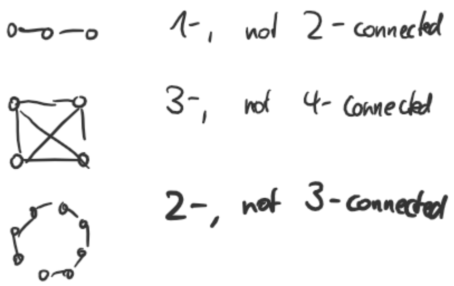
$\Rightarrow$by Hajnal-Szemeredis Theorem, $\exists K_k$-factor in $\bar{G}$.



8.3 Connectivity

Definition: A graph G is k −CONNECTED, $k \in \mathbb{N}$ if $|V(G)| \geq k+1$ and $G - X$ is connected $\forall X \subseteq V(G), |X| \leq k-1$.

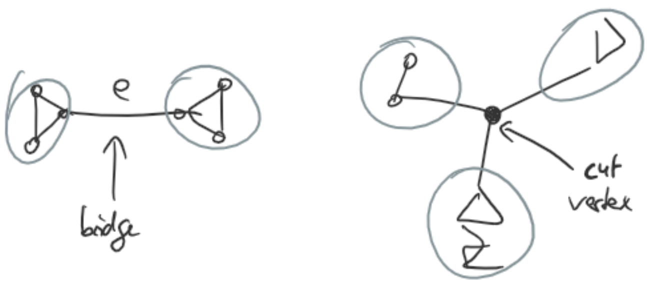
Example:



Definition: $k(G) \coloneqq \max\{k: G \text{ is } k\text{-connected}\}$ is called CONNECTIVITY of G. $k(K_n) = n-1, k(C_n) = 2, k(tree) = 1$.

Definition: $X \subseteq V(G) \cup E(G)$ is cut-set if $G - X$ has more components than G. If $X = \{v\}, v \in V$, v is called CUT-VERTEX. If $X = \{e\}, e \in E$, e is called BRIDGE.

Example:



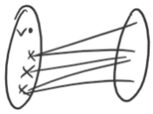
Definition: EDGE-CONNECTIVITY. $K'(G) = \max\{e: G \text{ } is \text{ } e - edge - connected\}$, G is E-EDGE-CONNECTED if $\forall E' \subseteq E$, $|E'| \leq e-1, G - E'$ is connected.

Example: $K'(C_n) = 2, K'(K_n) = n-1$.

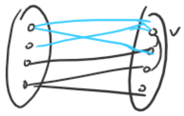
Lemma: $\forall G: k(G) \leq K'(G) \leq \delta(G)$.

Proof: $K'(G) \leq \delta(GG)$? All edges incident to a vertex of minimum degree form an edge-cut, thus $K'(G) \leq \delta(G)$. $k(G) \leq K'(G)$? Note $k(K_n) = n-1 = K'(K_n)$. Assume G is not complete. Consider $F \subseteq E(G)$, $|F| = K'(G)$, F-cutset. Want to find vertex-cut of size $|F|$.

- Case 1: $\exists v \in V(G)$ not incident to F.



  Let A be a connected component of $G - F$ containing v. Endpoints of F in A form a vertex-cut.
- Case 2: $\forall v \in V(G)$: v is incident to F. Let v be a vertex of degree less than $|V(G)| - 1$, exists since G is not completed. Claim that $N(v)$ forms a cut of size $\leq |F|$.



  Consider edges of F incident to $N(v) \cap A$, A is an vertex-set of connected components of $G - F$ containing v and consider $N(v) \backslash A$, each of these is incident to F (distinct edges respectively), i.e. $|N(v)| \leq |F|$.

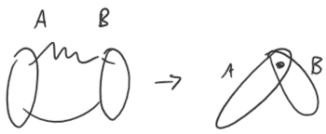
Qed.

8.4 Planar graphs

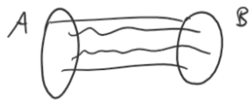
Preperation for Mengers theorem:

Let $p = x_1, x_2, \dots, x_n$ be a path, then $P_{x_i} \coloneqq x_1, x_2, \dots, x_i$, $x_i P \coloneqq x_i x_{+i1} \dots x_n$, $x_i P_{x_j} \coloneqq x_i, x_{i+1}, \dots, x_{j-1}, x_j$. Let F be a set of graphs, and $V(F)$ the set of all vertices in these graphs.

$A, B \subseteq V(G)$, an $A, B - path$ is a path with one endpoint in A, another in B and no other vertices in $A \cup B$.



Notation: $p(A_1 B)$ ≔largest member of pairwise vertex-disjoint A-B-paths.



9 Lecture 8.11.

TODO Abbildungen

$A, B \subseteq V(G)$, a A,B-path in P is a path such that one endpoint is in A and another in B, and no other vertices are in $A \cup B$.

For $A, B \subseteq V(G)$, define $p(A, B) \coloneqq \max$ #pairwise vertex-disjoint A-B-paths. Also $s(A, B) \coloneqq \min$ #vertices that separate $A \& B$.

Definition: $X \subseteq V \cup E$ SEPERATES $A\&B$ if $\forall A - B - path \text{ } P$ it holds that P contains an $e|t$ (?) of X.

If P is a family of paths, we define $Ends(P)$ ≔set of endpoints of paths in P.

Theorem (MENGER'S): $\forall graph \text{ } G, \forall A, B \subseteq V(G), p(A, B) = s(A, B)$.

Proof: Assume that $A \cap B = \emptyset$. Let P be a family of pairwise vertex-disjoint $A-B$-paths, $|P| = p(A,B)$. Observe that each vertex-A-B-seperator must have at least one vertex from each $p \in P$. [ABB]. Thus $s(A,B) \geq p(A,B)$.

Now we want to show that $s(A,B) \leq p(A,B)$.

Claim: If P is any set of *less* than than $s(A,B)$ pairwise-vertex-disjoint A-B-paths, then there exists a family Q of pair.v.d.A-B-paths with $|Q| > |P|$ and $Ends(P) \subseteq Ends(P)$.

Abb

Proof of claim: Fix G, fix A, apply induction on $|G| - |B|$. Basis: $|B| = |G| - |A|$. Apply Königs Theorem. $p(A,B)$ ≔size of largest A-B-match. $s(A,B)$ ≔ size of minimum vertex cover. Step: Assume that Claim holds for $\forall B: |G| - |B| < q$. Let $|G| - |B| = q$. (Idea: Abb) Let P be our $A-B$-path-family, $|P| < s(A,B)$. $\exists R$, an $A-B$-path not containing $V(P) \cap B$. (Abb)

Case 1: R is disjoint from P. $Q := P \cup \{R\}$,then the claim is proofed.

Case 2: Not Case 1: Let x be last vertex in R that is in P. Let $p \in P$ such that $x \in V(P)$. Let $B' = B \cup V(xp) \cup V(xR)$. Let $P' := P \backslash \{p\} \cup \{px\}$. $|P'| = |P| < s(A,B) \leq s(A,B') \Rightarrow |P'| < s(A,B')$, $|G| - |B'| < |G| - |B|$. By induction $\exists$ a path system $Q': |Q'| > |P'|$, $Ends(Q') \geq Ends(P')$, Q'-pairw.-vert.disj.A-B'-paths.

Abb

Let $y \in B, y \in Ends(Q') \backslash Ends(P')$. Let $q, q' \in Q'$ with endpoints x, y, respectively.

Abb (3 cases)

Case 1: $Q := Q' \backslash \{q\} \cup \{q \cup xp\}$

Case 2: $Q := Q' - \{q, q'\} \cup \{q \cup xp\} \cup q' \cup yR\}$

Case 3: $Q := Q' \backslash \{q, q'\} \cup \{q \cup xR\} \cup \{q' \cup yp\}$

Qed.

Corollary: For $a, b \in V(G), ab \notin E(G)$. Minimum number of vertices separating a and b = max number of independent a-b-paths, where "independent" means sharing only endpoints.

Proof: Apply Mengers Theorem to $N(a)$ and $N(b)$. Qed.

Global version of Mengers Theorem: A graph G is k-connected $\Leftrightarrow \forall a \neq b \in V(G) \exists k$ independent a-b-paths.

Proof:

- $\Rightarrow$: Let $\kappa(G) \geq k$. Then $|V(G)| \geq k+1$. Pick $a, b \in V(G)$, assume $\exists \leq (k-1)$ independent a-b-paths.
  - Case 1: $ab \notin E(G)$. We have $\leq (k-1)$ pairwise vertex-disjoint $N[a] - N[b]$-paths. ($N[a] := N(a) \cup \{a\}$, "closed Neighbourhood"). By Menger $\exists \leq (k-1)$ vertices separating a and b, which is a contradiction (Corollary also shows that).
  - Case 2: $ab \in E(G)$. Let $G' = G - \{ab\}$. There are $\leq (k-2)$ vertex-disjoint $N[a] - N[b]$-paths. By Menger there exists $X \subseteq V(G')$ s.t. $|X| \leq k-2$, X seperates $N[a]$ and $N[b]$ in G'. $|V(G)| \geq k+1 \Rightarrow \exists v \in V(G) \backslash (X \cup \{a,b\})$. Then X seperates v from either a or b in G'. Assume that X seperates v from a in G'. Then $X \cup \{b\}$ seperates v from a in G. Since $|X \cup \{b\}| \leq k-1$, which is a contradiction.
- $\Leftarrow$: Deleting <k vertices does not destroy all independent a-b-paths for any vertices a,b; thus $\kappa(G) \geq k$.

Qed.

Definition: G is **k-linked** if $\forall$set X of 2k vertices and any labeling of vertices in X: $s_1, \dots, s_k, t, \dots, t_k$, $\exists$pairwise-vertex-disjoint $s_i - t_i - paths, i = 1, \dots, k$.

Theorem* (Thomas-Wollan 2005): G is 10k-connected $\Rightarrow G$ is k-linked.

For edge-connectivity, apply Menger to line-graph L(G) of G. The Line-graph is defined as $L(G) := \left(E(G), \{\{e, e'\}; e, e' are\ adjacent\ in\ G\}\right)$.

Theorem (Berheke): G is a line graph if it does not contain any of the following as induced subgraphs: Abb (Siehe Buch)

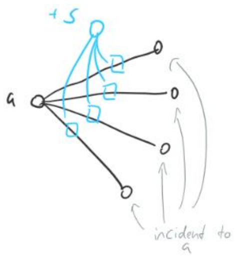
10 Lecture 11.11.

Corollary 1: If G is a graph then

- The min. number of edges separating a from b is equal to the max. number of pairwise edge-disjoint a-b-paths
- G is k-edge-connected $\Leftrightarrow u, v \in V(G)$ $\exists k$ edge-disjoint u-v-paths.

Proof (Sketch):

- Given G, two vertices $a, b \in V(G)$. Consider the graph G', first consider $L(G)$, add two new vertices s, t and join s to all edges incident to a, join t to all edges incident to b.



  We know that the min. number of vertices in $V(G')\{s,t\}$ separating s from t is equal to the max. number of pairwise vertex-disjoint s-t-paths in G'.
- Second statement directly follows from first.

Question: Is there a "simple" procedure which constructs all k-connected graphs for $k \geq 2$?

What about 2-connected graphs? A cycle is 2-connected.

lbach@outlook.de
lukasbach.com

Definition: Let H be a graph. An **H-path** is a path P that meets H exactly in its endpoints. ($E(P) \cap E(H) = \emptyset$).

An **Ear Decomposition** of G is a sequence

$$G_0 \subseteq G_1 \subseteq \cdots \subseteq G_k$$

such that

G_0 is a cycle

For each $i = 1, \dots, k$, $G_i$is obtained from G_{i-1} by adding a G_{i-1}-path to G_{i-1}.

$G_k = G$.

Theorem 1: A graph G is 2-connected if and only if G admits an Ear Decomposition.

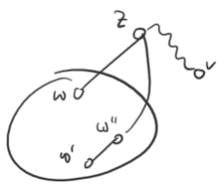
Proof: Let's suppose that G has an Ear Decomposition $G_0 \subseteq G_1 \subseteq \cdots \subseteq G_k = G$. G_0 is a cycle, so it is 2-connected.

Assume G_i is 2-connected, $i \geq 0$. Any cut-vertex of G_{i+1} is on P. But any vertex of P is contained in a cycle in G_{i+1}, so cannot be a cut-vertex.

Assume G is 2-connected. G must have a cycle C. Let H be the largest subgraph of G that can be built from C via Ear Decomposition. First, notice that H is induced: if $xy \in E(G) \backslash E(H), x, y \in V(H)$. Then xy defines an H-path, which contradicts the maximality of H.

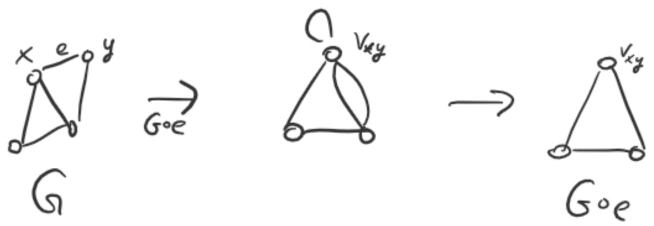
Aim: H=G.

Suppose not, $\exists v \in V(G) \backslash V(H)$. By connectivity, $\exists zw \in E(G)$ such that $z \in V(G) \backslash V(H), w \in V(H)$. As G is 2-connected, $G - w$ is connected, there is a $z - w'$ path P avoiding w. Let w'' be the first vertex of P in H. But this defines an H-path, which contradicts the maximality of H. So H=G. Qed.



Question: Is there a simple procedure for building 3-connected graphs? Yes!

Lemma 1 (Tutte): Suppose G is 3-connected and $G \neq K_4$. Then $\exists e \in E(G)$ such that $G \circ e$ is also 3-connected.



Proof: Suppose not. Then $\forall e \in E(G)$ it holds that $G \circ e$ has a 2-cut. Let $xy \in E(G)$, let $v_{xy} \in V(G \circ xy)$ be the vertex that x,y are identified with. Let S be a 2-cut in $G \circ xy$. Then S contains v_{xy} (o.w. (?) get a 2-cut in G) and it contains some other vertex z.

Then $\{x, y, z\}$ defines a 3-cut in G.

Let C be the smallest component in $G \circ \{x, y, z\}$. Pick $xy \in E(G)$, z, and C such that $|C|$ is minimized. Every vertex in S has ≥ 1 neighbor in every component. Let $v \in C$ be a neighbor of z.

$G \circ zv$ has a 2-cut, defines a 3-cut {z,v,w} in G. v has neighbors in C'. However, $N(v) \subseteq C \cup \{x, y, z\} \Rightarrow C \cap C' \neq \emptyset \Rightarrow C' \subseteq C$. C' does not contain v, so $|C'| < |C|$, contradicting minimality(?).

Theorem 2 (Tutte): G is 3-connected if and only if there is a sequence $G_0 \subseteq G_1 \subseteq \cdots \subseteq G_k$ such that

- $G_0 = K_4$.
- For each $x, y \in V(G_i)$ such that $d(x), d(y) \geq 3$ and $G_{i-1} = G_i \circ xy$.
- $G_k = G$.

Proof: Suppose G is 3-connected, if $G \neq K_4$ then apply Lemma 1 to G to find an edge xy with $G \circ xy$ 3-connected.

(Note: $d(x), d(y) \geq 3$, as G is 3-connected)

The number of vertices drops by 1 at each stage, so we repeat until we reach $|G_0| = 4$; as $\delta(G_0) \geq 3, G_0 \cong K_4$.

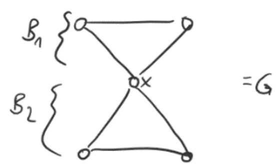
The other direction is left to the reader.

Can every graph be decomposed into maximal 2-connected subgraphs? Not quite.

Example:

Definition: Let G be a graph. A **block of G** is a maximal subgraph with no cut-vertex (with respect to the subgraph, not the whole graph G).

Example:



X is a cut-vertex of G, but not of B_1, B_2.

lbach@outlook.de
lukasbach.com

Observation: If B is a block of G, then either it is a maximal 2-connected subgraph or $|V(B)| \leq 2$. B is either an edge or an vertex.

Proposition 1: If B_1, B_2 are blocks that intersect, then $|V(B_1) \cap V(B_2)| = 1$.

Proof: Suppose $V(B_1) \cap V(B_2) = \{x, y\}$. We know that $\forall v \in B_1 \cup B_2$, $B_1 - v, B_2 - v$ are connected. But then, $B = B_1 \cup B_2$, $B - v = (B_1 - v) \cup (B_2 - v)$ is connected, because $B_1 - v, B_2 - v$ intersects in ≥ 1 vertex. So $B_1 \cup B_2$ has no cut-vertex and this contradicts the maximality of B_1.

11 Problem class 13.11.

Let G' be a bipartite graph with parts A and B. Suppose $|N(S)| \geq |S| \; \forall S \subseteq A$.

Suppose there is no matching covering A.

Question: At most how many independent s-t-paths can there be? There are $< |A|$ of these since there is no matching covering A. By Menger, $\exists S \subseteq A \cup B$ separating s from t. Write $S_A = S \cap A, S_B = S \cap B$. So we have $S(A \backslash S_A) \subseteq S_B$. Then $|N(A \backslash S_A)| \leq |S_B| < |A| - |S_A| = |A \backslash S_A|$, which contradicts Halls theorem.

12 Lecture 15.11. TODO

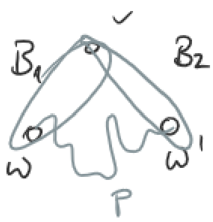
TODO eigener Aufschrieb

Last time: Let G be a graph. A block in G is a maximal connected subgraph with no cutvertex.

Proposition: Any two distinct blocks intersect in ≤ 1 vertex.

Proposition: If B_1, B_2 are distinct blocks in G and $V(B_1) \cap V(B_2) = \{v\}$, then v is a cutvertex of G.

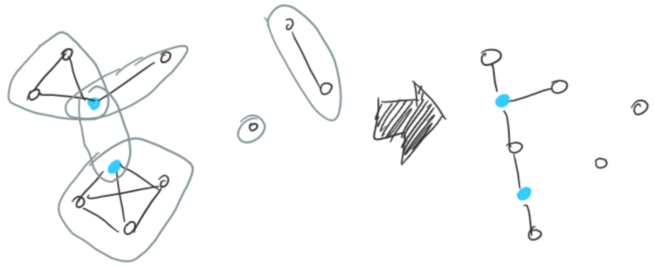
Proof:



If v is not a cutvertex, obtain a $w - w'$ $-$path p in $G - v$. p contains a B_1-path, say Q. But $B_1 \cup Q$ is a larger block than B_1, contradicting the maximality of B_1.

We know that every edge of G lies in a unique block.

Example:



Definition: A Block-cutvertex graph of G is a bipartite graph with partitions B and C, where

C = cut-vertices of G

B = blocks of G

Join $c \sim B$ if and only if $c \in B$.

Theorem: If G is connected, then its block-cutvertex-graph is a tree.

Hint: Suppose it was acyclic and consider a shortest cycle.

Theorem (Mader): If G is a graph with average degree $\geq k$, then G has a subgraph that is k-connected.

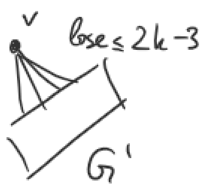
Proof: $k = 1$ works. We prove the following by induction on $n = |G|$.

$(*)$: $n \geq 2k - 1$ and $||G|| \geq (2k-3)(n-k+1)+1$, then G has a k-connected subgraph.

(Check: $(*)$ implies Mader.)

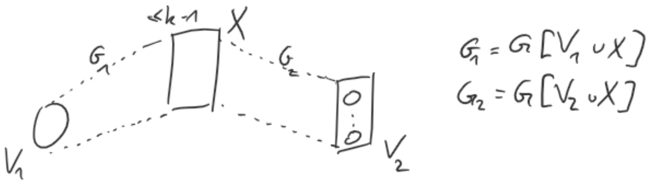
Base case: $n = 2k - 1$, calculations show that $G = K_{2k-1}$, so it's a tree.

Let's assume $n \geq 2k$, and the result holds for smaller value sof n. If $\exists v \in V(G)$ $d_G(v) \leq 2k - 3$, consider $G' := G - v$, apply induction hypothesis to G'.



So we may assume that $d(v) \geq 2k - 2 \;\; \forall v \in V(G)$. If G is k $-$connected, then we are done. Otherwise $\exists X \subset V(G), |X| \leq k - 1$ such that $G - X$ has ≥ 2 components.

lbach@outlook.de
lukasbach.com



By the minimum degree condition, $|G_1| \geq 2k-1, |G_2| \geq 2k-1$. So if $||G_i|| \geq (2k-3)(|G_i| - k + 1) + 1$ for some $i \in \{1,2\}$, then apply our Induction Hypothesis to G_i. Otherwise $||G|| \leq ||G_1|| + ||G_2|| \leq (2k-3)[|G_1| + |G_2| - 2k + 2] = (**)$.

We have $n \geq |G_1 \cup G_2| = |G_1| + |G_2| - \overbrace{|G_1 \cap G_2|}^{\leq k-1} \Rightarrow |G_1| + |G_2| \leq n + (k-1)$.

$$(**) \leq (2k-3)[n + (k-1) - 2k + 2]$$
$$= (2k-3)[n - k + 1]$$

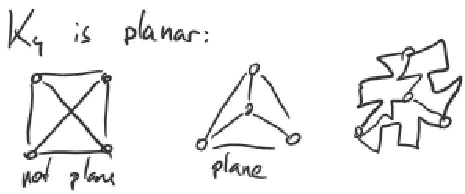
Which is a contradiction. Qed.

12.1 Planar graph

Motivation: A graph drawn in a plane without crossing edges is a **PLANE GRAPH**.

A graph is **PLANAR** if it can be drawn in such a way.

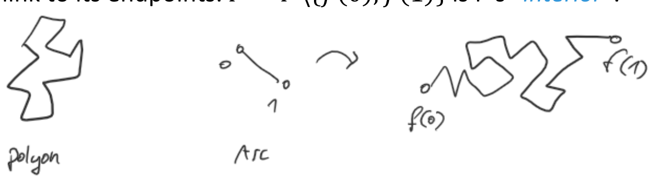
Example:



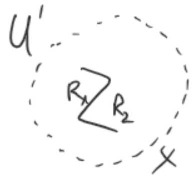
Question: When can you "tell" that an abstract graph is planar?

12.2 Topological Definitions

- $p, q \in \mathbb{R}^2$, the straight line segment between them is $\{(1-\lambda)p + \lambda q : \lambda \in [0,1]\}$.
- Let $A, B \subseteq \mathbb{R}^2$. We say they are homomorphic if $\exists f: A \to B$ such that f is a bijection, and f, f^{-1} are continuous [B can be obtained from A by continuously deforming A].
- A polygon $P \subseteq \mathbb{R}^2$ is a union of finitely many line segments, which is homeomorphic to $S' \coloneqq \{x \in \mathbb{R}^2 : ||x|| = 1\}$.
- An arc is a union of finitely many line segments homeomorphic to $[0, 1]$.
  $f(0), f(1)$ are the endpoints of the arc. This arc is said to link to its endpoints. $\dot{P} = P \setminus \{f(0), f(1)\}$ is P's "*interior*".



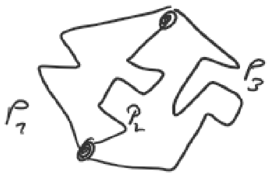
- Let $U \subseteq \mathbb{R}^2$ be an open set.
  $\Rightarrow$ being linked by an arc in U is an equivalence relation on U.
  $\Rightarrow$ Equivalence classes we call "regions" (they are also open)
- Closed set $X \subset \mathbb{R}^2$ seperates a region U' of U, if $U' \setminus X$ has more than one region.



- Frontier of a set $X \subset \mathbb{R}^2$ is the set $Y = \{y \in \mathbb{R}^2$: Every neighbourhood of y intersects X and $\mathbb{R}^2 \setminus X\}$.

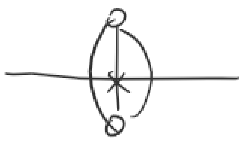
Theorem (Jordan Curve for Polygons): For any polygon $P \subset \mathbb{R}^2$, $\mathbb{R}^2 \setminus P$ has exactly two regions, both of which have P as frontiers.

Lemma ("Three paths lemma"): Let P_1, P_2, P_3 be internally disjoint arcs with the same endoints.



Then

1. $\mathbb{R}^2 \setminus \{P_1, P_2, P_3\}$ has 3 regions with frontiers $P_1 \cup P_2, P_2 \cup P_3, P_3 \cup P_1$.
2. If P is an arc between the interior in $\dot{P}_1$ and $\dot{P}_3$ that runs through the region containing P_2, then $\dot{P}_1 \cap \dot{P}_2 \neq \emptyset$.

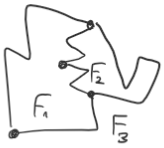


Definition: A **PLANE GRAPH** is a pair (V, E) of sets such that

- $V \subset \mathbb{R}^2$
- Every edge in E is an arc between two vertices.
- Different edges have different sets of endpoints ("no multiple edges")
- The interior of an edge contains no vertex and no point of any other edge.

A plane graph (V, E) defines a graph G on V in the obvious way ("G", "(V,E)").

Suppose G is a plane graph. $\mathbb{R}^2 \setminus G$ is open, and we call its regions the *faces* of G.

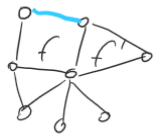


➔ The unbounded face is the *outer face* of G.
➔ The other faces are the *inner faces*.
➔ $F(G)$ denotes the set of faces in G.
➔ A planar embedding of an abstract graph $G = (V, E)$ is a bijection $f: V \to V'$, where $G' = (V', E')$ is a plane graph such that $uv \in E(G)$ if and only if there is an arc in G' between $f(u), f(v)$.
"G'" is a *drawing* of G.

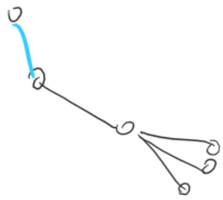
G is planar if it has a planar embedding.

Lemma: Let G be a plane graph, $e \in E(G)$. Then

1. If X is the frontier of a face of G, either $e \subseteq X$ or $\dot{e} \cap X = \emptyset$.



2. If e lies on a cycle $C \subseteq G$, then e lies on the frontier of exactly two faces, and these are contained in distinct faces of C.
3. If e lies on no cycle, then e lies on the frontier of exactly one face.



Corollary: The frontier of a face f is always the point-set of a subgraph.

[TODO letzte abbildung aus eigenem Aufschrieb]

13 Lecture 18.11.

Today: Eulers formular, corollaries minors, topological minors, Kuratowski's theorem.

- Planar graphs: Graphs
- Plane graphs: Not graphs ($V \subseteq \mathbb{R}^2, \neg E \subseteq \mathbb{R}^2 \times \mathbb{R}^2$, edges are not pairs of vertices but arcs!)

Goal:

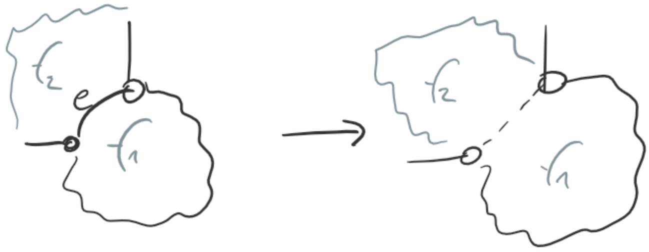
- Kuratowski's theorem – structural character
- Euler's theorem - #vertices, edges, faces
- $\chi(G) \leq 5, \chi_l(G) \leq 5$

We use the notation $\#vertices = n, \#edges = m, \#faces = f$.

Theorem (**Euler**): Let G be a connected plane graph, then $n(G) - m(G) + f(G) = 2$.

Proof: Induction on m.

- Basis: $m = n - 1$, then G is a tree, thus there are no cycles, thus there is exactly one face. Then $n - m + f = n - (n-1) + 1 = 2$.
- Step: $(m-1) \to (m)$: Let $|E(G)| = m$ and stat. (?) holds for all plane graphs with at most $m - 1$ edges. Since $m \geq n$, G is not a tree, thus there exists a cycle, $C \subseteq G$. Let $e \in E(C)$, e is on the boundary of f_1, f_2 –distinct faces of G.



Let G' be obtained from G by deleting e.
Then $n(G') = n(G), m(G') = m(G') - 1, f(G') - 1$, the set of faces of G' is equal to the set of faces of G minus $\{f_1, f_2\} \cup \{f_1 \cup f_2 \cup \dot{e}\}$.
We have $n(G) - m(G) + f(G) = n(G') - (m(G') + 1) + (f(G') + 1) = n(G') - m(G') + f(G') = 2$ by induction. Qed.

Corollary 1: A plane graph on n vertices has at most $3n - 6$ edges. (Equality is achieved for triangulations). ($n(G) \geq 3$)

Proof: Let G be a plane triangulation. Let $X = \{(f, e): f - face\ of\ G, e - edge\ on\ the\ boundary\ of\ f\}$.

Then $|X| = \sum_{f\ is\ a\ face\ of\ G} 3 = 3 \cdot f(G)$, $|X| = \sum_{e\ is\ an\ edge\ of\ G} 2 = 2 \cdot m(G)$. Thus $3f = 2m, f = \frac{2m}{3}$. Plug in $n - m + f = 2 \Rightarrow n - \frac{m}{3} = 2$. Qed.

Corollary 2: If G is a bipartite plane graph, then $m(G) \leq 2n(G) - 4$. ($n(G) \geq 4$)

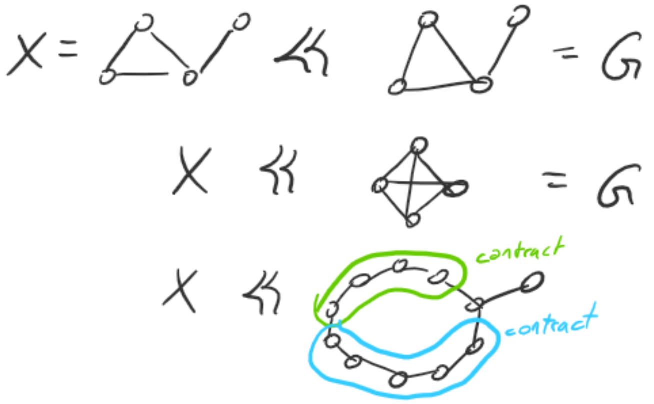
Proof (Outline): Note that each face has a length of at least 4.

13.1 Minors

(Notation: G denotes a large graph, X denotes a small graph)

X is a **minor** of G if X is obtained from G by edge deletions, contractions or vertex deletions, write $X \ll G$.

Examples:



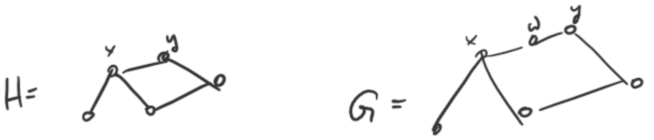
$X \ll G$ (assume connectivity) if $V(X) = \{x_1, \ldots, x_n\}, V(G) = X_1 \dot\cup \ldots \dot\cup X_n$ ($\dot\cup$ indicates disjoint union or partition) such that $G[X_i]$ is connected for $i = 1, \ldots, n$ if $x_i x_j \in E(X) \Rightarrow \exists$an edge between $X_i \& X_j$.

$X_1, \ldots, X_n$ are called branch sets. Example:

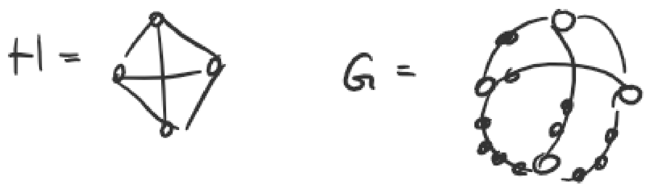
We say: $X \ll G$, X is a minor of G, G is an X-minor, G contains X as a minor, $G = MX$.

13.2 Topological minors

A graph G is obtained by a **SINGLE EDGE-SUBDIVISION** from graph H if $V(G) = V(H) \cup \{w\}, w \notin V(H); E(G) = E(H) \setminus \{xy\} \cup \{xw, wy\}$.



G is a **SUBDIVISION** of H if G is obtained by a series of edge-subdivisons from H. We write $G = TH$.



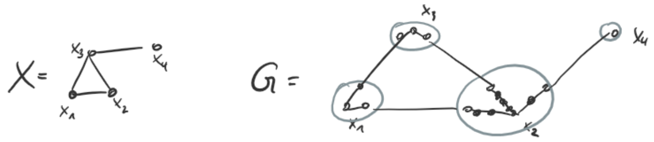
X is a **TOPOLOGICAL MINOR** of G if $TX \subseteq G$, i.e. if G contains a subdivision of X as a subgraph.

Note: $G = TX \Rightarrow G = MX, G = MX \neg\Rightarrow G = TX$.

KURATOWSKI'S THEOREM (Wagner's theorem): The following statements are equivalent.

- F is a class of planar graphs
- F is a class of graphs with no MK_5 $MK_{3,3}$
- F is a class of graphs with no $TK_5, TK_{3,3}$

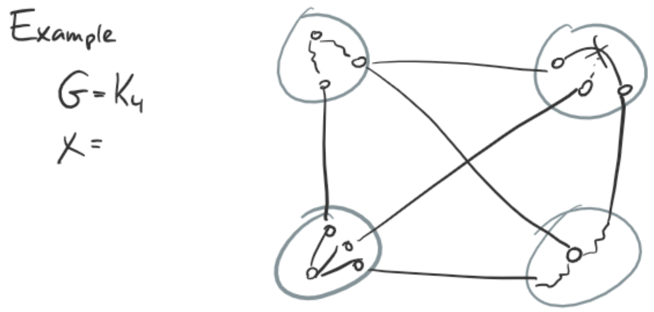
Obseration: Let $G = MX$ such that G is edge-minimal with respect to this property. If $X_1, \ldots, X_n$-branch sets of G, $G[X_i]$ is a tree and there is exactly one edge between X_i and X_j if $x_i x_j \in E(X)$ and no edges between X_i and X_j if $x_i x_j \notin E(X)$.



Moreover $G[X_i]$ either has one vertex or it has $\leq \deg_X(x_i)$ leaves.

Observation: If X-graph, $\Delta(X) \leq 3$ and $G = MX$ then $TX \subseteq G$.

Proof outline: Consider $G' \subseteq G$ such that $G' = MX$, G'-minimal.



Let H_i be the union of $G'[X_i]$ and all edges leaving X_i for a branch set X_i. Since $G[X_i]$ is a tree with at most 3 leaves, H_i is a spider with at most 3 legs. Then $w_1, \ldots, w_n$ form the branch vertices of TX. Qed.

14 Lecture 22.11.

Theorem (Kuratowski): The following statements are equivalent:

- G is planar
- $MK_{3,3}, MK_5 \neg\subseteq G$
- $TK_5, TK_{3,3} \neg\subseteq G$

Last time:

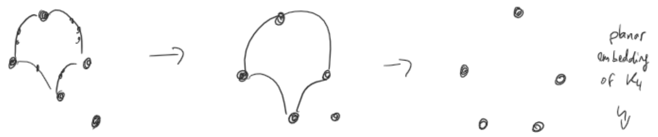
- Observation 1: $G = MX$, G is edge-minimal and $X_1, \ldots, X_n$ are branch sets of G corresponding to $\{X_1, \ldots, X_n\} = V(X)$, then $G[X_i]$ are trees and there is exactly one edge between X_i and X_j if and only if $x_i x_j \in E(X)$.
- Observation 2: $G = MX, \Delta(X) \leq 3$, then $TX \subseteq G$.
- Observation 3: $TK_5 \subseteq G$ or $TK_{3,3} \subseteq G \Rightarrow$ G is not planar
- Lemma 37: TK_5 or $TK_{3,3} \subseteq G \Leftrightarrow MK_5$ or $MK_{3,3} \subseteq G$
- Lemma 38: $k(G) \geq 3$ and $MK_5, MK_{3,3} \neg\subseteq G \Rightarrow$ G is planar

- Lemma 40: $|G| > 4$ and G is edge maximal with regards to not having $TK_5, TK_{3,3} \Rightarrow k(G) \geq 3$.

Proof for Observation 3:

Note that $K_5, K_{3,3}$ are not planar since Eulers formular ($n - m + f = 2$) fails. We had $m \leq 3n - 6$ (planar), $m \leq 2n - 4$ (planar no triangle). In K_4: $n = 5, m = 10, 3n - 6 = 9$, which is a contradiction.

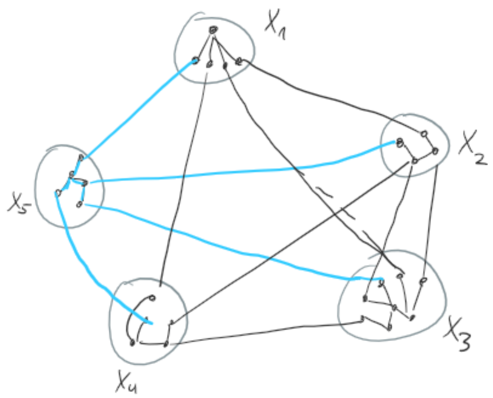
In $K_{3,3}$: $n = 6, m = 9, 2n - 4 = 8$ which is a contradiction. Thus $TK_5, TK_{3,3}$ are also not planar, otherwise embed TK_5, take a union of edges on a path between branch vertices to form an arc. (merge subdivisions).



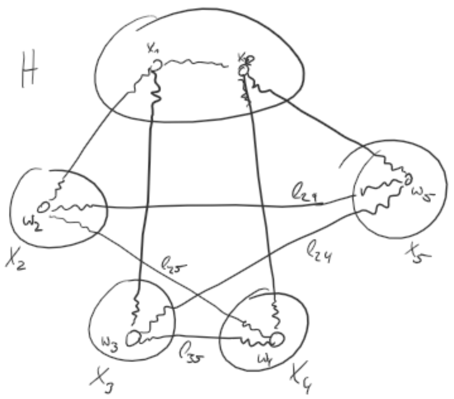
We end with a planar embedding of K_4, which is not possible and thus a contradiction. Qed.

Proof for Lemma 37:

- "⇒": Let G with $TK_5 \subseteq G$ or $TK_{3,3} \subseteq G$. We know that $TX = MX$.
- "⇐": Let G with $MK_5 \subseteq G$ or $MK_{3,3} \subseteq G$. If $MK_{3,3} \subseteq G$ then $TK_{3,3} \subseteq G$ since $\Delta(K_{3,3}) = 3$ (Observation 2). If $MK_5 \subseteq G$, let G' be edge-minimal, $G' \subseteq G$. $G' = MK_5$ with branch sets $X_1, \ldots, X\_5$.



Let H_i be a graph consisting of all edges incident to X_i. $H_i \in \{TW, TW'\}$.
Case 1: All $H_i's$ are TW then $G' = TK_5$.
Case 2: Without loss of generality $H_1 = TW'$.

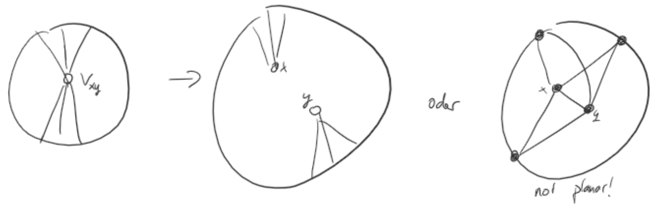


Let e_{ij} be an edge between X_i and X_j, $i \in \{2,3\}, j \in \{4,5\}$. Let $w_i, i = 2,3,4,5$ be in X_i such that there are three independent paths from x_i to e_{ij} and neighbor of x_0 or x_1. Then $\{x_0, w_2, w_3\} \cup \{x_1, w_4, w_5\}$ form branch vertices of $TK_{3,3}$.

Qed.

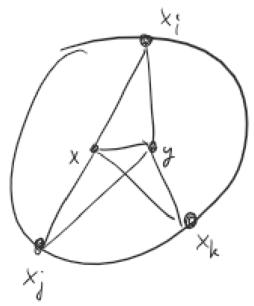
Proof of Lemma 38:

Induction on $|V(G)|$. Basis $|V(G)| = 4$ (any such graph is planar since K_4 is planar). Assume $|V(G)| > 4$. By Tutte's Lemma $\exists edge\ e = xy$ such that $k(G' = G \circ e) \geq 3$. Since $MK_5 \neg\subseteq G'$ and $MK_{3,3} \neg\subseteq G'$ and $|V(G')| < |V(G)|$ by induction, G' is planar. Consider plane embedding of G' with v_{xy} being the vertex obtained by contracting $e = xy$. Let's "uncontract" v_{xy}.

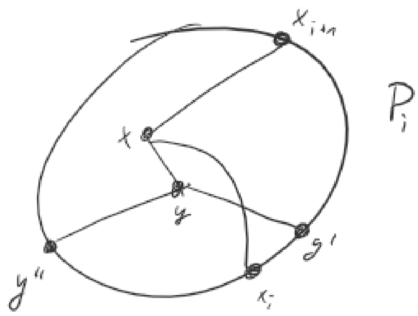


Let the face of $G' - v_{xy}$ be bounded by a cycle C, $N(v_{xy}) \cap V(C) = \{x_0, x_1, \ldots, x_{m-1}\}$. Let $Y = N(y) \cap V(C)$. Let P_i be a path on C from x_i to $x_{i+1} (mod\ m)$.

- Case 1: $|Y \cap X| \geq 3$.

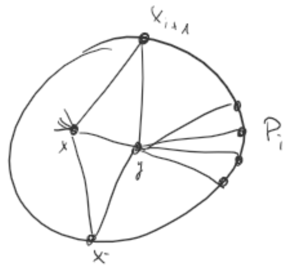


Then there exists TK_5 with branch vertices x, y, x_i, x_j, x_k with $x_i, x_j, x_k \in X \cap Y$, which is a contradiction.
- Case 2: $\exists y' \in Y \cap V(\dot{P}_i), y'' \in Y \cap (V(C) \setminus V(P_i))$ (with $\dot{P}_i$ being the interior of P_i)



There is $TK_{3,3}$ with branch vertices $\{y, x_i, x_{i+1}\} \cup \{x, y', y''\}$.
- Case 3: $Y \subseteq V(P_i)$ for some i.

lbach@outlook.de
lukasbach.com



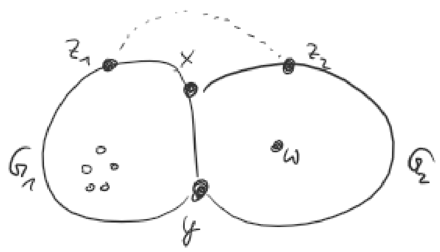
This actually creates a planar embedding of G.

Qed.

Lemma 39: Let X be a graph with $k(X) \geq 3$, G is edge-maximal with respect to not containing TX. Let $S \subseteq V(G), |S| \leq 2$ be a vertex-cut, $G = G_1 \cup G_2, V(G_1) \cap V(G_2) = S$. Then $\forall i = 1,2$ G_i is edge-maximal with respect to $TX \neg\subseteq G_i$ and S contains an edge.

Proof of Lemma 40:

Induction on $|V(G)|$. If $|V(G)| = 4 \Rightarrow G = K_4, k(K_4) = 3$. Let $|V(G)| > 4$. Assume $K(G) < 3$, i.e. $\exists$ vertex-cut $S', |S| \leq 2$. Let $S = \{x, y\}$, $G = G_1 \cup G_2, V(G_1) \cap V(G_2) = S$. We know that $TK_5, TK_{3,3} \neg\subseteq G_i, i = 1,2 \Rightarrow MK_5, MK_{3,3} \neg\subseteq G_i, i = 1,2$ (implies Lemma 37). By Lemma 39, G_i - edge-maximal does not contain $TK_5, TK_{3,3}$ and $xy \in E(G)$. By induction $k(G_i) \geq 3, i = 1,2$. By Lemma 38, G_i is planar, $i = 1,2$. Consider planar embeddings of G_i such that $e = xy$ is an unbounded face.



Case 1: $H \in \{TK_5, TK_{3,3}\}, H \subseteq G \cup z_1z_2$. $\exists$ at most two internally disjoint path of H outside of G_i. Then $G_i \cup$ these paths $\{TK_5, TK_{3,3}\} \ni H \subseteq$(planar graph of two touching cycles where one path on one cycle goes through the other cycle).

Case 2: $V(G_1)\backslash V(G_2)$ and $V(G_2)\backslash V(G_1)$ contain branch vertices of H. Since $k(K_5), k(K_{3,3}) \geq 3$, we see that $H \neq K_5$. (We cannot pass subdivided edges of K_5 through z, z_2, x, y). Thus $H = TK_{3,3}$ and exactly one branch vertex, say $w \in V(G_2)\backslash V(G_1)$. There are 3 internally disjoint $w - G_1$ paths in H. But $G_1 \cup$ these paths $\supseteq H = TK_{3,3}$.

Proof of Kuratowskis theorem:

1. G is planar
2. $G \neg\supseteq MK_5, MK_{3,3}$
3. $G \neg\supseteq TK_5, TK_{3,3}$

$2 \Leftrightarrow 3$: Lemma 37

$1 \Rightarrow 3$: Observation 3

$2 \Rightarrow 1$: $G \neg\supseteq MK_5, MK_{3,3}$. Add edges preserving this property and get G'. Lemma 40 and 37 implies $k(G') \geq 3$. Lemma 38 $\Rightarrow$ G' is planar, then $G \subseteq G'$ is also planar.

Qed.

15 Lecture 25.11.

[TODO incomplete, see Anne]

Note 1: If G is maximally plane, then $k(G) \geq 3$.

Note 2: If G is plane, $k(G) \geq 2$, then each face is bounded by a cycle.

Note 3: If G is planar then $\exists v \in V(G) : \deg(v) \leq 5$.

Proof: By Eulers theorem: $|E(G)| | \leq 3|V(G)| - 6$. $|E(G)| = \sum_{v \in V(G)} \deg(v) \Rightarrow \sum_{v \in V(G)} \deg(v) \leq 6|V(G)| - 12 < 6|V(G)| \Rightarrow \exists v \in V(G) : \deg(v) < 6$, i.e. $\deg(v) \leq 5$.

Fary's theorem: If G is planar, $k(G) \geq 3$, then G can be embedded in the plane such that all edges are straight line segments.

Proof outline: Induction on $|V(G)|$.

Base case: $|V(G)| = 4 \Rightarrow G = K_4$.

Step: Let $v \in V(G) : \deg(v) \leq 5$. Let $G' := G - v\ (+)\ triangle$. Apply Induction to G'. We have $k(G') \geq 2$

Abb1

Apply induction to G' to get that a face cont. (?) v in G is a polygon with at most 5 vetices. Insert v such that it can be joined to the corners by straight lines segments.

Qed.

Definition: *Poset – partially ordered set*. Let X be a set, a relation "$\leq$" is a subset of $X \times X$. Example: $X := \{1,2,3\}, \leq = \{(1,2), (3,2)\}$. A relation "$\leq$" is a partially ordered set if it is reflexive, antisymmetric and transitive.

- Reflexive: $x \leq x$
- Antisymmetric: $x \leq y \wedge x \neq y \Rightarrow y \nleq x$
- Transitive: $x \leq y \wedge y \leq z \Rightarrow x \leq z$

Example: "$\leq$"$= \emptyset, X = \{1,2,3\}$

Example: "$\leq$" is a total order or a chain if $x \leq y$ or $y \leq x\ \forall x, y = X$.

The *incidence poset* $(V \cup E, \leq)$ on a graph $G = (V, E)$ is given by $v \leq e \Leftrightarrow e$ is incident to v $\forall v \in V, e \in E$.

15.1 Cover Relation diagram

TODO

lbach@outlook.de
lukasbach.com

15.2 Dimension of a poset P

TODO

Schnyders theorem: A graph G is planar $\Leftrightarrow \dim(P(G)) \leq 3$, with $P(G)$ denoting G's incidence poset.

4-color-theorem: If G is a planar graph, then $\chi(G) \leq 4$.

5-color theorem: If G is a planar graph, then $\chi(G) \leq 5$.

Proof: Induction on $|V(G)|$.

Basis: $|V(G)| \leq 5$, works.

Step: Assume the statement is true for any planar graph on less than n vertices. Consider G as planar graph with $|V(G)| = n$. Assume that G is maximally planar. Then $k(G) \geq 3$.

Let $v \in V(G)$ such that $\deg(v) \leq 5$. Consider a plane embedding of G and $N(v)$.

Abb2

Let $c: V(G - v) \to \{1, 2, 3, 4, 5\}$ be a proper coloring. It exists by induction.

Case 1: $|c(N(v))| \leq 4$. Let $c': V(G) \to \{1,2,3,4,5\}$. Set $c'(w) = c(w)$ if $w \neq v, c'(v) \in \{1,2,3,4,5\} \backslash c(N(v))$. Then c' is a proper coloring.

Case 2: $|c(N(v))| = 5$. Without loss of generality, $c(v_i) = i, i = 1, \dots, 5$.

Abb3

Let $G_{ij}'(w)$ be a connected component of a subgraph of G' spanned by colors $i \& j$ and containing w.

Observe: If $v_3 \notin V(G_{13}'(v_1))$, swap colors $i \& j$ in $G_{13}'(v_1)$, color v with 1. Thus $v_3 \in V(G_{13}(v_1)) \Rightarrow v_1 - v_3$ $-$path with colors 1&3 on its vertices.

Similarly we could assume that $\exists v_2 - v_4$ $-$path with colors 2&4 on its vertices.

Since this path does not share a vertex, there is an edge crossing, contradicting planarity. Qed.

Abb4

Wrong proof of Kempe: TODO

16 Problem class 27.11.

Proof techniques

1. Induction
2. Extremal Principle (Contradiction, …)
3. Counting
    a. Dable counting
    b. Pigeonhole Principle
    c. Parity arguments
4. Use a result from the lecture
5. Algorithmic/Iterative
6. Hopeless proof by contradiction

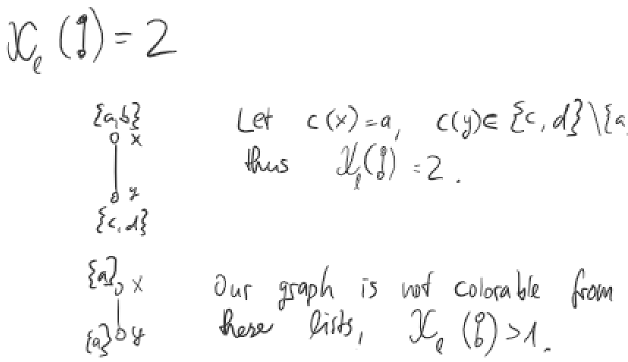
17 Lecture 29.11.

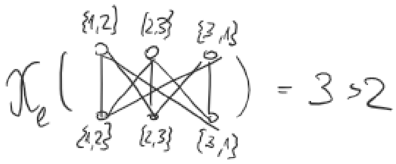
Today: $\chi_l(G) \leq 5$ for planar graphs G, Heawood formular, general coloring results (Brooks theorem).

For a graph G, the **CHOOSABILITY** $ch(G)$ or **LIST CHROMATIC NUMBER**, $\chi_l(G)$, is $\min\{k: \forall list\ assignment\ L: V(G) \to 2^{\mathbb{N}}, s.t. |L(v)| = k\ \forall v \in V(G)\ \exists coloring\ c: V(G) \to \mathbb{N}\ s.t. c\ is\ proper \wedge c(v) \in L(v)\ \forall v \in V(G)\}$

I.e. $\chi_l(G) = k$ if for any lists of size k assigned to vertices $\exists$ proper coloring from these lists and $\exists$ list assignment with lists of sizes $k - 1$ such that no proper coloring from these lists exists.

Example: $\chi_l(K_2) = 2, \chi_l(K_{3,3}) = 3$.



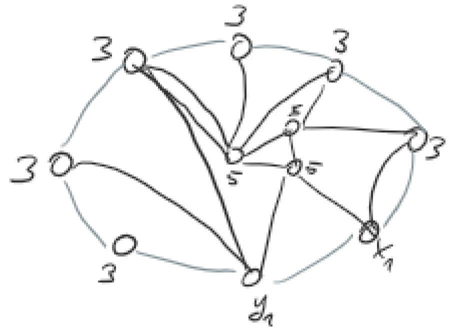


Note: $\chi_l(G) \geq \chi(G)$ since we can choose the lists to be {1, 2, …, k}. If $k < \chi(G)$ it means we cannot color from these lists.

Theorem (**THOMASSEN**): If G is planar, then $\chi_l(G) \leq 5$.

Note (**MIZRAKHANI**): $\exists$ planar graph G, $\chi(G) = 3, \chi_l(G) = 5 > 4$.

Proof of Thomassen: Assume that G is a inner triangulation, i.e. all bounded faces are triangles and its bounded face is bounded by a cycle.



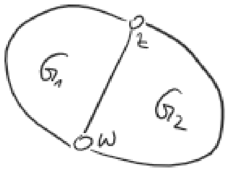
Stronger statement: Let C be the cycle bounding the unbounded face of G, $x, y \in V(C)$, $xy \in E(C)$. Let $L: V(G) \to 2^{\mathbb{N}}$ such that $|L(x)| = |L(y)| = 1, L(x) \neq L(y), |L(z)| = 3\ \forall z \in V(C)\backslash\{x,y\}, |L(v)| = 5$ otherwise.

Then G is colorable from these lists. We prove the stronger statement by induction on $|V(G)|$.

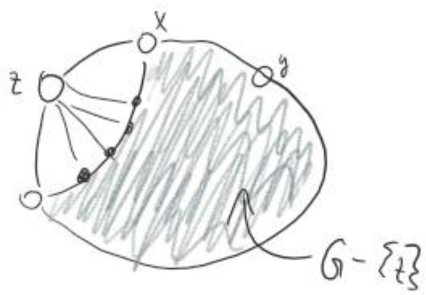
Basis: $|V(G)| = 3$, thus $G = K_3$ with $V(G) = \{x,y,z\}$. $L(x) = \{a\}, L(y) = \{b\}, L(z) = \{\cdot,\cdot,\cdot\}$. Let $c(x) = a, c(y) = b, c(z) \in L(z)\backslash\{a,b\}$.

Step: Assume $|V(G)| \geq 4$ and the statement holds for smaller graphs.

- Case 1: C has a chord.

  I.e. an edge between non-adjacent vertices (on C), i.e. $G = G_1 \cup G_2, V(G_1) \cap V(G_2) = \{z,w\}$, zw as the chord. $|G_1| < |G|$. Let w.l.o.g. $x, y \in V(G_1)$. By induction $\exists$ proper coloring c_1 of G_1 from L.
  Let $c(z) = \alpha, c(w) = \beta, \alpha \neq \beta$.
  Let $\alpha': V(G_2) \to 2^{\mathbb{N}}$ such that $L'(z) = \{\alpha\}, L'(w) = \beta, L'(v) = L(v)$ otherwise.
  By induction $\exists c_2: V(G_2) \to \mathbb{N}$ from lists L'. This is also an L-coloring. Let $c(v) := \{c_1(v), c_2(v) | v \in V(G_1), v \in V(G_2)\}$ be a proper coloring of G from lists L.
- Case 2: C has no chord.

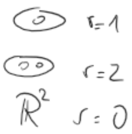
  

  Let $z \in \big(N(x) \cap V(C)\big)\backslash\{y\}$. Apply induction to $G' := G - \{z\}$.
  Let $L(x) = \{a\}, L(y) = \{b\}, L(z) = \{c, d, \star\}, c \neq a, d \neq a$.
  Let $L'(y) = \begin{cases} L(v)\{c,d\} & v \in N(z)\backslash V(C) \\ L(v) & otherwise \end{cases}$.

  By induction, $\exists c': \forall c(G') \to \mathbb{N}$ is a proper coloring from L'.
  Let $z' \in \big(N(z) \cap V(C)\big)\backslash\{x\}$.
  Let $c(z) \in \{c,d\}\backslash c'(z')$.
  Let $c(v) := c'(v), v \neq z$.
  Then c is a proper coloring of G from L, which proofs the stronger statements.

Qed.

> **Heawood formular** 1890: Let G be a graph. 2-cell-embeddable on a surface S with Euler characteristic $2 - 2\gamma$ ($\gamma = \#holes$)



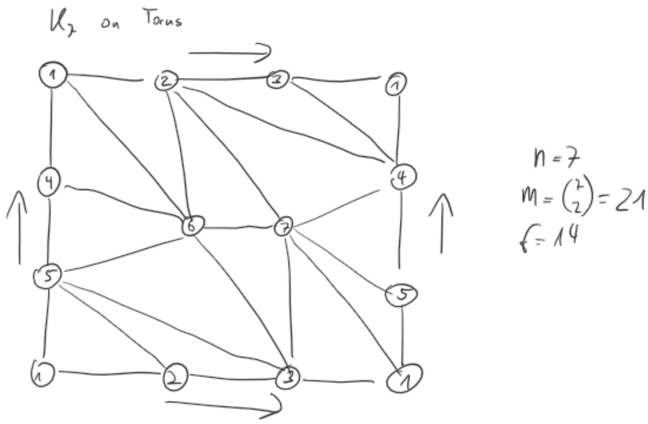
Let $\gamma > 0$, then $\chi(G) \leq \left\lfloor \frac{7+\sqrt{1+48\gamma}}{2} \right\rfloor =: f(\gamma)$.

Moreover $K_{f(\gamma)}$ is embeddable on S unless S is a Klein bottle ($f(\gamma) = 7, K_6$ *is embeddable*).

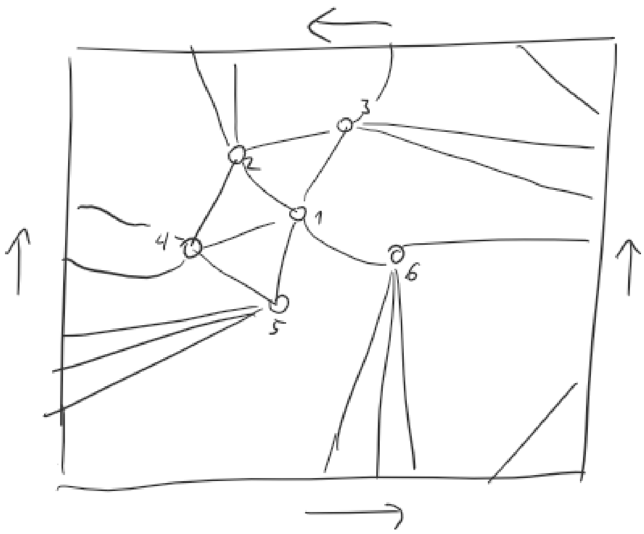
> Eulers formular: $n - m + f = 2 - 2\gamma$.

Examples:

- K_7 on a torus



- K_6 on a Klein Bottle



17.1 Colorings

Basic facts:

- $\chi(G) \leq \Delta(G) + 1$ (with greedy coloring)
- $\chi(G) \geq \omega(G)$ (ω being the clique number, i.e. the max k such that $K_k \subseteq G$)
- $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$ ($\alpha(G)$ being the indpeendence number, i.e. max $k: K_k \subseteq \bar{G}$.
- "Tight" examples:
$$\chi(K_n) = n = \Delta(K_n) + 1$$
$$\chi(C_{2n+1}) = 3 = 2 + 1 = (\Delta(C_{2n+1}) + 1)$$
$$\chi(K_n) = n = \omega(K_n)$$
$$\chi(K_{n,n}) = 2 = w(K_{n,n})$$
$$\chi(K_n) = n = \frac{n}{\alpha} = \frac{n}{1}$$
$$\chi(K_{n,n}) = 2 = \frac{2n}{2} = \frac{2n}{n}$$
- "Bad" examples:
$\chi(Star) = 2, \Delta = n$ if the star has n leafs.
$$\forall n \, \exists G: \omega(G) = 2, \chi(G) = n$$

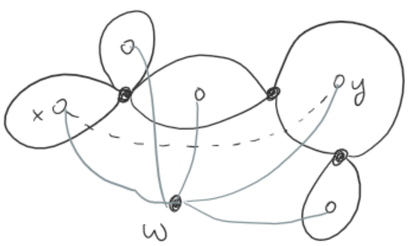
18 Lecture 02.12.

Observation 1: Let $G = G_1 \cup G_2, |V(G_1)| \cap |V(G_2)| = 1$, then $\chi(G) = \max\{\chi(G_1), \chi(G_2)\}$. Why? Color G_1 properly with $\chi(G_1)$ colors, color G_2 properly with $\chi(G_2)$ colors, permute colors in G_1 such that color of $v \in V(G_1) \cap V(G_2)$ is the same in both colorings.

Lemma 1: If G is a 2-connected graph, G is not a complete graph, $\delta(G) \geq 3$, then $\exists v, x, y \in V(G)$ such that $xy \notin E(G), xv \in E(G), yv \in E(G), G - \{x, y\}$ is connected.

Proof of Lemma 1: Let $w \in V(G)$ such that $\deg(w) \leq |V(G)| - 2$ (not a full degree).

- Case 1: $G - w$ has no cutvertex. Since $\deg(w) \leq |V(G)| - 2$, there is a vertex non-adjacent to w. Thus there exists a vertex y such that $dist(w, y) = 2$. Since y is not a cutvertex of $G - w$, $G - \{w, y\}$ is connected. Let $x := w, y := y, v :=$ a vertex in $N(x) \cap N(y)$.
- Case 2: $G - w$ has a cut-vertex. Note w is adjacent tp a vertex (that is not a cutvertex of $G - w$) in each leaf-block of $G - w$.



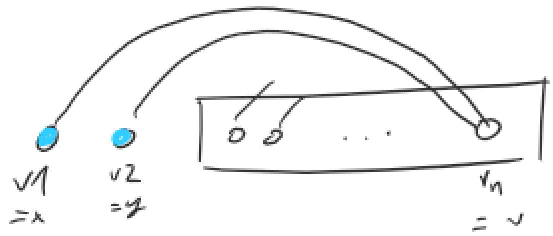
Let x, y be such neighbors of w in distinct leaf blocks, $v := w$. Note $(G - w) - \{x, y\}$ is connected. Since $\deg(w) \geq \delta(G) \geq 3$, there is a neighbor of w in $G - \{w, x, y\}$. Thus $G - \{x, y\}$ is connected.

Qed.

Brooks theorem: Let G be a connected graph, G is not a complete graph, and G is not an odd cycle, then $\chi(G) \leq \Delta(G)$.

Proof: Induction on $|V(G)| > 3$, assume that the theorem holds for graphs on less than $|V(G)|$ vertices.

- Case 1: G has a cutvertex v. $G = G_1 \cup G_2$, $V(G_1) \cap V(G_2) = \{v\}$.
If G_i is not clique, not an odd cycle, then $\chi(G_i) \leq \Delta(G_i)$ by induction.
If G_i is a clique or an odd cycle, then $\Delta(G_i) < \Delta(G)$. Then $\chi(G_i) \leq \Delta(G_i) + 1 \leq \Delta(G)$. In any case $\chi(G_i) \leq \Delta(G)$. Thus $\chi(G) \leq \max\{\chi(G_1), \chi(G_2)\} \leq \Delta(G)$.
- Case 2: $\Delta(G) \leq 2$. G is an even cycle or a path, $\chi(G) \leq 2$.
- Case 3: $\Delta(G) \geq 3, k(G) \geq 2$.
- Case 3.1: $\exists v \in V(G), \deg(v) \leq \Delta - 1$. Order vertices of G: $v_1, v_2, \dots, v_n$ such that $v_n = v$. Each vertex $v_i, i \neq n$, has a neighbor $v_j, j > i$. (we can do this by trimming leaves in a spanning tree with root v).
Use greedy coloring. The number of colored neighbors at each step is at most $\Delta - 1$, so there is Δ^{th} color available.
- Case 3.2: G is Δ-regular, i.e. $\Delta(G) = \delta(G) \geq 3$. $k(G) \geq 2$.
By Lemma 1, $\exists$graph {x,y,z} where $G - \{x, y\}$ is connected. Let us order $V(G)$ as $v_1, v_2, \dots, v_n: v_1 = x, v_2 = y, v_n = v$ such that $\forall v_i, 3 \leq i \leq n - 1 \, \exists v: j: v_i v_j \in E(G), j > i$.



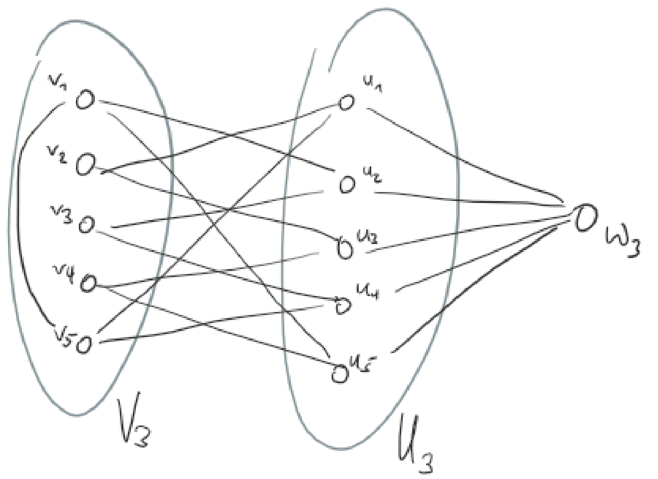
We can do this since $G - \{x, y\}$ is connected. Use greedy coloring. Since $c(v_1) = c(v_2)$, $|v(N(v))| < \Delta$, there is an available color for v.

Qed.

Mycielski's construction: Let $G_1, G_2, \dots$ be graphs: $G_1 = K_1, G_2 = K_2$, given $G_K = (V_K, E_K)$, construction $G_{K+1} = (V_{K+1}, E_{K+1}), V(G_{K+1}) = V_K \uplus U_K \uplus \{w_K\}$ ($\uplus$ being disjoint unions), $V_K = \{v_1, \dots, v_n\}, U_K = \{u_1, \dots, u_n\}$, $E(G_{K+1}) = E(G_K) \cup \{w_K u_i : u_i \in U_K\} \cup \{u_i v_j : v_i v_j \in E(G_K)\}$.

Claim: G_K has no triangles, $\chi(G_K) = K$.

lbach@outlook.de
lukasbach.com



Note: $\chi(G_K) \leq k$. True: G_1, G_2. To color G_K: color V_{k-1}, using $\leq k-1$ colors, mimic this in U_{K-1}, use new color in w.

Claim 1: $\omega(G_K) \leq 2$. Assume $\exists$triangle in $G_K, K \geq 3$. W is not in the triangle since $N(w)$ is an independent set. Inductively, the triangle is not induced by V_{k-1}. Let our triangle be $v_i v_j u_e$. Since $u_e v_i \in E(G_K) \Rightarrow v_e v_i \in E(G_K)$, similarly $v_e v_j \in E(G_K)$. Then $v_i v_j v_e$ form a triangle in G_{k-1}, which is a contradiction.

Claim 2: $\chi(G_K) \geq k$. By induction on k, $\chi(G_K) = k, k = 1, 2, 3$. Assume $k \geq 4$. Assume that $\chi(G_K) \leq k-1$. Wlog let $c: V(G_K) \to [k-1]$ be a proper coloring, $c(w_{k-1}) = k-1$. Idea: Take $S \coloneqq \{v \in V_{k-1} : c(v) = k-1\}$ recolor in colors from $[k-2]$.

Let $c': V_{k-1} \to [k-2]: c'(v_i) = \begin{Bmatrix} c(v_i) & otherwise \\ c(u_i) & u_i \in S \end{Bmatrix}$. Note: $k-1 \notin c(U_{k-1})$. So c' uses only colors from [k-2].

Claim: c' is a proper coloring of $G_{k-1} = G_k[V_{k-1}]$. Assume not, i.e. $c'(v_i) = c'(v_j), v_i \in S, v_j \notin S$. $v_i v_j \in E(G_k) \Rightarrow$ since u_i-twin of $v_i, u_i v_j \in E(G_k)$, $c(u_i) = c(v_j)$ which is a contradiction. Thus c' is a proper coloring of G_{k-1} with k-2 colors, a contradiction to the assumption $\chi(G_{k-1}) \geq k-1$. Qed.

18.1 Perfect graphs

A graph G is *perfect*, if for any induced subgraph H of G it holds that $\chi(H) = \omega(H)$.

Example: Any bipartite graph is perfect (both = 2). K_n is perfect (both = 2). The cycle C_5 is not perfect, $\chi(C_5) = 3, \omega(C_5) = 2$. Any cycle $C_{2k+1}, k \geq 2$ is not perfect.

Strong perfect graph theorem, proven by Chudnowsky, Seymour, Thomas, Robertson 2005: G is perfect $\Leftrightarrow \forall k \geq 2$ $C_{2k+1} \neg\subseteq G$ and $C_{2k+1} \neg\subseteq_{induced} \bar{G}$.

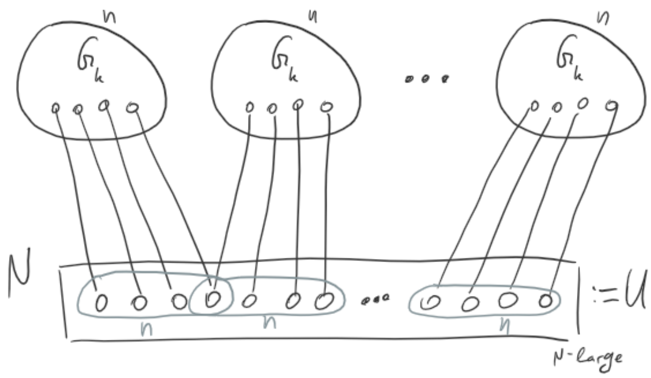
(Weak) perfect graph theorem, proven by Lov´asz: G is perfect $\Leftrightarrow$ $\bar{G}$ is perfect

19 Lecture 06.12.

Last time: Brooks theorem, Mycielski's construction, strong perfect graph theorem, perfect graph theorem (G is perfect $\Leftrightarrow$ $\bar{G}$ is perfect).

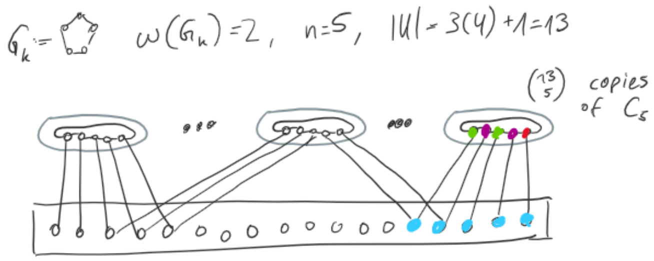
Today: Tutte's construction, properties of χ, edge colorings: König's & Vizing's theorem.

TUTTE'S CONSTRUCTION: Let G_k be a graph with $\chi(G_k) = k$, $\omega(G_k) = 2$ (e.g. there are no triangles). Construct a graph $G_{k+1}: \chi(G_{k+1}) \geq k+1, \omega(G_{k+1}) = 2$.



G_{k+1} is built of vertex disjoint copies of G_k on vertex sets $V_1, \dots, V_{\binom{N}{n}}$ and disjoint from these the set U with $|U| = k(n-1)+1$ and extra matchings between n elt (?) subsets of U and respective V_i's.

Example:

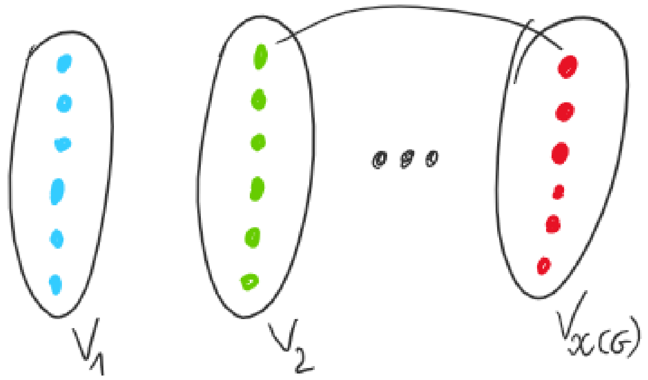


Assume $\chi(G_4) \leq 3$, i.e. 3 colors are used on U. If each color class in U has size ≤ 4, then we have ≤ 12 vertices in U, which is a contradiction. Thus there are 5 vertices in U of the same color. Thus the respective copy of G_k can not use this color, so the total number of colors is $\chi(G_k) + 1$.

Lemma 1: $|E(G)| \geq \binom{\chi(G)}{2}$.

Proof: Let $V_1, V_2, \dots, V_{\chi(G)}$ be color classes of a proper coloring of G.

lbach@outlook.de
lukasbach.com

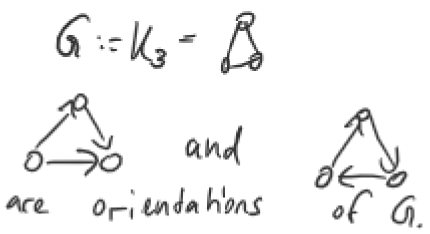


$\forall i,j: 1 \le i < j \le \chi(G)\ \exists$ edge between V_i and V_j. If not, make a new color class $V_i \cup V_j$, giving a proper coloring with less than $\chi(G)$ colors. Thus $|E(G)| \ge \binom{\chi(G)}{2}$. Qed.

Lemma 2: Let $f(D)$ be the length of longest directed path in directed graph D. Then $\chi(G) \le \min f(D) + 1$ where D is the orientation of G.

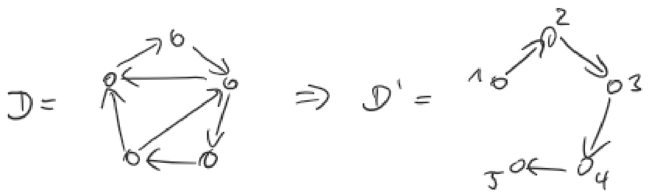
Definition: D is a **DIRECTED GRAPH** if $D = (V, \{(u,v): u,v \in V\})$. D is an orientation of G if $V(D) = V(G)$ and $\forall$ edge uv of D there is exactly one pair (u,v) or (v,u) in $E(D)$.

Example:



A directed graph is transitive if it does not contain directed cycles. Note: $\forall G\ \exists$transitive orientation D: Put arrows pointing to the right.

Proof of Lemma 2: We need to show that $\forall$ orientation D of G, $\chi(G) \le f(D) + 1$. Fix D. Let D' be the largest transitive (spanning) subgraph of D.



Introduce a vertex coloring c of G: $c(v) :=$ (length of longest directed path in D' ending at v) + 1.

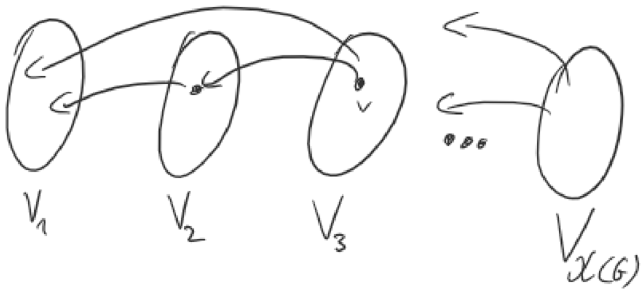
Claim: c is proper.

Proof of claim: If $(u,v) \in D'$ then $c(u) < c(v)$. If u, u' are joined by a directed path $p = (u, u_1, u_2, \dots, u')$ in D', then $c(u) < c(u_1) < c(u_2) < \dots < c(u')$, i.e. $c(u) \ne c(u')$. Let $uv \in E(G)$. If u, v are on a directed path in D', $c(u) \ne c(v)$ by above. If u, v are not on any directed path in D', i.e. $(u,v) \notin E(D'), (v,u) \notin E(D')$, we can add (u,v) to $E(D')$. If this addition created a directed cycle in the new D', then there was a directed path in D' containing u and v, which is a contradiction. Qed for claim.

Qed for lemma.

Note: $\forall G\ \exists orientation\ D$ s.t. $f(D) = \chi(G) - 1$.

Proof: Consider a proper coloring of G with color classes $V_1, \dots, V_{\chi(G)}$ such that $\forall i \ge 2\ \forall v \in V_i$ v can not be moved to $V_j, j < i$. Orientate edges from "right to left".



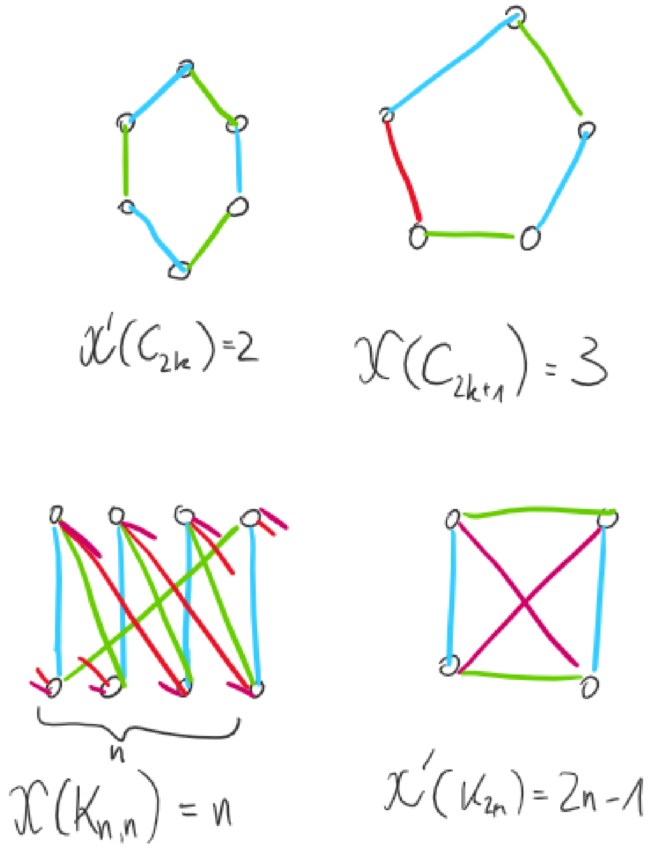
19.1 Edge-Colorings

$$\chi'(G) := \chi(L(G))$$
$$= \min\{k: \exists c: E(G) \to [k]\ such\ that\ c(e) \ne c(e')\ if\ e\ and\ e'\ share\ a\ vertex\}$$

where χ' is the edge chromatic number or chromatic index.

Example:

lbach@outlook.de
lukasbach.com



- $\chi'(C_{2k})=2$.
- $\chi'(C_{2k+1})=3$
- $\chi'(K_{n,n})=n$
- $\chi'(K_4)=3, \chi'(K_{2n})=2n-1$
- $\chi'(K_{2n+1})\geq 2n$ since degree of spiders is 2n, #edges in a color class on average is $=\frac{\binom{2n+1}{2n}}{2n}=\frac{(2n+1)(2n)}{2\cdot 2n}=\frac{2n+1}{2}\Rightarrow$ $\exists$ color class on n+1, thus $2n+2$ vertices $>2n+1$ which is a contradiction.

Note: $\chi'(G)\geq\Delta(G)$.

Königs theorem (1916): $\chi'(G)=\Delta(G)$ if G is bipartite.

Vizing's theorem: $\chi'(G)\in\{\Delta(G),\Delta(G)+1\}$ for any graph G.

Q: Are there 3-regular (cubic) graphs G with $\chi'(G)=4$? "Snarks". (e.g. Petersen graph).

Proof of König's theorem: Induction of $|E(G)|$. Basis $|E(G)|=0$. Step: Let $\Delta(G)=\Delta$, G is bipartite. We want $\chi'(G)\leq D$. By induction $\chi'(G-xy)\leq\Delta$ for an edge xy (because G is bipartite, $x\in X, y\in Y, G=X\cup Y$). Let $c: E(G-xy)\to[\Delta]=\{1,2,\dots,\Delta\}$ be a proper coloring. Let $Mis(x):=[\Delta]-\{c(xv): v\in N(x)\backslash\{y\}\}$ (missing colors). Let $Mis(y)$ be defined analogously.

Note: $Mis(x)\neq\emptyset, Mis(y\neq\emptyset,$ since $\deg_{G-xy}(x),\deg_{G-xy}(y)\leq\Delta-1$.

Case 1: $\exists\beta\in Mis(x)\cap Mis(y)$. Extend c to xy by $c(xy)=\beta$.

Case 2: $Mis(x)\cap Mis(y)=\emptyset$. Let $\alpha\in Mis(x)\backslash Mis(y), \beta\in Mis(y)\backslash Mis(x), \alpha\neq\beta$. If $\exists$ a maximal path containing x, colored β and α such that it does not contain y. Flip colors α and β on this path and color xy with β.

If such a path contains y, then it is $x_0=xy_0x_1y_1x_2y_2\dots x_my_m=y\dots$ with alternating coloring $\beta\alpha\beta\alpha\beta\dots\beta$, thus β is used on an edge incident to y, but $\beta\in Mis(y)$ which is a contradiction.

20 Lecture 09.12.

Recall Vizings theorem: $\chi'(G)\in\{\Delta(G),\Delta(G)+1\}$.

Recall $\chi'(G)\geq\Delta(G)$.

Proof: We want to show that $\chi'(G)\leq\Delta(G)+1$. Induction on $\Delta(G)=\Delta$. By induction hypothesis, any "proper" subgraph G' of G with $|E(G')|<|E(G)|$ satisfies $\chi'(G)>\Delta+1$.

Claim: $\forall xy\in E(G)\forall proper\ c: E(G-xy)\to[\Delta+1]\ \forall\alpha\in Mis_c(x)\forall\beta\in Mis_c(y)\ \exists x-y$ path colored and $Mis_c(x)\cap Mis_c(y)=\emptyset$.

Fix x. Let c_0 be a proper coloring of $G-\{xy_0\}$ from $[\Delta+1]$. Let $\alpha\in Mis_{c_0}(x)$. Let $y_0,y_1,\dots,y_k$ be the maximal sequence from $N(x)$ such that $c_0(xy_{i+1})\in Mis_{c_0}(y_i), 0\leq i<k$.

Let $c_i: E(G-xy_i)\to[\Delta+1], c_i(xy_i)=\begin{cases} c_0(xy_{j+1}) & 0\leq j\leq i-1 \\ c_i(e) & otherwise\end{cases}$. Note: $Mis_{c_i}(x)=Mis_{c_j}(x)$.

Claim 2: $\exists y=y_i, 0<i<k$ such that $c_0(yx)=\beta$ and $\beta\in Mis_{c_0}(y_k)$.

Proof of claim 2: If $\beta\in Mis_{c_0}(x)$, then $\beta\in Mis_{c_k}(x)$, so $\beta\in Mis_{c_k}(x)\cap Mis_{c_k}(y_k)$ which is a contradiction to the first claim.

Thus $c_0(xy)=\beta$ for some $y\neq y_i\ \forall 0<i<k$. In this case take $y_{k+1}=y$, extending the "fan" and contracting the maximality of k. (End of proof of Claim 2).

By claim in $c_k\ \exists\alpha-\beta$ path between x and y_k, call it P. In $c_i\ \exists\ \alpha-\beta$ path, call it P'. P is between y_{i-1} and y_i. P and $P'\backslash x$ have the same coloring in $c_0,c_1,\dots,c_k$. Note: $P\cup P'$ is a connected graph with vertices y_k,y_i,y_{i-1} of degree 1. Thus $P\cup P'$ has a vertex of degree 3, but $P\cup P'$ is properly edge-colored with 2 colors β and α_i which is a contradiction.

Lemma: $\chi_l\left(G:=K_{\binom{2k}{k},\binom{2k}{k}}\right)\geq k+1$.

Proof: ("The list chromatic number could be much larger than χ") Let the parts of G be X and Y. $X=\binom{[2k]}{k}=\{A: A\subseteq[2k],|A|=k\}, Y=\binom{[2k]}{k}$. Let $L(x)=x, x\in X$.

Claim: We cannot color this graph from these lists. Assume c is such a list coloring. Let wlog. $c(v_i)=a,\dots,v_i\in Y$, let $v_2\in$

lbach@outlook.de
lukasbach.com

Y be such that $a \notin L(v_2)$. Let $c(v_2) = a_2, a_1 \neq a_2$ …, Let $c(v_k) = a_k$; $a_1, a_2, \dots, a_k$ are distinct. Consider $u \in X$: Let $L(u) = \{a_1, \dots, a_k\}$. Then $c(u) = a_i, i = 1, \dots, k, c(u) = c(v_i)$ which is a contradiction.

[Notes: G above has $\binom{2k}{k}$ vertices in each part and distinct lists of sizes k from the set of colors $\{1, 2, \dots, 2k\}$.]

If $n = |V(G)|, \chi_l(G) \ge c \log n$, $n = \binom{2k}{k} \le 2^{2k}, \frac{\log_2 n}{2} \le k$.

20.1 Variants

Total colorings: $c: V \cup E \to [k]$ is proper if no two adjacent or incident objects have the same color. Minimum number of colors in G is denoted as $\chi''(G)$.

Vizing's conjecture: $\chi''(G) \le \Delta(G) + 2$.

Best known bound (Melloy-Reed): $\chi''(G) \le \Delta(G) + 10^{26}$.

In reality often better: $\chi''(G) \le \Delta(G) + 8 \ln 8\Delta(G)$.

21 Lecture 13.12.

TODO eigener Aufschrieb

22 Lecture 16.12.

Recall for a given $n \in \mathbb{N}$ and a graph H,

$$ex(n, H) := \max\{|E(G)| : |V(G)| = n, H \neg\subseteq G\}$$

$$EX(n, H) := \{G : |V(G)| = n, |E(G)| = ex(n, H), H \neg\subseteq G\}$$

where EX denotes the class of extremal graphs (for H).

$ex(n, P_3) = \left\lfloor \frac{n}{2} \right\rfloor, EX(n, P_3) = \{||||.\}$ for odd n, $EX(n, P_3) = \{||||\}$ for even n. $H = P_4, n \equiv 1 (mod\ 3), EX(n, P_4)$ is either a set of triangles or a star.

Turan theorem: $\forall n \in \mathbb{N}\ \forall r \in \mathbb{N}\ EX(n, K_r) = \{T(n, r-1)\}$.

Recall: $T(n, r)$ is a Turan graph, that is an n-vertex complete balanced r-partite graph.

Notations: $t(n, r) := |E(T(n, r))|, T(n, r) = T_r(n), t(n, r) = t_r(n)$.

Preparations for Turans theorem:

Lemma 58: $\forall n, r \in \mathbb{N}\ \ t_r(n + r) = t_r(n) + n(r-1) + \binom{r}{2}$.

Proof: Let parts of $T(n, r)$ be $V_1, \dots, V_r$ and parts of $T(n + 1, r)$ be $V_1 \cup \{v_1\}, \dots, V_r \cup \{v_r\}$ for new distinct $v_1, \dots, v_r$.

Abb1

Then $t_r(n + r) = |E(T(n + r, r))| = |E(T(n, r))| + \binom{r}{2} + n(r-1)$. Qed.

Lemma 59: Let G be an r-partite n-vertex graph with largest number of edges. Then $G \approx T(n, r)$.

Proof: G is completely r-partite, otherwise add edges. Assume G is not balanced, i.e. not $T(n, r)$. Then G has parts $V_1, V_2, \dots, V_r$ and $|V_i| \ge |V_j| + 2$ for some i, j. Move one vertex from V_i to V_j to get a complete r-partite G' with parts $V_i, \dots, V_i - v, \dots, V_j \cup \{v\}, \dots, V_r$. Then it holds that

$$|E(G'| = |E(G)| - |V_j| + |V_i| - 1 \ge |E(G)| + 1$$

Which is a contradiction, Qed.

Lemma 60: For a fixed r, $\lim_{n\to\infty} \frac{t_r(n)}{\binom{n}{2}} = 1 - \frac{1}{r}$.

Proof: Each part in $T(n, r)$ has size $\left\lfloor \frac{n}{r} \right\rfloor, \left\lceil \frac{n}{r} \right\rceil$, thus

$$\frac{\binom{n}{2} - r\binom{\lceil \frac{n}{r} \rceil}{2}}{\binom{n}{2}} \le \frac{t_r(n)}{\binom{n}{2}} \le \frac{\binom{n}{2} - r\binom{\lfloor \frac{n}{r} \rfloor}{2}}{\binom{n}{2}}$$

$$\Rightarrow 1 - \frac{r\frac{\left(\binom{n}{r} + 2\right)^2}{2}}{\binom{n}{2} \to_{n\to\infty} \frac{1}{r}} \le \frac{t_{r(n)}}{\binom{n}{2}} \le 1 - \frac{r\frac{\left(\binom{n}{r} - 2\right)^2}{2}}{\binom{n}{2} \to_{n\to\infty} \frac{1}{r}}$$

Qed.

Turan theorem: $\forall r, n \in \mathbb{N}\ EX(n, K_r) = \{T(n, r-1)\}$.

Proof: Fix r. Use induction on n. If $n \le r - 1$, then $K_r \neg\subseteq K_n$, thus $EX(n, K_r) = \{K_n\} = \{T(n, r-1)\}$.

Assume $n \ge r$ and statement is true for any smaller value of n. Let $G \in EX(n, K_r)$, we want to prove: $G = T(n, r-1)$. Then $K_{r-1} \cong K \subseteq G$ (if not, add edges to keep G being K_r free). Let $G' = G - V(K)$, $K_r \neg\subseteq V(K)$.

$$\begin{aligned}(*)|E(G)| &= |E(G')| + |E(K)| + |E(V(K), V(G'))| \\ &\le t_{r-1}(n - (r-1)) + \binom{r-1}{2} \\ &+ (r-2)(n - (r-1)) =_{Lem58} t_{r-1}(n)\end{aligned}$$

$(**)|E(G)| \ge t_{r-1}(n)$ since $K_r \neg\subseteq T(n, r-1)$.

$(*) \Rightarrow (**)$: $|E(G)| = t_{r-1}(n)$.

Since the bounds $(*)\&(**)$ match, we must have " = " in $(*)$, i.e. $|E(G')| = t_{r-1}(n - r + 1)$ and each vertex in G' sends exactly $r - 2$ edges to K. By induction $G' \approx T_{r-1}(n - r + 1)$ (since $|V(G')| < n$), i.e. $\forall v \in G'$, v is non-adjacent to exactly one vertex in K.

Assume $V_1, \dots, V_{r-1}$ are parts of G'. Assume $\exists v \in V_i, v' \in V_j, i \neq j; vu, v'u \notin E(G), \exists u \in V(K)$. Then v, v' and $(V(K) - u)$ form K_r, which is a contradiction.

Then one can order vertices of K as $v_1, \dots, v_{r-1}$ such that $v_i v \notin E(G)\ \forall v \in V_i$. Then $V_1 \cup \{v_1\}, V_2 \cup \{v_2\}, \dots, V_{r-1} \cup \{v_{r-1}\}$ are parts of $T(n, r-1)$ i.e. $G \approx T(n, r-1)$.

Qed.

lbach@outlook.de
lukasbach.com

22.1 Knowings on Extremal Theory

$\lim_{n\to\infty} \frac{ex(n,H)}{\binom{n}{2}}$ is known if $\chi(H) \geq 3$.

$ex(n,H) = ??$ if $\chi(H) = 2$ thus H bipartite.

$c'n \leq ex(n,H) \leq cn^2, ex(n,H) = o(n^2)\ if\ \chi(H) = 2$.

Conjencture (Erdös): $\forall r \in \mathbb{Q}, 1 \leq r \leq 2: \exists H: ex(n,H) \cong n^r$ and $\forall H\ \exists r \in \mathbb{Q}: ex(n,H) = n^r$.

Theorem: $ex(n,P_{k+1}) \leq \frac{k-1}{2}n$. If k divides n, the equality holds.

Proof: Induction on n. If $n \leq k, P_{k+1} \neg\subseteq K_n$, thus $ex(n,P_{k+1}) \leq |E(K_n)| = \binom{n}{2} = \frac{n(n-1)}{2} \leq \frac{n(k-1)}{2}$.

Let $n > k$. Let G: $|G| = n, P_{k+1} \neg\subseteq G$. We want: $|E(G)| \leq \frac{k-1}{2} \cdot n$.

- Case 1: G is disconnected, i.e. $G = G_1 \cup G_2, G_1 \& G_2$ are disjoint. Then $|E(G)| = |E(G_1)| + |E(G_2)| \leq_{ind} \frac{k-1}{2}(|V(G_1)| + \frac{k-1}{2}|V(G_2)| = \frac{k-1}{2}n$.
- Case 2: G is connected. Claim: $\exists v: \deg\ (v) < \frac{k-1}{2}$. If not, i.e. $\delta(G) \geq \frac{k-1}{2}$. Consider the longest path, arrive at a contradiction.
  Consider a longest path P on $m < k+1$ vertices. Since $|V(G)| \geq k+1$, $\exists$cycle of length m in G, consider $v \notin V(C), v \sim C$. Such a cycle exists because G is connected. But then exists a longer path P'.
  Let $A := N(v_1), B := N(v_m). |A| \geq \frac{k-1}{2}, |B| \geq \frac{k-1}{2}$. Let $A' = \{i: v_i \in A\}, B' = \{j: v_{j-1} \in B\}$. $|A'| \geq \frac{k-1}{2}, |B'| \geq \frac{k-1}{2}, |A' \cap B'| = 0$, which is a contradiction.

Claim: $\deg(v) < \frac{k-1}{2}$ for some v. $|E(G)| \leq |E(G-v)| + \frac{k-1}{2} \subseteq_{ind} (n-1)\left(\frac{k-1}{2}\right) + \left(\frac{k-1}{2}\right) = n\left(\frac{k-1}{2}\right)$.

Qed.

23 Lecture 20.12.

Last time:

Corollary Turans theorem: $ex(n,K_n) = t_{r-1} \cong \left(1 - \frac{1}{r-1}\right)\binom{n}{2}$

Observation 1: Let G be connected, $|V(G)| \geq k+1$ with its longest path having m vertices, then $C_m \neg\subseteq G$.

Proof: If not, i.e. $C = C_M \subseteq G$, $|V(G) \backslash V(C)| \neq \emptyset \Rightarrow \exists v \in V(G) \backslash V(C)\ \exists u \in V(C): uv \in E(C) \Rightarrow \exists$ path on m+1 vertices, which is a contradiction.

Abb1

Observation 2: If $P_m \subseteq G$ and $C_m \neg\subseteq G$ $\forall$path $v_1, \ldots, v_m$ if $v_i \in N(v_1)$ then $v_{i-1} \notin N(v_m)$.

Abb2

Observation 3: Let k divides n, then

Abb3

$$ex(n,T) \geq ||G|| = \frac{n}{k} \cdot ||K_k|| = \frac{n}{k}\binom{k}{2} = \frac{n(k-1)}{2}$$

$ex(n,T) \geq \frac{n(k-1)}{2}$ if $k|n$.

Theorem: $ex(n,P_{k+1}) \leq \frac{k-1}{2}n$.

Proof: We need to show $P_{k-1} \neg\subseteq G$ and $|G| = n \Rightarrow ||G|| \leq \frac{k-1}{2}n$. Use induction on n. If $n \leq k \Rightarrow P_{k+1} \neg\subseteq K_n, ||K_n|| \leq \frac{n(k-1)}{2}$. If G is a vertex-disjoint union of $G_1 \& G_2 \Rightarrow ||G|| = ||G_1|| + ||G_2|| \leq_{induction\ on\ n} \left(\frac{k-1}{2}\right) \cdot |G_1| + \left(\frac{k-1}{2}\right) \cdot |G_2| = \frac{k-1}{2}(|G_1| + |G_2|) = \frac{k-1}{2}n$.

If G has a vertex v of degree $\leq \frac{k-1}{2} \Rightarrow ||G|| = ||G-v|| + \deg v \leq_{induction} \frac{k-1}{2}(n-1) + \frac{k-1}{2} = \frac{k-1}{2}n$. Thus, assume G is connected, $n > k$, $\delta(G) > \frac{k-1}{2}$. Let $m = $#vertices in a longest path in G, assume that $m < k+1$.

Consider a path $v_1, \ldots, v_m$. We have $N(v_1) \subseteq V(P), N(v_m) \subseteq V(P)$, otherwise a longer path exists. By observation 1 and 2, if $v_i \in N(v_i) \Rightarrow v_{i-1} \notin N(v_m)$, i.e. out of $v_1, \ldots, v_{m-1}$, the possible neighbors of v_m, $d(v_1)$ vertices are not allowed, i.e. not in $N(v_m)$. Then $d(v_m) \leq (m-1) - d(v_1) < (m-1) - \frac{k-1}{2}$. On the other hand, $d(v_m) > \frac{k-1}{2} \Rightarrow \frac{k-1}{2} < (m-1) - \frac{k-1}{2} \Rightarrow m > k-1+1 = k$.

Qed.

Erdös-Süs-Conjencture: $|G| = n, ||G|| > \frac{k-1}{2}n \Rightarrow$ G contains any $(k+1)$ vertex-tree as a subgraph, i.e.$\forall tree\ T, |T| = k+1, ex(n,T) \leq \frac{k-1}{2}n$.

Theorem: Let $k, n \in \mathbb{N}, k < \frac{n}{2}$ $\forall$tree T on k edges: $ex(n,T) \leq kn$.

Proof: Claim: If $|G| = n, ||G|| = kn \Rightarrow \exists G' \subseteq G$ such that $\delta(G') \geq k$.If not, G is $(k-1)$-degenerate, i.e. $||G|| \leq (k-1)n$ which is a contradiction.

Greedily embed T in G', more formally: $\delta(G') \geq k \Rightarrow T \subseteq G'$. $\forall T\ tree: ||T|| = m \leq k$, we claim $T \subseteq G'$ by induction on m. Trivial basis is $m = 0$. Induction step for m: Let $|T| = m, T' := T - v$, where v is a leaf of T adjacent to u.

By induction $T' \subseteq G'$. We have $|N(u)| \geq k$, thus $\exists v' \in V(G') \backslash (N(u) \cup \{u\})$. So, $T' \cup \{v'\} \cong T \subseteq G'$. $|T'| = m, |V(T') \cap N(u)| \leq m-1 \leq k-1$. Qed.

Note: k fixed, $|T| = k$ as tree, $ex(n,T) = O(n)$.

Erdös-Stone-Simonovits theorem: $\forall H$ as graph: $\chi(H) \geq 3, \forall \epsilon > 0\ \exists n_0\ \forall n > n_0$: $(1 - \left(\frac{1}{\chi(H)-1}\right)\binom{n}{2} \leq ex(n,H) \leq \left(1 - \left(\frac{1}{\chi(H)}\right) + \epsilon\right)\binom{n}{2}$, $\quad ex(n,H) \cong \left(1 - \left(\frac{1}{3-1}\right)\right)\binom{n}{2} = \frac{1}{2}\binom{n}{2} \cong ex(n,K_3)$.

Recall Turans theorem, $ex(n,K_r) \cong \left(1 - \frac{1}{r-1}\right)\binom{n}{2} = \left(1 - \left(\frac{1}{\chi(K_r)-1}\right)\right)\binom{n}{2}$.

Idea of the proof:

Lower bound: $H \neg\subseteq T(n, \chi(H)-1)$, since $\chi\big(T(n,\chi(H)-1)\big) = \chi(H) - 1, \chi(H) = \chi(H)$.

Upper bound: Take a graph G, $|G| = n, ||G|| = \left(1 - \frac{1}{\chi(H)-1} + \epsilon\right)\binom{n}{2}$. Apply induction on the number of parts to show that $K^t_{\chi(H)} \subseteq G$, where $K^t_{\chi(H)}$ is the complete $\chi(H)$-partite graph with parts of sizes t, t is "large". Note that $K^t_{\chi(H)} = T\big(t\chi(H), \chi(H)\big), H \subseteq K^t_{\chi(H)}$ since any graph with $\chi = r$ is a subgraph of a large r-partite complete graph.

Chratal-Szemeredis theorem: $\forall \epsilon > 0\ \forall r \geq 3\ \forall G$: $|G| = n, ||G|| \geq \left(1 - \left(\frac{1}{r-1}\right) + \epsilon\right)\binom{n}{2}$, then $K^t_r \subseteq G, t = \left(\frac{\log n}{500\left(\log\left(\frac{1}{\epsilon}\right)\right)}\right)$. $\exists G: |G| = n, ||G|| = \left(1 - (1-\epsilon)\left(\frac{1}{r-1}\right)\right)\binom{n}{2}$ and $K^t_r \neg\subseteq G, t = \frac{5 \log n}{\log \frac{1}{\epsilon}}$.

Zarankiewicz function: $z(m,n;s,t) :=$ maximum number of edges in the subgraph of a complete bipartite graph with parts A, B with $|A| = m, |B| = n$ that contains no $K_{s,t}$ subgraph with a part of size s in A and a part of size t in B.

Abb4

Plan: $z(n,n;s,t) \leq \dots \Rightarrow z(n,n,t,t) \leq cn^{2-\frac{1}{t}} \Rightarrow ex(n,K_{t,t}) \leq \frac{z(n,n;t,t)}{2} \leq \frac{c}{n} n^{2-\frac{1}{t}} = o(n^2)$.

Kövari-Sös-Turan-Theorem: $z(m,n;s,t) \leq (s-1)^{\nu_t}(n - t + 1)m^{1-\frac{1}{t}} + (t-1)m \cong_{s,t\ small, s=t, n=m\ large} < n \cdot n^{1-\frac{1}{t}} + c'n \cong ch^{2-\frac{1}{t}}$.

Proof: $G \subseteq (A \cup B, E), |A| = m, |B| = n, K_{s,t} \neg\subseteq G$ with part of size s in A and part of size t in B. Let $T := \#stars\ of\ size\ t\ with\ center\ in\ A.$ $T = \sum_{v \in A} \binom{\deg(v)}{t}, T \leq_{(*)} \binom{n}{t} \cdot (s-1)$

$(*)$ holds since $\forall B' \subseteq B, |B'| = t$ there are at most $(s-1)$ stars with a leaf-set B', otherwise they form $K_{s,t}$. The number of such B'-s is $\binom{n}{t}$. Thus: $\sum_{v \in A} \binom{\deg v}{t} \leq \binom{n}{t}$.

(TODO next year)

24 Lecture 10.01.

Last time:

$$ex(n,H) := \max\{|E(G)|, |V(G)| =, H \neg\subseteq G\}$$

Turans theorem (alternative): $ex(n,K_r) = t_{r-1}(n) = \left(1 - \frac{1}{r-1}\right)\binom{n}{2} \leq \left(1 - \frac{1}{r-1}\right)\left(\frac{n^2}{2}\right)$

Erdös-Sös conjecture: $ex(n,T) \leq \frac{kn}{2}, |E(T)| = k$ and T-tree.

Erdös-Stone theorem (without proof): $\forall \epsilon > 0 \exists n_0 \forall n > n_0$:

$$\left(1 - \frac{1}{1-r} - \epsilon\right)\binom{n}{2} \leq ex(n, K^s_r) \leq \left(1 - \frac{1}{r-1} + \epsilon\right)\binom{n}{2}$$

Zaarankiewics function: $z(m,n;s,t) :=$ max #edges in a bipartite graph with parts A and B, $|A| = m, |B| = n$ such that no copy of $K_{s,t}$ with part of size s in A and part of size t in B.

Kövari-Sös-Turan: $z(m,n;s,t) \leq (s-1)^{\frac{1}{t}}(n - t + 1)m^{1-\frac{1}{t}} + (t-1)m$.

Proof: Let $G = (A \cup B, E), |A| = m, |B| = n$, no $K_{s,t}$, s in A, t in B. Let $T :=$ #stars of size t with centers in A. $T = \sum_{v \in A}\binom{\deg(v)}{t}, T \leq \binom{n}{t}(s-1)$ ($\binom{n}{t}$ describes the amount of ways to choose t vertices in B).

$$\Rightarrow \sum_{v \in A}\binom{\deg(v)}{t} \leq (s-1)\binom{n}{t}$$

$$\Rightarrow \sum_{v \in A}\binom{\frac{e}{m}}{t} \leq (s-1)\binom{n}{t}$$

$$\sum_{v \in A}\binom{\frac{e}{m}}{t} = m\binom{\frac{e}{m}}{t} \leq (s-1)\binom{n}{t}$$

$$\frac{m}{s-1} \leq \frac{\binom{n}{t}}{\binom{\frac{e}{m}}{t}} = \cdots$$

$$\Rightarrow \frac{m}{s-1} \leq$$

Notes: $e = \#edges\ in\ G, \frac{e}{m} =$avg. degree of a vertex in A. We want $e \leq ?$.

Lemma: $ex(n, K_{t,t}) \leq \frac{z(n,n;t,t)}{2}$.

Proof: Let $K_{t,t} \neg\subseteq G$, $|G| = n$. We want $|E(G)| < ?$, thus an upper bound. Let G' be bipartite with parts $V_1 = \{v(1): v \in V(G)\}, V_2 = \{v(2): v \in V(G)\}, E(G') = \{v(1)u(2): vu \in E(G)\}$.

Abb1

Claim: $K_{t,t} \neg\subseteq G'$, otherwise there is $K_{t,t}$ in G.

lbach@outlook.de
lukasbach.com

Then $|E(G)| = \frac{|E(G')|}{2} \leq \frac{z(n,n;t,t)}{2}$. Qed.

Corollary: $ex(n, K_{t,t}) \leq c \cdot n^{2-\frac{1}{t}}$, $ex(n, K_{2,2}) \leq c \cdot n^{2-\frac{1}{2}} = cn^{\frac{3}{2}}$.

Theorem: $ex(n, K_{t,t}) \geq cn^{2-\frac{2}{t}+1}$.

Proof: Consider n vertices, choose an edge randomly with probability $p = n^{-\frac{2}{t}+1}$ (independently). $EXP(\#edges) = \binom{n}{2}p, EXP(\#K_{t,t}'s) \leq \binom{n}{2t}\binom{2n}{t}p^{t^2}$.

($\binom{n}{2t}$ describes the ways to choose t2 vertices, p^{t^2} is the probability of a fixed $K_{t,t}$ to appear).

Let G' be obtained from G by deleting an edge from each copy of $K_{t,t}$ in G. Thus $K_{t,t} \neg\subseteq G$.

$$\begin{aligned} EXP(|E(G')|) &= EXP(|E(G)|) - EXP(\#deleted\ edges) \\ &\geq EXP(|E(G)|) - EXP(\#K_{t,t}\text{'s in } G) \end{aligned}$$

$$\geq \binom{n}{2}p - \binom{n}{2t}\binom{2t}{t}p^{t^2} \geq cn^{2-\frac{2}{t+1}}$$

Thus $\exists \mathcal{F}: |V(\mathcal{F})| = n, K_{t,t} \neg\subseteq \mathcal{F}, |E(\mathcal{F})| \geq cn^{2-\frac{2}{t+1}}$.

Qed.

Construction of $K_{2,2}$-free (or C_4-free) G on n vertices and $\cong n^{\frac{3}{2}}$ edges.

Let $G = Hp$, p being a prime. $V(Hp) \coloneqq (\mathbb{Z}_p \backslash \{0\} \times \mathbb{Z}_p), E(Hp) \coloneqq \{\{(a,b),(x,y)\}: ax = b + y (mod\ p)\}$.

Claim 1: $K_{2,2} \neg\subseteq Hp$. Assume not.

ABb2

$$\begin{Bmatrix} ax = b + y & for & (x,y) = (x',y') \\ cx = d + y & & (x,y) = (x'',y'') \end{Bmatrix}$$

i.e. fixed a, b, c, d the system has ≥ 2 distinct solutions.

$(a-c)x = b - d$ if $a \neq c \Rightarrow \exists$ unique x $\Rightarrow unique\ x \Rightarrow unique\ y \Rightarrow \leq 1\ solution$. If $a = c \Rightarrow b = d\ cont. (a,b) \neq (c,d)$, qed for Claim 1.

Claim 2: $|V(Hp)| = p(p-1), |E(Hp)| = \frac{p(p-1)(p-1)}{2}$. $(p = |Zp|, (p-1) = |Zp \backslash \{0\}|)$.

For a fixed $(a,b) \in V(Hp)$ the number of neighbors of (x,y) of (a,b) is the number of solutions of $ax = b + y$. I can choose x in $(p-1)$ ways, then y is uniquely defined. I.e. the $\#solutions = \deg((a,b)) = p - 1, ||Hp|| = |E(Hp)| = |V(Hp)| \cdot \frac{p-1}{2}$.

If p is large, $|V(Hp)| = n \cong p^2, |E(Hp)| \cong p^3 \cong \sqrt{n}^3 = n^{\frac{3}{2}}$. Qed.

Big questions: $ex(n,H) =?$

Conjencture (Erdös): $\forall r \in \mathbb{Q}, 1 \leq r \leq 2\ \exists H: ex(n,H) = O(n^r)$. Such r are called Turan exponents.

24.1 Known bounds

$$ex(n, K_{2,2}) = \frac{1}{2}n^{\frac{3}{2}} + o\left(n^{\frac{3}{2}}\right)$$

$$ex(n, K_{3,3}) = \frac{1}{2}n^{\frac{5}{3}} + o\left(n^{\frac{5}{3}}\right) \quad (Brown)$$

$$ex(n, K_{2,t+1}) \cong \frac{1}{2}\sqrt{t}n^{\frac{3}{2}} + o\left(n^{\frac{3}{2}}\right) \quad (Füderi)$$

$$ex(n, K_{r,s}) > cn^{2-\frac{1}{r}}, r \geq 4, s \geq r! + 1 \quad (Kollar)$$

$$ex(n, C_6) = O\left(n^{\frac{4}{3}}\right)$$

$$ex(n, C_{10}) = O\left(n^{\frac{6}{5}}\right)$$

$$c'n^{1+\frac{1}{3k-2}} \leq ex(n, C_{2k}) \leq cn^{1+\frac{1}{k}}$$

$$ex(n, C_{2k+1}) = O(n^2) \quad (Erdös - Stone\ theorem)$$

$ex(n, C_{2k+1}) = \left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil \cong \frac{n^2}{4} = ex(n, C_3)$ for large n.

Recent findings (Tao Jang, Yu Qiv 2019+): $\forall p, q, q > p^2, \left(1 + \frac{p}{q}\right)$ is a Turan exponent.

Oliver Janzer (2019): $\forall s, k \geq 2$ integers $1 + \frac{s-1}{sk}$ is a Turan exponent. Specifically $\exists t_0 \forall s, k \geq 2\ \forall t \geq t_0\ ex\left(n, \widetilde{K_{s,t}^k}\right) = O\left(n^{1+\frac{s-1}{sk}}\right)$ where $K_{s,t}^k$ is a $(k-1)$-subdivision of $K_{s,t}$.

25 Lecture 13.01.

Szemeredis regularity lemma: For a given graph G, $X, Y \subseteq V(G), X \cap Y = \emptyset$, let $||X,Y|| \coloneqq$ #edges between X and Y. Let $d(X,Y) = \frac{||X,Y||}{|X| \cdot |Y|}$ – density of the pair $(X,Y), 0 \leq d(X,Y) \leq 1$.

For $\epsilon > 0$, the pair (X,Y) is ϵ-regular if $|d(A,B) - d(X,Y)| \leq \epsilon\ \forall A \subseteq X, B \subseteq Y, |A| \geq \epsilon|X|, |B| \geq \epsilon|Y|$.

Example: $d(X,Y) = \frac{1}{2}, \epsilon = \frac{1}{10}, \frac{1}{2} - \frac{1}{10} \leq d(A,B) \leq \frac{1}{2} + \frac{1}{10}$

$\forall A, B: |A| \geq \frac{|X|}{10}, |B| \geq \frac{|Y|}{10}$.

Let G be a graph, $V = V(G)$, let $V = V_1 \cup V_2 \cup \ldots \cup V_k \cup V_0$ be a partition of V. This partition is ϵ-regular if

1. $|V_0| \leq \epsilon|V|$
2. $|V_1| = |V_2| = \cdots = |V_k|$
3. All but at most ϵk^2 pairs $(v_i, v_j), 1 \leq i < j \leq k$ are ϵ-regular.

Note that $\#(v_i, v_j) = \binom{k}{2} \leq k^2$.

Theorem: Szemeredis Regularity Lemma: $\forall\epsilon > 0 \forall integer\ m \geq 1\ \exists M \in \mathbb{N}$ such that any graph of order at least m has an ϵ-regular partition $V_0, V_1, \dots, V_k, m \leq k \leq M$.

Note: $\#parts$ is constant and independent on $\#vertices$.

$$\epsilon^{-\frac{1}{16}} \left\{ 2^{2^{2^{(\dots)^2}}} \leq M(\epsilon) \leq 2^{2^{2^{(\dots)^2}}} \right\} \epsilon^{-5}$$

Proof idea:

Mean square density of a partition $V_1, \dots, V_k =: \Pi,\ n := |V(G)|$

$$d_2(\Pi) := \sum_{1 \leq i < j \leq k} \frac{|V_i| \cdot |V_j|}{n^2} d^2(V_i, V_j)$$

$d_2(\Pi) \leq 1$ because $\sum \frac{|V_i| \cdot |V_j|}{n^2} \leq 1$. $\sum |V_i| \cdot |V_j| \leq \binom{n}{2} \cong \frac{n^2}{2}$.

Idea of the proof:

- Start with an arbitrary partition.
- If not ϵ-regular, refine the partition in doing so, $d_2(\Pi)$ increases by $f(\epsilon)$
- Repeat…
- Stop because $d_2(\pi) \leq 1$.

Embedding (blowup) lemma: Let G be a graph. If R^s is an s-blowup of $R = R_d$ and $H \subseteq R_s \Rightarrow H \subseteq G$.

Outline of proof of Erdös-Stöne theorem:

We want: Given G, $|V(G)| = n, |E(G)| = t_{r-1}(n) + \gamma n^2$, then $K_r^5 \subseteq G$.

Apply the regularity lemma to G with R-reduced graph $R = R_d$ $(\epsilon \ll \gamma, d \ll \gamma)$.

$|E(R)| >^* t_{(r-1)}(k) \Rightarrow^{Turan} K_r \subseteq R \Rightarrow^{blowup} K_r^5 \subseteq R^5 \Rightarrow^{Embedding\ Lemma} K_r^5(G)$.

Proof of *: done in lecture.

25.1 Ramsey theory

Fact: any coloring of $E(K_G)$ in red, blue contains a red or blue triangle $\Delta = K_3$.

Proof: Let $c: E(K_6) \to \{r, b\}, x \in v(K_6)$. Assume wlog. X has three red incident edges.

So, xv_1, xv_2, xv_3 are red. If $v_i v_j$ is red for $i, j \in \{1,2,3\}, i \neq j$, then $xv_i v_j$ is a red triangle. Otherwise $v_1 v_2 v_3$ is a blue triangle.

Ramsey numbers:

$$R(k) := \min\{n: \forall c: E(K_n) \to \{r, b\}\ \exists red\ K_k\ or\ blue\ K_k\}$$

$red\ K_k \cong$ all edges are red.

Monochromatik H $\cong$ H with all edges of the same color.

Theorem: $\sqrt{2}^k \leq R(k) \leq 4^k$.

Lemma: $R(k) \leq 4^k$

Proof: Let $N = 4k$. Let $c: E(K_N) \to \{r, b\}$. Let $G = K_N$. We construct a sequence $x, \dots, x_{2k}$ of vertices and $V_1, \dots, V_{2k}$ of subsets of vertices such that $\forall i = 1, \dots, 2k$ colors of $x_i v, v \in V_i$ are the same.

Let x_i be chosen arbitrarily…

26 Problem class 15.01.

Theorem: n, t are integers and $t < n$, then $\exists$ constant $C > 0$ such that $ex(n, K_{t,t}) \geq C n^{2-\frac{1}{t+1}}$.

Proof: Define a graph G on $[n]$ where we put an edge uniformly at random with probability p, these $\binom{n}{2}$ choices being independent. We want a graph with “many” edges and “few” $K_{t,t}'s$. $X :=$ random variable counting #edges. $Y :=$ random variable counting $\#K_{t,t}'s$. $X = \sum_{e \in \binom{[n]}{2}} 1_e, 1_e := \begin{cases} 1 & e \in E(G) \\ 0 & otherw. \end{cases}, EX = \sum_{e \in \binom{[n]}{2}} \mathbb{P}(e \in E(G)) = p\binom{n}{2} < \binom{n}{t}\binom{n}{t} \leq n^{2t}$.

$\mathbb{P}(K_{t,t}) = p^{t^2}$.

$EY \leq n^{2t} \cdot p^{t^2}$.

27 Lecture 17.01.

$$R(k) := \min\{N \in \mathbb{N}: \forall c: E(K_N) \to \{r, b\}\ there\ is\ a\ monochromatic\ K_k\}$$

$R(3) = 6$. Abb1

Theorem: $\sqrt{2}^k \leq R(k) \leq_{last\ time} 4^k$.

Proof: Lower bound.'

We need a coloring of $K_{2^{\frac{k}{2}}}$ into red and blue with no monochromatik K_k. Consider $E = E(K_N), N = \sqrt{2}^k = 2^{\frac{k}{2}} \ltimes$ color E randomly with red and blue, such that $Prob(edge\ e\ is\ red) = Prob(edge\ e\ is\ blue) = \frac{1}{2}$, color edges independently. Let $S \subseteq V(K_N), |S| = k$.

$Prob(S\ induces\ red\ clique) = \left(\frac{1}{2}\right)^{\binom{k}{2}}$.

$Prob(S\ induces\ monochromatic\ clique) \leq 2 \cdot \left(\frac{1}{2}\right)^{\binom{k}{2}}$.

$$\begin{aligned} Prob(\exists S \subseteq V(K_N): |S| &= k\ and\ S\ induces\ monochr.clique) \\ &\leq \binom{N}{k} \cdot 2 \cdot \left(\frac{1}{2}\right)^{\binom{k}{2}} \leq \frac{N^k}{k!} \cdot 2^{1-\frac{k^2}{2}+\frac{k}{2}} \\ &= \frac{2^{\left(\frac{k}{2}\right)k}}{k!} 2^{1-\frac{k^2}{2}+\frac{k}{2}} = \frac{1}{k!} 2^{\frac{k}{2}+1} \end{aligned}$$

For $k \geq 3$.

Thus

$Prob(\exists S \subseteq V(K_N): |S| = k \text{ and } S \text{ induces mon.clique}) < 1 \Rightarrow$ there is a coloring of $E(K_N)$ in red and blue with no monochromatik K_K.

Qed.

Some notations:

$R(k)$ – diagonal Ramsay number

$R(k,l)$ – off-diagonal Ramsey number

$$R(k,l) \coloneqq \min\{N : \forall c : E(K_N) \to \{r,b\} \exists red\ K_K\ or\ \exists blue\ K_K\}$$

$R(2,l) = l..$

Why: $c: E(K_l) \to \{r,b\}$. If $\exists$ red edge $\Rightarrow$ red K_2. If there is no red egde, we have blue K_l. Qed.

For the lower bound, color $E(K_{l-1})$ blue, i.e. no red K_2, no blue K_l, i.e. $R(2,l) \geq (l-1)+1$.

Lemma: $R(k,l) \leq R(k-1,l) + R(k,l-1)\ \forall k,l \geq 3$.

Proof: Let $N = R(k,l) - 1, c: E(K_N) \to \{r,b\}$ with no red K_k, no blue K_k. Let $x \in V(K_N)$. Let $A \coloneqq \{u : c(xu) = r\}, B \coloneqq \{u : c(xu) = b\}$.

Abb2

Claim: $|A| \leq R(k-1,l) - 1$.

If not A contains either red K_{k-1} or blue K_L. Note that red K_(k-1) with x gives red K_k, which is a contradiction. We also have no blue K_l. This proofs the claim.

Similarly $|V| \leq R(k,l-1) - 1$.

Thus $N = |B| + |A| + 1 \leq (R(k-1,l) - 1) + (R(k,l-1) - 1) + 1, N = R(k,l) - 1$.

Thus $R(k,l) - 1 \leq R(k-1,l) + R(k,l-1) - 1 - 1 + 1$. Qed.

Lemma: $R(k,l) \leq \binom{k+l-1}{k-1},\ k,l \geq 2$.

Proof: Induction on $k+l$, basis $k+l = 2+2 = 4$, $R(2,2) = 2 \leq^? \binom{2+2-2}{1} = 2$.

Step: $R(k,l) \leq_{Lemma1} R(k-1,l) + R(k,l-1) \leq_{induction} \binom{k+l-3}{k-2} + \binom{k+l-3}{k-1} = \binom{k+l-2}{k-1}$.

Qed.

Corollary: $k \ll n, R(k,n) \leq \binom{k+n-2}{k-1} \cong c_k n^{k-1}$.

27.1 Graph Ramsey number

Let G and H be graphs. $R(G,H) \coloneqq \min\{N : \forall c : E(K_N) \to \{r,b\}\ \exists red\ G\ or\ blue\ H\}$.

Note $R(k,l) = R(K_k, K_l), R(G,H) \leq R(K_{|V(G)|}, K_{|V(H)|}), R(K_2, H) = |V(H)|$.

Abb3

Lemma 3: $R(sK_2, tK_2) = 2s + t - 1, s \geq t \geq 1$.

Proof:

Lower bound: Color $K_{2s+t-2}, V(sK_2) = 2s \Rightarrow$ no red sK_2. In this coloring any blue matching must have an vertex of each edge in B. But $|B| = t - 1 < |E(tK_2)|$. Thus no blue tK_2.

Upper bound: Induction of min {s, t}.

$t = 1: R(sK_2, 1 \cdot K_2) =^* 2s = 2s + t - 1$.

Step: Consider $c: E(K_{2s+t-1}) \to \{r,b\}$, we want to find red sK_2 or blue tK_2. Note $|V(K_{2s+t-1})| = 2s + t - 1 \geq 2s \geq 2t$

If c is monochromatic, i.e. all edges are either red or blue, then we have either red sK_2 or blue tK_2. Thus c is not monochromatic. I.e. we have Abb4

Consider our colored graph with deleted $\{x,y,z\}$. This graph G' has $(2s+t-1) - 3$ vertices, i.e. $|V(G')| = 2s + t - 4 = 2(s-1) + (t-1) - 1 =_{induction} R((s-1)K_2, (t-1)K_2)$. Thus we have either red $(s-1)K_2$ or blue $(t-1)K_2$ in G'. Then, together with either xy or yz we have red sK_2 or blue tK_2 in K_{2s+t-1}. Qed.

27.2 Multicolor Ramsey Numbers

$$R(G_1, G_2, \ldots, G_k) \coloneqq \min\{N : c : E(K_N) \to \{1,2,\ldots,k\}\ \exists i \in \{1,2,\ldots,k\} and\ copy\ of\ G_i\ in\ color\ i\}$$

With graphs G_i.

$$R(G_1, G_2, \ldots, G_k) \leq R\left(\underbrace{K_{|V(G_1)|}}_{x_1}, \ldots, \underbrace{K_{|V(G_k)|}}_{x_k}\right) =:^{def} R(x_1, x_2, \ldots, x_k)$$

Lemma 4: $2^k \leq R\left(\underbrace{3,3,\ldots,3}_{k}\right) \leq 3k!$.

Proof:

Lower bound: Color edges between bipartite parts with one color, then color the edges between bipartite subparts of each of the bipartite parts with next color, and so on.

Upper bound: Induction. $k = 2: R(3,3) = 6 = 3 \cdot 2!$. Consider $c: E(K_{3k!}) \to \{1,\ldots,k\}, x \in V$. $\exists$ color, say K, x is incident to $\geq \left\lceil \frac{3k!-1}{k} \right\rceil$ edges of this color.

$S \coloneqq \{u \in V : c(xn) = k\}, |S| \geq 3(k-1)!$.

If S induces an edge of color K, we have K_3 in color K. Otherwise S uses only colors $1, \ldots, k-1$.

$|S| = 3(k-1)! \geq_{ind} R(\underbrace{3,\ldots,3}_{k-1})$, there is a monochromatic triangle induced by S. Qed.

27.3 Hypergraph Ramsey Numbers

Notation: For a set X, $r \in \mathbb{N}, r \geq 2$:

$$\binom{X}{r} := \{x': X' \subseteq X, |X'| = r\}$$

$$\binom{X}{2} = E(complete\ graph\ on\ vertex\ set\ X)$$

$\binom{X}{r}$ is an r-clique of order $|X|$ on vertex set X.

28 Lecture 20.01.

Definition: $R(p,q;r) := \min\{N: \forall c: \binom{[N]}{r} \to \{0,1\}$

$$\exists A \subseteq [N]: |A| = p \,\&\, \forall A' \in \binom{A}{r}, c(A') = 0\ or$$

$$\exists B \subseteq [N]: |B| = q \,\&\, \forall B' \in \binom{B}{r}, c(B') = 1\}$$

Recall $\binom{X}{r}$ -r-clique, $\binom{X}{z}$ – complete graph, clique.

Theorem (Hypergraph Ramsey): $\forall r \geq 3\ \forall p, q \geq 3$ integers: $R(p,q;r) \leq R\left(\underbrace{R(p-1,q;r)}_{p_1}, \underbrace{R(p,q-1;r)}_{q_1}; r-1\right) + 1.$

Proof: Let $c: \binom{X}{r} \to \{r,b\}, |X| = R(p_1, q_1; r-1) + 1$. Let $x \in X$, let $c': \binom{X-x}{r-1} \to \{r,b\}$ such that $\forall A \subseteq X - x, |A| = r - 1, c'(A) := c(A' \cup x)$.

We have $|X - x| = R(p_1, q_1; r-1)$. Apply "Ramsey" to c' in $X - x$ inductively.

We have that either ∃red $(r-1)$-clique on p_1 vertices in $X - x$ under c' or ∃blue $(r-1)$-clique on q_1 vertices in $X - x$ under c'. Assume the former. Recall $p_1 = R(p-1,q;r)$. Apply "Ramsey" inductively (on $p+q$) in X' under c. We have that either (1) ∃ red r –clique in X' on $p-1$ vertices or (2) ∃ blue r –clique in X' in q vertices.

If (2) happens, we are done. Assume (1) holds. Thus $\binom{X'' \cup \{x\}}{r}$ is a red r-clique on p vertices.

Application 1 (Erdös-Szekerös): $\forall m \geq 3, m \in \mathbb{Z}, \exists N = N(m) \in \mathbb{Z}$ such that if X is a set of N points in the plane (no three on a line), then X contains vertices of convex m-gon.

Proof: Let $N = R(m, 5; \underbrace{4}_{uniformity=size\ of\ hyperedges})$.

Let X be a set of N points in $\mathbb{R}^2$ (generic position). Let $c: \binom{X}{4} \to \{r,b\}$ such that $\forall U \subseteq X, |U| = 4, c(U) = \begin{cases} red & if\ convex\ hull\ of\ U\ is\ a\ 4-gon \\ blue & if\ convex\ jull\ of\ U\ is\ a\ 3-gon \end{cases}$.

By definition of Ramsey numbers, we have either

1. $\exists x' \subseteq X, |x'| = m, \binom{X'}{4}$ is red, or
2. $\exists x' \subseteq X, |x'| = 5, \binom{X'}{4}$ is blue.

If (2) happens, then the convex hull of X' is a triangle. Let l be a line through two "internal" vertices z, z'. Let x, y be vertices of the convex hull of X' on the same side of l. Then $c(\{x,y,z,z'\}) =$ red, which is a contradiction. Thus (2) is impossible.

If (1) holds, $|X'| = m$, $\binom{X'}{4}$ is red. We want to show that X' forms a vertex set of a convex m-gon. Let $\tilde{X} := convexHull(X')$, if $\tilde{X}$-convex m-gon we are done.

Otherwise $\tilde{X}$ corr. to a convex k-gon, $k < m$. $\tilde{X}$ is a triangle. If z-internal for $\tilde{X}$ elf of $X' \Rightarrow z$ is internal to some of the triangles. Say xyw. Then $c(\{x,y,w,z\}) = blue$, which is a contradiction. Qed.

Andrew Suk 2016: smallest $N(m) = 2^{n+o(n)}$.

Another Erdös-Szekeres theorem: Any list of more than n^2 numbers contains a non-decreasing or non-increasing sublist of more than n numbers.

Example: $n = 2, n^2 + 1, (2\ 0\ 1\ 3\ 1)$.

Proof: Let $a_1, \dots, a_{n^2+1}$ be a list of reals. Let $u_i :=$ the length of a longest nonincreasing sublist ending at a_i, and $d_i :=$ the length of a longest nondecreasing sublist ending at a_i.

Assume no (monoton) sublist on more than n elfs. Thus $n_i \leq n, d_i \leq n\ \forall i$ #disttinct pairs $(u_i, d_i) \leq n^2$. But we have $a_1, \dots, a_{n^2+1}$. Thus $\exists i, j, i < j: (u_i, d_i) = (u_j, d_j)$.

If $a_i \leq a_j \Rightarrow d_i \leq d_j$. If $a_i \geq a_j \Rightarrow u_i < u_j$, which is a contradiction, Qed.

Application 2 (Schurs theorem): $k \in \mathbb{N}\ \exists N \in \mathbb{N}\ \forall c: [N] \to \{1, 2, \dots, k\}\ \exists x, y, z \in [N]: c(x) = c(y) = c(z) \,\&\, x + y = z$.

(∃ monochromatic solution of this linear equation)

Proof: Let $c: [N] \to [k]$, where $N := R(\underbrace{3, 3, \dots, 3}_{k})$. Let $c': E(K_N) \to [k]$ such that if $V(K_N) = [N]$, $c'(i,j) = c(|i - j|)$. We know (by definition of R) that ∃ monochromatic triangle under $c', i < j < m$, i.e. $c(j-i) = c(m-j) = c(m-i) = red$. Let $x = m - j, y = j - i, z = m - i$, then $x + y = z, c(x) = c(y) = c(z) = red$. Qed.

Generalization to systems of equations: Let $r \in \mathbb{N}, A \in \mathbb{Z}^{n \times k}$, A is called r-regular if ∃ monochromtic solution of $\bar{A}x = \bar{o}$ for any coloring $c: \mathbb{N} \to [r]$.

A matrix A fulfills column condition if ∃partition of set of columns $C_1 \cup \dots \cup C_m$ such that if $\bar{s_i} = \sum_{c \in C_i} \bar{c}$, then (1) $\bar{s_1} = \bar{o}$, (2) $\forall i = 2, \dots, m: \bar{s_i}$ is a rational linear combination of columns from $C_1 \cup \dots \cup C_{i-1}$.

Rados theorem: If a matrix A satisfies column condition, then it is r – regular $\forall r \in \mathbb{N}$.

lbach@outlook.de
lukasbach.com

Theorem (Ray-Chaudhuri and Wilson): $\mathcal{F} \subseteq \binom{[n]}{k}, |\{|F \cap F'|: F, F' \in \mathcal{F}\}| \subseteq s$. Then $|\mathcal{F}| \leq \binom{n}{s}$.

Theorem (Frankl-Wilson): $\mathcal{F} \subseteq \binom{[n]}{k}, |F \cap F'| \not\equiv k (mod\ q) \Rightarrow |\mathcal{F}| \leq \binom{n}{q-1}$.

29 Lecture 24.01.

$(*)$: $\mathcal{F} \subseteq \binom{[n]}{k}, |\{|F \cap F'|: F, F' \in \mathcal{F}\}| \leq s \Rightarrow |\mathcal{F}| \leq \binom{n}{s}$.

$(**)$: $\mathcal{F} \subseteq \binom{[n]}{k}, |F \cap F'| \not\equiv k (mod\ q) \Rightarrow |\mathcal{F}| \leq \binom{n}{q-1}$, q-prime power.

$$R(k) = \min\{N: \forall c: E(K_N) \to \{r, b\}\ \exists monochr.\, K_K\}$$

Theorem (Frankl & Wilson): $R(k) \geq \underbrace{e^{\frac{\log^2 k}{20 \log\log k}}}_{\xi}$, k-large.

Moreover there is an explicit construction giving this bound.

Proof: We shall construct a graoh G on ξ vertices such that $w(G) < k, \alpha(G) < k$. Later, color $E(G)$ red, rest of vertex paris blue. This coloring has no red K_K, has no blue K_K.

$V(G) := \binom{X}{q^2-1}, |X| = q^3$, q is a large prime power.

$E(G) := \{\{F, F'\}: F, F' \in V(G), |F \cap F'| \not\equiv -1(\text{mod q})\}$.

If $K_m \subseteq G \Rightarrow m \leq \binom{q^3}{q-1}$ by $(**)$.

$$|F \cap F'| \not\equiv -1(\text{mod q}) \equiv q^2 - 1 (mod\ q)$$

If G contains an independent set of m vertices. Thus$\forall F, F': |F \cap F'| \not\equiv -1(\text{mod q})$, i.e. $|F \cap F'| \in \{\text{q} - 1, 2\text{q} - 1, 3\text{q} - 1, \dots, \text{q}^2 - q - 1\}$.

By $(*)$: $m \leq \binom{q^3}{q-1}$.

So we have G on $\binom{q^3}{q^2-1} = n$ vertices such that no clique or independent set on more than m elements, $k := m + 1, k \cong \binom{q^3}{q-1}$. We want: $n \geq f(k)$.

We use $\left(\frac{n}{x}\right)^x \leq^1 \binom{n}{x} \leq^2 n^x$. First bound k in terms of q, then bound n terms of k. We have $k = \binom{q^3}{q-1} \Rightarrow q^{3q} >^2 k >^1 q^q$.

Take $log \Rightarrow 3q \log q >_2 \log k >_3 q \log q \Rightarrow \frac{\log k}{3 \log q} <^7_{from\ 4} q <^5_{from\ 3} \frac{\log k}{\log q} < \log k \Rightarrow$
$\log q <^6_{from\ 5} \log\log k \Rightarrow_{plug\ into\ 7} q > \frac{\log k}{3 \log\log k}$.

$$n = |V(G)| = \binom{q^3}{q^2-1}$$
$$> q^{q^{\frac{2}{2}}} >_{plug\ LB\ on\ q} \left(\frac{\log k}{3 \log\log k}\right)^{\frac{\log^2 k}{9(\log\log k)^2 \cdot 2}}$$
$$= e^{\frac{\log^2 k}{18(\log\log k)^2} \cdot (\log\log k - \log 3 \log\log k)} \geq e^{\frac{\log^2 k}{2o(\log logk)}}$$

Note: If k-large, then $e^{\frac{\log^2 k}{20 \log\log k}} < c^k$ for any $c > 1$. Take log: $\frac{\log^2 k}{20 \log\log k} < k \log c$. On the other hand, $e^{\frac{\log^2 k}{20 \log\log k}} > k^m$ for any m-fixed, take log $\frac{\log^2 k}{20 \log\log k} > m \log k$.

29.1 Induced Ramsey Numbers

$IR(H)$
$:= \min\{N: \exists a\ graph\ G\ on\ n\ vertices\ s.t. \forall c: E(G)$
$\to \{r, b\}\ \exists monochr.\, induced\ subgraph\ isomorphic\ to\ H\}$.

Example: H=Squaregraph, we want G.

Note: $IR(K_k) = R(K_k) = R(k)$. Take $G = K_{R(k)}$.

Conjencture (Erdös): $IR(H) \leq 2^{ck}, k = |V(H)|$. Best known $IR(H) \leq 2^{ck \cdot \log k}, k = |V(H)|$.

Goal: $IR(H)$ exists for bipartite H.

Definition: Incidence graph $I(X, k) = G\left(X \cup \binom{x}{k}, a \in X, a \in A\right)$.

$$E = \left\{\{A, a\}: A \in \binom{X}{k}, a \in X, a \in A\right\}.$$

Lemma 1: Any bipartite graph is an induced subgraph of an appropriate incidence graph. Specifically, if $B = (\{a_1, \dots, a_n\} \cup \{b_1, \dots, b_n\}, E)$, then $B \subseteq_{induced} I, I = \left(X \cup \binom{X}{n+1}, E\right), |X| = 2n + m$.

Proof: Let $X := \{x_1, \dots, x_n, y_1, \dots, y_n, \dots, z_1, \dots, z_m\}$.

Embed B into I with γ. Let $\gamma(a_i) = x_i, i = 1, \dots, n; \gamma(b_i) = \{z_i\} \cup \left\{x_i, \dots, x_{i_a}\right\} \cup Y'$, where $\left\{a_i, \dots, a_{i_q}\right\} = N(b_i), |\gamma_i| = n + 1 - 1 - q$.

Lemma 2: Any coloring of edges of $\left(X' \cup \binom{X'}{2k-1}, E\right)$ contains an induced monochromatic $\left(X \cup \binom{X}{k}, E\right)$, where $|X'|$ is multicolor (2^{2k-1}) colors hypergraph Ramsey number with uniformity $2k - 1$ and unavoidable clique size $k \cdot |X| + l - 1$.

Proof: Let $X' = \{x, \dots\}, Y' = \binom{X'}{2k-1}, c: E' \to \{r, b\}$.

Color vertices of Y' with $c': c'(y) = \left(c(yx_i), c\left(yx_{i_2}\right), \dots, c\left(yx_{i_{2k-1}}\right)\right)$. #colors in c' is $\leq 2^{2k-1}$. By Ramsey theorem $\exists Z \subseteq X'$ such that $|Z| = k|X| + k - 1, \binom{Z}{2k-1}$ have the same color c_0, ex. $c_0 = (r, r, b, r, b)$.

lbach@outlook.de
lukasbach.com

Assume red is the majority color in c_0, i.e. each vertex in $\binom{Z}{2k-1}$ send $\geq k$ red edges to Z. We shall find $(X \cup \binom{X}{k}, E)$ in red.

We shall embed X into each k^{th} vertex of Z.

Let $X \subseteq Z, \forall y \in \binom{X}{k}$ let $\gamma(y) = y' \in \binom{Z}{2k-1}$ such that $y' \geq y, y' \backslash y \subseteq Z \backslash X$. Then X and $\gamma(y), y \in \binom{X}{k}$ form a red induced copy of $\left(X \cup \binom{X}{k}, E\right)$. Qed.

30 Lecture 27.01.

$R(H) = R(H,H)$. We know $\sqrt{2}^k \leq R(K_k) = R(k) \leq 4^k$.

Chratal-Rüdl-Szemeredi-Trotter: $\forall$positive $\Delta\ \exists c = c(\Delta)\ \forall H: \underbrace{\Delta(H) = \Delta}_{\max degree}, R(H) \leq c\ \underbrace{|V(H)|}_{k}$

Choonbun Lee (2015): $\forall$ positive $d\ \exists c = c(d)\ \forall H\ \ H - d - degenerate\ \ R(H) \leq c\,\underbrace{|V(H)|}_{k}$.

30.1 Random graphs

Consider a set of all graphs on n vertices.

> $G(n,p)$ is the Erdös-Renyi random graph, and is created by choosing edges independently with probability $P_{\binom{n}{2}-m}$, In particular $Prob(G) = p^m(1-p)$ with a given graph G on n vertices and m edges.

Abb1

Note: $\sum_{G\ graph\ on\ [n]} Prob(G) = \sum_{m=0}^{\binom{n}{2}} \overbrace{\binom{\binom{n}{2}}{m}}^{\#graphs\ on\ m\ edges} \underbrace{p^m(1-p)^{\binom{n}{2}-m}}_{prob\ of\ occurence\ of\ fixed\ m-edge\ graph} = (p + (1-p))^{\binom{n}{2}} = 1$, with m being the number of edges of G.

Lemma 1: Let $0 < p < 1$, p constant, H be a fixed graph, $G \in G(n,p)$, $Prob(H \subseteq_{ind} G) \to_{n\to\infty} 1$.

Proof: $k := |V(H)|$, $n = tk + \varepsilon, t \in \mathbb{N}, 0 \leq \varepsilon \leq k$. Let $V(G) = A_1 \cup \ldots \cup A_t \cup A_\varepsilon, |A_i| = k, 1 \leq i \leq t$.

$$Prob(H \neg\subseteq_{ind} G) \leq Prob(H \neg\subseteq_{ind} G[A_1]\ \&\ H \neg\subseteq_{ind} G[A_2]\ \&\ \ldots\ \&\ H \neg\subseteq_{ind} G[A_t])$$

$$=_{A_i's\ pairwise\ disjoint} Prob(H \neg\subseteq_{ind} G[A_1]) \cdot \ldots \cdot Prob\ (H \neg\subseteq_{ind} G[A_t])$$

$= (1-r)^t$, where r is the $Prob\ (H \subseteq_{ind} G[A_i])$ and $r > 0$ and r is independent on n. We have $t = \left\lfloor \frac{n}{k} \right\rfloor \to_{n\to\infty} \infty$.

Thus $Prob\ (H \subseteq_{ind} G) = 1 - Prob(H \neg\subseteq_{ind} G) \to_{n\to\infty} \infty$.

Lemma 2: $n \geq k \geq 2$ integers, $G = G(n,p)$. $Prob\ (\alpha(G) \geq k) \leq \binom{n}{k}(1-p)^{\binom{k}{2}}$, $Prob(\omega(G) \geq k) \leq \binom{n}{k} p^{\binom{k}{2}}$.

Proof: $Prob(\alpha(G) \geq k) = Prob\left(\exists U \subseteq \binom{[n]}{k}, G[U] \cong E_k\right) \leq Prob\left(\bigvee_{U \in \binom{[n]}{k}} G[U] \cong E_k\right) \leq \sum_{U \in \binom{[n]}{k}} Prob(G[U] \cong E_k) \leq \binom{n}{k}(1-p)^{\binom{k}{2}}$ with E_k being the empty graph. Qed.

Lemma 3: $G = G(n,p)$, then $Exp(\#cycles\ of\ length\ k\ in\ G) = \frac{(n)_k}{2k} p^k$, where $(n)_k = n \cdot (n-1) \cdot \ldots \cdot (n-k+1)$.

Proof of Lemma 3: Let C_k be a set of all cycles of length k in K_n. For $C \in C_k$, let $X_c := \begin{cases} 1 & C \subseteq G \\ 0 & otherwise \end{cases}$. Then $X = \sum_{C \in C_K} X_c =$ #cycles of length k in G.

$Exp(X) = |C_k| \cdot Prob(X_c = 1) = \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{2k} p^k$. Qed.

Theorem (Erdös, Hajnal): $\forall k \in \mathbb{Z},\ k \geq 3\ \exists$graph $G, girth(G) > k$ and $\chi(G) > k$.

Idea: # cycles of length $\leq k$ is $< \frac{n}{2}$. Delete a vertex rom each of these, get a graph G' with $girth(G') > k$. We know that $\chi(G') \geq \frac{|V(G')|}{\alpha(G')}$.

Proof: Fix $\varepsilon, 0 < \varepsilon < \frac{1}{k}, p := n^{\varepsilon - 1}$, let $G = G(n,p)$. Let Y be # cycles of length $\leq k$ in G. By Lemma 3, $Exp(Y) = \sum_{i=3}^k \frac{(n)i}{2i} p^i \leq \frac{1}{2} \sum_{i=3}^k n^i p^i \leq \frac{1}{2} n^k p^k \cdot k$. $np = n \cdot n^{\varepsilon-1} = n^\varepsilon > 1$.

By Markov's inequality $Prob\left(Y \geq \frac{n}{2}\right) \leq \frac{Exp(Y)}{\frac{n}{2}} \leq n^{k-1} p^k k = kn^{k-1} n^{(\varepsilon-1)k} = kn^{\varepsilon k - 1} \to_{n\to\infty} 0$ because $\varepsilon k < 1$, thus $\varepsilon k - 1 < 0$.

By Lemma 2, $Prob\left(\alpha(G) \geq \frac{n}{2k}\right) \leq \binom{n}{\frac{n}{2k}} \cdot (1-p)^{\binom{\frac{n}{2k}}{2}} \to_{n\to\infty} 0$. Indeed $\binom{n}{\frac{n}{2k}}(1-p)^{\binom{\frac{n}{2k}}{2}} \leq n^{\frac{n}{2k}}\left(1 - \frac{1}{n^{1-\varepsilon}}\right)^{\frac{\left(\frac{n}{2k}\right)^2}{4}} \underbrace{\cong (1-x) \cong e^{-x}}_{small\ x} \cong n^{\frac{n}{2k}} e^{-n \cdot \frac{\frac{n^2}{(2k)^2}}{4}} \cong \frac{e^{\frac{n}{2k} nn - n^{1+\varepsilon}}}{(2k)^2 4} \to_{n\to\infty} 0$.

Choose n large enough such that $Prob\left(Y \geq \frac{n}{2}\right) < \frac{1}{2}$ and $Prob\left(\alpha(G) \geq \frac{n}{2k}\right) < \frac{1}{2}$. Thus $\exists$ a graph G^* on n vertices with $< \frac{n}{2}$ cycles of length $\leq k$ and $\alpha(G^*) < \frac{n}{2k}$. Let G' be obtained from G^* by deleting a vertex from each cycle of length $\leq k$, i.e. $|V(G')| \geq \frac{n}{2}$, $girth(G') > k$, $\alpha(G') \leq \alpha(G^*) < \frac{n}{2k}$. Thus $\chi(G') \geq \frac{|V(G')|}{\alpha(G')} > k$. Qed.

Expand $\chi(G') \geq \frac{|V(G')|}{\alpha(G')} > \left(\frac{\frac{n}{2}}{\frac{n}{2k}}\right) = k$. Thus G' is the desired graph.

30.2 Graph properties and threshold functions

A graph property is a set of graphs.

Example: $\mathcal{P} = \{G: G \text{ is connected}\}$.

Let $(p_n) \in [0,1]^{\mathbb{N}}$, we say that $G = G(n, p_n)$ almost always has the property $\mathcal{P}$ if $Prob(G \in \mathcal{P}) \to_{n\to\infty} 1$.

A function $f(n): \mathbb{N} \to [0,1]$ is a threshold function for $\mathcal{P}$ if

1. $\forall p_n: \frac{p_n}{f(n)} \to_{n\to\infty} 0$, then $G = G(n, p_n)$ almost always does not have $\mathcal{P}$.
2. $\forall p_n: \frac{p_n}{f(n)} \to_{n\to\infty} \infty$, then $G = G(n, p_n)$ almost always has $\mathcal{P}$.

1. $p_n = o(f(n)) \to$ no $\mathcal{P}$
2. $p_n = \omega(f(n)) \to \mathcal{P}$

If $f(n) = \frac{1}{n} - threshold$ for $\mathcal{P}$

$G(n,p)$ almost always has $\mathcal{P}$.

If $p = \frac{1}{2}$ or $p = \frac{1}{4}, p = \frac{1}{1000}, p = \frac{1}{\sqrt{n}}, p = \frac{1}{\log n}$

$G(n,p)$ almost alwas has no $\mathcal{P}$.

If $p = \frac{1}{n^2}, p = \frac{1}{n\sqrt{n}}, p = \frac{1}{n \log n}$

(?)

31 Problem class 29.01.

31.1 The Probabilistic Method

Example: Any graph G with m edges has a bipartite subgraph with $\geq \frac{m}{2}$ edges. (Trivial proof: Consider bipartite subgraph with maximal edges)

Probabilistic proof: Define a subset $T \subset V(G)$ by including a vertex $v \in T$ by including a vertex $v \in T$ with probability $\frac{1}{2}$.

For every edge $xy \in E(G)$: $X_{xy} = \begin{Bmatrix} 1 & \text{if } xy \text{ are crossing} \\ 0 & \text{otherwise} \end{Bmatrix}$

(Crossing edge: edge that goes from T to $V \backslash T$).

Let $\sum_{xy\in E(G)} X_{xy}$; $\mathbb{E}X \overset{linearity}{\cong} \sum_{xy\in E(G)} \mathbb{E}[X_{xy}] = \sum_{xy\in E(G)} \underbrace{\mathbb{P}(xy \text{ is crossing})}_{\frac{1}{4}+\frac{1}{4}=\frac{1}{2}} = \frac{m}{2}$.

$\Rightarrow$ There is at least one choice of T such that $X \geq \mathbb{E}X = \frac{m}{2}$. Qed.

$X: \Omega \to \mathbb{R}$ is a discrete random variable. Suppose $X(\omega) < \mathbb{E}X, \forall \omega \in \Omega$: $\mathbb{E}X \coloneqq \sum_{\omega\in\Omega} X(\omega)\mathbb{P}(\omega) < \mathbb{E}X \cdot \underbrace{\sum_{\omega\in\Omega}\mathbb{P}(\omega)}_{=1} = \mathbb{E}X$.

Example: In Tournaments. Tournament is obtained by orienting the edges of the complete graph (choose directions for the edges).

Consider the property S_k of a graph T. For every k-element subset $S \subset V(T)$ there exists a vertex y outside of T that "beats all of S" (that has k edges leading *into* S).

Let's show that such tournaments exist. Let T be a tournament on n vertices choosen uniformly at random (by orienting edges with probability $\frac{1}{2}$). $\forall S \subset V(T), |S| = k$; consider B_S as "S violates property S_K".

$$\mathbb{P}(B_S) = (1 - 2^{-k})^{n-k}$$

$$\mathbb{P}\left(\bigcup_S B_S\right) \leq \binom{n}{k}(1-2^{-k})^{n-k} \qquad \underbrace{< 1}_{choose\ n\ usfficiently\ large\ depending\ on\ k}$$

Then with positive probability none of the bad events occur, i.e. there is a tournament with property S_K.

32 Lecture 31.01.

Recall on Threshold functions: For a property $\mathcal{P}$, $f(n)$ is a structural function for $\mathcal{P}$, if the following conditions hold:

1. $\forall p_n = \omega(f(n))$: $Prob(G(n, p_n) \in \mathcal{P}) \to_{n\to\infty} 1$.
2. $\forall p_n = o(f(n))$: $Prob(G(n, p_n) \in \mathcal{P}) \to_{n\to\infty} 0$.

Recall: Markov's inequality $t > 0$ for a random variable $X, X > 0, Prob(X \geq t) \leq Exp(x)$

[TODO ANNE]

33 Lecture 03.02.

33.1 Hamilton Cylcles

Hamilton Cycles were introduced by Sir William Rovan Hamilton 1857 via a game "Icosian game" (Traveling Salesperson problem, min. weight of a Hamilton Cycle, edges are weighted).

A Hamilton Cycle is a cycle C in a graph G that is spanning, thus if it contains all vertices. A graph that has a Hamilton Cycle is called Hamiltonian graph.

Lemma (necessary condition for Hamiltonicity): If G is Hamiltonian, then $\forall S \subseteq V(G), S \neq \emptyset$: #components of $G - S$ is at most $|S|$.

Proof: Let C be a Hamiltonian Cycle of G. Let $S \subseteq V(G), S \neq \emptyset, t \coloneqq$ #components of $G - S$. There are at least 2 edges of C between each component of $G - S$ and S. If $e =$ #edges

lbach@outlook.de
lukasbach.com

of C between S and $V - S$, we have $e \geq t - 2$ and $e \leq |S| \cdot 2$ (C is 2-regular). Then $t \leq |S|$. Qed.

Theorem (Dirac): If $|V(G)| = n, n \geq 3, \delta(G) \geq \frac{n}{2}$, then G is Hamiltonian.

Proof: Let $\delta(G) \geq \frac{n}{2}, |V(G)| = n \geq 3$. Let k be the number of vertices in a longest path.

Claim: Any path on k vertices span a cycle. Assume not, consider a path $p = (v_0, \dots, v_{k-1})$. Then

1. $v_0 v_{k-1} \notin E(G)$
2. If $v_i \in N(v_0) \Rightarrow v_{i-1} \notin N(v_{k-1})$
3. $N(v_0), N(v_{k-1}) \subseteq V(P)$ (maximality of P)

Then #non-neighbors of v_{k-1} in $V(P)$ ≥#neighbors of v_0 in $V(P) \geq \delta(G) \geq \frac{n}{2}$.

#neighbors of $v_{k-1} \leq (k-1) -$ (#nonneighbors of v_{k-1} in $V(P)) \leq (k-1) - \frac{n}{2} \leq n - 1 - \frac{n}{2} \leq \frac{n}{2} - 1$, which is a contradiction to the Claim.

Case 1: $k = n \Rightarrow$ by Claim, $V(P)$ spans a cycle on n vertices, i.e. Hamiltonian cycles.

Case 2: $k < n$. Note that G is connected, otherwise a vertex in a smallest component has degree $\leq \frac{n}{2} - 1 - y$. By claim, $P = (v_0, \dots, v_{p-1})$ spans a cycle. There is an edge between a cycle C of length k and a vertex outside of the cycle. Then C and e span a path on $k + 1$ vertices, which is a contradiction.

Other degree conditions:

Ore's theorem: A graph G on $n \geq 3$ vertices is Hamiltonian $\Leftrightarrow \forall u, v \in V(G), uv \notin E(G), d(u) + d(v) \geq n$.

Komlós-Sárkközy, Szemerédi (gen. of Dirac): $\delta(G) \geq \frac{k}{k+1} n$, then G has a k^{th} power of a Hamiltonian Cycle, that is a subgraph obtained from a Hamiltonian Cycle by joining all vertices at distance $\leq k$ on the cycle by an edge.

Csába, Kühn, Osthus, Lo, Treglown 2014: For sufficiently large n, each d-regular graph with $d \geq \left\lfloor \frac{n}{2} \right\rfloor$ has an edge-decomposition into Hamilton cycles and at most one matching.

Theorem: Let $\alpha(G) \leq k(G) \wedge |V(G)| \geq 3 \Rightarrow G$ is Hamiltonian, with $k(G)$ denoting G's vertex connectivity.

Proof: Let C be a longest cycle in G. $C := (v_0, v_1, \dots, v_{m-1}, v_0)$. If C is not Hamiltonian, $\exists v \in V(G) \setminus V(C)$. Let F be a $C - R$-fan, i.e. $F = \{P_i : P_i \textit{ is a } v_i - v - \textit{path}, i \in I\}$. P_i's share only v pairwise. Moreover, let F be of maximal cardinality. By Menger's theorem, $|F| \geq \min\{k, |C|\}$. We have $\forall i \in I : i + 1 \ (mod\ m) \notin I$, otherwise C is not longest. $\forall i, j \in I, i \neq j : v_{i+1}, v_{j+1} \notin E(G)$.

Thus $\forall i \in I, v_{i+1} v \notin E(G)$ and $\{v_i : i \in I\} \cup \{v\}$ is an independent set on $|I| + 1 \geq k + 1$ elements, which is a contraidction. ($|F| < |C| \Rightarrow |F| \geq r$). Qed.

Tutte: $k(G) \geq 4$, G is planar ⇒ G is Hamiltonian.

Thomassen 1983: $k(G) \geq 4$, G is planar ⇒ G is Hamiltonian-connected, i.e. $\forall u, v \in G$ there is a $u - v -$path that is spanning, i.e. a Hamiltonian path.

33.2 Network flows

Let G be a graph (multigraph, no loops), $s, t \in V(G), s \neq t$, source s, sink t.

$T := \{(x, e, y) : e = xy \in E(G)\}$.

Let $c : T \to \mathbb{N} \cup \{0\}$ is a capacity function.

A network is a quadruple $N = (G, s, t, c)$.

A function $f : T \to \mathbb{R}$ is a network flow if the following conditions hold:

1. $f(x, e, y) = -f(y, e, x) \ \forall (x, e, y) \in T$
2. $f(x, V(G)) := \sum_{v \in V \setminus x, (x,e,v) \in T} f(x, e, v) = 0 \ \forall x \in V(G) \setminus \{s, t\}$

A cut $(S, \bar{S})$ is a pair $S \subseteq V(G), s \in S, t \notin S, \bar{S} = V(G) - S$.

Capacity of a cut $c(S, \bar{S}) := \sum_{x \in S, y \in \bar{S}, (x,e,y) \in T} c(x, e, y)$.

34 Problem class 05.02.

34.1 Proof Techniques v2

1. Induction
2. Extremal Principle/Contradiction
3. Counting arguments
   - ➔ $\left[ex(n, K_{t,t}) \leq cn^{2-\frac{1}{t}} : Double\ Counting \right]$
   - ➔ Pigeonhole Principle
   - ➔ Parity Arguments (even vs odd)
4. Algorithmic/ Iterative ("Just do it")
5. "Dichotomy"/Ramsey
6. Probabilistic methods
   - ➔ $\mathbb{P}(\cup\ "BadEvent") < 1$
   - ➔ Computing $\mathbb{E}X$
   - ➔ Alterations [e.g. $ex(n, K_{t,t}) \geq Cn^{2-\frac{1}{t+1}}$]. Choose $G \in G(n, p)$ for appropriate p. Compute $\mathbb{E}\#K_{t,t}$'s. Delete an edge from each copy of $K_{t,t}$. (Erdös) Construction of graph G with $\chi(G) > k, g(G) > k$.
7. Apply a theorem

35 Lecture 07.02.

Definition: Given $g : T \to \mathfrak{S}. \forall X, Y \subseteq V(G) : g(X, Y) := \sum_{x \in X, y \in Y, x \neq y, (x,e,y) \in T} g(x, e, y)$.

Notation: $f(x, V) = f(\{x\}, V) = \sum_{xey \in T} f(x, e, y)$

Lemma 1: $\forall$cut $(S,\bar{S})$: $f(S,\bar{S}) = f(s,V)$ where f is a network flow.

(Recall $(S,\bar{S})$ is a cut if $S \subseteq V(G), s \in S, t \notin S, \bar{S} = V \backslash \bar{S}$)

Proof of Lemma 1: $f(S,\bar{S}) = f(S, V-S) = f(S,V) - f(S,S) = \left(f(s,V) + \underbrace{\sum_{v \in S\backslash\{s\}} f(v,V)}_{=0,} \right) - \underbrace{f(S,S)}_{=0} = f(s,V)$

Qed.

Let the value of the flow f be $f(s,V)$, denote it $|f|$.

Theorem (Ford-Fulkerson): Let $N = (G,s,t,c)$ be a network, then $|\max \underbrace{\{|f|: f-N-flow\}}_{:=x} = \underbrace{\min\{c(S,\bar{S}): S,\bar{S}-cut\}}_{:=y}$.

Proof:

- $x \le y$: $\forall f-N-flow: |f| = f(s,V) =^{\forall cut}_{Lemma 1} f(S,\bar{S}) \le c(S,\bar{S})$
- $y \le x$: We shall construct an $f-N-flow$: $|f| = c(S,\bar{S})$ for some cut. We shall construct N-flows $f_0, f_1, \dots$ such that $f_0 \equiv 0, |f_{i+1}| \ge |f_i| + 1$. (The sequence is finite since $|f| \le (S,\bar{S})$.
  Suppose f_n has been constructed.
  Case 1: $\exists$ augmented path: $s = x_0 e_0 x_1 e_1 \dots x_m = t$. $f_n(x_{i-1}e_i x_i) < c(x_{i-1}e_i x_i) \forall i = 1, \dots, m$. Let $\varepsilon := \min\{c(x_{i-1}e_i x_i) - f(x_{i-1}e_i x_i)\}$.
  $$f_{n+1}(xey) := \begin{Bmatrix} f_n(x_{i-1}e_i x_i) + \varepsilon & i = 1, \dots, m \\ f_{n(x_i e_i x_{i-1})} - \varepsilon & i = 1, \dots, m \\ f_n(xey) & otherwise \end{Bmatrix}$$
  Case 2: There exists no such path. Let $S := \{v \in V: \exists path\ s = x_0 e_0 x_1 \dots v;\ (v = x_m), f(x_{i-1}e_{i-1}x_i) < c(x_{i-1}e_{i-1}x_i), i = 1, \dots, m\}, s \in S, t \notin S$.
  $f_n(x,e,y) = c(x,e,y)\ \forall x \in S, y \in \bar{S}, (x,e,y) \in T \Rightarrow f_n(S,\bar{S}) = c(S',\bar{S})$. In thise case, let $f := f_n$.

Qed.

35.1 Group valued flows

Let G be a multigraph with loops allowed, T as above. Let $f: T \to H$, where H as an abelian group is a circulation if

1. $f(x,e,y) = -f(y,e,x)\ \ \forall (x,e,y) \in T,\ \ x \ne y$
2. $f(x,V) = 0\ \ \forall x \in V$.

A circulation f is an H-flow if it is nonzero on each triple.

If an H-flow exists for f, in particular $\mathbb{Z}$-flow exists, then $\gamma(G) := \min\{k: G \text{ has a k-flow, i.e. } \mathbb{Z}\text{-flow } |f(x,e,y)| < k\ \ \forall (x,e,y) \in T\}$, where γ denotes the flow value of G.

By Lemma 1, $f(S,\bar{S}) = f(v,V)\ \forall v \in V\ \ \forall S \subseteq V, S \ne \emptyset, S \ne V$.

This implies that $f(S,\bar{S}) = 0$ (which does not hold if there exists a bridge) $\Rightarrow$ G is bridgeless.

Theorem (Seymour): If G is bridgeless, then it has a nowhere zero $\mathbb{Z}_6$-flow.

Theorem (Tutte): $\forall$multigraph $G = (V,E,T)$ $\exists$polynomial $P \in \mathbb{Z}[x]$ such that $\forall$abelian group H, the number of zero H-flows on G is equal to $P(|H|-1)$.

(I.e. the number of flows depends on the order of H and not the structure of H)

~~Proof: Induction on the number of non-loop edges = x.~~

~~Basis: $x = 0$, Multigraph consists only of loops on single vertices. We can assign any nonzero value to any triple. #such assignments = $(|H|-1)^{\|G\|}$.~~

~~Step: $x = k \to k+1$: Assume $\exists$ non-loop edge $e_0 = xy$.~~

Tutte: For a plane graph G, G^* is dual, then $\chi(G) = \gamma(G^*)$.