31310 E20 Exam Questionnaire

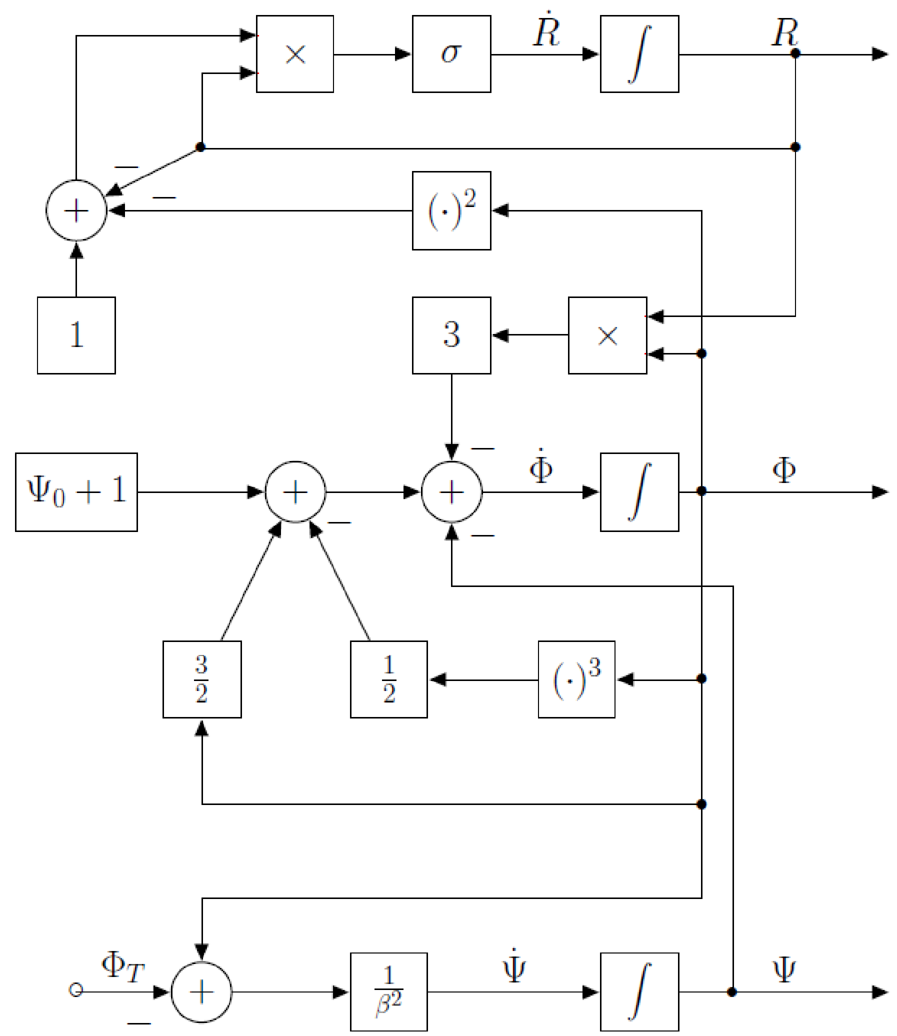
- [x] Show correct answers
- [ ] Hide correct answers

Page 1

Analysis of open loop systems - Part 1

Question 1

Consider the block diagram of a jet engine compression system shown in the following figure, where Φ is the mass flow, Ψ is the pressure rise and $R \geq 0$ is the normalized stall cell squared amplitude. Φ_T is the mass flow through the throttle, and σ and β are constant positive parameters. Let $\mathbf{x} = [\Phi, \Psi, R]^{\mathrm{T}}$ be the state vector. Which of the following state space models is associated with the block diagram? (In the diagram the black dots address intersection of signal flows.)



The state space model of the system is:

☐ $$\Sigma : \begin{cases} \dot{x}_1 = -x_2 - 3x_1x_3 \\ \dot{x}_2 = \dfrac{1}{\beta^2}(x_2 - \Phi_T) \\ \dot{x}_3 = \sigma x_3(1 - x_1 - x_3) \end{cases}$$

The state space model of the system is:

☐ $$\Sigma : \begin{cases} \dot{x}_1 = -x_2 + \Psi_c(x_1) - 3x_1x_3 \\ \dot{x}_2 = \dfrac{1}{\beta^2}(x_2 - \Phi_T) \\ \dot{x}_3 = \sigma x_3(1 - x_1^2 - x_3) \\ \dot{x}_4 = x_3 \end{cases}$$

where $\Psi_c(x_1) = \Psi_0 + 1 + \frac{3}{2}x_1 - \frac{1}{2}x_1^3$

- [ ] The state space model of the system is:

$$\Sigma : \begin{cases} \dot{x}_1 = -x_2 + \Psi_c(x_1) - 3(x_1 + x_3) \\ \dot{x}_2 = \dfrac{1}{\beta^2}(x_2 - \Phi_T) \\ \dot{x}_3 = \sigma x_3(1 - x_1^2 - x_3) \end{cases}$$

where $\Psi_c(x_1) = \Psi_0 + 1 + \dfrac{3}{2}x_1 - \dfrac{1}{2}x_1^3$

- [x] The state space model of the system is:

$$\Sigma : \begin{cases} \dot{x}_1 = -x_2 + \Psi_c(x_1) - 3x_1x_3 \\ \dot{x}_2 = \dfrac{1}{\beta^2}(x_1 - \Phi_T) \\ \dot{x}_3 = \sigma x_3(1 - x_1^2 - x_3) \end{cases}$$

where $\Psi_c(x_1) = \Psi_0 + 1 + \dfrac{3}{2}x_1 - \dfrac{1}{2}x_1^3$

- [ ] The state space model of the system is:

$$\Sigma : \begin{cases} \dot{x}_1 = -x_2 + \Psi_c(x_1) - 3x_1x_3 \\ \dot{x}_2 = \dfrac{1}{\beta^2}(x_2 - \Phi_T) \end{cases}$$

where $\Psi_c(x_1) = \Psi_0 + 1 + \dfrac{3}{2}x_1 - \dfrac{1}{2}x_1^3$

Question 2

A synchronous generator connected to an infinite bus can be represented by

$$M\ddot{\delta} = P - D\dot{\delta} - \eta_1 E_q \sin\delta$$
$$\tau\dot{E}_q = -\eta_2 E_q + \eta_3 \cos\delta + E_F$$

where δ is the angular position of the generator's shaft, E_q is the voltage, P is the mechanical input power, E_F is the input field voltage, D is a damping coefficient, M is an inertial coefficient, τ is the electrical time constant, and $\eta_1,\ \eta_2,\ \eta_3$ are constant parameters.

Let $\delta = \delta_0 \neq 0$ and $P = P_0 \neq 0$ be the steady state values of the generator shaft position and mechanical input power. What is the steady state solution of the system?

- [ ] The steady state solution of the system is:
  $$\cdot\,\text{States}:\quad \delta = \delta_0;\ \dot{\delta} = 0;\ E_q = \frac{P_0}{\eta_1 \sin\delta_0}$$
  $$\cdot\,\text{Inputs}:\quad P = P_0;\ E_F = 0$$

- [ ] The steady state solution of the system is:
  $$\cdot\,\text{States}:\quad \delta = \delta_0;\ \dot{\delta} = 0;\ E_q = \frac{P_0}{\eta_1 \sin\delta_0}$$
  $$\cdot\,\text{Inputs}:\quad P = P_0;\ E_F = \frac{\eta_2 P_0}{\eta_1 \sin\delta_0}$$

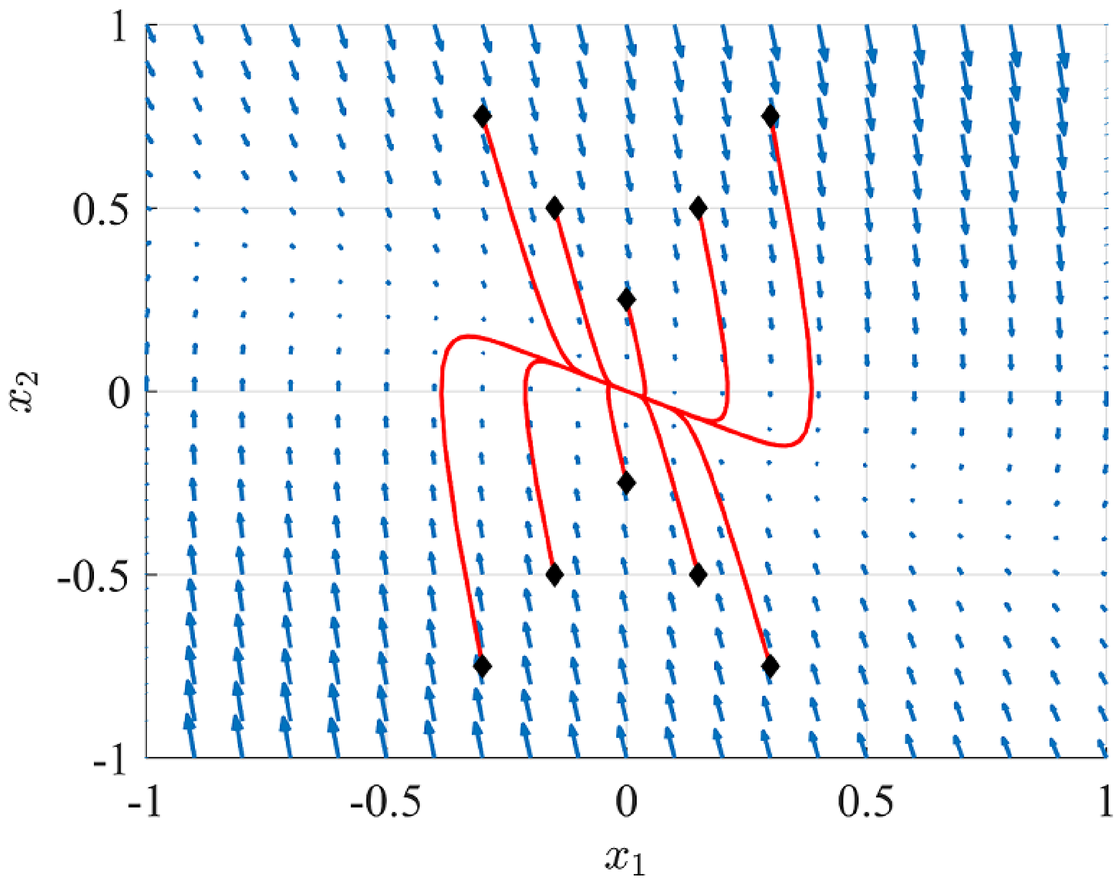
- [ ] The steady state solution of the system is:
  $$\cdot\,\text{States}:\quad \delta = \delta_0;\ E_q = \frac{P_0}{\eta_1 \sin\delta_0}$$
  $$\cdot\,\text{Inputs}:\quad P = P_0;\ E_F = \frac{\eta_2 P_0}{\eta_1 \sin\delta_0} - \eta_3 \cos\delta_0$$

- [x] The steady state solution of the system is:
  $$\cdot\,\text{States}:\quad \delta = \delta_0;\ \dot{\delta} = 0;\ E_q = \frac{P_0}{\eta_1 \sin\delta_0}$$
  $$\cdot\,\text{Inputs}:\quad P = P_0;\ E_F = \frac{\eta_2 P_0}{\eta_1 \sin\delta_0} - \eta_3 \cos\delta_0$$

- [ ] The steady state solution of the system is:
  $$\cdot\,\text{States}:\quad \delta = \delta_0;\ E_q = \frac{P_0}{\eta_1 \sin\delta_0}$$
  $$\cdot\,\text{Inputs}:\quad P = P_0;\ E_F = -\eta_3 \cos\delta_0$$

Question 3

The phase portrait of a second order continuous time LTI system is shown in the following figure (in the given phase portraits each black diamond represents an initial condition $\mathbf{x}(0) = [x_{10}, x_{20}]^{\mathrm{T}}$ for the system; each red line is a trajectory of the system originated from the initial condition; the blue arrows represent the direction of the vector field in the neighborhood of the origin.)



Which of the following statement is correct?

- [x] The equilibrium point is a stable node.
- [ ] The equilibrium point is an unstable focus.
- [ ] The equilibrium point is a saddle point.
- [ ] The equilibrium point is an unstable node.
- [ ] The equilibrium point is a centre.

Question 4

Consider the 4-th order LTI discrete time system

$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{G}\mathbf{u}(k), \quad \mathbf{x} \in \mathbb{R}^4, \mathbf{u} \in \mathbb{R}^2$$
$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k), \quad \mathbf{y} \in \mathbb{R}^2$$

Assume that the reachable subspace has dimension 3

$$\dim(\mathcal{R}) = 3,$$

the observable subspace has dimension 2

$$\dim(\mathcal{O}) = 2,$$

and that the intersection of the two subspaces has dimension 1

$$\dim(\mathcal{R} \cap \mathcal{O}) = 1.$$

Which of the following are the state and output responses? ($\mathbf{x}_0$ is the system initial condition at time $k = 0$)

- [ ] The state and output responses are:

$$\begin{aligned}\mathbf{x}(k) = {} & \left(\mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0\lambda_1^k\right)\mathbf{v}_1 + \left(\mathbf{w}_2^{\mathrm{T}}\mathbf{x}_0\lambda_2^k\right)\mathbf{v}_2 + \left(\mathbf{w}_3^{\mathrm{T}}\mathbf{x}_0\lambda_3^k\right)\mathbf{v}_3 + \left(\mathbf{w}_4^{\mathrm{T}}\mathbf{x}_0\lambda_4^k\right)\mathbf{v}_4 \\ & + \left(\sum_{i=0}^{k-1}\lambda_1^{k-1+i}\mathbf{w}_1^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_1 + \left(\sum_{i=0}^{k-1}\lambda_2^{k-1+i}\mathbf{w}_2^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_2 \\ & + \left(\sum_{i=0}^{k-1}\lambda_3^{k-1+i}\mathbf{w}_3^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_3 + \left(\sum_{i=0}^{k-1}\lambda_4^{k-1+i}\mathbf{w}_4^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_4\end{aligned}$$

$$\begin{aligned}\mathbf{y}(k) = {} & \left(\mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0\lambda_1^k\right)\mathbf{C}\mathbf{v}_1 + \left(\mathbf{w}_2^{\mathrm{T}}\mathbf{x}_0\lambda_2^k\right)\mathbf{C}\mathbf{v}_2 + \left(\mathbf{w}_3^{\mathrm{T}}\mathbf{x}_0\lambda_3^k\right)\mathbf{C}\mathbf{v}_3 + \left(\mathbf{w}_4^{\mathrm{T}}\mathbf{x}_0\lambda_4^k\right)\mathbf{C}\mathbf{v}_4 \\ & + \left(\sum_{i=0}^{k-1}\lambda_1^{k-1+i}\mathbf{w}_1^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{C}\mathbf{v}_1 + \left(\sum_{i=0}^{k-1}\lambda_2^{k-1+i}\mathbf{w}_2^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{C}\mathbf{v}_2 \\ & + \left(\sum_{i=0}^{k-1}\lambda_3^{k-1+i}\mathbf{w}_3^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{C}\mathbf{v}_3 + \left(\sum_{i=0}^{k-1}\lambda_4^{k-1+i}\mathbf{w}_4^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{C}\mathbf{v}_4\end{aligned}$$

- [ ] The state and output responses are:

$$\begin{aligned}\mathbf{x}(k) = {} & \left(\mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0\lambda_1^k\right)\mathbf{v}_1 + \left(\mathbf{w}_2^{\mathrm{T}}\mathbf{x}_0\lambda_2^k\right)\mathbf{v}_2 + \left(\mathbf{w}_3^{\mathrm{T}}\mathbf{x}_0\lambda_3^k\right)\mathbf{v}_3 + \left(\mathbf{w}_4^{\mathrm{T}}\mathbf{x}_0\lambda_4^k\right)\mathbf{v}_4 \\ & + \left(\sum_{i=0}^{k-1}\lambda_1^{k-1+i}\mathbf{w}_1^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_1 + \left(\sum_{i=0}^{k-1}\lambda_3^{k-1+i}\mathbf{w}_3^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_3 \\ & + \left(\sum_{i=0}^{k-1}\lambda_4^{k-1+i}\mathbf{w}_4^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_4\end{aligned}$$

$$\begin{aligned}\mathbf{y}(k) = {} & \left(\mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0\lambda_1^k\right)\mathbf{C}\mathbf{v}_1 + \left(\mathbf{w}_2^{\mathrm{T}}\mathbf{x}_0\lambda_2^k\right)\mathbf{C}\mathbf{v}_2 + \left(\mathbf{w}_3^{\mathrm{T}}\mathbf{x}_0\lambda_3^k\right)\mathbf{C}\mathbf{v}_3 \\ & + \left(\sum_{i=0}^{k-1}\lambda_1^{k-1+i}\mathbf{w}_1^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{C}\mathbf{v}_1 + \left(\sum_{i=0}^{k-1}\lambda_3^{k-1+i}\mathbf{w}_3^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_3\end{aligned}$$

- [x] The state and output responses are:

$$\mathbf{x}(k) = \left(\mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0\lambda_1^k\right)\mathbf{v}_1 + \left(\mathbf{w}_2^{\mathrm{T}}\mathbf{x}_0\lambda_2^k\right)\mathbf{v}_2 + \left(\mathbf{w}_3^{\mathrm{T}}\mathbf{x}_0\lambda_3^k\right)\mathbf{v}_3 + \left(\mathbf{w}_4^{\mathrm{T}}\mathbf{x}_0\lambda_4^k\right)\mathbf{v}_4 + \left(\sum_{i=0}^{k-1}\lambda_2^{k-1+i}\mathbf{w}_2^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_2 + \left(\sum_{i=0}^{k-1}\lambda_3^{k-1+i}\mathbf{w}_3^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_3 + \left(\sum_{i=0}^{k-1}\lambda_4^{k-1+i}\mathbf{w}_4^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_4$$

$$\mathbf{y}(k) = \left(\mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0\lambda_1^k\right)\mathbf{C}\mathbf{v}_1 + \left(\mathbf{w}_2^{\mathrm{T}}\mathbf{x}_0\lambda_2^k\right)\mathbf{C}\mathbf{v}_2 + \left(\sum_{i=0}^{k-1}\lambda_2^{k-1+i}\mathbf{w}_2^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{C}\mathbf{v}_2$$

- [ ] The state and output responses are:

$$\mathbf{x}(k) = \left(\mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0\lambda_1^k\right)\mathbf{v}_1 + \left(\mathbf{w}_2^{\mathrm{T}}\mathbf{x}_0\lambda_2^k\right)\mathbf{v}_2 + \left(\mathbf{w}_3^{\mathrm{T}}\mathbf{x}_0\lambda_3^k\right)\mathbf{v}_3 + \left(\mathbf{w}_4^{\mathrm{T}}\mathbf{x}_0\lambda_4^k\right)\mathbf{v}_4$$

$$\mathbf{y}(k) = \mathbf{0}$$

- [ ] The state and output responses are:

$$\mathbf{x}(k) = \left(\mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0\lambda_1^k\right)\mathbf{v}_1 + \left(\mathbf{w}_2^{\mathrm{T}}\mathbf{x}_0\lambda_2^k\right)\mathbf{v}_2 + \left(\mathbf{w}_3^{\mathrm{T}}\mathbf{x}_0\lambda_3^k\right)\mathbf{v}_3 + \left(\mathbf{w}_4^{\mathrm{T}}\mathbf{x}_0\lambda_4^k\right)\mathbf{v}_4 + \left(\sum_{i=0}^{k-1}\lambda_1^{k-1+i}\mathbf{w}_1^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_1 + \left(\sum_{i=0}^{k-1}\lambda_3^{k-1+i}\mathbf{w}_3^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_3 + \left(\sum_{i=0}^{k-1}\lambda_4^{k-1+i}\mathbf{w}_4^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_4$$

$$\mathbf{y}(k) = \left(\mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0\lambda_1^k\right)\mathbf{C}\mathbf{v}_1 + \left(\mathbf{w}_3^{\mathrm{T}}\mathbf{x}_0\lambda_3^k\right)\mathbf{C}\mathbf{v}_3 + \left(\mathbf{w}_4^{\mathrm{T}}\mathbf{x}_0\lambda_4^k\right)\mathbf{C}\mathbf{v}_4 + \left(\sum_{i=0}^{k-1}\lambda_1^{k-1+i}\mathbf{w}_1^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{C}\mathbf{v}_1 + \left(\sum_{i=0}^{k-1}\lambda_3^{k-1+i}\mathbf{w}_3^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{C}\mathbf{v}_3 + \left(\sum_{i=0}^{k-1}\lambda_4^{k-1+i}\mathbf{w}_4^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{C}\mathbf{v}_4$$

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Analysis of open loop systems - Part 2

Question 5

Given the 3rd order LTI discrete time SISO system

$$\mathbf{x}(k+1) = \begin{bmatrix} -\alpha & 0 & 0 \\ 0 & \gamma & \beta \\ 0 & -\beta & \gamma \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(k), \qquad \mathbf{x} \in \mathbb{R}^3,\ u \in \mathbb{R}$$

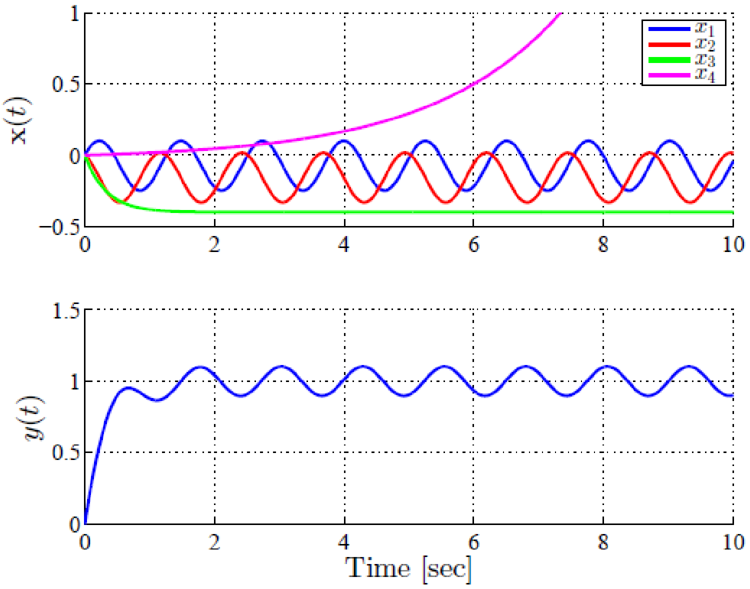
$$y(k) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \mathbf{x}(k), \qquad y \in \mathbb{R}$$

where $\alpha,\ \beta,\ \gamma$ are real coefficients. Which of the following statements is correct?

- [ ] The system is asymptotically stable for any triple $(\alpha, \beta, \gamma) \in \mathbb{R}^3$.
- [x] The system is asymptotically stable if $|\alpha| < 1 \wedge |\gamma| < 1 \wedge \beta < \sqrt{1-\gamma^2}$
- [ ] The system is asymptotically stable if $\alpha > 0 \wedge \gamma < 0 \wedge \forall \beta \in \mathbb{R}$
- [ ] The system is asymptotically stable if $|\alpha| < 1 \wedge |\gamma| < 1 \wedge \forall \beta \in \mathbb{R}$
- [ ] There is no triple $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ that renders the system asymptotically stable.

Question 6

Consider the step response of a 4th order LTI continuous time SISO system with zero initial conditions shown in the following figure. Which of the following statements is correct?



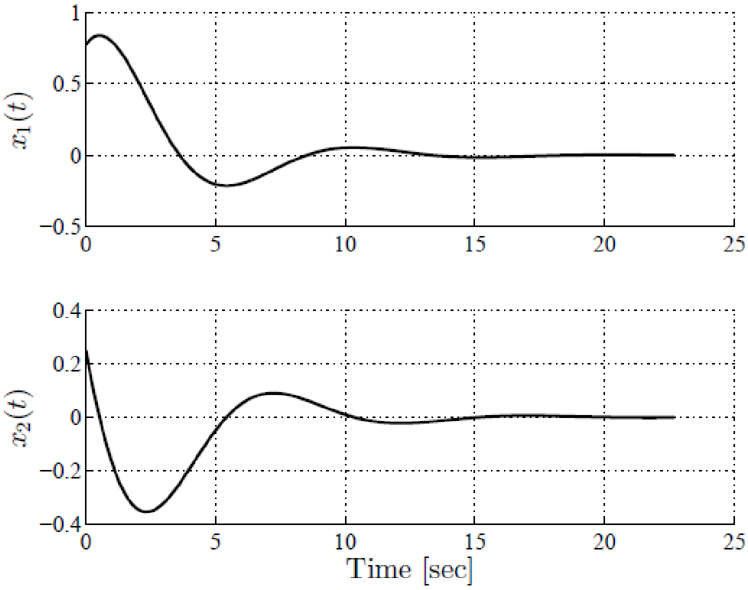
- [ ] The system is internally asymptotically stable and BIBO stable.
- [ ] The system has an unstable eigenmode, which is not controllable.
- [ ] The system is internally asymptotically stable but not observable.
- [x] The system has an unstable eigenmode, which is not observable.
- [ ] The system is internally marginally stable but not BIBO stable.

Question 7

Consider the 2nd order LTI continuous time SISO system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x}$$

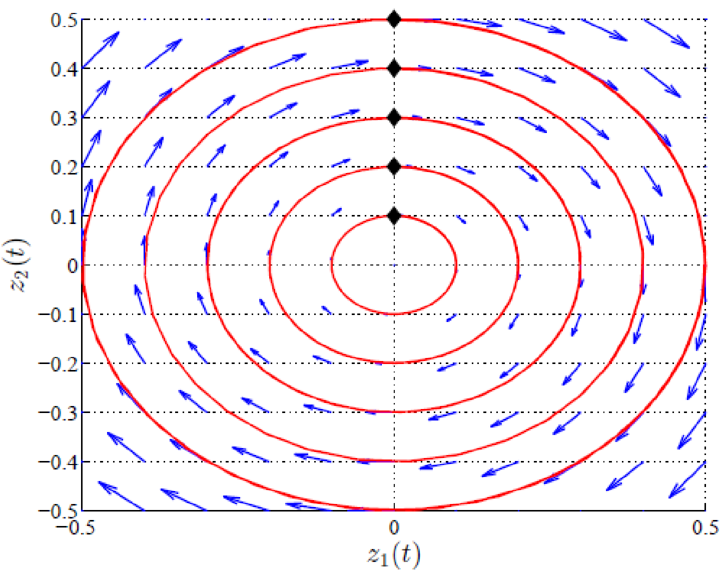
where $\mathbf{x} \in \mathbb{R}^2$, $u \in \mathbb{R}$, and $y \in \mathbb{R}$. The zero-input response of the system to a given initial condition $\mathbf{x} = [x_{10}, x_{20}]^{\mathrm{T}} \neq [0, 0]^{\mathrm{T}}$ is shown in the following figure.

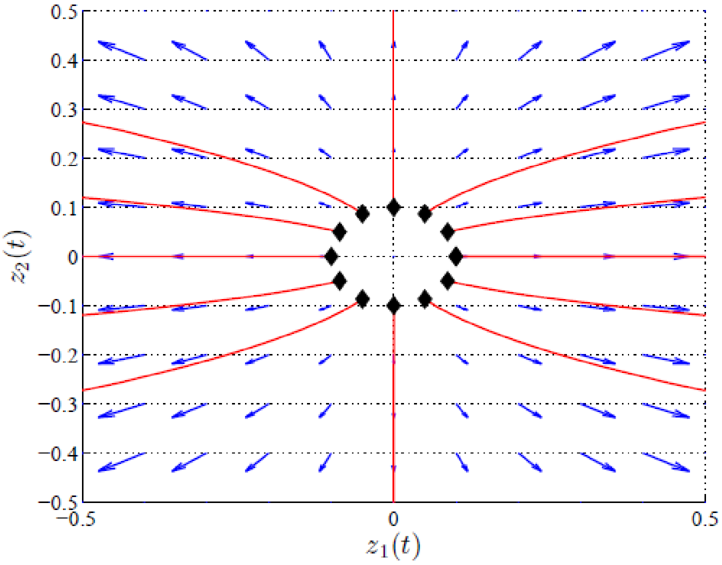


Consider the system Σ_z

$$\Sigma_z : \begin{cases} \dot{\mathbf{z}} = \mathbf{\Lambda}\mathbf{z} + \mathbf{B}_z u \\ y = \mathbf{C}_z \mathbf{z} \end{cases}$$

obtained through the similarity transformation $\mathbf{z} = \mathbf{M}^{-1}\mathbf{x}$, where $\mathbf{M}$ is the modal matrix. Which of the following phase portraits is that one associated with the dynamics of the system Σ_z? (In the given phase portraits each black diamond represents an initial condition $\mathbf{z}(0) = [z_{10}, z_{20}]^{\mathrm{T}}$ for the system Σ_z ; each red line is a trajectory of the system originated from the initial condition; the blue arrows represent the direction of the vector field in the neighbourhood of the origin.)

☐ 

☐ 
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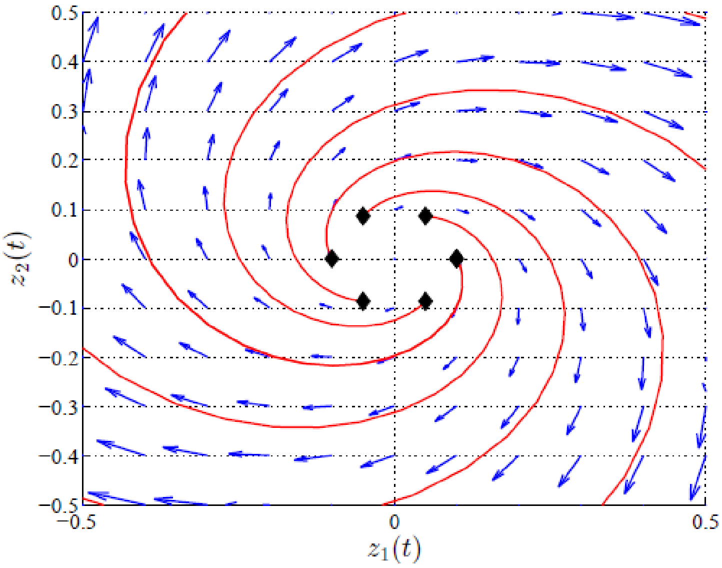
$z_1(t)$
$z_2(t)$

☐ 
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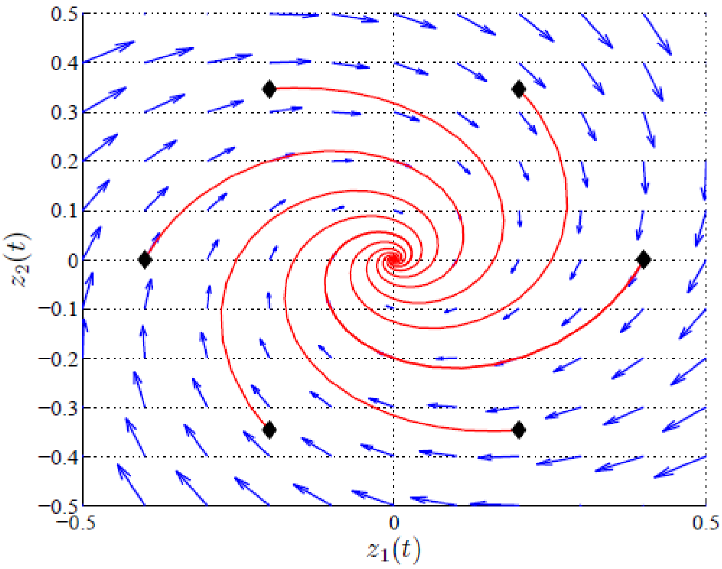
☑ 
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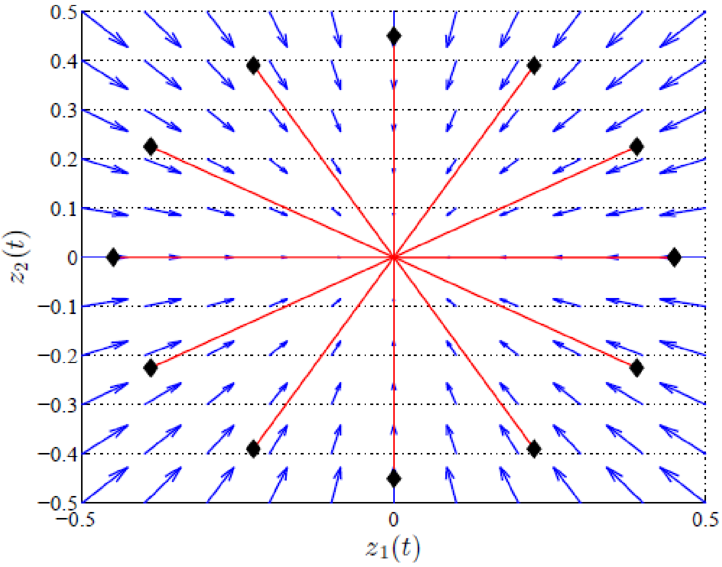
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$z_2(t)$
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0
0.5
$z_1(t)$

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Analysis of closed-loop systems - Part 1

Question 8

Consider the 2nd order LTI continuous time SISO system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} u + \begin{bmatrix} 0 \\ \alpha \end{bmatrix} d, \qquad \mathbf{x} \in \mathbb{R}^2, u \in \mathbb{R}, d \in \mathbb{R}$$
$$y = \begin{bmatrix} \gamma & 0 \end{bmatrix} \mathbf{x}, \qquad y \in \mathbb{R}$$

where $\omega_n > 0,\ \zeta > 0,\ \alpha > 0,\ \gamma > 0$ are constant parameters. The system is subject to the unknown constant disturbance d.

An output feedback control law is designed as

$$u = -\mathbf{K}\hat{\mathbf{x}} - K_d\hat{d} + Nr$$

where $\mathbf{K} = [K_1\ K_2]$ is the vector of controller gains, $K_d \in \mathbb{R}$ is the disturbance rejection gain, and $N \in \mathbb{R}$ is the reference feedforward gain. $\hat{\mathbf{x}}$ and $\hat{d}$ are the estimates of the state and disturbance provided by the observer. Said y_{des} the desired value of the output y of the closed-loop system, set $r = y_{\mathrm{des}}$.

What are the values of K_d and N that guarantee the fulfillment of the control objective

$$\lim_{t\to\infty} y(t) = r\,?$$

- [ ] $\begin{cases} K_d = \alpha \\ N = \dfrac{\omega_n^2(1+K_1)}{\gamma(1+K_2)} \end{cases}$

- [ ] $\begin{cases} K_d = \alpha \\ N = 1 \end{cases}$

- [x] $\begin{cases} K_d = \dfrac{\alpha}{\omega_n^2} \\ N = \dfrac{(1+K_1)}{\gamma} \end{cases}$

- [ ] $\begin{cases} K_d = \dfrac{\alpha}{\omega_n^2} \\ N = \dfrac{\omega_n^2+K_1}{\gamma\omega_n^2} \end{cases}$

- [ ] There are no values of K_d and N that can fulfil the control objective because the controller does not have integral action.

Question 9

Consider the second order LTI continuous time system

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u + \mathbf{B}_v v_1 \quad && \mathbf{x} \in \mathbb{R}^2,\ u \in \mathbb{R},\ v_1 \in \mathbb{R} \\
y &= \mathbf{C}\mathbf{x} + v_2 && y \in \mathbb{R},\ v_2 \in \mathbb{R}
\end{aligned}
$$

v_1 is a disturbance acting on the system, and it is given by the sum of an unknown constant component $\bar{v}_1$ with a stochastic component $\tilde{v}_1$. $\tilde{v}_1$ is zero mean band-limited noise described by the following autocorrelation function

$$R_{\tilde{v}_1}(\tau) = \sigma_1^2 \, e^{-\beta|\tau|}$$

with $\sigma_v^2 > 0$ and $\beta > 0$. v_2 is zero mean white noise with intensity σ_2^2.

Under the assumption that the pair $(\mathbf{A}, \mathbf{C})$ is observable, a Kalman filter is to be designed to estimate the state $\mathbf{x}$ as well as the disturbance v_1. Which of the following state space models provides the correct architecture for the design of the Kalman filter?

- [x] $$\begin{cases} \dot{\mathbf{x}}_{kf} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_v & \mathbf{B}_v \\ \mathbf{0} & 0 & 0 \\ 0 & 0 & -\beta \end{bmatrix} \mathbf{x}_{kf} + \begin{bmatrix} \mathbf{B} \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ 1 & 0 \\ 0 & \sqrt{2\beta}\sigma_1 \end{bmatrix} \boldsymbol{\eta} \\ y_{kf} = \begin{bmatrix} \mathbf{C} & 0 & 0 \end{bmatrix} \mathbf{x}_{kf} + v_2 \end{cases}$$

  where $\mathbf{x}_{kf} = \left[\mathbf{x}^{\mathrm{T}}, \bar{v}_1, \tilde{v}_1\right]^{\mathrm{T}}$ is the state of the Kalman filter and $\boldsymbol{\eta} = [\eta_1, \eta_2]^{\mathrm{T}}$ is the process noise.

- [ ] $$\begin{cases} \dot{\mathbf{x}}_{kf} = \begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 0 \\ 0 & 0 & -\beta \end{bmatrix} \mathbf{x}_{kf} + \begin{bmatrix} \mathbf{B} \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ 1 & 0 \\ 0 & \sqrt{2\beta}\sigma_1 \end{bmatrix} \boldsymbol{\eta} \\ y_{kf} = \begin{bmatrix} \mathbf{C} & 0 & 0 \end{bmatrix} \mathbf{x}_{kf} + v_2 \end{cases}$$

  where $\mathbf{x}_{kf} = \left[\mathbf{x}^{\mathrm{T}}, \bar{v}_1, \tilde{v}_1\right]^{\mathrm{T}}$ is the state of the Kalman filter and $\boldsymbol{\eta} = [\eta_1, \eta_2]^{\mathrm{T}}$ is the process noise.

- [ ] $$\begin{cases} \dot{\mathbf{x}}_{kf} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_v & \mathbf{B}_v \\ \mathbf{0} & 0 & 0 \\ 0 & 0 & -\beta \end{bmatrix} \mathbf{x}_{kf} + \begin{bmatrix} \mathbf{B} \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{\eta} \\ y_{kf} = \begin{bmatrix} \mathbf{C} & 0 & 0 \end{bmatrix} \mathbf{x}_{kf} + v_2 \end{cases}$$

  where $\mathbf{x}_{kf} = \left[\mathbf{x}^{\mathrm{T}}, \bar{v}_1, \tilde{v}_1\right]^{\mathrm{T}}$ is the state of the Kalman filter and $\boldsymbol{\eta} = [\eta_1, \eta_2]^{\mathrm{T}}$ is the process noise.

- [ ] $$\begin{cases} \dot{\mathbf{x}}_{kf} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_v \\ \mathbf{0} & 0 \end{bmatrix} \mathbf{x}_{kf} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} \eta \\ y_{kf} = \begin{bmatrix} \mathbf{C} & 0 \end{bmatrix} \mathbf{x}_{kf} + v_2 \end{cases}$$

  where $\mathbf{x}_{kf} = \left[\mathbf{x}^{\mathrm{T}}, \bar{v}_1\right]^{\mathrm{T}}$ is the state of the Kalman filter and η is the process noise.

- [ ] $$\begin{cases} \dot{\mathbf{x}}_{kf} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_v \\ \mathbf{0} & -\beta \end{bmatrix} \mathbf{x}_{kf} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u + \begin{bmatrix} \mathbf{0} \\ \sqrt{2\beta}\sigma_1 \end{bmatrix} \eta \\ y_{kf} = \begin{bmatrix} \mathbf{C} & 0 \end{bmatrix} \mathbf{x}_{kf} + v_2 \end{cases}$$

  where $\mathbf{x}_{kf} = \left[\mathbf{x}^{\mathrm{T}}, \tilde{v}_1\right]^{\mathrm{T}}$ is the state of the Kalman filter and η is the process noise.

Question 10

Consider the 3rd order LTI discrete time SISO system

$$\mathbf{x}(k+1) = \begin{bmatrix} -\alpha & \beta & 0 \\ -\beta & -\alpha & \gamma \\ 0 & \gamma & \delta \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ \varepsilon \\ 0 \end{bmatrix} u(k), \qquad \mathbf{x} \in \mathbb{R}^3, u \in \mathbb{R}$$
$$y(k) = \begin{bmatrix} 0 & 0 & \nu \end{bmatrix} \mathbf{x}(k), \qquad y \in \mathbb{R}$$

where $\alpha, \beta, \gamma, \delta, \varepsilon,$ and ν are real and positive coefficients.
A full order observer is designed for the given system as

$$\hat{\mathbf{x}}(k+1) = \begin{bmatrix} -\alpha & \beta & 0 \\ -\beta & -\alpha & \gamma \\ 0 & \gamma & \delta \end{bmatrix} \hat{\mathbf{x}}(k) + \begin{bmatrix} 0 \\ \varepsilon \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} (y(k) - \hat{y}(k))$$
$$\hat{y}(k) = \begin{bmatrix} 0 & 0 & \nu \end{bmatrix} \hat{\mathbf{x}}(k)$$

Define the estimation error as $\mathbf{e}_e(k) = \mathbf{x}(k) - \hat{\mathbf{x}}(k)$. Which values of the observer gains render the estimation error exactly equal to zero for $k \geq 3$ regardless of the estimation error initial condition?

- [ ] The observer gains are:
  $l_1 = 0;\ l_2 = 0;\ l_3 = 0$

- [ ] The observer gains are:
  $l_1 = \dfrac{\alpha}{\nu};\ l_2 = \dfrac{\alpha}{\nu};\ l_3 = \dfrac{\delta}{\nu}$

- [x] The observer gains are:
  $l_1 = \dfrac{\alpha^3 - 3\alpha\beta^2}{\beta\gamma\nu};\ l_2 = \dfrac{3\alpha^2 - \beta^2 + \gamma^2}{\gamma\nu};$
  $l_3 = \dfrac{-2\alpha + \delta}{\nu}$

- [ ] The observer gains are:
  $l_1 = \dfrac{\alpha^3}{\beta\gamma\nu};\ l_2 = \dfrac{3\alpha^2 - \beta^2}{\gamma\nu};\ l_3 = \dfrac{-2\alpha}{\nu}$

- [ ] The observer gains are:
  $l_1 = \dfrac{\alpha(1 - 3\beta^2)}{\gamma\nu};\ l_2 = \dfrac{3\alpha^2 + \gamma^2}{\gamma\nu};$
  $l_3 = \dfrac{-2\alpha + \delta}{\beta\nu}$

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Analysis of closed-loop systems - Part 2

Question 11

Consider the n-th order LTI continuous time SISO system

$$\Sigma_x : \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u + \mathbf{B}_d d, & \mathbf{x} \in \mathbb{R}^n, u \in \mathbb{R}, d \in \mathbb{R} \\ y = \mathbf{C}\mathbf{x}, & y \in \mathbb{R} \end{cases}$$

The observability matrix satisfies the following rank condition

$$\mathrm{rank}(\mathbf{M}_o) = n$$

The system is subject to a time-varying not measurable disturbance given by

$$d(t) = d_1 \sin(\beta_1 t) + d_2 \sin(\beta_2 t)$$

where d_1, d_2, β_1 and β_2 are positive real constants. Which of the following state estimators estimates the disturbance $d(t)$?

- [x] The dynamics of the state estimator is

$$\begin{bmatrix} \dot{\hat{\mathbf{x}}} \\ \dot{\hat{\mathbf{w}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_d\mathbf{C}_w \\ \mathbf{0} & \mathbf{A}_w \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} u + \begin{bmatrix} \mathbf{L}_x \\ \mathbf{L}_w \end{bmatrix} (y - \hat{y})$$

$$\hat{y} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix}$$

where

$$\mathbf{A}_w = \begin{bmatrix} 0 & -\beta_1 & 0 & 0 \\ \beta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta_2 \\ 0 & 0 & \beta_2 & 0 \end{bmatrix} \quad , \quad \mathbf{C}_w = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$$

- [ ] The dynamics of the state estimator is

$$\begin{bmatrix} \dot{\hat{\mathbf{x}}} \\ \dot{\hat{\mathbf{w}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_d\mathbf{C}_w \\ \mathbf{0} & \mathbf{A}_w \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} u + \begin{bmatrix} \mathbf{L}_x \\ \mathbf{L}_w \end{bmatrix} (y - \hat{y})$$

$$\hat{y} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix}$$

where

$$\mathbf{A}_w = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad \mathbf{C}_w = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

- [ ] The dynamics of the state estimator is

$$\begin{bmatrix} \dot{\hat{\mathbf{x}}} \\ \dot{\hat{\mathbf{w}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_d\mathbf{C}_w \\ \mathbf{0} & \mathbf{A}_w \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} u + \begin{bmatrix} \mathbf{L}_x \\ \mathbf{L}_w \end{bmatrix} (y - \hat{y})$$

$$\hat{y} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix}$$

where

$$\mathbf{A}_w = \begin{bmatrix} -\beta_1 & 0 \\ 0 & -\beta_2 \end{bmatrix} \quad , \quad \mathbf{C}_w = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

The dynamics of the state estimator is

$$\begin{bmatrix} \dot{\hat{\mathbf{x}}} \\ \dot{\hat{\mathbf{w}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_d\mathbf{C}_w \\ \mathbf{0} & \mathbf{A}_w \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} u + \begin{bmatrix} \mathbf{L}_x \\ \mathbf{L}_w \end{bmatrix} (y - \hat{y})$$

- [ ] $$\hat{y} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix}$$

where

$$\mathbf{A}_w = \begin{bmatrix} 0 & -\beta_1 & 0 \\ \beta_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad , \quad \mathbf{C}_w = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

- [ ] It is not possible to design a state estimator to reconstruct the disturbance $d(t)$ because the system Σ_x is not observable.

Question 12

Consider the 2nd order LTI continuous time SISO system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} u, \qquad \mathbf{x} \in \mathbb{R}^2, u \in \mathbb{R}$$
$$y = \begin{bmatrix} \gamma & 0 \end{bmatrix} \mathbf{x}, \qquad y \in \mathbb{R}$$

where $\omega_n > 0,\ \zeta > 0,\ \gamma > 0$ are constant parameters.

Three steady state optimal controllers are designed using the performance index

$$J(\mathbf{u}) = \int_0^{+\infty} \mathbf{x}^{\mathrm{T}}(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^{\mathrm{T}}(t)\mathbf{R}\mathbf{u}(t)\,\mathrm{d}t$$

where the weighting matrices have the following structure

$$\mathbf{Q} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} > 0 \quad , \quad \mathbf{R} = \rho > 0$$

The first controller $\mathbf{K}_1$ is designed choosing (the symbol $\gg$ means "much larger than")

$$\alpha > \beta \gg \rho.$$

The second controller $\mathbf{K}_2$ is designed choosing

$$\beta > \alpha \gg \rho.$$

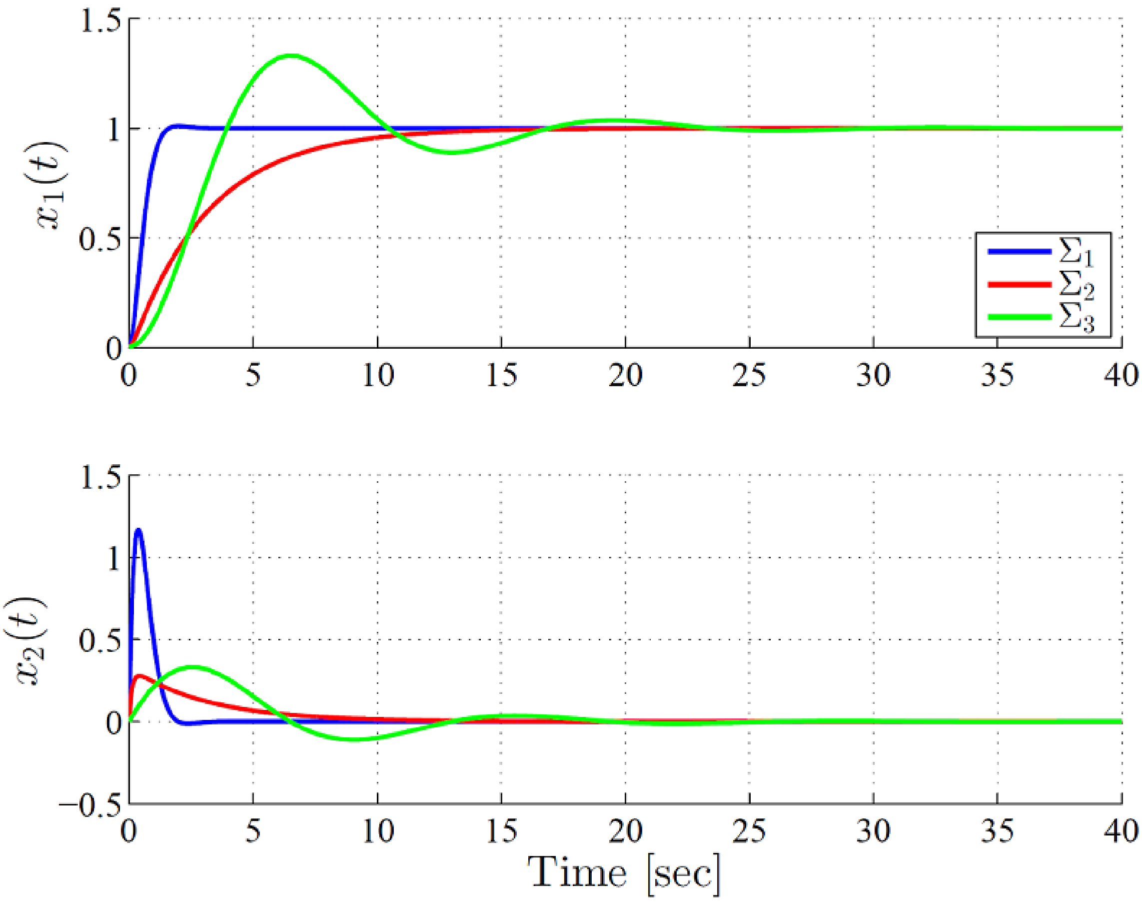
The third controller $\mathbf{K}_3$ is designed choosing

$$\rho \gg \alpha > \beta.$$

Said $u_i = -\mathbf{K}_i x + N_i r$ the control signal with N_i the reference feedforward gain, the following figure shows the state response of the three closed-loop systems ($i = 1, 2, 3$)

$$\Sigma_i : \begin{cases} \dot{\mathbf{x}} = \left(\begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} - \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} \mathbf{K}_i \right) \mathbf{x} + \begin{bmatrix} 0 \\ \omega_n^2 N_i \end{bmatrix} r \\ y = \begin{bmatrix} \gamma & 0 \end{bmatrix} \mathbf{x} \end{cases}$$

when a step of amplitude one is sent through the reference r.



The control effort (CE) of the controller u_i over the time horizon $t \in [0, +\infty)$ is defined as

$$\mathrm{CE}(u_i) = \int_0^{+\infty} u_i^2(t)\,\mathrm{d}t$$

Taking into account the design choices for the controllers $\mathbf{K}_i$, and considering the step responses in the figure above, which of the following statements is correct?

- [ ] $\mathrm{CE}(u_1) > \mathrm{CE}(u_3) > \mathrm{CE}(u_2)$
- [ ] $\mathrm{CE}(u_2) > \mathrm{CE}(u_1) > \mathrm{CE}(u_3)$
- [x] $\mathrm{CE}(u_1) > \mathrm{CE}(u_2) > \mathrm{CE}(u_3)$
- [ ] $\mathrm{CE}(u_1) = \mathrm{CE}(u_3) = \mathrm{CE}(u_2)$
- [ ] $\mathrm{CE}(u_3) > \mathrm{CE}(u_2) > \mathrm{CE}(u_1)$