Evaluating the Robustness of Energy-Based Swing-Up Control for Inverted Pendulum Systems

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***Abstract*—Problem Statement: The inverted pendulum on a cart is a highly nonlinear, multivariable, and inherently unstable system. It serves as a benchmark for testing advanced control strategies and teaching modern control theory. The objective of this study was to investigate how varying initial conditions affect the robustness of an energy-based swing-up controller for the inverted pendulum system, combined with a Linear Quadratic Regulator (LQR) for stabilization.**

Approach: The study proposes a hybrid control strategy combining an energy-based swing-up controller and an LQR for stabilization. The energy-based controller regulates the pendulum's total mechanical energy to swing it from the downward to the near-upright position. Once near the upright equilibrium, the LQR stabilizes the pendulum by minimizing deviations in state variables. Simulations were conducted to validate the robustness of the swing-up controller across various initial conditions and the performance of the hybrid control strategy.

Results: The simulation results demonstrate the robustness of the energy-based swing-up controller in transitioning the pendulum from a range of initial angles to the upright position. The LQR achieves effective stabilization with minimal oscillations and efficient control effort. Smooth transitions between the controllers were observed across all test scenarios.

Conclusion: The proposed hybrid control strategy effectively addresses the nonlinear dynamics and stabilization challenges of the inverted pendulum system while demonstrating robustness to varying initial conditions. This work provides a framework for further experimental validation and the development of advanced control techniques for similar unstable systems.

I. Introduction

The inverted pendulum on a cart is a benchmark problem in control theory, renowned for illustrating the challenges of stabilizing inherently unstable systems. Since the 1960s, such systems have been employed to explain concepts in linear control. From the 1990s onwards, their nonlinear dynamics have made them useful for exploring advanced control methods such as passivity-based control, backstepping, and task-oriented techniques like swing-up control [1] [2]. Among the approaches in the literature, energy-based control methods have gained particular attention for their simplicity and effectiveness in addressing the nonlinearities of the swing-up phase.

Åström and Furuta [1] formalized the dynamics of the inverted pendulum and introduced energy-based strategies for swing-up control. Their method regulates the pendulum's total mechanical energy to drive it toward the upright position, offering a robust framework for nonlinear control. Building on these principles, Chung and Hauser [3] proposed a strategy that integrates energy regulation with cart position control, demonstrating effective stabilization. Wei et al. [4] further developed swing-up strategies to accommodate systems with limited cart travel.

Research on related systems, such as the rotary inverted pendulum, has demonstrated the effectiveness of combining energy-based and optimal control techniques. Sukontanakarn and Parnichkun [5] applied an energy-based proportional-derivative (PD) controller for swing-up and a linear quadratic regulator (LQR) for stabilization. Their results, validated through simulations and real-time experiments, highlighted the robustness of these control techniques in handling nonlinear dynamics and maintaining stability. While their work focused on rotary systems, the insights are applicable to cart-based inverted pendulum setups, particularly the use of energy-based methods for swing-up and state-feedback control for stabilization.

Practical considerations in real-world systems, such as rail length and cart velocity limitations, introduce additional challenges to control design. Muskinja and Tovornik [6] addressed these constraints and compared the performance of energy-based and fuzzy control methods in achieving swing-up and stabilization. Their findings underscore the importance of accounting for physical limitations and highlight the adaptability of energy-based methods in various experimental settings.

The inverted pendulum system presents two primary challenges:

1) Swinging the pendulum from the downward to the upright position requires overcoming the system's nonlinear dynamics while respecting physical constraints.
2) Stabilizing the pendulum at the upright equilibrium demands robust linear control capable of rejecting disturbances and minimizing oscillations.

This work investigates how varying initial conditions affect the performance and robustness of a hybrid control strategy that combines an energy-based swing-up algorithm and a Linear Quadratic Regulator (LQR) for stabilization. The swing-up controller drives the system toward the upright position by regulating the pendulum's total energy, while the LQR ensures stabilization near the equilibrium using optimal control. This method builds on the principles outlined by Åström and Furuta [1], incorporating insights from Sukontanakarn and Parnichkun [5] and Muskinja and Tovornik [6] to address both theoretical and practical aspects of inverted pendulum control.

II. Mathematical Preliminaries

We use the Lagrangian approach to describe the dynamics of the inverted pendulum system. This section details the kinematics, potential energy, kinetic energy, and the resulting equations of motion.

A. Equations of Motion

1) System Parameters: The parameters used to model the cart-pendulum system, as shown in Figure 1, are:

- m_c: Mass of the cart
- m_p: Mass of the pendulum
- l: Length of the pendulum (from pivot to end)
- x_c: Horizontal position of the cart
- $\dot{x}_c$: Velocity of the cart
- θ: Angle of the pendulum from the vertical
- $\dot{\theta}$: Angular velocity of the pendulum
- g: Acceleration due to gravity
- F: Force applied to the cart
- COM: Center of mass

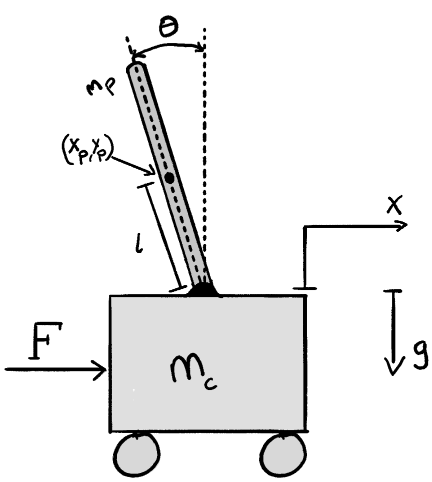

Fig. 1. Pendulum model with parameters.

2) Kinematics: The position of the pendulum's centre of mass (COM) is given by:

$$x_p = x_c - l\sin(\theta), \tag{1}$$

$$y_p = l\cos(\theta). \tag{2}$$

The velocity of the pendulum's centre of mass (COM) is given by:

$$\dot{x}_p = \dot{x}_c - l\dot{\theta}\cos(\theta), \tag{3}$$

$$\dot{y}_p = -l\dot{\theta}\sin(\theta). \tag{4}$$

3) Potential Energy: The potential energy of the system is derived from the height of the pendulum's centre of mass:

$$V = m_p g y_p = m_p g l\cos(\theta). \tag{5}$$

4) Kinetic Energy: The kinetic energy of the pendulum is derived using the point mass assumption, which simplifies the calculation by avoiding the need to account for rotational inertia. The kinetic energy consists of the translational contributions from both the cart and the pendulum's centre of mass:

$$T = \frac{1}{2}m_c\dot{x}_c^2 + \frac{1}{2}m_p\left(\dot{x}_p^2 + \dot{y}_p^2\right). \tag{6}$$

Expanding and simplifying:

$$T = \frac{1}{2}(m_c + m_p)\dot{x}_c^2 + \frac{1}{2}m_p l^2\dot{\theta}^2 - m_p l\dot{x}_c\dot{\theta}\cos(\theta). \tag{7}$$

This approach models the pendulum as a point mass, focusing on the translational motion of its centre of mass rather than considering rotational dynamics.

5) Lagrangian and Equations of Motion: The Lagrangian is defined as:

$$L = T - V. \tag{8}$$

Substituting T and V:

$$L = \frac{1}{2}(m_c+m_p)\dot{x}_c^2+\frac{1}{2}m_p l^2\dot{\theta}^2-m_p l\dot{x}_c\dot{\theta}\cos(\theta)-m_p gl\cos(\theta). \tag{9}$$

Using the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i, \tag{10}$$

we derive the equations of motion for x_c and θ.

For x_c:

$$(m_c + m_p)\ddot{x}_c + m_p l\ddot{\theta}\cos(\theta) + m_p l\dot{\theta}^2\sin(\theta) = F. \tag{11}$$

For θ:

$$m_p l^2\ddot{\theta} + m_p l\ddot{x}_c\cos(\theta) - m_p gl\sin(\theta) = 0. \tag{12}$$

B. State Space and LQR

1) State-Space Representation: To design controllers, the system is linearized around the upright equilibrium ($\theta \approx 0$ and $\dot{\theta} \approx 0$). The resulting state-space representation is:

$$\dot{X} = AX + BU, \tag{13}$$

Where the state vector is:

$$X = \begin{bmatrix} x_c \\ \dot{x}_c \\ \theta \\ \dot{\theta} \end{bmatrix}. \tag{14}$$

The A matrix is:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{m_p g}{m_c+m_p}\cdot\frac{1}{\left(\frac{4}{3}-\frac{m_p}{m_c+m_p}\right)} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{l\left(\frac{4}{3}-\frac{m_p}{m_c+m_p}\right)} & 0 \end{bmatrix}. \tag{15}$$

The B matrix is:

$$B = \begin{bmatrix} 0 \\ \frac{1}{m_c+m_p} \cdot \left(1 + \frac{m_p l}{l\left(\frac{4}{3} - \frac{m_p}{m_c+m_p}\right)}\right) \\ 0 \\ -\frac{1}{l\left(\frac{4}{3} - \frac{m_p}{m_c+m_p}\right)} \cdot \frac{1}{m_c+m_p} \end{bmatrix}. \quad (16)$$

These matrices provide the foundation for designing the LQR for stabilization.

2) Linear Quadratic Regulator (LQR): The Linear Quadratic Regulator (LQR) is an optimal control strategy used to stabilize linear systems by minimizing a quadratic cost function. [7] The cost function is typically of the form:

$$J = \int_0^\infty \left(X^T Q X + U^T R U\right) dt, \quad (17)$$

where:

- X is the state vector,
- U is the control input,
- Q is a positive semi-definite matrix that penalizes deviations in the state variables,
- R is a positive definite scalar or matrix that penalizes the control effort.

The LQR computes the optimal feedback gain K such that the control input:

$$U = -KX, \quad (18)$$

minimizes the cost function J.

The feedback gain K is computed by solving the Algebraic Riccati Equation (ARE):

$$A^T P + PA - PBR^{-1}B^T P + Q = 0, \quad (19)$$

where P is the solution to the ARE.

The eigenvalues of the closed-loop system matrix $(A-BK)$ determine the stability and performance of the controlled system.

3) Tuning the LQR Gains: The matrices Q and R are tuning parameters that influence the behaviour of the controller:

- The Q matrix assigns weights to the state variables, prioritizing which variables the controller should focus on. For example, a higher weight on θ ensures the pendulum angle is stabilized faster.
- The R matrix penalizes the control effort, preventing excessive control actions.

In the implementation, the Q matrix is defined as:

$$Q = \text{diag}(1, 1, 10, 1), \quad (20)$$

which prioritizes the pendulum angle θ for stabilization. The scalar R is set to:

$$R = 0.1, \quad (21)$$

to ensure moderate control effort while maintaining performance.

The resulting feedback gain K is used to compute the control input:

$$U = -KX, \quad (22)$$

where X is the current state vector.

4) Performance and Stability: The LQR controller ensures optimal stabilization by minimizing the cost function J. The closed-loop eigenvalues of $(A - BK)$ determine the system's stability and transient response. By appropriately tuning Q and R, the controller can achieve a balance between fast stabilization and minimal control effort.

C. Energy Controller

The energy controller regulates the pendulum's total mechanical energy to achieve the desired upright equilibrium position. This section details the energy calculation, control law, and how the energy dynamics influence the applied control force. [8] [9]

1) System Energy: The total energy of the cart-pendulum system consists of potential energy and kinetic energy, given by:

- **Potential Energy:** The pendulum's potential energy is determined by the height of its centre of mass:
$$V = m_p g l(1 - \cos\theta), \quad (23)$$
where θ is the angle of the pendulum relative to the vertical.
- **Kinetic Energy:** The kinetic energy includes the contributions from the pendulum's angular velocity and the cart's linear velocity:
$$T = \frac{1}{2} m_c \dot{x}_c^2 + \frac{1}{2} m_p l^2 \dot{\theta}^2 + m_p l \dot{x}_c \dot{\theta} \cos\theta. \quad (24)$$

The total energy of the system is then:

$$E = T+V = \frac{1}{2} m_c \dot{x}_c^2 + \frac{1}{2} m_p l^2 \dot{\theta}^2 + m_p l \dot{x}_c \dot{\theta}\cos\theta + m_p g l(1-\cos\theta). \quad (25)$$

2) Energy Dynamics: The rate of change of energy is influenced by the control force u applied to the cart:

$$\frac{dE}{dt} = m_p l \dot{\theta} \ddot{\theta} - m_p g l \dot{\theta} \sin(\theta) = -u\dot{\theta}\cos(\theta), \quad (26)$$

where the term $-u\dot{\theta}\cos(\theta)$ represents the contribution of the control force to the energy dynamics.

3) Control Law: The control law is designed to regulate the system's energy by driving it toward the desired energy E_0 corresponding to the upright position. The applied force consists of two components:

- **Swing-Up Control:** This term adds energy to the pendulum when it is near the downward position ($|\theta| \approx \pi$) by applying a force proportional to the energy deviation:
$$u_{\text{swing-up}} = k_{\text{energy}} \cdot \text{sign}(\dot{\theta}), \quad (27)$$

where k_{energy} is a proportional gain that scales the applied force.

- **Stabilization Control:** To prevent the cart from drifting away during the swing-up process, a stabilization term is added based on the cart's position and velocity:

$$u_{\text{stabilization}} = -k_{\text{pos}} x_c - k_{\text{vel}} \dot{x}_c, \tag{28}$$

where k_{pos} and k_{vel} are proportional and derivative gains, respectively.

The total control force is the sum of these components:

$$u = u_{\text{swing-up}} + u_{\text{stabilization}}. \tag{29}$$

The force is clipped to remain within the physical limits of the system:

$$u = \text{clip}(u, u_{\text{min}}, u_{\text{max}}). \tag{30}$$

4) Connection to Reference Energy Control: In comparison to the control law derived in the reference study:

$$u = \text{sat}_u(k(E - E_0))\text{sign}(\dot{\theta}), \tag{31}$$

The implemented controller simplifies the computation by directly using the pendulum's angular velocity $\dot{\theta}$ and a fixed gain k_{energy}. The stabilization term also provides additional robustness by actively maintaining the cart near the centre of the track throughout the swing-up process.

III. Problem Formulation

The inverted pendulum on a cart is a benchmark problem in control theory, known for its nonlinear dynamics and inherent instability. The system comprises a cart of mass m_c that moves along a horizontal track and a pendulum of mass m_p attached to the cart via a frictionless pivot. The pendulum's center of mass is located at a distance l from the pivot point. The goal is to control the horizontal motion of the cart such that the pendulum transitions from its downward position to the upright equilibrium point and remains stabilized at that position.

A. Research Question

This study seeks to answer the question:

> *How do different initial conditions of the inverted pendulum affect the performance and robustness of the hybrid energy-based swing-up and LQR stabilization control strategy?*

B. System Dynamics

The dynamics of the cart-pendulum system are nonlinear and highly coupled. The equations of motion, derived using the Lagrangian approach, are:

$$(m_c + m_p)\ddot{x}_c + m_p l \ddot{\theta}\cos(\theta) - m_p l \dot{\theta}^2 \sin(\theta) = F, \tag{32}$$

$$m_p l^2 \ddot{\theta} + m_p l \ddot{x}_c \cos(\theta) - m_p g l \sin(\theta) = 0, \tag{33}$$

where:

- x_c: Horizontal position of the cart,
- $\ddot{x}_c$: Horizontal acceleration of the cart,
- θ: Angle of the pendulum from the vertical,
- $\ddot{\theta}$: Angular acceleration of the pendulum,
- g: Gravitational acceleration,
- F: Force applied to the cart.

C. Control Objectives

The control problem is divided into two distinct phases:

1) **Swing-Up Phase:** The goal is to move the pendulum from its downward position ($\theta = \pi$) to a near-upright position ($\theta \approx 0$). Due to the nonlinear nature of the system, an energy-based control function modulates the total mechanical energy of the pendulum to achieve the swing-up while respecting physical constraints.
2) **Stabilization Phase:** Once the pendulum is sufficiently close to the upright equilibrium, a Linear Quadratic Regulator (LQR) stabilizes the system. The LQR minimizes deviations in the state variables by solving an optimal control problem based on a quadratic cost function.

D. State-Space Representation

For the stabilization phase, the system is linearized around the upright equilibrium ($\theta \approx 0$, $\dot{\theta} \approx 0$). The linearized state-space representation is:

$$\dot{X} = AX + BU, \tag{34}$$

where the state vector X and control input U are:

$$X = \begin{bmatrix} x_c \\ \dot{x}_c \\ \theta \\ \dot{\theta} \end{bmatrix}, \quad U = F. \tag{35}$$

The matrices A and B are derived by linearizing the equations of motion and are used to design the LQR controller.

E. Switching Mechanism

A smooth transition between the energy-based swing-up control and LQR-based stabilization control is crucial for seamless performance. The switching mechanism is based on predefined thresholds for the pendulum angle (θ) and angular velocity ($\dot{\theta}$):

$$|\theta| < \theta_{\text{threshold}} \quad \text{and} \quad |\dot{\theta}| < \dot{\theta}_{\text{threshold}}. \tag{36}$$

When these conditions are satisfied, the control system switches from the energy-based swing-up function to the LQR controller. This ensures that the pendulum is sufficiently close to the upright equilibrium before stabilization begins.

F. Scope of Research

To address the research question, this study explores how varying initial conditions, such as pendulum angles and cart positions, influence the robustness of the proposed hybrid control strategy. These conditions simulate real-world scenarios, where the pendulum might start from different configurations or disturbances.

IV. Results

The mathematical preliminaries discussed earlier laid the foundation for the simulation environment. With these in place, it was possible to evaluate the robustness of the energy-based swing-up controller under varying initial angles and the performance of the implemented control laws. Both control algorithms required tuning parameters, and the successful implementation largely relied on iterative trial and error to optimize these parameters. The source code for our testing and simulation environment is listed in the bibliography at the end of this paper. [10]

For the LQR controller, the tuning process involved two key matrices. The Q matrix determines the relative importance of different states, representing the control effort allocated to each. As a diagonal matrix, it is straightforward to adjust its elements to tailor the regulator's behaviour. Our experiments indicated that the pendulum angle required a higher control effort compared to other states, ensuring stabilization under challenging conditions. The R matrix, on the other hand, imposes a penalty on the overall control effort. Since the system was not constrained by energy-saving requirements, the weight of this penalty was reduced, allowing the controller to operate effectively without significant energy constraints.

The swing-up controller, based on energy feedback, employed a different tuning approach. Similar to a proportional controller, simple gains were introduced for both the energy input and the position correction terms. These gains were determined through experimentation, with the input energy gain incrementally increased until satisfactory performance was achieved. The tuning process also considered the ability of the swing-up controller to handle a variety of initial pendulum angles, ensuring robustness across a wide range of scenarios.

The following section presents the outcomes of these tuning efforts, showcasing the behaviour of the system under the final parameter settings and evaluating the swing-up controller's robustness to different initial conditions.

V. Numerical Examples/Simulations

This section presents simulation results to validate the robustness of the energy-based swing-up function and the LQR stabilization. The system is initialized in different configurations to test the LQR stabilization and energy-based swing-up function. The simulations were performed using Python, and the results are presented in the following figures.

A. Simulation Results

The simulation results for each scenario are presented below.

Scenario 1: Figure 2 shows the stabilization of the pendulum when starting upright at a 10-degree angle. The LQR controller effectively reduces the angle to zero, achieving stabilization without overshooting.

Scenario 2: Figure 3 demonstrates the stabilization performance when the pendulum starts upright at a 10-degree angle and is displaced 2 units horizontally. The LQR controller brings the pendulum to the upright position while minimizing oscillations and quickly reducing the horizontal displacement.

Scenario 3: Figure 4 illustrates the complete swing-up and stabilization process. The energy-based control function successfully brings the pendulum from the downward position to near-upright. The LQR controller then takes over, stabilizing the pendulum at the upright position with minimal oscillations and control effort.

Scenario 4: Figure 5 illustrates the complete swing-up and stabilization process subjected to many starting angle configurations. The energy-based control function successfully brings the pendulum to near-upright. The LQR controller then takes over, stabilizing the pendulum at the upright position with minimal oscillations and control effort. The result displays the robustness and simplicity of the swing-up algorthm

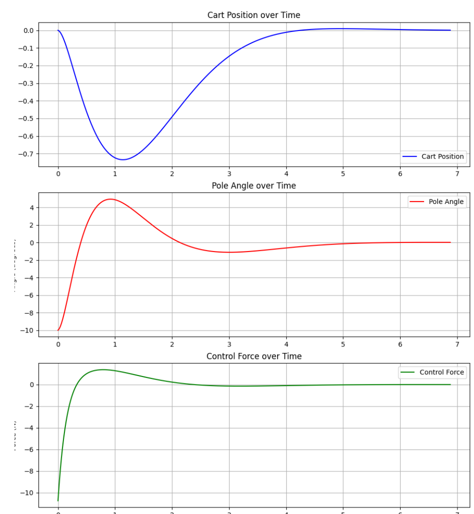

Fig. 2. Pendulum stabilization with LQR starting from an upright position with a 10-degree angle.

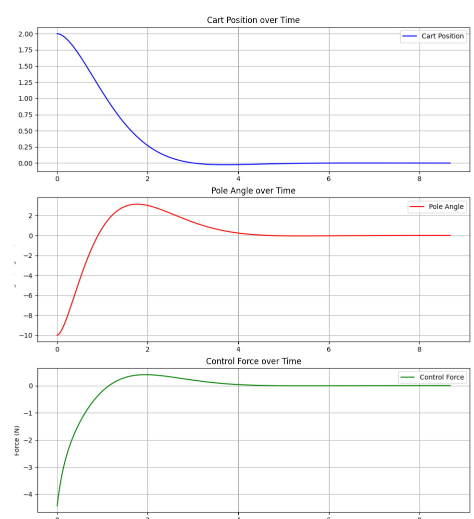

Fig. 3. Pendulum stabilization with LQR starting from an upright position with a 10-degree angle and a 2-unit horizontal displacement.

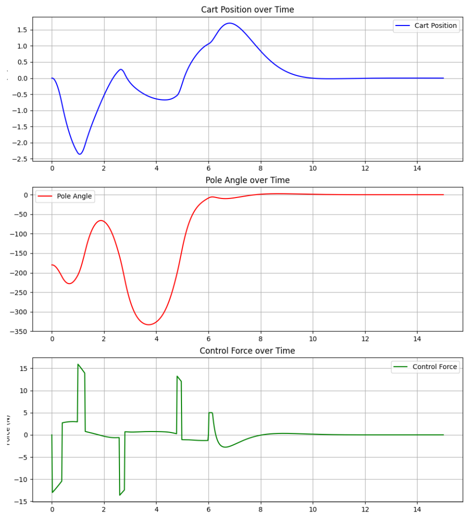

Fig. 4. Pendulum swing-up and stabilization using the energy-based control function and LQR.

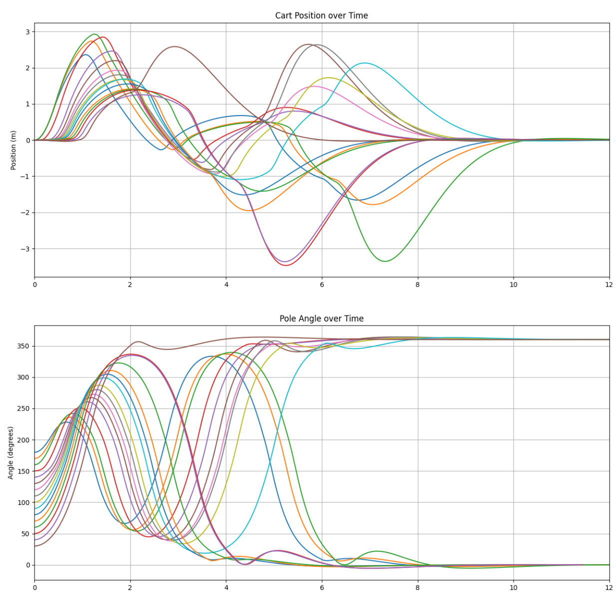

Fig. 5. Robustness of the hybrid control strategy under varying initial conditions, demonstrating effective swing-up and stabilization across multiple starting angles.

B. Discussion

The results highlight the robustness and effectiveness of the proposed control strategy:

- The LQR controller efficiently stabilizes the pendulum in scenarios where it starts close to the upright position, even with horizontal displacement.
- The energy-based swing-up controller demonstrates robustness by successfully transitioning the pendulum to the upright position across a wide range of initial angles.
- All scenarios confirm smooth control transitions and minimal control effort, validating the ability of the hybrid control approach to handle varying initial conditions effectively.

These findings demonstrate that the hybrid control strategy is robust across a variety of initial conditions, confirming the hypothesis that energy-based swing-up control effectively transitions the pendulum to an upright position even under challenging starting scenarios.

C. Comments on Results

In Figure 5, it is important to highlight that the two stabilization solutions correspond to the same final pose of the pendulum. Due to the symmetric nature of angles, both $0°$ and $360°$ represent the same upright position.

VI. Conclusion and Future Work

This paper evaluated the robustness of an energy-based swing-up controller for the inverted pendulum system under varying initial angles, in combination with a Linear Quadratic Regulator (LQR) for stabilization. The results indicate that the energy-based controller reliably handles varying initial conditions, including different pendulum angles and cart displacements, to transition the pendulum to a near-upright position. The LQR then stabilizes the system at the upright equilibrium, ensuring robust performance across a range of starting scenarios. The robustness of the swing-up approach was validated through simulations that tested multiple initial angles, demonstrating smooth transitions between the controllers, minimal oscillations, and effective stabilization.

Future work should explore enhancements to the energy-based swing-up strategy to handle even more extreme initial conditions and disturbances. Additionally, implementing a Nonlinear Model Predictive Controller (NMPC) for the swing-up phase could improve performance by accounting for system constraints and providing more precise control over nonlinear dynamics. [11] Testing the control strategy on physical hardware and incorporating disturbance rejection capabilities would further establish the practical utility of the proposed hybrid approach, paving the way for more advanced control applications in unstable systems.

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