Cafourek

1) a) nad $\mathbb{R}$ $\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -5 & -3 & 1 \end{array}\right)^{R_1}_{R_2-3R_1} \sim \left(\begin{array}{cc|cc} 5 & 0 & -1 & 2 \\ 0 & -5 & -3 & 1 \end{array}\right)^{5R_1+2R_2}_{R_2} \sim$

$\sim \left(\begin{array}{cc|cc} 1 & 0 & -\frac{1}{5} & \frac{2}{5} \\ 0 & 1 & \frac{3}{5} & -\frac{1}{5} \end{array}\right)^{\frac{1}{5}\cdot R_1}_{-\frac{1}{5}\cdot R_2} \longrightarrow A^{-1} = \begin{pmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{3}{5} & -\frac{1}{5} \end{pmatrix} = \frac{1}{5}\cdot\begin{pmatrix} -1 & 2 \\ 3 & -1 \end{pmatrix}$

nad $\mathbb{F}_5$: $\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right)^{R_1}_{R_2+2R_1}$ ✗

b) $\mathbb{Z}_7$: $\left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 3 & 2 & 6 & 0 & 1 & 0 \\ 1 & 0 & 5 & 0 & 0 & 1 \end{array}\right) \sim$ ↖ singulární $\left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 3 & 1 & 4 & 1 & 0 \\ 0 & 5 & 1 & 6 & 0 & 1 \end{array}\right)^{R_1}_{R_2+4R_1}_{R_3-R_1} \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 3 & 1 & 4 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right)^{R_1}_{R_2}_{R_3+R_1} \sim$

$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 6 & 1 & 0 \\ 0 & 6 & 0 & 1 & 2 & 1 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right)^{2R_1+R_2}_{2R_2+R_3}_{R_3} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & 1 & 1 \\ 0 & 1 & 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{5} \end{array}\right)^{R_1+R_3}_{\frac{1}{6}\cdot R_2}_{\frac{1}{5}\cdot R_3}$

c) $\left(\begin{array}{ccccc|ccccc} 1 & -1 & 1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$

2) $B = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} & 0 \end{pmatrix}$ najít — úprava nelze, protože $\operatorname{rank}(A) = 2 \neq \operatorname{rank}(E_3) = 3$

3) a) $g: (\mathbb{Z}_3)^2 \to (\mathbb{Z}_3)^2$ $\quad K_2 = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$

$e_1 \mapsto e_1 + 2\cdot e_2$ $\quad \begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}1\\2\end{pmatrix}$

$e_2 \mapsto 2e_1 + e_2$ $\quad \begin{pmatrix}0\\1\end{pmatrix} \mapsto \begin{pmatrix}2\\1\end{pmatrix}$

$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$

$\operatorname{rank}(A) = \operatorname{def}(A) = 1$

b) $\ker(A) = \operatorname{span}\left(\begin{pmatrix}1\\1\end{pmatrix}\right)$

$\operatorname{im}(A) = \operatorname{span}\left(\begin{pmatrix}1\\2\end{pmatrix}\right)$

c) $B = \left(\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right)$

$M = T_{K_2 \to B} \cdot A \cdot T_{B \to K_2} =$

$\begin{pmatrix}0&2\\1&1\end{pmatrix}$ $\begin{pmatrix}1&2\\2&1\end{pmatrix} \cdot \begin{pmatrix}1&1\\2&0\end{pmatrix} = \begin{pmatrix}0&2\\1&1\end{pmatrix} \cdot \begin{pmatrix}2&1\\1&2\end{pmatrix} = \begin{pmatrix}2&1\\0&0\end{pmatrix}$

$\sim \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right)$

d) $\ker(M) = \operatorname{span}(\vec{w})$, $\operatorname{Ide\,coord}_B(\vec{w}) = \begin{pmatrix}1\\1\end{pmatrix}$

$\operatorname{im}(M) = \operatorname{span}(\vec{v})$, $\operatorname{Ide\,coord}_B(\vec{v}) = \begin{pmatrix}1\\0\end{pmatrix}$

4) $h: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ je to iso, protože $\operatorname{rank}(A) = 2$, $\operatorname{def}(A) = 0$

a) $\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$ b) $\begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix}$ c) $\begin{pmatrix} -1/5 & 2/5 \\ 3/5 & -1/5 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ 6 & 7 \end{pmatrix} \sim \begin{pmatrix} 7 & 4 \\ 0 & 25/7 \end{pmatrix}_{7R_2 - 6R_1}$

$\hookrightarrow \sim \begin{pmatrix} 1 & 2 \\ 0 & -5 \end{pmatrix}_{R_2 - 3R_1}$ $\quad \to$ taky iso, taky $\operatorname{rank}(A^2) = 2$ a $\operatorname{def}(A^2) = 0$

e) $\frac{1}{5} \cdot \begin{pmatrix} -1 & 2 \\ 3 & -1 \end{pmatrix} \cdot \frac{1}{5}\begin{pmatrix} -1 & 2 \\ 3 & -1 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 7 & -4 \\ -6 & 7 \end{pmatrix}$

5) $A[1,1], B[2,2]$

a) $ax^4 + bx^3 + cx^2 + dx + e$, $a \neq 0$, $\ker f = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \operatorname{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ -15 \\ 14 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -7 \\ 6 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -3 \\ 2 \end{pmatrix} \right) \right\}$

b) $\begin{pmatrix} 14\alpha + 6\beta + 2\gamma \\ -15\alpha - 7\beta - 3\gamma + 1 \\ \gamma \\ \beta \\ \alpha \end{pmatrix}$ c) $\begin{pmatrix} -\alpha - \beta - \gamma + 1 \\ -15\alpha - 7\beta - 2\gamma + 1 \\ \gamma + \beta \\ \beta + \alpha \\ \alpha \end{pmatrix}$

$B = \{1, x, x^2, x^3, x^4\}$ $\quad C = \{1, x-1, x^2 - x + 1, x^3 - x^2 + x - 1, x^4 - x^3 + x^2 - x + 1\}$

6) $\begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix} + \operatorname{span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right)$

$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right)_{R_2 - R_1}$ $\quad \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \operatorname{span}\left(\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$

$(-1\ 1\ 0\ 0)\begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix} = -1 + 2 + 0 + 0 = 1 \longrightarrow -x + y = 1$

$(0\ 0\ 0\ 1)\begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix} = 0 + 0 + 0 + 3 = 3 \longrightarrow t = 3$

7)