LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 2.1

a. Here the positions and velocities are selected as the states:

$$\mathbf{x} = \begin{bmatrix} \theta \\ \dot{\theta} \\ p \\ \dot{p} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

The state space model is then:

$$\dot{\mathbf{x}} = \begin{bmatrix} x_2 \\ \dfrac{\frac{3g}{4L}(M+m)\sin x_1 - \frac{3m}{4}\sin x_1 \cos x_1 \cdot x_2^2 - \frac{3}{4L}\cos x_1 \cdot u}{M + m - \frac{3}{4}m\cos^2 x_1} \\ x_4 \\ \dfrac{mL\sin x_1 \cdot x_2^2 - \frac{3}{4}mg\sin x_1 \cos x_1 + u}{M + m - \frac{3}{4}m\cos^2 x_1} \end{bmatrix}.$$

$$y = x_3$$

b. The stationary state is selected as $\mathbf{x}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$ and it is assumed that:

$$\sin\theta \cong \theta,\ \cos\theta \cong 1 \text{ and } \dot{\theta}^2 \cong 0.$$

This leads to

$$\dot{x}_2 = \ddot{\theta} \cong \frac{\frac{3g}{4L}(M+m)\theta - \frac{3}{4L}u}{M + \frac{m}{4}},$$

$$\dot{x}_4 = \ddot{p} \cong \frac{-\frac{3}{4}mg\theta + u}{M + \frac{m}{4}}.$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 2.1

The linearized state equation then becomes:

$$\Delta\dot{\mathbf{x}} = 0\begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{3g(M+m)}{\left(M+\frac{m}{4}\right)4L} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{-\frac{3}{4}mg}{M+\frac{m}{4}} & 0 & 0 & 0 \end{bmatrix}\Delta\mathbf{x} + \begin{bmatrix} 0 \\ \dfrac{-\frac{3}{4L}}{M+\frac{m}{4}} \\ 0 \\ \dfrac{1}{M+\frac{m}{4}} \end{bmatrix}\Delta u$$

$$\Delta y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}\Delta\mathbf{x}$$

Note that the incremental variables (the Δ-variables) are the same as the absolute (large signal) variables because $\mathbf{x}_0 = \mathbf{0}$.

c. Defining

$$\Delta\mathbf{x} = \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} \text{ where } \mathbf{z}_1 = \begin{bmatrix} \Delta\theta \\ \Delta\dot{\theta} \end{bmatrix} \text{ and } \mathbf{z}_2 = \begin{bmatrix} \Delta p \\ \Delta\dot{p} \end{bmatrix},$$

one obtains:

$$\dot{\mathbf{z}}_1 = \begin{bmatrix} 0 & 1 \\ \dfrac{3g(M+m)}{\left(M+\frac{m}{4}\right)4L} & 0 \end{bmatrix}\mathbf{z}_1 + \begin{bmatrix} 0 \\ \dfrac{-\frac{3}{4L}}{M+\frac{m}{4}} \end{bmatrix}\Delta u .$$

Note that the angle system with state vector $\mathbf{z}_1$ is independent of the position system with state vector $\mathbf{z}_2$. The opposite is not true.

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 2.1

Therefore the model for $\mathbf{z}_2$ can only be considered a valid state model if θ is assumed to be a disturbance:

$$\dot{\mathbf{z}}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{z}_2 + \begin{bmatrix} 0 \\ \dfrac{1}{M + \dfrac{m}{4}} \end{bmatrix} \Delta u + \begin{bmatrix} 0 & 0 \\ \dfrac{-\dfrac{3}{4}mg}{M + \dfrac{m}{4}} & 0 \end{bmatrix} \mathbf{z}_1$$

or

$$\dot{\mathbf{z}}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{z}_2 + \begin{bmatrix} 0 \\ \dfrac{1}{M + \dfrac{m}{4}} \end{bmatrix} \Delta u + \begin{bmatrix} 0 \\ \dfrac{-\dfrac{3}{4}mg}{M + \dfrac{m}{4}} \end{bmatrix} \Delta\theta,$$

where the disturbance is $v = \Delta\theta$ and

$$\mathbf{B}_v = \begin{bmatrix} 0 \\ \dfrac{-\dfrac{3}{4}mg}{M + \dfrac{m}{4}} \end{bmatrix}$$

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LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 2.10

a. The heat conduction power flows are:

$$q_c = k_c(T_s - T_a)$$
$$q_g = k_g(T_s - T_g)$$

The radiation power flow is:

$$q_r = k_r(T_s^4 - T_a^4)$$

From the conservation of energy:

$$C_s\dot{T}_s = q - q_g - q_c - q_r$$
$$C_g\dot{T}_g = q_g$$

where C_s and C_g are the heat capacities.

The states of the system are: T_s , $T_g \Rightarrow \mathbf{x} = \begin{bmatrix} T_s \\ T_g \end{bmatrix}$.

With the input, *u*, the output, $T_g = y$ and the disturbance, $T_a = v$.

The nonlinear state equation for the production oven is:

$$\begin{bmatrix} \dot{T}_s \\ \dot{T}_g \end{bmatrix} = \begin{bmatrix} \frac{1}{C_s}(ku - k_g(T_s - T_g) - k_c(T_s - T_a) - k_r(T_s^4 - T_a^4)) \\ \frac{1}{C_g}k_g(T_s - T_g) \end{bmatrix}$$

b. To find the stationary state one has to define: $\dot{T}_s = 0$ and $\dot{T}_g = 0$. This implies that

$$\begin{cases} ku_0 - k_g(T_{so} - T_{go}) - k_c(T_{so} - T_{ao}) - k_r(T_{so}^4 - T_{ao}^4) = 0 \\ T_{so} = T_{go} \end{cases}$$

Problem 2.10

or

$$ku_o - k_c(x_{10} - v_0) - k_r(x_{10}^4 - v_0^4) = 0$$

c. To find the linearized model of the system the following incremental variables are defined:

$$\begin{aligned} T_s &= x_1 = x_{10} + \Delta x_1 \qquad & u &= u_0 + \Delta u \\ T_g &= x_2 = x_{20} + \Delta x_2 \qquad & T_a &= v = v_0 + \Delta v \end{aligned}$$

$$\dot{x}_1 = \frac{1}{C_s}(ku - (k_g + k_c)x_1 + k_g x_2 + k_c v - k_r x_1^4 + k_r v^4) = f_1$$

$$\dot{x}_2 = \frac{1}{C_g} k_g (x_1 - x_2) = f_2$$

Now the Jacobians of the state equation can be calculated:

$$\mathbf{A} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix}_0 = \begin{bmatrix} -\dfrac{k_g + k_c}{C_s} - \dfrac{4k_r}{C_s} x_{10}^3 & \dfrac{k_g}{C_s} \\ \dfrac{k_g}{C_g} & -\dfrac{k_g}{C_g} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \dfrac{\partial f_1}{\partial u} \\ \dfrac{\partial f_2}{\partial u} \end{bmatrix}_0 = \begin{bmatrix} \dfrac{k}{C_s} \\ 0 \end{bmatrix}, \quad \mathbf{B}_v = \begin{bmatrix} \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial v} \end{bmatrix}_0 = \begin{bmatrix} \dfrac{k_c}{C_s} + \dfrac{4k_r}{C_s} v_0^3 \\ 0 \end{bmatrix}$$

$$\mathbf{C} = [0 \quad 1]$$

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LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 2.3

a. The stationary state is defined by the expression: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{0}$

Every vector: $\mathbf{x}_0 = \{0 \ \ 0 \ \ \mathrm{p}_0 \ \ 0\}^T$ is a solution if $u_0 = 0$ is selected.

For the given numerical values given in the problem text the non-linear state equation are:

$$\dot{\mathbf{x}} = \begin{bmatrix} x_2 \\ \dfrac{45\sin x_1 - 0.75\sin x_1 \cos x_1 \cdot x_2^2 - 1.5\cos x_1 \cdot u}{3 - 0.75\cos^2 x_1} \\ x_4 \\ \dfrac{0.5\sin x_1 \cdot x_2^2 - 7.5\sin x_1 \cos x_1 + u}{3 - 0.75\cos^2 x_1} \end{bmatrix}$$

The Jacobian of the state equation above with the parameter values can be calculated from:

$$\mathbf{A} = \left\{\frac{\partial f_i}{\partial x_j}\right\}_{x_1 = 0,\ \mathrm{x}_2 = 0},$$

with the following results:

$$a = (3 - 0.75\cos^2 x_1)(45\cos x_1 - 0.75\, x_2^2(\cos^2 x_1 - \sin^2 x_1) + 1.5\sin x_1 \cdot u)$$

$$b = (45\sin x_1 - 0.75\sin x_1 \cos x_1 \cdot x_2^2 - 1.5\cos x_1 \cdot u)(2 \cdot 0.75\cos x_1 \sin x_1)$$

$$\frac{\partial f_2}{\partial x_1} = \frac{a - b}{(3 - 0.75\cos^2 x_1)^2}$$

$$\frac{\partial f_2}{\partial x_2} = \frac{-2 \cdot 0.75\sin x_1 \cos x_1 \cdot x_2}{3 - 0.75\cos^2 x_1}$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 2.3 (continued)

$$c = (3 - 0.75\cos^2 x_1)(0.5\, x_2^2 \cos x_1 + 7.5\,(\sin^2 x_1 - \cos^2 x_1))$$

$$d = (0.5\, \sin x_1 \cdot x_2^2 - 7.5\, \sin x_1 \cos x_1 + u)(2 \cdot 0.75 \cos x_1 \sin x_1)$$

$$\frac{\partial f_4}{\partial x_1} = \frac{c - d}{(3 - 0.75\, \cos^2 x_1)^2}$$

$$\frac{\partial f_4}{\partial x_2} = \frac{2 \cdot 0,5 \sin x_1 \cdot x_2}{3 - 0.75 \cos^2 x_1}$$

After insertion of numerical values for x_0 one obtains:

$$\left.\frac{\partial f_2}{\partial x_1}\right|_0 = \frac{2.5\; 45}{(2.25)^2} = 20$$

$$\left.\frac{\partial f_2}{\partial x_2}\right|_0 = 0$$

$$\left.\frac{\partial f_4}{\partial x_1}\right|_0 = \frac{2.25\; (-7.5)}{(2.25)^2} = -3.3$$

$$\left.\frac{\partial f_4}{\partial x_2}\right|_0 = 0$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3.33 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{B} = \left\{\frac{\partial f_i}{\partial u}\right\}_0 = \begin{bmatrix} 0 \\ \dfrac{-1.5\, \cos x_1}{3 - 0.75\, \cos^2 x_1} \\ 0 \\ \dfrac{1}{3 - 0.75\, \cos^2 x_1} \end{bmatrix}_{x_1 = 0,\, x_2 = 0} = \begin{bmatrix} 0 \\ -0.667 \\ 0 \\ 0.444 \end{bmatrix}$$

Problem 2.3 (continued)

b. The same result can be obtained from problem 2.1.

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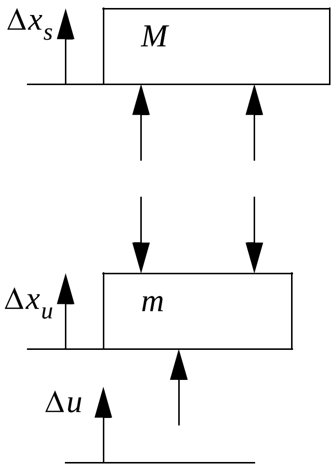
LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 2.4



State convention:
$\Delta u > \Delta x_u > \Delta x_s$
$\Delta\dot{x}_u > \Delta\dot{x}_s$

a. Using the sign convention defined on the figure above, Newton's second law applied to M gives:

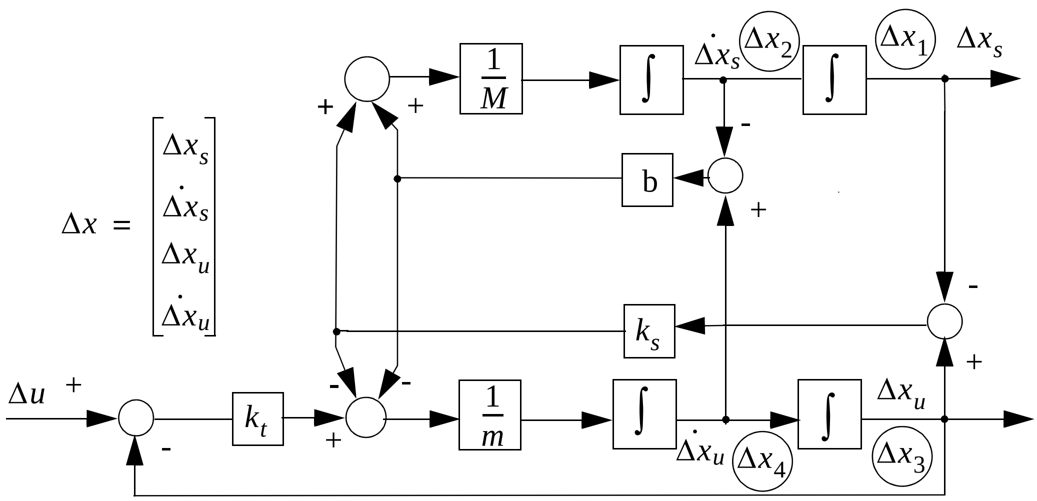
$$M\Delta\ddot{x}_s = k_s(\Delta x_u - \Delta x_s) + b(\Delta\dot{x}_u - \Delta\dot{x}_s)$$

Newton's second law applied to m gives:

$$m\Delta\ddot{x}_u = k_t(\Delta u - \Delta x_u) - k_s(\Delta x_u - \Delta x_s) - b(\Delta\dot{x}_u - \Delta\dot{x}_s)$$

b. The block diagram of the overall system is then as below:

$$\Delta x = \begin{bmatrix} \Delta x_s \\ \Delta\dot{x}_s \\ \Delta x_u \\ \Delta\dot{x}_u \end{bmatrix}$$



LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 2.4

c. The linearized state equations are then:

$$\begin{aligned}
\Delta\dot{x}_1 &= x_2 \\
\Delta\dot{x}_2 &= \frac{1}{M}(k_s(-\Delta x_1 + \Delta x_3) + b(-\Delta x_2 + \Delta x_4)) \\
\Delta\dot{x}_3 &= x_4 \\
\Delta\dot{x}_4 &= \frac{1}{m}(k_t(-\Delta x_3 + \Delta u) - k_s(\Delta x_3 - \Delta x_1) - b(\Delta x_4 - \Delta x_2))
\end{aligned}$$

or in matrix form:

$$\Delta\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_s}{M} & -\frac{b}{M} & \frac{k_s}{M} & \frac{b}{M} \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m} & \frac{b}{m} & -\frac{k_t + k_s}{m} & -\frac{b}{m} \end{bmatrix} \Delta\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_t}{m} \end{bmatrix} \Delta u = \mathbf{A}\Delta\mathbf{x} + \mathbf{B}\Delta u$$

with the output equations:

$$\Delta\mathbf{y} = \begin{bmatrix} \Delta x_s \\ \Delta x_u \\ \Delta f_t \\ \Delta\ddot{x}_s \end{bmatrix} = \begin{bmatrix} \Delta x_1 \\ \Delta x_3 \\ k_t(\Delta u - \Delta x_3) \\ -\frac{k_s}{M}\Delta x_1 - \frac{b}{M}\Delta x_2 + \frac{k_s}{M}\Delta x_3 + \frac{b}{M}\Delta x_4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -k_t & 0 \\ -\frac{k_s}{m} & -\frac{b}{M} & \frac{k_s}{M} & \frac{b}{m} \end{bmatrix} \Delta\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ k_t \\ 0 \end{bmatrix} \Delta u = \mathbf{C}\Delta\mathbf{x} + \mathbf{D}\Delta u$$

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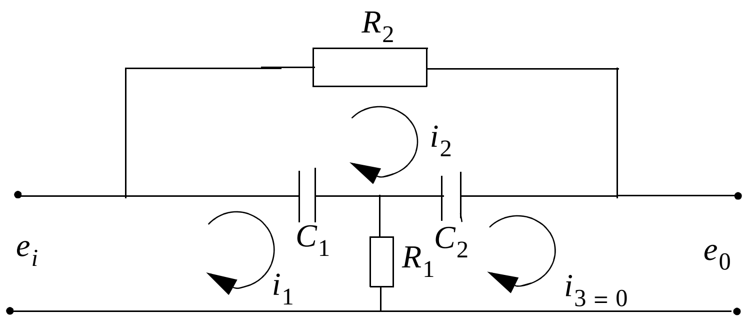
LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 2.5

a. The first filter (a.) is:



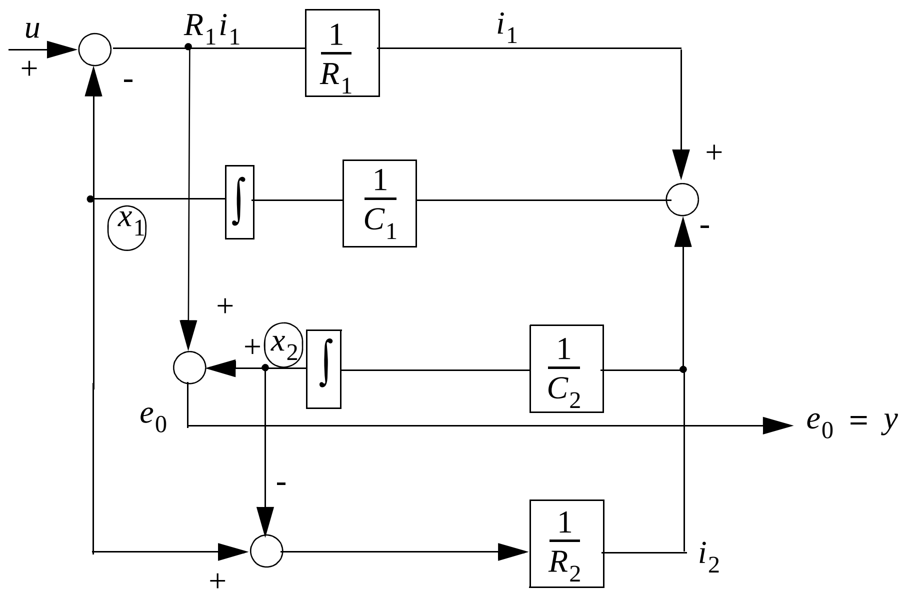
With the mask currents as drawn and using Ohm's law, one obtains:

$$e_i = \frac{1}{C_1}\int(i_1 - i_2)dt + R_1 i_1$$

$$0 = \frac{1}{C_1}\int(i_2 - i_1)dt + R_2 i_2 + \frac{1}{C_2}\int i_2 dt$$

$$-e_0 = -R_1 i_1 - \frac{1}{C_2}\int i_2 dt$$

b. These equations have the block diagram below:



Problem 2.5 (continued)

c. The state equations for bridged T filter (a.) are:

$$\dot{x}_1 = \frac{1}{C_1}\left(\frac{1}{R_1}(u - x_1) - \frac{1}{R_2}(x_1 - x_2)\right)$$

$$\dot{x}_2 = \frac{1}{C_2}\frac{1}{R_2}(x_1 - x_2)$$

$$y = e_0 = x_2 - x_1 + u$$

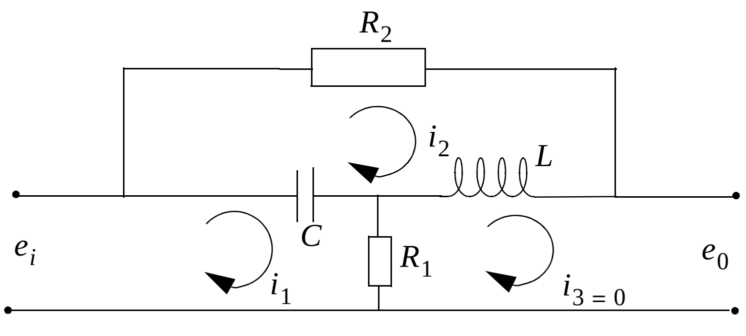
$$\dot{x}_1 = \frac{1}{C_1R_1}u - \frac{1}{C_1R_1}x_1 - \frac{1}{C_1R_2}x_1 + \frac{1}{C_1R_2}x_2$$

$$\dot{x}_2 = \frac{1}{C_2R_2}x_1 - \frac{1}{C_2R_2}x_2$$

$$\dot{\mathbf{x}} = \begin{bmatrix} -\dfrac{R_1 + R_2}{C_1R_1R_2} & \dfrac{1}{C_1R_2} \\ \dfrac{1}{C_2R_2} & \dfrac{1}{C_2R_2} \end{bmatrix} + \begin{bmatrix} \dfrac{1}{C_1R_1} \\ 0 \end{bmatrix} u$$

$$\mathbf{y} = [-1 \quad 1]\mathbf{x} + u$$

The second filter (b.) is:



This filter is decribed by the mask equations:

LINEAR SYSTEMS CONTROL

Solutions to Problems

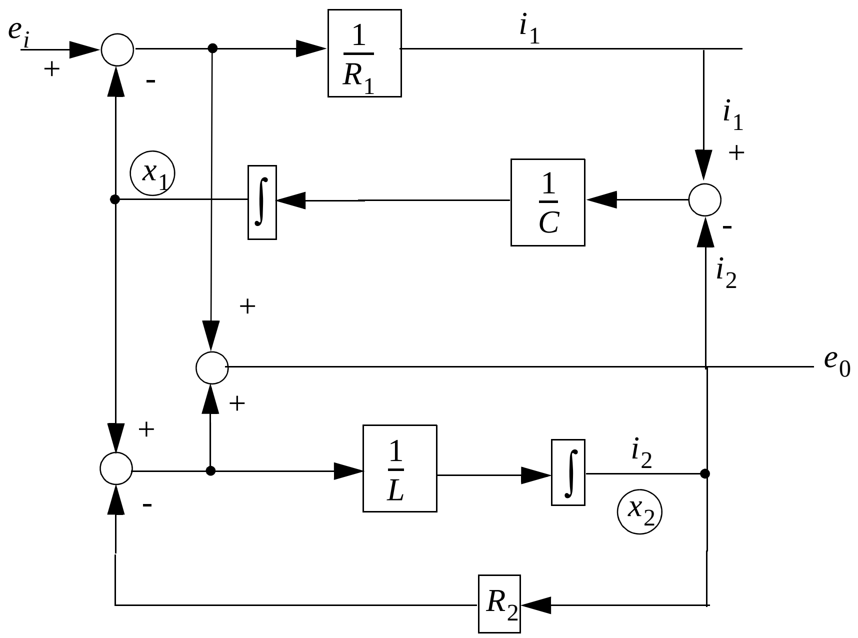
Problem 2.5 (continued)

$$e_i = \frac{1}{C}\int(i_1 - i_2)dt + R_1 i_1$$

$$0 = \frac{1}{C}\int(i_2 - i_1)dt + R_2 i_2 + L\frac{di_2}{dt}$$

$$-e_0 = -R_1 i_1 - L\frac{di_2}{dt}$$

With the block diagram:



This block diagram can be translated into the state equations:

$$\dot{x}_1 = \frac{1}{C}\left(\frac{1}{R_1}(e_i - x_1) - x_2\right) = \frac{1}{CR_1}e_i - \frac{1}{CR_1}x_1 - \frac{1}{C}x_2$$

$$\dot{x}_2 = \frac{1}{L}(-R_2 x_2 + x_1) = -\frac{R_2}{L}x_2 + \frac{1}{L}x_1$$

$$y = e_i - x_1 + x_1 - R_2 x_2$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 2.5 (continued)

$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{1}{CR_1} & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R_2}{L} \end{bmatrix}\mathbf{x} + \begin{bmatrix} \frac{1}{CR_1} \\ 0 \end{bmatrix} e_i$$

$$\mathbf{y} = [0 \quad -R_2]\mathbf{x} + e_i$$

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LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 2.6

a. For the volume flows one has:

$$\left.\begin{aligned} q_1 &= A_c\dot{x} + \frac{V_1}{\beta}\dot{p}_1 + C_l(p_1 - p_2) \\ q_2 &= A_c\dot{x} - \frac{V_1}{\beta}\dot{p}_2 + C_l(p_1 - p_2) \end{aligned}\right\}$$

$$\Rightarrow \qquad \left.\begin{aligned} \dot{p}_1 &= \frac{\beta}{V_1}(q_1 - A_c\dot{x} - C_l(p_1 - p_2)) \\ \dot{p}_2 &= \frac{\beta}{V_2}(-q_2 + A_c\dot{x} + C_l(p_1 - p_2)) \end{aligned}\right\}.$$

The flow are controlled by the input relation:

$$q_1 = q_2 = ku\,.$$

Newton's second law applied to M gives:

$$\begin{aligned} M\ddot{x} &= f + A_c(p_1 - p_2) - C_f\ \dot{x} \\ \Rightarrow \dot{x} &= \frac{1}{M}(f + A_c(p_1 - p_2) - C_f\ \dot{x}). \end{aligned}$$

b. The state vector for the system is (see the block diagram below):

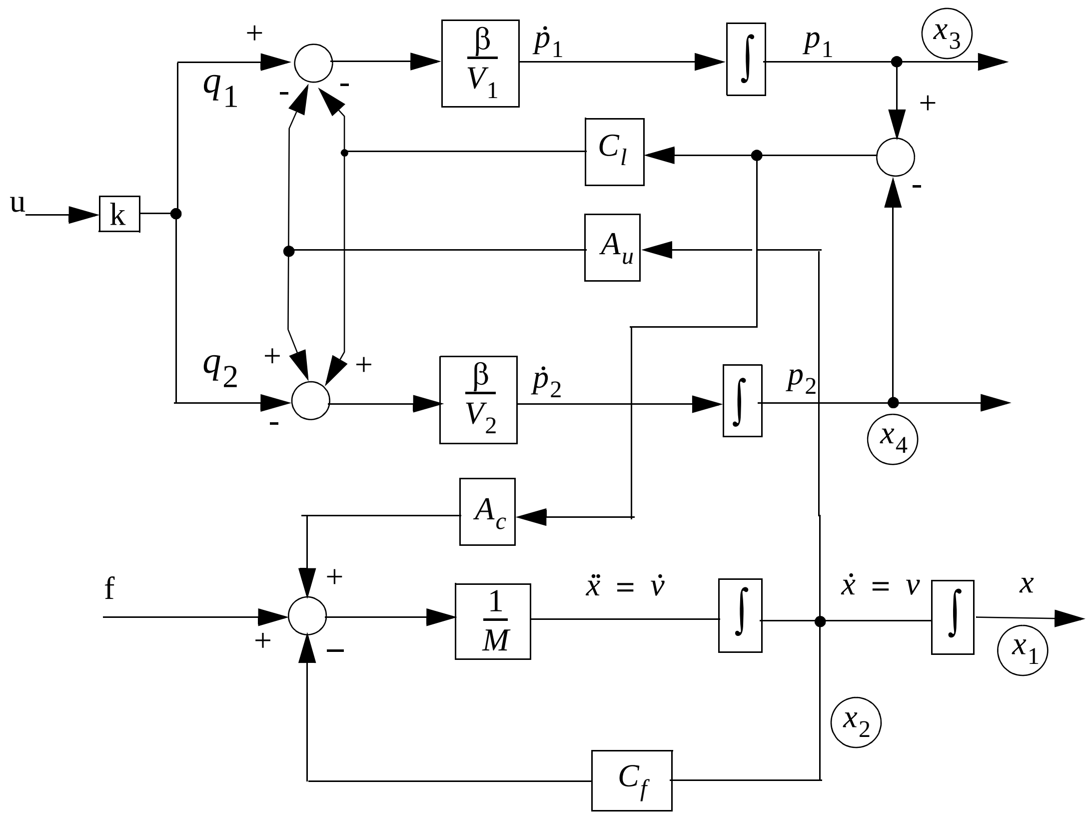
$$\mathbf{x} = \begin{bmatrix} x \\ \dot{x} \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} x \\ v \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 2.6 (continued)

Blockdiagram:



The state equations for the system can be written in two different ways:

$$
\begin{aligned}
\dot{x} &= v \\
\dot{v} &= \frac{1}{M}(-C_f v + A_c(p_1 - p_2) + f) \\
\dot{p}_1 &= \frac{\beta}{V_1}(ku - A_c v - C_l(p_1 - p_2)) \\
\dot{p}_2 &= \frac{\beta}{V_2}(-ku + A_c v + C_l(p_1 - p_2))
\end{aligned}
\quad \text{or} \quad
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{M}(-C_f x_2 + A_c(x_3 - x_4) + f) \\
\dot{x}_3 &= \frac{\beta}{V_1}(ku - A_c x_2 - C_l(x_3 - x_4)) \\
\dot{x}_4 &= \frac{\beta}{V_2}(-ku + A_c x_2 + C_l(x_3 - x_4))
\end{aligned}
$$

This leads to:

Problem 2.6 (continued)

$$\Rightarrow \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\dfrac{C_f}{M} & \dfrac{A_c}{M} & -\dfrac{A_c}{M} \\ 0 & -\dfrac{A_c\beta}{V_1} & -\dfrac{C_l\beta}{V_1} & \dfrac{C_l\beta}{V_1} \\ 0 & \dfrac{A_c\beta}{V_2} & \dfrac{C_l\beta}{V_2} & -\dfrac{C_l\beta}{V_2} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \dfrac{k\beta}{V_1} \\ -\dfrac{k\beta}{V_2} \end{bmatrix}$$

$$\mathbf{B}_v = \begin{bmatrix} 0 \\ \dfrac{1}{M} \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{C} = [1 \;\; 0 \;\; 0 \;\; 0]$$

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LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 2.7

Dividing numerator and denominator through by 4 makes polynomial division easier:

$$H(z) = \frac{0.75\,z^3 + 0.5\,z^2 + 0.25\,z + 1.25}{z^3 + z^2 + 0.5\,z + 2}$$

Now using polynomial division:

$$(0.75\,z^3 + 0.5\,z^2 + 0.25\,z + 1.25) \div (z^3 + z^2 + 0.5\,z + 2) = 0.75$$

$$-0.75\,z^3 - 0.75\,z^2 - 0.375\,z - 1.5$$

$$\overline{\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$-0.25\,z^2 - 0.125\,z - 0.25$$

$$\Rightarrow H(z) = 0.75 + \frac{-0.25\,z^2 - 1.25\,z - 0.25}{z^3 + z^2 + 0.5\,z + 2}$$

Companion form 1 is then:

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -0.5 & -1 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{C}_1 = [-0.25 \quad -1.25 \quad -0.25], \quad \mathbf{D}_1 = [0.75]$$

and companion form 2:

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & -0.5 \\ 0 & 1 & -1 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} -0.25 \\ -0.125 \\ -0.25 \end{bmatrix}$$

$$\mathbf{C}_2 = [0 \quad 0 \quad 1], \quad \mathbf{D}_2 = [0.75]$$

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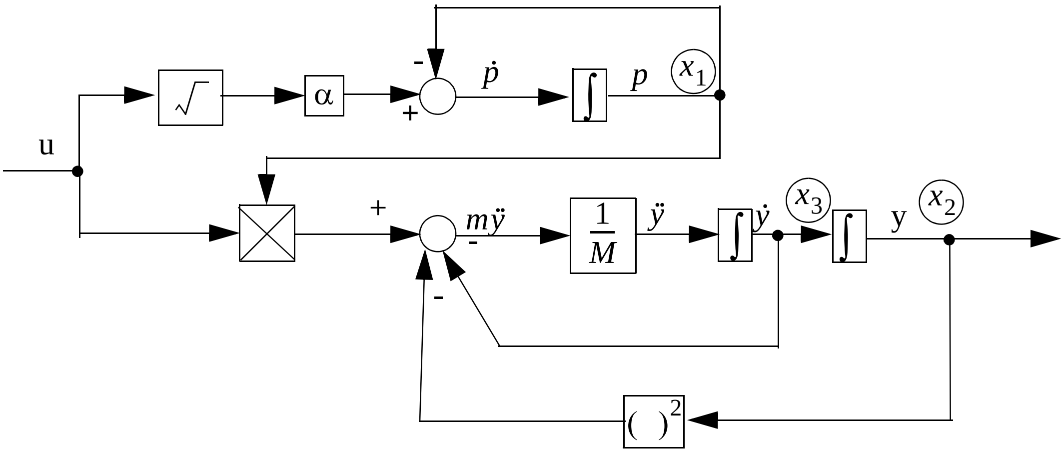
LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 2.8

a. A block diagram for the system can be drawn from the given equations.



b. The state equations for the nonlinear system can be derived as follows by choosing the state vector: $\mathbf{x} = [p \quad \dot{y} \quad y]^T = [x_1 \quad x_2 \quad x_3]^T$

$$\begin{aligned} \dot{x}_1 &= -x_1 + \alpha\sqrt{u} \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= \frac{1}{m}(-x_2^2 - x_3 + x_1 u) \\ y &= x_2 \end{aligned}$$

The stationary stated can be found by setting $\dot{\mathbf{x}} = 0$:

$$\left.\begin{aligned} 0 &= -x_{10} + \alpha\sqrt{u_0} \\ 0 &= x_{30} \\ 0 &= -x_{20}^2 - x_{30} + x_{10}u_0 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} x_{30} &= 0 \\ x_{10} &= a\sqrt{u_0} \\ x_{20} &= \pm\sqrt{\alpha u_0^{\frac{3}{2}}} \end{aligned}\right.$$

c. Now defining:

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 2.8

$$x_1 = x_{10} + \Delta x_1 \qquad u = u_0 + \Delta u$$
$$x_2 = x_{20} + \Delta x_2 \qquad y = y_0 + \Delta y$$
$$x_3 = x_{30} + \Delta x_3$$

the linearized state space model is:

$$\Delta\dot{\mathbf{x}} = \mathbf{A}\Delta\mathbf{x} + \mathbf{B}\Delta\mathbf{u}, \quad \Delta\mathbf{y} = [0 \quad 1 \quad 0]\Delta\mathbf{x}$$

where:

$$\mathbf{A} = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} & \frac{\partial \dot{x}_1}{\partial x_3} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} & \frac{\partial \dot{x}_2}{\partial x_3} \\ \frac{\partial \dot{x}_3}{\partial x_1} & \frac{\partial \dot{x}_3}{\partial x_2} & \frac{\partial \dot{x}_3}{\partial x_3} \end{bmatrix}_0, \quad \mathbf{B} = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial u} \\ \frac{\partial \dot{x}_2}{\partial u} \\ \frac{\partial \dot{x}_3}{\partial u} \end{bmatrix}_0$$

Choosing $x_{20} = \sqrt{\alpha u_0^{\frac{3}{2}}}$ one obtains:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ \frac{u_0}{m} & -\frac{2}{m}\sqrt{\alpha u_0^{\frac{3}{2}}} & -\frac{1}{m} \end{bmatrix}, B = \begin{bmatrix} \frac{\alpha}{2\sqrt{u_0}} \\ 0 \\ \frac{\alpha}{m}\sqrt{u_0} \end{bmatrix}$$

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LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 2.9

a. Using the volume conservation law:

$$\left.\begin{aligned} A_1\dot{x}_1 &= ku_1 - q_1 \\ A_2\dot{x}_2 &= q_1 - q_0 + q_2 \\ A_3\dot{x}_3 &= -q_2 + ku_2 \end{aligned}\right\} \qquad (1)$$

The corresponding flow equations are:

$$\left.\begin{aligned} q_1 &= c_1\sqrt{x_1 - x_2} \\ q_2 &= c_2\sqrt{x_3 - x_2} \\ q_0 &= c_0\sqrt{x_2} \end{aligned}\right\} \qquad (2)$$

Inserting equation (2) into (1):

$$\dot{x}_1 = \frac{1}{A_1}(-c_1\sqrt{x_1 - x_2} + ku_1) = f_1(x, u)$$

$$\dot{x}_2 = \frac{1}{A_2}(c_1\sqrt{x_1 - x_2} + c_2\sqrt{x_3 - x_2} - c_0\sqrt{x_2}) = f_2(x, u)$$

$$\dot{x}_3 = \frac{1}{A_3}(ku_2 - c_2\sqrt{x_3 - x_2}) = f_3(x, u)$$

$$y = q_0 = c_0\sqrt{x_2} = g(x, u)$$

b. The stationary states are found by setting:

$$\dot{x}_1 = 0, \qquad \dot{x}_2 = 0, \qquad \dot{x}_3 = 0.$$

$$ku_{10} = c_1\sqrt{x_{10} - x_{20}} \qquad (3)$$

$$c_1\sqrt{x_{10} - x_{20}} + c_2\sqrt{x_{30} - x_{20}} = c_0\sqrt{x_{20}} \qquad (4)$$

$$ku_{20} = c_2\sqrt{x_{30} - x_{20}} \qquad (5)$$

Problem 2.9

When equations (3) and (5) are inserted into (4):

$$
\begin{aligned}
ku_{10} + ku_{20} &= c_0\sqrt{x_{20}} \\
\Rightarrow x_{20} &= \left(\frac{k}{c_0}(u_{10}+u_{20})\right)^2
\end{aligned}
$$

From (3) and (5) the following result is obtained:

$$
\begin{aligned}
x_{10} &= \left(\frac{k}{c_1}u_{10}\right)^2 + x_{20} \\
x_{30} &= \left(\frac{k}{c_2}u_{20}\right)^2 + x_{20}
\end{aligned}
$$

with the output:

$$
y_0 = c_0\sqrt{x_{20}}
$$

Now define the following incremental variables:

$$
\begin{aligned}
x_1 &= x_{10} + \Delta x_1 \qquad & u_1 &= u_{10} + \Delta u_1 \\
x_2 &= x_{20} + \Delta x_2 \qquad & u_2 &= u_{20} + \Delta u_2 \\
x_3 &= x_{30} + \Delta x_3 \qquad & y &= y_0 + \Delta y
\end{aligned}
$$

$$
\mathbf{A} = \left\{\frac{\partial f_i}{\partial x_j}\right\}_0
$$

$$
= \begin{bmatrix}
\dfrac{-c_1}{2A_1\sqrt{x_{10}-x_{20}}} & \dfrac{c_1}{2A_1\sqrt{x_{10}-x_{20}}} & 0 \\
\dfrac{c_1}{2A_2\sqrt{x_{10}-x_{20}}} & -\dfrac{c_1}{2A_2\sqrt{x_{10}-x_{20}}} - \dfrac{c_0}{2A_2\sqrt{x_{20}}} - \dfrac{c_2}{2A_2\sqrt{x_{30}-x_{20}}} & \dfrac{c_2}{2A_2\sqrt{x_{30}-x_{20}}} \\
0 & \dfrac{c_2}{2A_3\sqrt{x_{30}-x_{20}}} & -\dfrac{c_2}{2A_3\sqrt{x_{30}-x_{20}}}
\end{bmatrix}
$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 2.9

$$\mathbf{B} = \left\{ \frac{\partial f_i}{\partial u_j} \right\}_0 =$$

$$\begin{bmatrix} \frac{k}{A_1} & 0 \\ 0 & 0 \\ 0 & \frac{k}{A_3} \end{bmatrix}$$

$$\mathbf{C} = \left\{ \frac{\partial g_i}{\partial x_j} \right\}_0 = \left\{ 0 \quad \frac{c_0}{2\sqrt{x_{20}}} \quad 0 \right\}$$

$$\mathbf{D} = \left\{ \frac{\partial g_i}{\partial u_j} \right\}_0 = 0$$

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LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 3.1

a. The time varying differential equation in the problem has two states and for an input, $u(t) = 0$, the transition matrix can be written:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \Phi(t, t_0)x_0 = \begin{bmatrix} \Phi_{11}(t, t_0) & \Phi_{12}(t, t_0) \\ \Phi_{21}(t, t_0) & \Phi_{22}(t, t_0) \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \text{ or}$$

$$\begin{cases} x_1(t) = \Phi_{11}(t, t_0)x_{10} + \Phi_{12}(t, t_0)x_{20} \\ x_2(t) = \Phi_{21}(t, t_0)x_{10} + \Phi_{22}(t, t_0)x_{20} \end{cases} \qquad (1)$$

From section 3.1.2 one has:

$$\begin{cases} \dot{x}_1(t) = t\, x_2(t) \\ \dot{x}_2(t) = 0 \end{cases}$$

$$\begin{cases} x_2(t) = x_{20} \\ x_1(t) = \int_{t_0}^{t} \tau x_{20} d\tau + x_{10} = \frac{1}{2}\tau^2 x_{20}\Big|_{t_0}^{t} + x_{10} \end{cases}$$

$$= \frac{1}{2}t^2 x_{20} - \frac{1}{2}t_0^2 x_{20} + x_{10}$$

From (1) one obtains:

$$\Phi_{21} = 0, \quad \Phi_{22} = 1$$
$$\Phi_{11} = 1, \quad \Phi_{12} = \frac{1}{2}(t^2 - t_0^2)$$

$$\Rightarrow \Phi(t, t_0) = \begin{bmatrix} 1 & \frac{1}{2}(t^2 - t_0^2) \\ 0 & 1 \end{bmatrix}$$

b. From equation (3.21):

$$\frac{\partial \Phi(t, t_0)}{\partial t} = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 3.1

which leads to:

$$\mathbf{A}\cdot\Phi(t,t_0) = \begin{bmatrix}0 & t\\ 0 & 0\end{bmatrix}\Phi(t,t_0) = \begin{bmatrix}0 & t\\ 0 & 0\end{bmatrix}$$

c. The zero state solution is:

$$\begin{aligned}
\mathbf{x}_{zs}(t) &= \int_{t_0}^{t}\Phi(t,\tau)\mathbf{B}(\tau)u(\tau)d\tau \\
&= \int_{t_0}^{t}\begin{bmatrix}1 & \frac{1}{2}(\tau^2-\tau^2)\\ 0 & 1\end{bmatrix}\begin{bmatrix}0\\ \frac{1}{\tau}\end{bmatrix}\cdot 1\cdot d\tau = \int_{t_0}^{t}\begin{bmatrix}\frac{1}{2}\left(\frac{t^2}{\tau}-\tau\right)\\ \frac{1}{\tau}\end{bmatrix}d\tau \\
&= \begin{bmatrix}\frac{t^2}{2}\ln\tau-\frac{1}{4}\tau^2\\ \ln\tau\end{bmatrix}\Bigg|_{t_0}^{t} = \begin{bmatrix}\frac{t^2}{2}\ln\tau-\frac{1}{4}t^2-\frac{t^2}{2}\ln\tau_0+\frac{1}{4}t_0^2\\ \ln\tau-\ln t_0\end{bmatrix} \\
&= \begin{bmatrix}\frac{1}{4}(t_0^2-t^2)+\frac{t^2}{2}\ln\frac{t}{t_0}\\ \ln\frac{t}{t_0}\end{bmatrix}
\end{aligned}$$

The overall solution is then:

$$\begin{aligned}
\mathbf{x}(t) &= \Phi(t,t_0)x_0+\int_{t_0}^{t}\Phi(t,\tau)\mathbf{B}(\tau)\mathbf{u}(\tau)d\tau \\
&= \begin{bmatrix}x_{10}+\left(\frac{x_{20}}{2}-\frac{1}{4}\right)(t^2-t_0^2)+\frac{t^2}{2}\ln\frac{t}{t_0}\\ x_{20}+\ln\frac{t}{t_0}\end{bmatrix} \qquad (2)
\end{aligned}$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 3.1

d. From (2) one finds:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \left(\frac{x_{20}}{2} - \frac{1}{4}\right)2t + \frac{t^2}{2}\frac{1}{t} + t \ln\frac{t}{t_0} \\ \frac{1}{t} \end{bmatrix} = \begin{bmatrix} tx_{20} + t \ln\frac{t}{t_0} \\ \frac{1}{t} \end{bmatrix}$$

$$\mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t) \cdot \mathbf{u}(t) = \begin{bmatrix} tx_{20} + t \ln\frac{t}{t_0} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{t} \end{bmatrix} = \begin{bmatrix} tx_{20} + t \ln\frac{t}{t_0} \\ \frac{1}{t} \end{bmatrix}$$

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Linear Systems Control

Solutions to problems

Problem 3.10

$$x_{kf1} = \begin{bmatrix} 0 & 1 \\ -1 & \frac{5}{2} \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k \qquad y_k = [-2 \;\; 1] x_k$$

Eigenvalues:

$$\begin{bmatrix} z & -1 \\ 1 & z - \frac{5}{2} \end{bmatrix} = z^2 - \frac{5}{2}z + 1 = 0 \Rightarrow z = \begin{cases} 2 \\ \frac{1}{2} \end{cases}$$

Natural modes:

$$m_i = \begin{cases} 2^k \\ \frac{1}{2}k \end{cases}$$

The system has an eigenvalue outside the unit circle and it is not asymptically internally stable.

Transfer furnction

$$\begin{aligned} H(z) &= C(zI - A)^{-1}B \\ &= [-2 \;\; 1] \begin{bmatrix} z & -1 \\ 1 & z - \frac{5}{2} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{z^2 - \frac{5}{2}z + 1} [-2 \;\; 1] \begin{bmatrix} z - \frac{5}{2} & 1 \\ -1 & z \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{z - \frac{1}{2}} \end{aligned}$$

Only one pole (within the unit circle).

$\Rightarrow$ The system is BIBO-stable (externally stable)

The unstable pole/eigenvalue $z = 2$ is cancelled in the transfer function.

NOTE: BIBO-stability is not the same as asymptotic internal stability.

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 3.11

a. The system is given by the state equation:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u \quad , \quad \mathbf{y} = [a \quad -1]\mathbf{x}$$

The eigenvalues of the system are determined from the equation:

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \det\begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix} = \lambda^2 - 1 = 0 \quad \text{for}$$

$$\lambda = \begin{cases} 1 \\ -1 \end{cases}$$

One eigenvalue in the right half plane: the system is thus unstable

b. The transfer function of the system can be found from:

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix} = \frac{1}{s^2 - 1}\begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix}$$

$$\mathbf{G}(s) = [a \;\; -1]\frac{1}{s^2 - 1}\begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= [a \;\; -1]\begin{bmatrix} \dfrac{1}{s^2 - 1} \\ \dfrac{s}{s^2 - 1} \end{bmatrix} = \frac{a}{s^2 - 1} - \frac{s}{s^2 - 1} = \frac{a - s}{(s + 1)(s - 1)}$$

For a = 1 one has:

$$G(s) = -\frac{1}{s + 1} \quad \text{one pole} \;\; s = -1$$

$$\Rightarrow \text{BIBO-stable}$$

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LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 3.12

a. The state equations for the target system are:

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\mathbf{u}, \qquad \mathbf{y} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}\mathbf{x}$$

To investigate the controllability of the system the rank of the controllability matrix is tested.

$$\mathbf{M}_c = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \mathrm{rank}(\mathbf{M}_c) = 1$$

$\Rightarrow$ the system is not controllable

To investigate the observability of the system the rank of the observability matrix is tested.

$$\mathbf{M}_0 = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \\ -1 & 2 \\ 1 & -2 \end{bmatrix} \quad \mathrm{rank}(\mathbf{M}_0) = 1$$

$\Rightarrow$ the system is not observable

b. Now the input matrix of the system is changed:

$$\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$$

$$\mathbf{M}_c = \begin{bmatrix} 1 & 1 & 5 & 1 \\ 2 & 1 & 4 & 1 \end{bmatrix} \quad \mathrm{rank}(\mathbf{M}_c) = 2$$

$\Rightarrow$ the system is controllable

$$\mathbf{M}_o = \begin{bmatrix} 1 & 2 \\ 1 & -2 \\ -7 & 10 \\ 1 & -2 \end{bmatrix} \quad \mathrm{rank}(\mathbf{M}_o) = 2 \quad \Rightarrow \text{the system is observable}$$

❐

Problem 3.13

a)

$$\dot{x} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} x$$

$$\begin{vmatrix} \lambda + 1 & -1 & -1 \\ 0 & \lambda & -1 \\ 0 & 2 & \lambda + 3 \end{vmatrix} = \begin{matrix} \lambda(\lambda+1)(\lambda+3) + 2(\lambda+1) = \\ (\lambda+1)(\lambda^2 + 3\lambda + 2) = 0 \end{matrix}$$

Eigenvalues:
$$\lambda = \begin{cases} -1 \\ -1 \\ -2 \end{cases}$$

$$Av_i = \lambda_i v_i \qquad , \qquad \lambda = -1$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} \Rightarrow \quad = -1 \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} \Rightarrow \begin{cases} -v_{11} + v_{12} + v_{13} = -v_{11} \\ v_{13} = -v_{12} \\ -2v_{12} - 3v_{13} = -v_{13} \end{cases}$$

For $v_{11} = 1$ we can choose $v_{12} = 0 \quad , \quad v_{13} = 0$

$$\Rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Similarly:
$$v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad , \qquad v_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

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Solutions to problems

Problem 3.13

b)

$$M = [v_1 \;\; v_2 \;\; v_3] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix} \quad , M^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\Lambda = M^{-1}AM = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

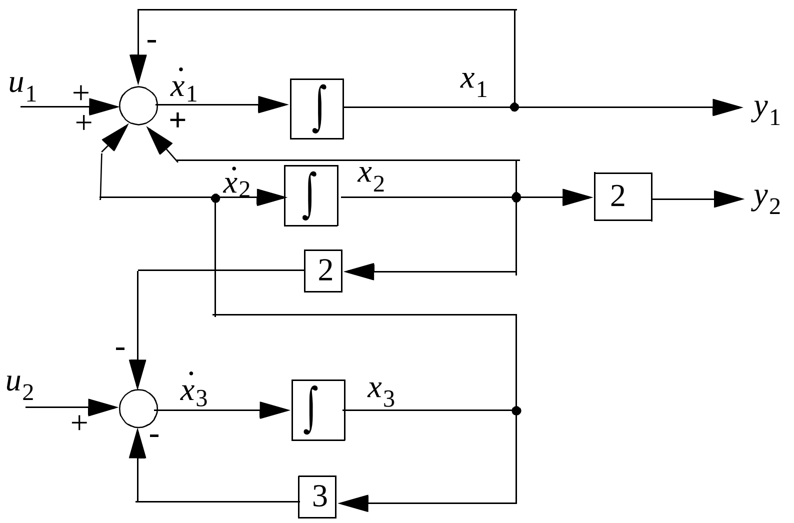
$$B_\Lambda = M^{-1}B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \quad , \quad C_\Lambda = CM = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

c)

$$\dot{x}_1 = -x_1 + x_2 + x_3 + u_1$$

$$\dot{x}_2 = x_3 \qquad\qquad y_1 = x_1$$

$$\dot{x}_3 = -2x_2 - 3x_3 + u_2 \qquad\qquad y_2 = 2x_2$$



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Linear Systems Control

Solutions to problems

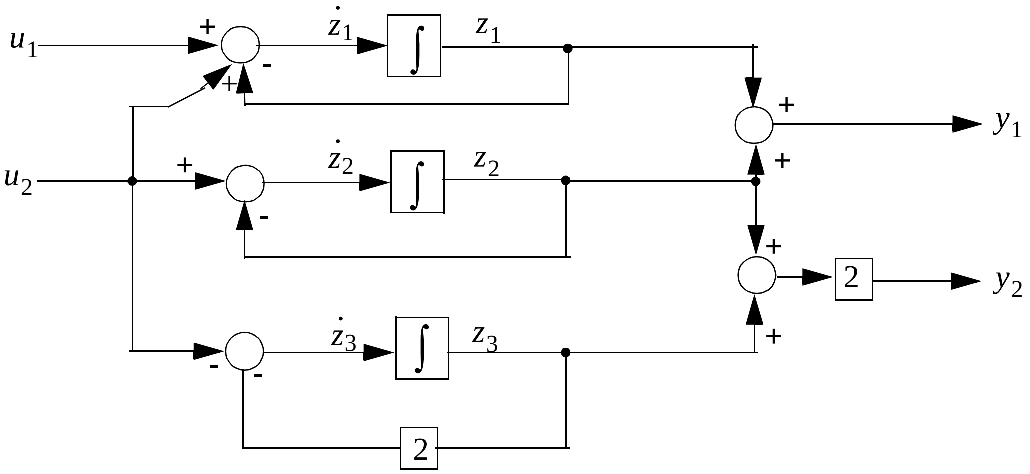
Problem 3.13

The diagonal form:

$$\dot{z}_1 – z_1 + u_1 + u_2$$

$$\dot{z}_2 = -z_2 + u_2 \qquad\qquad y_1 = z_1 + z_2$$

$$\dot{z}_3 = -2z_3 - u_2 \qquad\qquad y_2 = 2z_2 + 2z_3$$



d)

-

$$G(s) = C(sI - A)^{-1}B$$

$$(sI - A)^{-1} = \begin{bmatrix} s+1 & -1 & -1 \\ 0 & s & -1 \\ 0 & z & s+3 \end{bmatrix}^{-1} = \frac{1}{|sI - A|}\begin{bmatrix} s(s+3)+2 & 0 & 0 \\ s+1 & (s+1)(s+3) & -2(s+1) \\ s+1 & s+1 & s(s+1) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s+1} & \frac{s+1}{(s+1)(s^2+3s+2)} & \frac{1}{s^2+3s+2} \\ 0 & \frac{s+3}{s^2+3s+2} & \frac{1}{s^2+3s+2} \\ 0 & \frac{-2}{s^2+3s+2} & \frac{s}{s^2+3s+2} \end{bmatrix}$$

Problem 3.13

$$G(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}(sI - A)^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{1}{s+1} & \dfrac{1}{s^2 3s + 2} \\ 0 & \dfrac{2}{s^2 + 3s + 2} \end{bmatrix}$$

Diagnonal form:

$$(sI - \Lambda)^{-1} = \begin{bmatrix} (s+1)^{-1} & 0 & 0 \\ 0 & (s+1)^{-1} & 0 \\ 0 & 0 & (s+2)^{-1} \end{bmatrix}$$

$$G(s) = C_\Lambda(sI - \Lambda)^{-1}B_\Lambda$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}(sI - \Lambda)^{-1}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{1}{s^2 + 3s + 2} \\ 0 & \dfrac{2}{s^2 + 3s + 2} \end{bmatrix}$$

e)

Left eigenvectors are easiest found by taking the w_i^T vectors as the rows of M^{-1} (see equation (3.245):

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Linear Systems Control

Solutions to problems

Problem 3.13

$$w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{for} \quad \lambda_1 = -1$$

$$w_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \quad \text{for} \quad \lambda_2 = -1$$

$$w_3 = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \quad \text{for} \quad \lambda_3 = -2$$

If we find w_i from the definition (3.342)

$$w_i^T A = \lambda_i w_i^T \quad \text{or} \quad A^T w_i = \lambda_i w_i$$

it is important to norm the w_i's such that the condition

$$\begin{bmatrix} w_1^T \\ w_2^T \\ w_3^T \end{bmatrix} [v_1 \;\; v_2 \;\; v_3] = I \qquad (1)$$

(see (3.245))

is fulfilled. If we let

$$w_1^T = [p \;\; p \;\; p] \quad , \quad w_2^T = [0 \;\; 2q \;\; q] \quad \text{and} \quad w_3^T = [0 \;\; r \;\; r]$$

we can find from 1 that

$$p = 1 \quad , q = 1 \quad , r = -1$$

Problem 3.13

f)

$$M_C = [B \;\; AB \;\; A^2B] = \begin{bmatrix} 1 & 0 & -1 & 1 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 1 & 0 & 3 & 0 & 7 \end{bmatrix}$$

$$\Rightarrow rank(M_C) = 3 \Rightarrow \text{The system is controllable}$$

$$M_0 = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 2 \\ 1 & -3 & -3 \\ 0 & -4 & -6 \end{bmatrix} \quad \text{rank}(M_0) = 3$$

$$\Rightarrow \text{The system is observable}$$

g) PHS-test:

The system is controllable if $w_i^T B \neq 0$ for all i

$$w_1^T B = [1 \;\; 1 \;\; 1]\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = [1 \;\; 1] \neq 0$$

$$w_2^T B = [0 \;\; 1] \neq 0, \qquad w_3^T B = [0 \;\; -1] \neq 0 =$$

$$\Rightarrow \text{The system is controllable}$$

Similarly, the system is observable if $w_i^T B \neq 0$ for all i

$$Cv_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad Cv_2 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \qquad Cv_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\Rightarrow \text{The system is observable}$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 3.14

a. The characteristic polynomial for the control object can be found as:

$$P_{ch,\mathbf{A}} = \det(\lambda\mathbf{I} - \mathbf{A}) = \det\begin{bmatrix} \lambda + 2 & 3 & -5 \\ -4 & \lambda - 5 & 5 \\ -3 & -4 & \lambda + 3 \end{bmatrix} = \lambda^3 - 2\lambda - 4$$

$$\Rightarrow a_0 = -4; \quad a_1 = -2; \quad a_2 = 0$$

b. The system is controllable if $\det(\mathrm{M}_c) \neq 0$.

$$\mathbf{p}_1 = \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{p}_2 = \mathbf{A}\mathbf{p}_1 + 0 \cdot \mathbf{p}_1 = \mathbf{AB} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{p}_3 = \mathbf{A}\mathbf{p}_2 - 2\mathbf{p}_1 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{P} = [\mathbf{p}_3 \quad \mathbf{p}_2 \quad \mathbf{p}_1] = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow \mathbf{P}^{-1} = \begin{bmatrix} 1 & 2 & -2 \\ 0 & -1 & 1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\mathbf{A}_t = \mathbf{A}_{cc} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 2 & 0 \end{bmatrix}, \qquad \mathbf{B}_t = \mathbf{B}_{cc} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{C}_t = \mathbf{C}_{cc} = [-2 \quad 1 \quad 0]$$

The system is not observable as $(\det(\mathrm{M}_o) = 0)$.

If one attempts to set up the transformation matrix Q , one will has:

Problem 3.14 (continued)

$$\mathbf{q}_1^T = \mathbf{C} = [-2 \quad -5 \quad 5]$$
$$\mathbf{q}_2^T = \mathbf{CA} + 0 \cdot \mathbf{C} = [-1 \quad -1 \quad 0]$$
$$\mathbf{q}_3^T = \mathbf{CA}^2 + 0 \cdot \mathbf{CA} - 2 \cdot \mathbf{C} = [10 \quad 18 \quad 20]$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{q}_3^T \\ \mathbf{q}_2^T \\ \mathbf{q}_1^T \end{bmatrix} = \begin{bmatrix} 10 & 18 & -20 \\ -1 & 1 & 0 \\ -2 & -5 & 5 \end{bmatrix}$$

Q is singular and therefore the observer canonical form cannot be found by a similarity tranformation. However, the form can be found by duality as mentioned in section 3.9.3.

One finds that:

$$\mathbf{A}_{oc} = \mathbf{A}_{cc}^T = \begin{bmatrix} 0 & 0 & 4 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{B}_{oc} = \mathbf{C}_{cc}^T = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$
$$\mathbf{C}_{oc} = \mathbf{B}_{cc}^T = [0 \quad 0 \quad 1]$$

Note that system $S(\mathbf{A}_{oc}, \ \mathbf{B}_{oc}, \ \mathbf{C}_{oc})$ is observable but not controllable.

The system $S(\mathbf{A}_{cc}, \ \mathbf{B}_{cc}, \ \mathbf{C}_{occ})$ is controllable but not observable.

c. The transfer function can be determined directly from the controller canonical form:

$$G(s) = \frac{s-2}{s^3-2s-4} = \frac{s-2}{(s-2)(s+1+j)(s+1-j)}$$
$$= \frac{1}{s^2+2s+2}$$

d. In systems which are not controllable as well as observable it is always possible to cancel zeros/poles in the transfer function.

Note that this is only true for SISO-systems, see for exanple section 3.9.4. ❐

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 3.15

a. The differential equation in the problem text can be written:

$$\dddot{y} + \ddot{y} - 4\dot{y} - 4y = \dot{u} - 2u$$

After Laplace tranformation this equation is :

$$s^3 y + s^2 y - 4sy - 4y = su - 2u$$
$$\Rightarrow G(s) = \frac{y(s)}{u(s)} = \frac{s-2}{s^3 + s^2 - 4s - 4}$$

The controller canonical form is:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 4 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \qquad (1)$$
$$\mathbf{y} = [-2 \quad 1 \quad 0]\mathbf{x}$$

b. The eigenvalues may be found from: $\det(\lambda\mathbf{I} - \mathbf{A}) = 0 \Rightarrow \lambda = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$

One eigenvalue in the right half plane: the system is unstable.

$\mathbf{M}_s = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}]$, $\quad det(\mathbf{M}_s) \neq 0$ because the controller canonical form is always controllable.

$$\mathbf{M}_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 4 & 4 & -3 \end{bmatrix}, \quad det(\mathbf{M}_o) = 0$$

$\Rightarrow$ The system is not observable and therefore it is not minimal either.

Problem 3.15 (continued)

c. The transfer function is:

$$G(s) = \frac{s-2}{(s-2)(s+1)(s+2)} = \frac{1}{s^2+3s+2}$$

and the the controller canonical form is:

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u, \qquad \mathbf{y} = [1 \quad 0]\mathbf{z} \tag{2}$$

There is no cancellation of poles/zeros in the second order transfer function, so one can be sure that the second order state space model is controllable and observable and therefore also minimal.

d. (1) is controllable and therefore also stabilizable. (2) has two stable eigenvalues

$$\lambda = \begin{bmatrix} -1 \\ -2 \end{bmatrix}.$$

The unstable eigenvalue $\lambda = 2$ in (1) must therefore belong to the non-observable subspace.

$\Rightarrow$ (1) is not detectable.

❐

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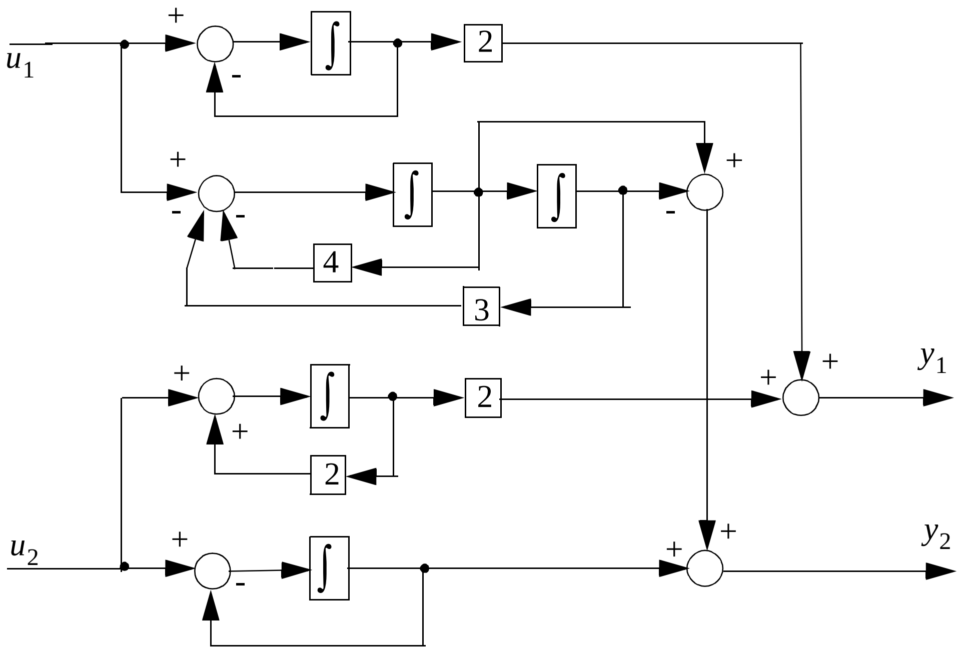
Linear Systems Control

Solutions to problems

Problem 3.16

a)



From this block diagram we see that:

$$\frac{q_1}{u_1} = \frac{2}{s+1} \quad , \quad \frac{q_2}{u_1} = \frac{s-1}{s^2+4s+3}$$

$$\frac{q_3}{u_2} = \frac{2}{s-2} \quad , \quad \frac{q_4}{u_2} = \frac{1}{s+1}$$

$$y_1 = q_1 + q_3 = \frac{2}{s+1}u_1 + \frac{2}{s-2}u_2$$

$$y_2 = q_2 + q_4 = \frac{s-1}{(s+1)(s+3)}u_1 + \frac{1}{s+1}u_2$$

$$\Rightarrow \begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{2}{s+1} & \frac{2}{s-2} \\ \frac{s-1}{(s+1)(s+3)} & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}$$

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Linear Systems Control

Solutions to problems

Problem 3.16

b)

$$
\begin{aligned}
\dot{x}_1 &= -x_1 + u_1 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= 3x_2 - 4x_3 + u_1 \\
\dot{x}_4 &= 2x_4 + u_2 \\
\dot{x}_5 &= -x_5 + u_2
\end{aligned}
\qquad
\begin{aligned}
y_1 &= 2x_1 + 2x_4 \\
y_2 &= -x_2 + x_3 + x_5
\end{aligned}
$$

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 0 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 & 1 \end{bmatrix} x$$

or:
$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
b &= Cx
\end{aligned}
$$

The state space model above is minimal if it is controllable as well as observable.

$$M_c = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -4 & 0 & 13 & 0 & -40 & 0 \\ 1 & 0 & -4 & 0 & 13 & 0 & -40 & 0 & 121 & 0 \\ 0 & 1 & 0 & 2 & 0 & 4 & 0 & 8 & 0 & 16 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Problem 3.16

$$M_o = \begin{bmatrix} 2 & 0 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ -2 & 0 & 0 & 4 & 0 \\ 0 & -3 & -5 & 0 & -1 \\ 2 & 0 & 0 & 8 & 6 \\ 0 & 15 & 17 & 0 & 1 \\ -2 & 0 & 0 & 16 & 0 \\ 0 & -51 & -53 & 0 & -1 \\ 2 & 0 & 0 & 32 & 0 \\ 0 & 159 & 161 & 0 & 1 \end{bmatrix}$$

Using Matlab it is easy to see that both matrices have rank 4. So the system is neither controllable nor observable. Therefore it is not minimal either.

d) Denominator polynomium containing all poles of the transfer function matrix:

$$d(s) = (s+1)(s+3)(s-2) = s^3 + 2s^2 - 5s - 6$$

Then the system matrix (3.384) can be found directly

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 6 & 0 & 5 & 0 & -2 & 0 \\ 0 & 6 & 0 & 5 & 0 & -2 \end{bmatrix} \qquad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Problem 3.16

$$d(s)G(s) = \begin{bmatrix} 2(s+3)(s-2) & 2(s+1)(s+3) \\ (s-1)(s-2) & (s+3)(s-2) \end{bmatrix}$$

$$= \begin{bmatrix} 2s^2+2s-12 & 2s^2+8s+6 \\ s^2-3s+2 & s^2+s-6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} s^2 + \begin{bmatrix} 2 & 8 \\ -3 & 1 \end{bmatrix} s + \begin{bmatrix} -12 & 6 \\ 2 & -6 \end{bmatrix}$$

$$\Rightarrow C_1 = \begin{bmatrix} -12 & 6 & 2 & 8 & 2 & 2 \\ 2 & -6 & -3 & 1 & 1 & 1 \end{bmatrix}$$

e) The fifth order system $[A, \quad B, \quad C]$ is not minimal so the sixth order system $[A_1, \quad B_1, \quad C_1]$ can obviously not be minimal either.

f) First we must find 4 linearly independent columns in M_c. We choose the first 4 columns. The transformation matrix Q is composed of these 4 columns and a fifth one chosen such that Q becomes regular. We can choose for instance:

$$Q = \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -4 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right] \qquad \det(Q) = -3$$

and

$$Q^{-1} = \begin{bmatrix} 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 1 & -3 & -1 & 0 & 0 \end{bmatrix}$$

Now Matlab provide us with the result:

Problem 3.16

$$A_t = Q^{-1}AQ = \left[\begin{array}{cccc|c} 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & -4 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 \end{array}\right] \quad ; \quad B_t = Q^{-1}B = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ \hline 0 & 0 \end{array}\right]$$

$$C_t = CQ = \left[\begin{array}{cccc|c} 2 & 2 & -2 & 4 & 2 \\ 1 & 1 & -5 & -1 & 0 \end{array}\right]$$

Matrix partition:

$$A_t = \left[\begin{array}{c|c} A_c & A_{12} \\ \hline 0 & A_{nc} \end{array}\right] \quad , \quad B_t = \left[\begin{array}{c} B_c \\ \hline 0 \end{array}\right] \quad , \quad C_t[C_c \mid C_{nc}]$$

The controllable subsystem becomes

$$A_c = \begin{bmatrix} 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 0 & -4 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad , \quad B_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C_c = \begin{bmatrix} 2 & 2 & -2 & 4 \\ 1 & 1 & -5 & -1 \end{bmatrix}$$

This system is of course of fourth order because the original system [A, B, C] has a controllability matrix of rank 4. The system $[A_c, B_c, C_c]$ is not only controllable but also observable, and consequently it is minimal.

A has the eigenvalues:

$$\lambda_A = \begin{cases} -3 \\ -1 \\ -1 \\ -1 \\ 2 \end{cases}$$

$$A_c \text{ has } \lambda_{A_c} = \begin{cases} -3 \\ -1 \\ -1 \\ 2 \end{cases}$$

Problem 3.16

The system natrix A_{nc} of the non-controllable part of the system has the eigenvalue $\lambda = -1$. Since this eigenvalue is asymptotically stable, the system [A, B, C] is stabilizable.

The system $[A_c, B_c, C_c]$ is observable, so the fifth (stable) eigenvalue must belong to the non-observable subspace, and the system [A, B, C] is detectable.

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 3.2

a. Given is the continuous system below:

$$\dot{x} = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ b \end{bmatrix} u, \qquad y = [1 \;\; 0]x$$

with the stable eigenvalues:

$$\begin{vmatrix} \lambda+3 & -2 \\ -1 & \lambda+2 \end{vmatrix} = (\lambda+3)(\lambda+2) - 2 = 0$$

$$\Rightarrow \lambda^2 + 5\lambda + 4 = 0 \quad \Rightarrow \quad \lambda = \begin{cases} -1 \\ -4 \end{cases}$$

The resolvent matrix is:

$$(s\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} 3+s & -2 \\ -1 & s+2 \end{bmatrix}^{-1} = \frac{1}{s^2+5s+4}\begin{bmatrix} s+2 & 2 \\ 1 & s+3 \end{bmatrix}$$

$$\Phi(s) = (s\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} \frac{s+2}{(s+1)(s+4)} & \frac{2}{(s+1)(s+4)} \\ \frac{1}{(s+1)(s+4)} & \frac{s+3}{(s+1)(s+4)} \end{bmatrix}$$

The transfer function is then given by:

$$G(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} = [1 \;\; 0](s\mathbf{I}-\mathbf{A})^{-1}\begin{bmatrix} 1 \\ b \end{bmatrix} = \frac{s+2+2b}{(s+1)(s+4)}$$

The state transition matrix can be calculated via the Laplace transform:

$$\Phi(t) = L^{-1}\{\Phi(s)\} = L^{-1}\begin{bmatrix} \frac{\frac{1}{3}}{s+1} + \frac{\frac{2}{3}}{s+4} & \frac{\frac{2}{3}}{s+1} - \frac{\frac{2}{3}}{s+4} \\ \frac{\frac{1}{3}}{s+1} - \frac{\frac{1}{3}}{s+4} & \frac{\frac{2}{3}}{s+1} + \frac{\frac{1}{3}}{s+4} \end{bmatrix}$$

Problem 3.2 (continued)

Inverse Laplace transforming the expression above gives the transition matrix:

$$\Phi(t) = e^{\mathbf{A}t} = \begin{bmatrix} \frac{1}{3}e^{-t} + \frac{2}{3}e^{-4t} & \frac{2}{3}e^{-t} - \frac{2}{3}e^{-4t} \\ \frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t} & \frac{2}{3}e^{-t} + \frac{1}{3}e^{-4t} \end{bmatrix}$$

The impulse response can be found as:

First method:

$$y(t) = L^{-1}\{G(s)\} = L^{-1}\left\{\frac{2b+1}{3}\ \frac{1}{s+1} + \frac{2b-2}{-3}\ \frac{1}{s+4}\right\}$$

$$y(t) = \frac{2b+1}{3}e^{-t} + \frac{2-2b}{3}e^{-4t}$$

Second method:

$$y(t) = \mathbf{C}\Phi(t)\mathbf{B} = [1\ \ 0]\Phi(t)\begin{bmatrix} 1 \\ b \end{bmatrix}$$

$$= \frac{1+2b}{3}e^{-t} + \frac{2-2b}{3}e^{-4t}$$

b. The eigenvalue farthest away from the origin is: $\lambda = -4$. For $b = -\frac{1}{2}$ the response will only contain the corresponding natural mode, e^{-4t}:

$$y(t) = e^{-4t}$$

c. Note that the system is asymptotically internally stable because both eigenvalues are in the open left half plane.

$$For \quad u(t) = \begin{cases} 0 & \text{for} \quad t < 0) \\ 2e^{t} & \text{for} \quad t \geq 0) \end{cases}$$ the response is:

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 3.2 (continued)

$$y(t) = \mathbf{C}\, e^{\mathbf{A}t}\mathbf{x}_0 + \mathbf{C}\int_0^t e^{A(t-\tau)}\mathbf{B}\,\mathbf{u}(\tau)d\tau = y_1 + y_2$$

x_0 does not influence $\lim_{t\to\infty} y(t)$:

$$\mathbf{C}\, e^{\mathbf{A}(t-\tau)}\mathbf{B}\,\mathbf{u}(\tau) = [1\ \ 0]e^{A(t-\tau)}\begin{bmatrix} 2e^{\tau} \\ 2be^{\tau} \end{bmatrix}$$

$$= \left[\frac{1}{3}e^{-(t-\tau)} + \frac{2}{3}e^{-4(t-\tau)} \quad \frac{2}{3}e^{-(t-\tau)} - \frac{2}{3}e^{-4(t-\tau)}\right]\begin{bmatrix} 2e^{\tau} \\ 2be^{t} \end{bmatrix}$$

$$= \frac{2+4b}{3}e^{-t+2\tau} + \frac{4-4b}{3}e^{-4t+5\tau}$$

$$\Rightarrow y_2 = \int_0^t(\ldots\ldots)d\tau = -\frac{2+4b}{6}e^{-t} - \frac{4-4b}{15}e^{-4t} + \frac{2+4b}{6}e^{t} + \frac{4-4b}{15}e^{t}$$

$$\lim_{t\to\infty}(y(t)) = 0 \quad for$$

$$\frac{2+4b}{6} + \frac{4-4b}{15} = 0 \Rightarrow b = -\frac{3}{2}$$

For $b = -\frac{3}{2}$ one obtains:

$$y_2 = \frac{2}{3}e^{-t} - \frac{2}{3}e^{-4t}$$

$$y_1 = \mathbf{C}\, e^{\mathbf{A}t}x_0 = \left[\ \frac{1}{3}e^{-t} + \frac{2}{3}e^{-4t} \quad \frac{2}{3}e^{-t} - \frac{2}{3}e^{-4}\ \right]\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= -\frac{1}{3}e^{-t} + \frac{4}{3}e^{-4t}$$

$$y = y_1 + y_2 = \frac{1}{3}e^{-t} + \frac{2}{3}e^{-4t}$$

❐

Problem 3.3

Given the system dynamical matrix

$$\mathbf{A} = \begin{bmatrix} -4 & \frac{1}{2} & 0 \\ 0 & -1 & 8 \\ 0 & 0 & -3 \end{bmatrix}$$

the resolvent matrix $\Phi(s)$ is given by

$$\begin{aligned} \Phi(s) &= (s\mathbf{I} - \mathbf{A})^{-1} \\ &= \left(s \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & \frac{1}{2} & 0 \\ 0 & -1 & 8 \\ 0 & 0 & -3 \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} \frac{1}{s+4} & \frac{1}{2(s+4)(s+1)} & \frac{4}{(s+4)(s+3)(s+1)} \\ 0 & \frac{1}{s+1} & \frac{8}{(s+3)(s+1)} \\ 0 & 0 & \frac{1}{s+3} \end{bmatrix} \end{aligned}$$

The state transition matrix $\phi(t)$ can be computed as the inverse Laplace transform of the resolvent matrix $\Phi(s)$ or directly as the exponential matrix of the system dynamical matrix $\mathbf{A}$. Applying the inverse Laplace transform we obtain

$$\begin{aligned} \phi(t) &= \mathcal{L}^{-1}\{\Phi(s)\} \\ &= \begin{bmatrix} e^{-4t} & \frac{1}{6}e^{-t} - \frac{1}{6}e^{-4t} & \frac{2}{3}e^{-t} - 2e^{-3t} + \frac{4}{3}e^{-4t} \\ 0 & e^{-t} & 4e^{-t} - 4e^{-3t} \\ 0 & 0 & e^{-3t} \end{bmatrix} \end{aligned}$$

Calculating the exponential matrix we obtain

$$\begin{aligned} \phi(t) &= e^{\mathbf{A}t} \\ &= \begin{bmatrix} e^{-4t} & \frac{1}{6}e^{-t} - \frac{1}{6}e^{-4t} & \frac{2}{3}e^{-t} - 2e^{-3t} + \frac{4}{3}e^{-4t} \\ 0 & e^{-t} & 4e^{-t} - 4e^{-3t} \\ 0 & 0 & e^{-3t} \end{bmatrix} \end{aligned}$$

As expected the two approaches produce the same result.

The zero input solution of the associated 3^{rd} order LTI system with initial condition $\mathbf{x}_0 = [0, 1, 0]^{\mathrm{T}}$ is given by

$$\begin{aligned} \mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}_0 \\ &= \begin{bmatrix} e^{-4t} & \frac{1}{6}e^{-t} - \frac{1}{6}e^{-4t} & \frac{2}{3}e^{-t} - 2e^{-3t} + \frac{4}{3}e^{-4t} \\ 0 & e^{-t} & 4e^{-t} - 4e^{-3t} \\ 0 & 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{6}e^{-t} - \frac{1}{6}e^{-4t} \\ e^{-t} \\ 0 \end{bmatrix} \end{aligned}$$

The solution shows that the effect of the non-zero initial condition x_{20} naturally produces a dynamic response in the state variable $x_2(t)$ but also in the state variable $x_1(t)$. However the state variable $x_3(t)$ is not affected at all. This can be explained by looking at the structure of the system dynamical matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} -4 & \frac{1}{2} & 0 \\ 0 & -1 & 8 \\ 0 & 0 & -3 \end{bmatrix}$$

which shows that the dynamics of $x_1(t)$ is coupled with the dynamics of $x_2(t)$ through the coefficient $a_{12} = 1/2$, but the dynamics of $x_3(t)$ is totally uncoupled from the dynamics of the other two state variables since $a_{31} = a_{32} = 0$.

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 3.4

a. The resolvent of the discrete time system is:

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -\frac{1}{8} & \frac{3}{4} \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(k) = \mathbf{F}\ \mathbf{x}(k) + \mathbf{G}\ u(k)$$

$$\mathbf{y}(k) = \begin{bmatrix} -\frac{1}{8} & -\frac{1}{4} \end{bmatrix}u(k), \qquad x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad u(k) = 1$$

$$x_1(k+1) = x_2(k)$$

$$x_2(k+1) = -\frac{1}{8}x_1(k) + \frac{3}{4}x_2(k) + u(k)$$

$$y(k) = -\frac{1}{8}x_1(k) - \frac{1}{4}x_2(k)$$

$$\underline{y(0) = 0}$$

$$x_1(1) = 0, \qquad x_2(1) = 1, \qquad \underline{y(1) = -\frac{1}{4}}$$

$$x_1(2) = 1, \qquad x_2(2) = 0 + \frac{3}{4} + 1 = \frac{7}{4},$$

$$y(2) = -\frac{1}{8}\cdot 1 - \frac{1}{4}\cdot\frac{7}{4} = \underline{-\frac{9}{16}}$$

b. The eigenvalues of the system can be computed as:

$$\begin{vmatrix} \lambda & -1 \\ \frac{1}{8} & \lambda - \frac{3}{4} \end{vmatrix} = \lambda^2 - \frac{3}{4}\lambda + \frac{1}{8} = 0 \Rightarrow \lambda = \begin{cases} \frac{1}{2} \\ \frac{1}{4} \end{cases}$$

The natural modes are then:
$$m_i = \begin{cases} \left(\frac{1}{2}\right)^k \\ \left(\frac{1}{4}\right)^k \end{cases}$$

LINEAR SYSTEM CONTROL

Solutions to Problems

Problem 3.4

The discrete resolvent matrix is:

$$\Psi(z) = (zI - F)^{-1} z = \begin{bmatrix} z & -1 \\ \frac{1}{8} & z - \frac{3}{4} \end{bmatrix}^{-1} z$$

$$\Rightarrow \Psi(z) = \begin{bmatrix} z\dfrac{z - \frac{3}{4}}{z^2 - \frac{3}{4}z + \frac{1}{8}} & z\dfrac{1}{z^2 - \frac{3}{4}z + \frac{1}{8}} \\ z\dfrac{-\frac{1}{8}}{z^2 - \frac{3}{4}z + \frac{1}{8}} & z\dfrac{z}{z^2 - \frac{3}{4}z + \frac{1}{8}} \end{bmatrix}$$

c.

$$F^k = Z^{-1}\{z(zI - F)^{-1}\} = Z^{-1}\begin{bmatrix} z\left(-\dfrac{1}{z - \frac{1}{2}} + \dfrac{2}{z - \frac{1}{4}}\right) & z\left(\dfrac{4}{z - \frac{1}{2}} - \dfrac{4}{z - \frac{1}{4}}\right) \\ z\left(-\dfrac{\frac{1}{2}}{z - \frac{1}{2}} + \dfrac{\frac{1}{2}}{z - \frac{1}{4}}\right) & z\left(\dfrac{2}{z - \frac{1}{2}} - \dfrac{1}{z - \frac{1}{4}}\right) \end{bmatrix}$$

Note that: $\quad Z\{a^k\} = \dfrac{z}{z - a}$

$$\Rightarrow F^k = \begin{bmatrix} 2 \cdot \frac{1}{4}^k - \frac{1}{2}^k & 4 \cdot \frac{1}{2}^k - 4 \cdot \frac{1}{4}^k \\ \frac{1}{2} \cdot \frac{1}{4}^k - \frac{1}{2} \cdot \frac{1}{2}^k & 2 \cdot \frac{1}{2}^k - \frac{1}{4}^k \end{bmatrix}$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 3.4

d. The transfer fuction of the system can be determined from:

$$H(z) = C(zI-F)^{-1}G = \begin{bmatrix} -\frac{1}{8} & -\frac{1}{4} \end{bmatrix} \Psi(z)\frac{1}{z}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{8} & -\frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} \dfrac{1}{z^2 - \frac{3}{4}z + \frac{1}{8}} \\ \dfrac{z}{z^2 - \frac{3}{4}z + \frac{1}{8}} \end{bmatrix} = \frac{-\frac{1}{4}z - \frac{1}{8}}{z^2 - \frac{3}{4}z + \frac{1}{8}}$$

e. The impulse response is: $y(k) = h(k)$ where

$$h(k) = \begin{cases} 0 & \text{for} \quad k = 0 \\ \mathbf{C}\mathbf{F}^{k-1}\mathbf{G} & \text{for} \quad k \geq 1 \end{cases} \text{ and}$$

$$\mathbf{C}\mathbf{F}^{k-1}\mathbf{G} = \begin{bmatrix} -\frac{1}{8} & -\frac{1}{4} \end{bmatrix} \mathbf{F}^{k-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{8} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 4 \cdot \frac{1}{2}^{k-1} - 4 \cdot \frac{1}{4}^{k-1} \\ 2 \cdot \frac{1}{2}^{k-1} - \frac{1}{4}^{k-1} \end{bmatrix} = -2 \cdot \frac{1}{2}^{k} + 3 \cdot \frac{1}{4}^{k}$$

$$\Rightarrow h(k) = \begin{cases} 0 & \text{for} \quad k = 0 \\ -2\frac{1}{2}^{k} + 3\frac{1}{4}^{k} & \text{for} \quad k \geq 1 \end{cases}$$

LINEAR SYSTEM CONTROL

Solutions to Problems

Problem 3.4

f. From the given initial conditions and input the outout can be calculated as follows:

$$y(k) = \sum_{i=0}^{k} h(k-i)u(i) = \sum_{i=0}^{k-1} h(k-i)u(i) + h(k-k)u(k)$$

$$= \sum_{i=0}^{k-1} h(k-i)u(i) + 0 \quad \text{(because } h(k-k) = h(0) = 0\text{)}$$

$$\Rightarrow y(k) = \sum_{i=0}^{k-1} \left(-2 \cdot \frac{1}{2}^{k-i} + 3 \cdot \frac{1}{4}^{k-i}\right) \quad \text{for} \quad k \geq 1$$

$$y(1) = -2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{4} = -\frac{1}{4}$$

$$y(2) = -2\frac{1}{2}^2 + 3\frac{1}{4}^2 - 2 \cdot \frac{1}{2} + 3\frac{1}{4} = -\frac{9}{16}$$

This agrees with the answer in point a.

❐

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 3.5

a. The eigenfrequencies of the matrix in problem 3.3 can be found from.

$$\det(\lambda \mathbf{I} - \mathbf{A}) = 0 \Rightarrow \lambda = \begin{cases} -1 \\ -3 \\ -4 \end{cases}$$

and the eigenvectors by solving the equations: $\mathbf{A}v_i = \lambda_i v_i$.

1. $\lambda_1 = -1$:

$$\begin{bmatrix} -4 & \frac{1}{2} & 0 \\ 0 & -1 & 8 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = -1 \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix}$$

$$\begin{aligned} -4v_{11} + \frac{1}{2}v_{12} &= -v_{11} & \Rightarrow \quad 3v_{11} &= \frac{1}{2}v_{12} \\ -v_{12} + 8v_{13} &= -v_{12} & \Rightarrow \quad v_{13} &= 0 \\ -3v_{13} &= v_{13} \end{aligned}$$

One can choose for example:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$$

and similarly:

2.

$$\lambda_2 = -3 \quad \Rightarrow \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ -4 \\ 1 \end{bmatrix}$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 3.5 (continued)

3.

$$\lambda_3 = -4 \quad \Rightarrow \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

b. The modal matrix is then given by:

$$\mathbf{M} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 6 & -4 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{M}^{-1} = \frac{1}{6}\begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 6 \\ 6 & -1 & 8 \end{bmatrix}$$

The diagonal transformation is: $\mathbf{z} = \mathbf{M}^{-1}\mathbf{x}$

$$\mathbf{A}_t = \Lambda = \mathbf{M}^{-1}\mathbf{AM} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

c. The transformed state vector then becomes:

$$\mathbf{z}(t) = \Phi_z(t)\mathbf{z}_0 = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-3t} & 0 \\ 0 & 0 & e^{-4t} \end{bmatrix} z_0$$

Problem 3.5 (continued)

$$\mathbf{z}_0 = \mathbf{M}^{-1}\mathbf{x}_0 = \begin{bmatrix} \frac{1}{6} \\ 0 \\ -\frac{1}{6} \end{bmatrix}$$

$$\Rightarrow \mathbf{z}(t) = \begin{bmatrix} \frac{1}{6}e^{-t} \\ 0 \\ -\frac{1}{6}e^{-4t} \end{bmatrix}$$

$$\mathbf{x}(t) = \mathbf{M}\mathbf{z}(t) = \begin{bmatrix} \frac{1}{6}(e^{-t} - e^{-4t}) \\ e^{-t} \\ 0 \end{bmatrix}$$

See for example problem 3.3.

❐

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 3.6

a. The eigenvalues of the matrix can be found as:

$$\begin{vmatrix} \lambda+4 & 3 \\ -1 & \lambda \end{vmatrix} = \lambda(\lambda+4)+3 = 0 \quad \text{for} \quad \lambda = \begin{cases} -1 \\ -3 \end{cases}$$

with the corresponding eigenvectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

b. The diagonal transformed system can be found using the modal matrix:

$$\mathbf{M} = \begin{bmatrix} 1 & -3 \\ -1 & 1 \end{bmatrix} \Rightarrow \mathbf{M}^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\mathbf{A}_t = \mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} = \Lambda$$

$$\mathbf{B}_t = \mathbf{M}^{-1}\mathbf{B} = \begin{bmatrix} -\frac{3}{2} \\ -\frac{3}{2} \end{bmatrix}, \qquad \mathbf{C}_t = \mathbf{C}\mathbf{M} = \{-1 \quad 1\}$$

c. The transformed state transition matrix is:

$$\Phi_t(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} = e^{\Lambda t}$$

d. To come back to the state transiton matrix of the original system use the expression:

$$e^{\mathbf{A}t} = \mathbf{M}e^{\Lambda t}\mathbf{M}^{-1} \qquad \text{(equation 3.133)}$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 3.6

$$e^{\mathbf{A}t} = \begin{bmatrix} 1 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t} & -\frac{3}{2}e^{-t} + \frac{3}{2}e^{-3t} \\ \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} & \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} \end{bmatrix}$$

❐

LINEAR SYSTEMS CONTROL

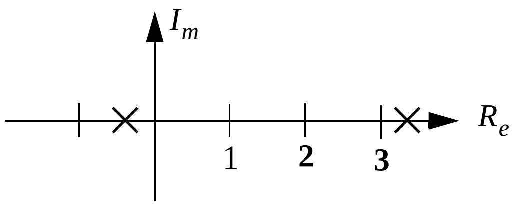
Solutions to Problems

Problem 3.8

1. The eigenvalues may be determined from:

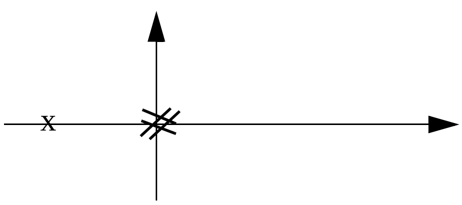
$$\det(\lambda \mathbf{I} - \mathbf{A}) = 0 \Rightarrow \begin{vmatrix} \lambda - 1 & 1 \\ 3 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 2) - 3$$

$$= \lambda^2 - 3\lambda - 1 = 0 \quad \text{for} \quad \lambda = \begin{cases} -0.303 \\ 3.30 \end{cases}$$



One eigenvalue in the right half plane and thus the system is unstable.

2. From Matlab one finds that: $\lambda = \begin{cases} 0 \\ 0 \\ -3 \end{cases}$.



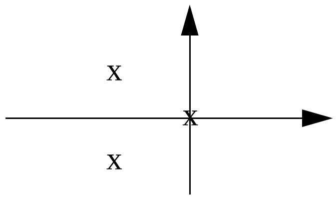
There is a double eigenvalue on the imagining axis. The system is unstable.

3. Again from Matlab one finds that $\lambda = \begin{cases} -3 \pm j \\ 0 \end{cases}$.

LINEAR SYSTEMS CONTROL

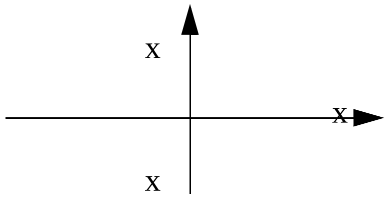
Solutions to Problems

Problem 3.8 (continued)



One eigenvalue on the imagining axis and two in the left half plane. The system is Lyapunov stable.

4. The eigenvalues in this case are: $\lambda = \begin{cases} -1 \pm j2 \\ 2 \end{cases}$.



One eigenvalue is in the right half plane and two in the left half plane. The system is unstable.

❐

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 3.9

1. The first matrix is:

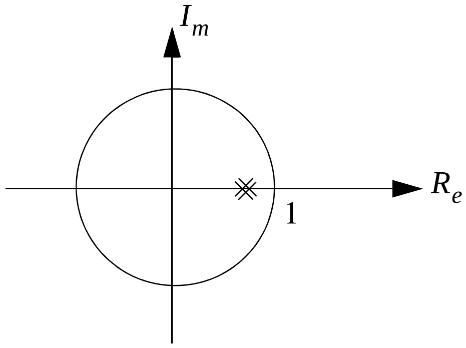
$$\mathbf{F} = \begin{bmatrix} \frac{1}{2} & \frac{1}{8} \\ -\frac{1}{2} & 1 \end{bmatrix}$$

The eigenvalues are determined from

$$\det(\lambda\mathbf{I} - \mathbf{F}) = 0$$

$$\Rightarrow \det\begin{bmatrix} \lambda - \frac{1}{2} & -\frac{1}{8} \\ \frac{1}{2} & \lambda - 1 \end{bmatrix} = \left(\lambda - \frac{1}{2}\right)(\lambda - 1) + \frac{1}{16} = \lambda^2 + \frac{3}{2}\lambda + \frac{9}{16}$$

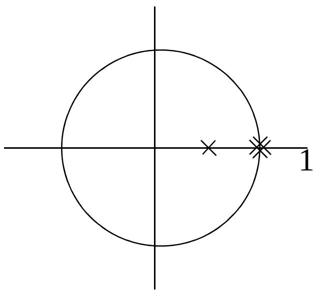
$$= 0 \quad \text{for} \quad \lambda = \begin{cases} \frac{3}{4} \\ \frac{3}{4} \end{cases}$$



Both of these poles are within the unit circle: this $\Rightarrow$ the system is symptotically stable.

2. Matlab produces the following result for the matrix:

$$\lambda = \begin{cases} 1 \\ 1 \\ 0,5 \end{cases}$$

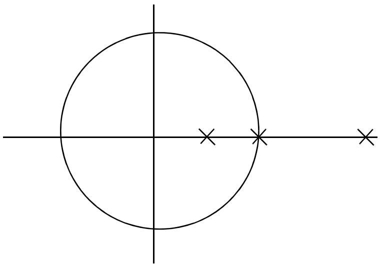


Problem 3.9 (continued)

Double eigenvalue on the unit circle: this $\Rightarrow$ the system is unstable .

3. Matlab gives the follow pole placement:

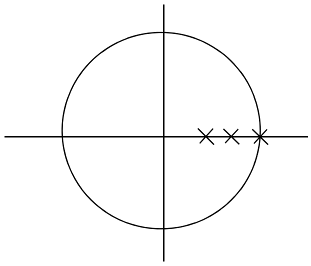
$$\lambda = \begin{cases} 0,5 \\ 1 \\ 2 \end{cases}$$



One eigenvalue outside the unit circle: this $\Rightarrow$ The system is unstable .

4. Matlab gives the follow pole placement:

$$\lambda = \begin{cases} 0,5 \\ 0,75 \\ 1 \end{cases}$$



One eigenvalue is on the unit circle and the rest within the circle:

this $\Rightarrow$ the system is Lyapunov stable .

❐

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 4.1

a. The state equations of the system are:

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & -2 \\ 0.5 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}\mathbf{u}$$

$$\mathbf{y} = [-1 \quad 1]\mathbf{x}$$

The eigenvalues are:

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \left|\begin{bmatrix} \lambda - 1 & 2 \\ -0.5 & \lambda + 1 \end{bmatrix}\right| = 0$$

$$\Rightarrow (\lambda - 1)(\lambda + 1) + 1 = 0 \Rightarrow \lambda^2 = 0 \qquad \lambda = \begin{cases} 0 \\ 0 \end{cases}$$

b. Testing the controllability of the system:

$$\mathbf{M}_c = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & -1 \end{bmatrix}, \qquad det(\mathbf{M}_c)$$

$\Rightarrow$ the system is controllable

$$\mathbf{A}_k = \mathbf{A} - \mathbf{BK} = \begin{bmatrix} 1 & -2 \\ 0.5 & -1 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - 2k_1 & -2 - 2k_2 \\ 0.5 - 2k_1 & -1 - 2k_2 \end{bmatrix}$$

$$\lambda_{Cl} = -\sqrt{2} \pm j\sqrt{2} \Rightarrow \begin{cases} \omega_n = 2 \\ \zeta = 0.707 \end{cases}$$

$$\det(\lambda\mathbf{I} - \mathbf{A}_K) = 0 \Rightarrow \det\begin{bmatrix} \lambda - 1 + 2k_1 & 2 + 2k_2 \\ -0.5 + 2k_1 & \lambda + 1 + 2k_2 \end{bmatrix} = 0$$

LINEAR SYSTEM CONTROL

Solutions to Problems

Problem 4.1

$$\Rightarrow \lambda^2 + 2(k_1 + k_2)\lambda - 2k_1 - k_2 = 0$$

$$\lambda = -(k_1 + k_2) \pm \sqrt{(k_1 + k_2)^2 + 2k_1 + k_2}$$

or:

$$\lambda_1 = -(k_1 + k_2) + \sqrt{(k_1 + k_2)^2 + 2k_1 + k_2} \qquad (1)$$

$$\lambda_2 = -(k_1 + k_2) - \sqrt{(k_1 + k_2)^2 + 2k_1 + k_2} \qquad (2)$$

(1) +(2) $\qquad \Rightarrow k_1 + k_2 = -\frac{1}{2}(\lambda_1 + \lambda_2) = \sqrt{2} \qquad (3)$

$$\Rightarrow k_1 = \sqrt{2} - k_2 \qquad (4)$$

(1)-(2) $\qquad \Rightarrow \lambda_1 - \lambda_2 = 2\sqrt{(k_1 + k_2)^2 + 2k_1 + k_2}$

Inserting (3) og (4) one has:

$$j2\sqrt{2} = 2\sqrt{2 + k_2 + 2(\sqrt{2} - k_2)}$$
$$\Rightarrow k_2 = 6.828$$

(4): $\qquad k_1 = -5.414$

c. One can use MATLAB with the commands below to make the plots below:

LINEAR SYSTEMS CONTROL

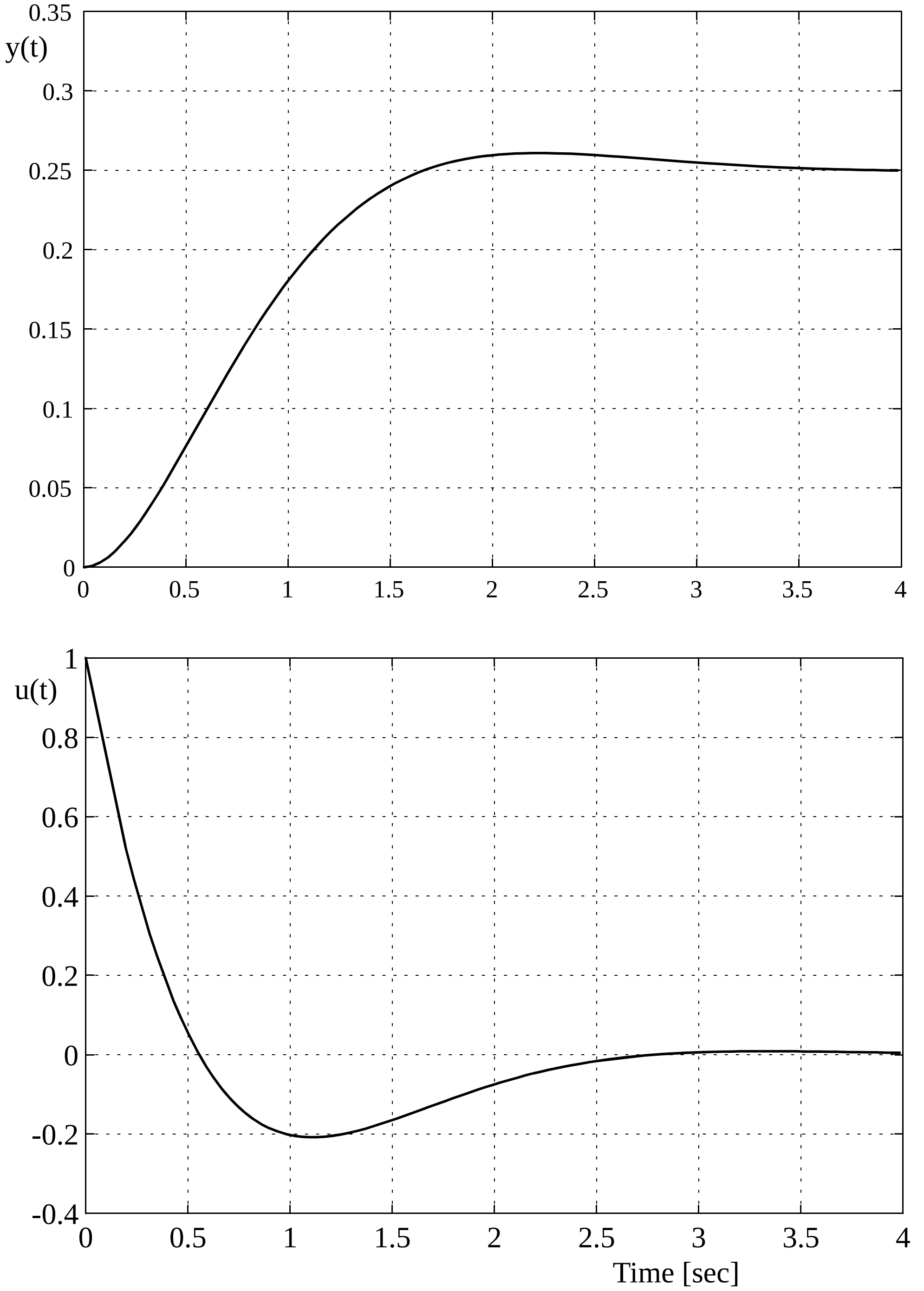
Solutions to Problems

$$\left.\begin{array}{l} \text{>>}[y, x, t] \;=\; \text{step(AK,B,C,0);} \\ \text{>>plot(t,y),grid on} \\ \text{>>}[u1, x, t] \;=\; \text{step(AK,B,-K,0);} \\ \text{>>plot(t, u1+1),grid on} \} \end{array}\right\} \quad \text{See the plots below.}$$

LINEAR SYSTEM CONTROL

Solutions to Problems

Problem 4.1



❐

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 4.10

From example 2.9 one has the state vector:

$$\mathbf{z} = \begin{bmatrix} H_1 \\ H_2 \\ T_1 \\ T_2 \end{bmatrix}$$

A transformation matrix **Q** is selected such that:

$$\mathbf{x} = \mathbf{Qz} = 0\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} H_1 \\ H_2 \\ T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} H_2 \\ T_2 \\ \hline H_1 \\ H_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \hline x_2 \end{bmatrix}$$

The matrices of the transformed system will then be:

$$\mathbf{A}_t = \mathbf{QAQ}^{-1} = \left[\begin{array}{cc|cc} -0.0667 & 0 & 0.0499 & 0 \\ 0 & -0.0335 & 0 & 0.0335 \\ \hline 0.0499 & 0 & -0.0499 & 0 \\ 0 & 0 & 0 & -0.0251 \end{array}\right]$$

$$= \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

$$\mathbf{B}_t = \mathbf{QB} = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 0.0051 & 0.0051 \\ 0.0377 & -0.0377 \end{array}\right] = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}$$

$$\mathbf{C}_t = \mathbf{CQ}^{-1} = \left[\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \end{array}\right] = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \end{bmatrix}$$

The matrix L is found by application of Matlab's place-function:

Problem 4.10

>> L = PLACE(A22′, (C1*A12)′, [–0, 4 + 0, 5J –0, 4 – 0, 5J]

and then M , N and P can be found with the statements:

```
>> M =  A22–L*C1*A12
>> N =  B2–L*C1*B1
>> P =  (A21–L*C1*A11)*inv(C1) + M*L
```

An m-file which do all the calculations is given on the next page.

The output from running this file is shown on pages 4-6.

The SIMULINK-diagram of the entire system is also shown (on page 7), and on the last page can be seen plots showing simulation results using this SIMULINK-model. Note that the estimates for the states x_3 and x_4 are initiated with “wrong” values. The estimation error vanishes within the first 12-15 sec.

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 4.10

m-file

```
% Calculates data for the SIMULINK-model opg410.mdl
%
%
a=[-.0499 .0499 0 0
.0499 -.0667 0 0
0 0 -.0251 0
0 0 .0335 -.0335]
b=[.00510 .00510
   0   0
   .0377 -.0377
   0 0]
c=[0 2 0 0
   0 0 0 .1]
q=[0 1 0 0
   0 0 0 1
   1 0 0 0
   0 0 1 0]
at=q*a*inv(q)
bt=q*b
ct=c*inv(q)
a22=at(3:4,3:4)
a12=at(1:2,3:4)
a11=at(1:2,1:2)
a21=at(3:4,1:2)
c1=ct(:,1:2)
b1=bt(1:2,:)
b2=bt(3:4,:)
eobs=[-.4+j*.5 -.4-j*.5]
l=place(a22',(c1*a12)',eobs)'
m=a22-l*c1*a12
n=b2-l*c1*b1
p=(a21-l*c1*a11)*inv(c1)+m*l
```

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 4.10

Data generated by the m-file

a =

-0.0499	0.0499	0	0
0.0499	-0.0667	0	0
0	0	-0.0251	0
0	0	0.0335	-0.0335

b =

0.0051	0.0051
0	0
0.0377	-0.0377
0	0

c =

0	2.0000	0	0
0	0	0	0.1000

q =

0	1	0	0
0	0	0	1
1	0	0	0
0	0	1	0

at =

-0.0667	0	0.0499	0
0	-0.0335	0	0.0335
0.0499	0	-0.0499	0
0	0	0	-0.0251

bt =

0	0
0	0
0.0051	0.0051
0.0377	-0.0377

ct =

2.0000	0	0	0
0	0.1000	0	0

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 4.10

```
a22 =
-0.0499         0
        0   -0.0251

a12 =
0.0499         0
        0    0.0335

a11 =
-0.0667         0
        0   -0.0335

a21 =
0.0499         0
        0         0

c1 =
2.0000         0
        0    0.1000

b1 =
0     0
    0     0

b2 =
0.0051    0.0051
    0.0377   -0.0377

eobs =
-0.4000 + 0.5000i  -0.4000 - 0.5000i
place: ndigits= 15

l =
3.5080    5.0100
 -149.2537  111.9104

m =
-0.4000   -0.0168
   14.8955   -0.4000
```

Problem 4.10

n =

0.0051	0.0051
0.0377	-0.0377

p =

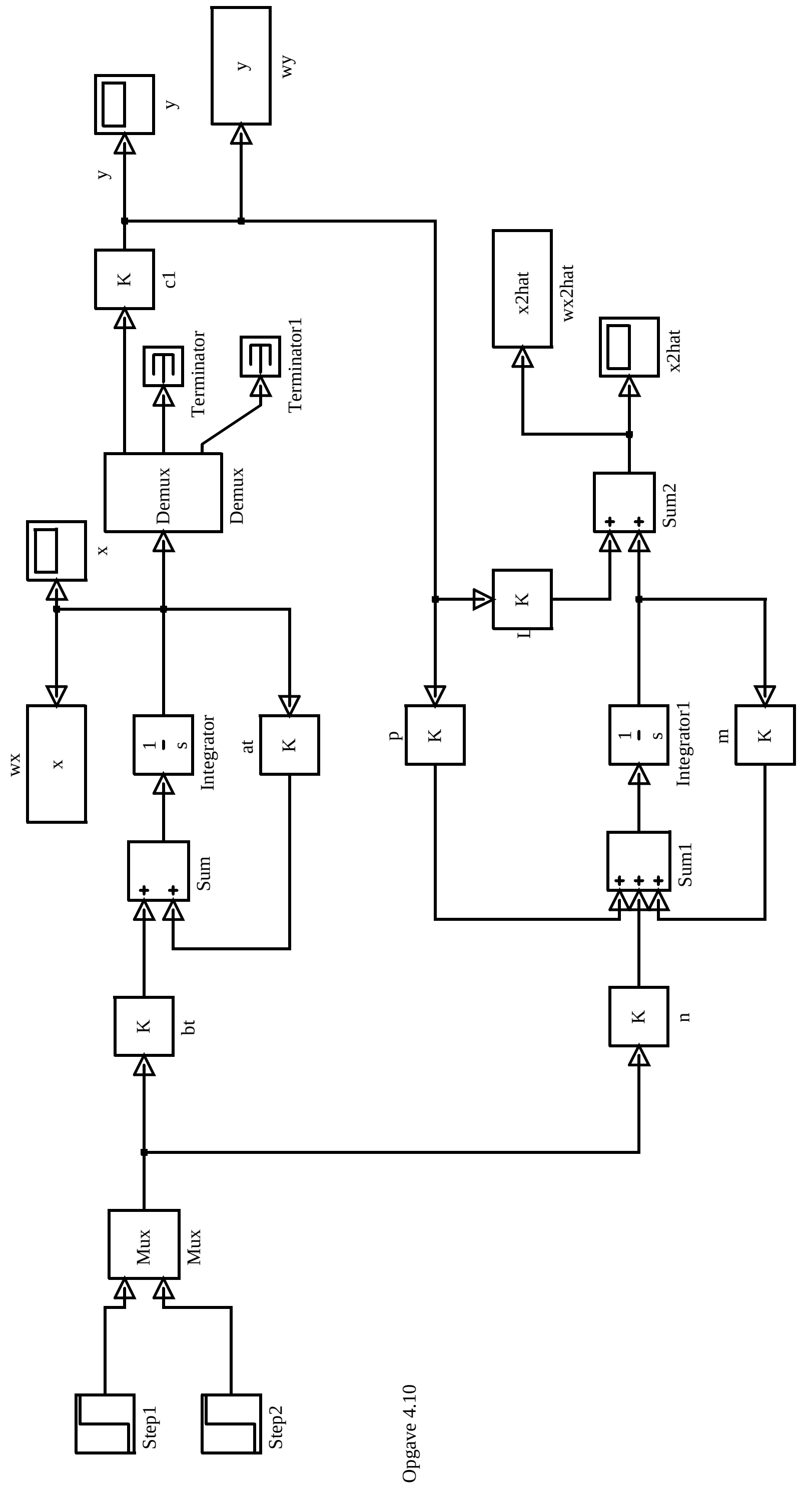
1.3607	-3.7144
102.0000	33.6117

LINEAR SYSTEMS CONTROL

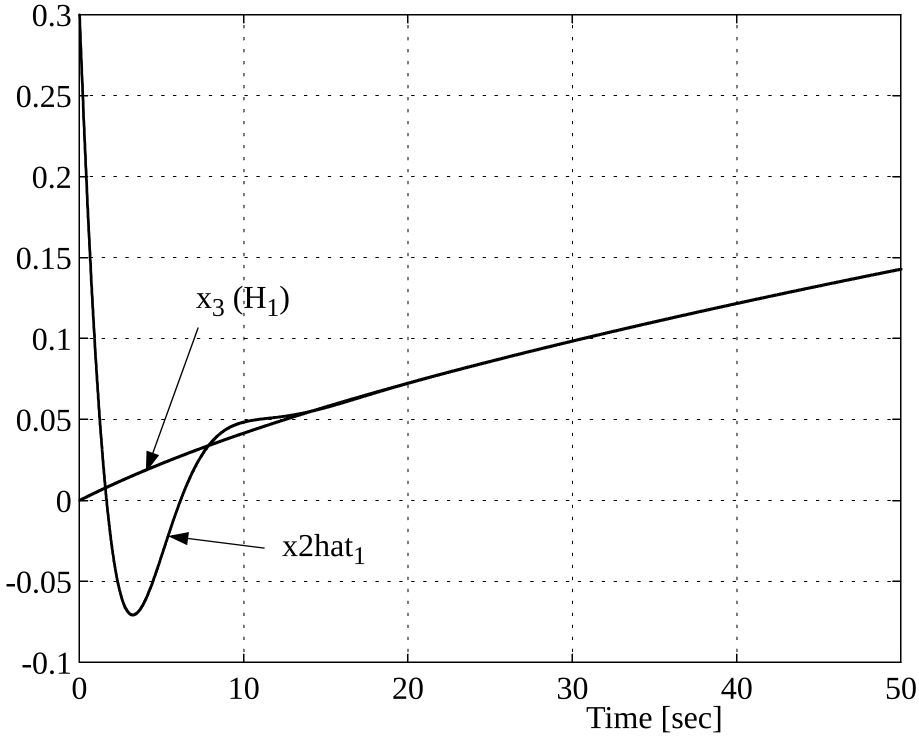
Solutions to Problems

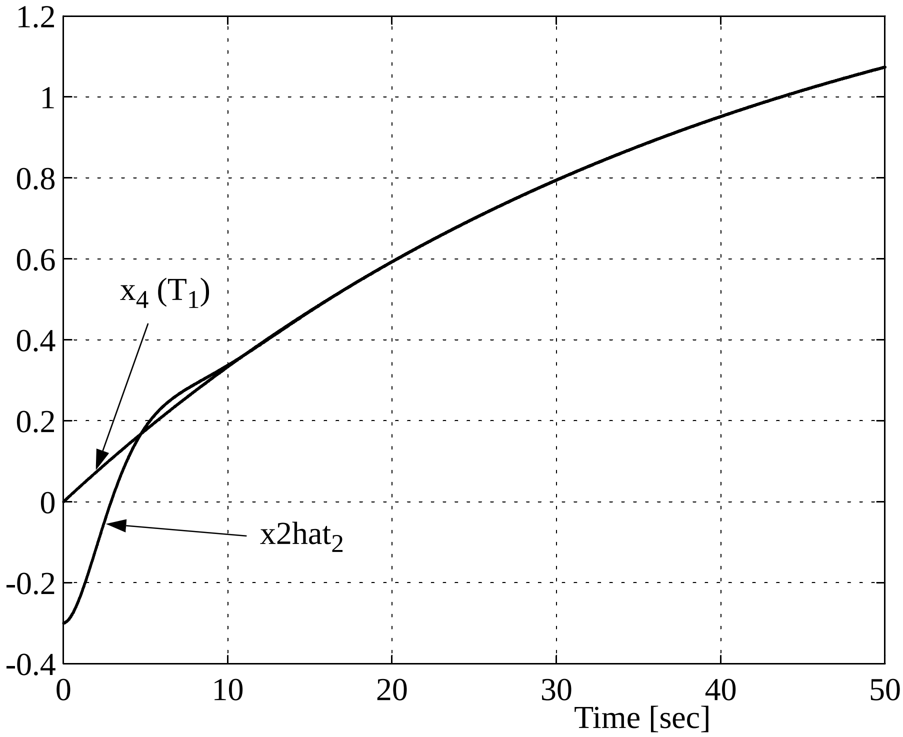
Problem 4.10

SIMULINK diagram



Problem 4.10


0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
-0.1
0
10
20
30
40
50
Time [sec]
x3 (H1)
x2hat1


1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
10
20
30
40
50
Time [sec]
x4 (T1)
x2hat2

❒

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 4.2

a. Here MATLAB can be used directly.

The controllability matrix is:

$$\mathbf{M}_c = \begin{bmatrix} 4.2 & 0.1 \\ 2 & 0.05 \end{bmatrix}, \quad det(\mathbf{M}_c) = 0.005.$$

With MATLAB's acker-function (Ackermann's formula) one can find $\mathbf{K}$:

```
>> ev = [–sqrt(2) + j* sqrt(2)    –sqrt(2)-j*sqrt(2)]
>> K = acker (A, B, ev)
```

The result is:

$$\mathbf{K} = [-1571.7 \quad 3223.4].$$

The gains are 300-500 times larger than in problem 4.1.

The plots of y(t) and u(t) are shown below.

The strange appearance of $y(t)$ is due to the fact that the system in this case has a zero in the right half plane ($s = 0.0238$) and therefore the sytem is not a minimum-phase. This cannot be changed by the controller. See the remark in the notes on page 230. $u(t)$ is identical with the control signal in problem 4.1 However, if one plots the to components which constitute $u(t)$:

$$u = -\mathbf{K}\mathbf{x} + r = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + r =$$

$$-k_1 x_1 - k_2 x_2 + r = -u_a - u_b + r$$

The result is seen on the last page.

Here it is clear what the large gains mean. Control signals with such large amplitudes may be very difficult to realize.

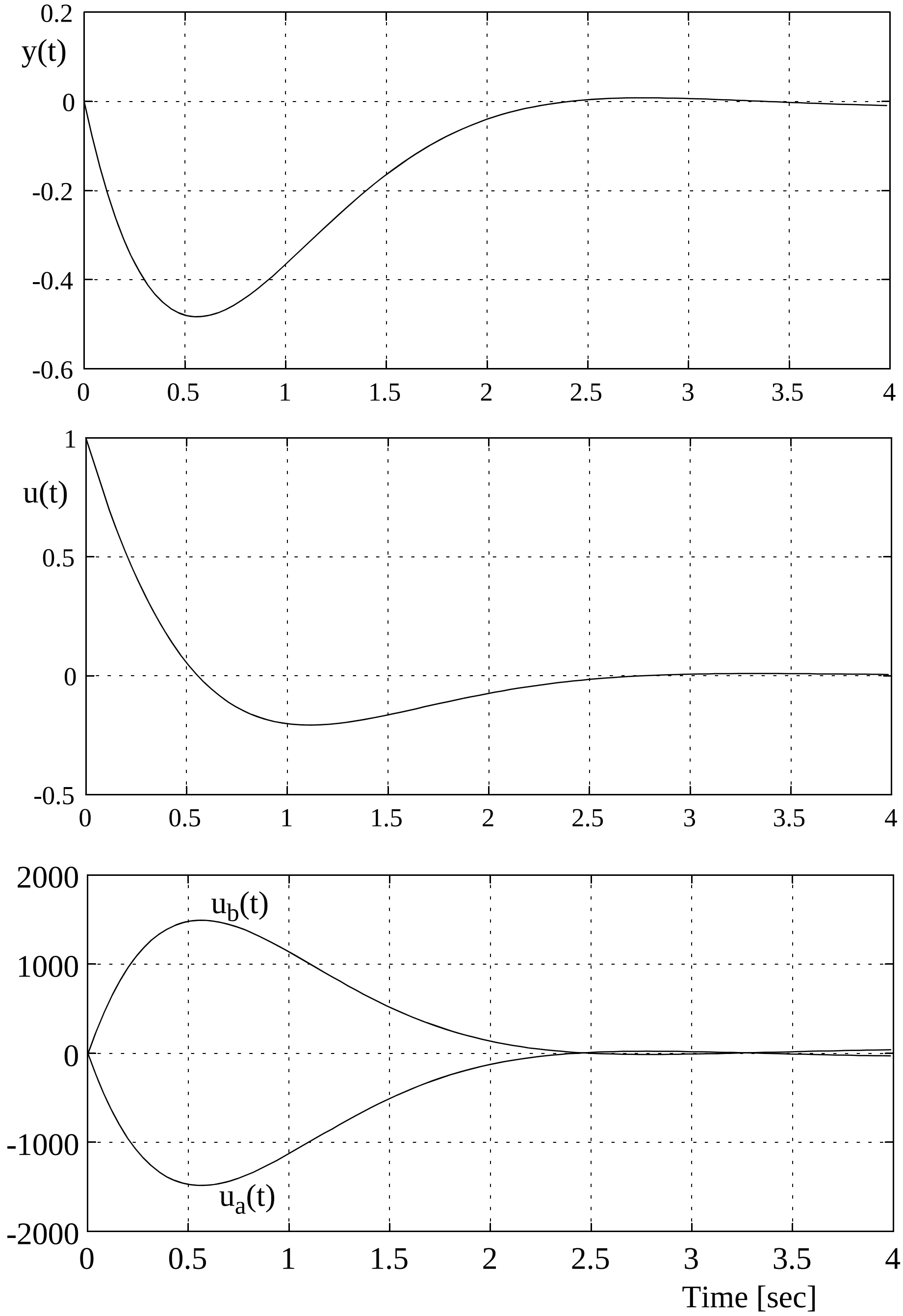
LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 4.2

b. The problem with this system is that it is close to being non-controllable. ($\mathbf{M}_c$ is almost singular).

In order to live up to the demands on the placement of the eigenvalues, the controller must generate control signals of an unrealistic size.


y(t)
0.2
0
-0.2
-0.4
-0.6
0 0.5 1 1.5 2 2.5 3 3.5 4
u(t)
1
0.5
0
-0.5
0 0.5 1 1.5 2 2.5 3 3.5 4
$u_b(t)$
$u_a(t)$
2000
1000
0
-1000
-2000
0 0.5 1 1.5 2 2.5 3 3.5 4
Time [sec]

❐

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 4.3

a. The system to be investigated is:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$

Its stability can be determined from its eigenvalues:

$$\det\begin{bmatrix} \lambda & -1 \\ 2 & \lambda - 2 \end{bmatrix} = \lambda^2 - 2\lambda + 2 = 0 \Rightarrow \lambda = 1 \pm j$$

Both eigenvalues in right half plane and this implies that the system is unstable.

The controllability matrix for the system is:

$$\mathbf{M}_c = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \text{ thus } det(\mathbf{M}_c) = -1 .$$

The controllabilitly matrix has rank 2 and hence the system is controllable. Therefore it can be stabilized using linear state feedback:

$$u = -\mathbf{K}x + r$$

b. As the system has only one output and one input, the feedback matrix for it can be calculated using Ackermann's formula. For this one needs the inverse of the controllability matrix:

$$\mathbf{M}_c^{-1} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$$

The desired closed loop eigenvalues are: $\lambda_{A_K} = -1 \pm j$

$$P_{Ch,\,\mathbf{A}_K}(\lambda) = (\lambda + 1 - j)(\lambda + 1 + j) = \lambda^2 + 2\lambda + 2$$

$$P_{Ch,\,\mathbf{A}_K}(\mathbf{A}) = \mathbf{A}^2 + 2\mathbf{A} + 2 \cdot \mathbf{I} = \begin{bmatrix} -2 & 2 \\ -4 & 2 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -8 & 8 \end{bmatrix}$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 4.3

The gain is then easily calculated as:

$$\mathbf{K} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ -8 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 4 \end{bmatrix}$$

To test this gain matrix one can calculate the closed loop eigenfrequencies:

$$\mathbf{A_K} = \mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$$

which has the closed loop eigenfrequencies:

$$det\begin{bmatrix} \lambda & -1 \\ 2 & \lambda + 2 \end{bmatrix} = \lambda^2 + 2\lambda + 2 = 0 \text{ for } \lambda = -1 \pm j$$

❐

LINEAR SYSTEMS CONTROL

Solutions to Problems

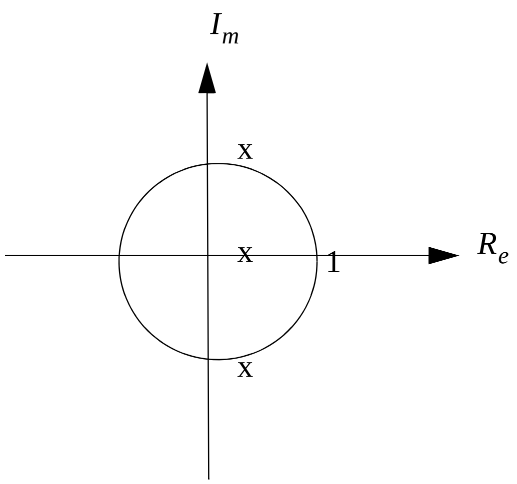
Problem 4.4

a. The characteristic equation for the system dynamic matrix is:

$$\det\begin{bmatrix} \lambda+2 & 3 & -5 \\ 0.875 & \lambda-0.5 & -1 \\ 1.875 & -0.5 & \lambda-3 \end{bmatrix} = \lambda^3 - 0.5\lambda^2 + 1.75\lambda - 0.625 = 0$$

$$\Rightarrow \lambda = \begin{cases} 0.5 \\ 0.5 \pm j \end{cases}$$

The pole placement is thus as shown below.



There are two eigenvalues outside the unit circle $\Rightarrow$ the system is unstable. The controllability matrix is:

$$\mathbf{M}_c = [\mathbf{G} \quad \mathbf{FG} \quad \mathbf{F}^2\mathbf{G}] = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 0.5 & 1.5 \\ 1 & 2.5 & 3 \end{bmatrix}$$

The determinent of the controllability matrix is:

$$det(\mathbf{M}_c) = 1 \quad \Rightarrow \quad rank(\mathbf{M}_c) = 3 \quad \text{and the system is controllable}$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 4.4

b. To convert the system into the controller canonical form first find:

$$P_{ch,\mathbf{F}} = \lambda^3 - 1.5\lambda^2 + 1.75\lambda - 0.625$$
$$\Rightarrow a_2 = -1.5, \qquad a_1 = 1.75, \qquad a_0 = -0.625$$

Using the method from sec. 3.9.1 in the book:

$$p_1 = \mathbf{G} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$p_2 = \mathbf{F}p_1 + a_{n-1}p_1 = \begin{bmatrix} 2 \\ 1.5 \\ 2.5 \end{bmatrix} + \begin{bmatrix} 0 \\ -1.5 \\ -1.5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$p_3 = \mathbf{F}p_2 + a_{n-2}p_1 = \begin{bmatrix} 1 \\ -0.75 \\ -0.75 \end{bmatrix} + \begin{bmatrix} 0 \\ 1.75 \\ 1.75 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{P} = [p_3 \ \ p_2 \ \ p_1] = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & 2 & -2 \\ 0 & -1 & 1 \\ -1 & -1 & 2 \end{bmatrix}, \qquad z = \mathbf{P}^{-1}x$$

$$\mathbf{F}_t = \mathbf{F}_{cc} = \mathbf{P}^{-1}\mathbf{F}\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.625 & -1.75 & 1.5 \end{bmatrix}$$

$$\mathbf{G}_t = \mathbf{G}_{cc} = \mathbf{P}^{-1}\mathbf{G} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 4.4

c. The desired continuous eigenvalues:

$$\lambda_{cont} = \begin{cases} -2 \\ -2 \pm j \end{cases} \qquad \lambda_F = e^{T \cdot \lambda_A} \Leftrightarrow \lambda_A = \frac{1}{T}\ln\lambda_F$$

$$\lambda_{F_K} = \begin{cases} e^{0.2(-2 \pm j)} = e^{-0.4}(\cos 0.2 \pm j\sin 0.2 = 0.657 \pm j0.133 \\ e^{0.2(-2)} = 0.67 \end{cases}$$

$$P_{ch,\mathbf{F_K}}(\lambda) = (\lambda - 0.67)(\lambda - 0.657 + j0.133)(\lambda - (0.657 + j0.133)$$
$$= \lambda^3 - 1.984\lambda^2 + 1.33\lambda - 0.301$$

The closed loop system matrix (in controller canonical form):

$$\mathbf{F}_{Kt} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.301 & -1.33 & 1.984 \end{bmatrix}$$

$$\mathbf{F}_{Kt} = \mathbf{F}_{cc} - \mathbf{G}_{cc}\mathbf{K}_t, \qquad \mathbf{K}_t = \begin{bmatrix} k_{1t} & k_{2t} & k_{3t} \end{bmatrix}$$

$$\mathbf{F}_{Kt} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.625 - k_{1t} & -1.75 - k_{2t} & 1.5 - k_{3t} \end{bmatrix}$$

$$\Rightarrow \begin{cases} 0.301 = 0.625k_{1t} \\ -1.33 = -1.75 - k_{2t} \\ 1.984 = 1.5 - k_{3t} \end{cases} \quad \Rightarrow \quad \begin{cases} k_{1t} = 0.324 \\ k_{2t} = -0.42 \\ k_{3t} = -0.484 \end{cases}$$

$$\mathbf{K} = \mathbf{K}_t\mathbf{P}^{-1} = [0.808 \quad 1.552 \quad -2.036]$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 4.4

d.

$$P_{ch,\,\mathbf{F_K}}(\mathbf{F}) = \mathbf{F}^3 - 1.984\mathbf{F}^2 + 1.33\mathbf{F} - 0.301\mathbf{I}$$

$$= \begin{bmatrix} 2.495 & 0.292 & -3.068 \\ 0.6398 & -1.036 & 0.0035 \\ 1.483 & -1.666 & -0.5125 \end{bmatrix}$$

$$\mathbf{M}_c^{-1} = \begin{bmatrix} 0.75 & 4 & -3 \\ -1.5 & -4 & 4 \\ 1 & 2 & -2 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{M}_c^{-1} P_{ch,\,\mathbf{F_K}}(\mathbf{F}) = [0.808 \quad 1.552 \quad -2.036]$$

❐

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 4.6

a. The state vector for the hydraulic servo is:

$$\mathbf{x} = \begin{bmatrix} x \\ \dot{x} \\ p_1 - p_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \\ \Delta p \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The equations (3.413) and (3.414) are:

$$q_1 = A_c\dot{x} + \frac{V}{\beta}\dot{p}_1 + C_l(p_1 - p_2)$$
$$q_2 = A_c\dot{x} + \frac{V}{\beta}\dot{p}_1 + C_l(p_1 - p_2)$$

Adding these equations and introducing the new variable: $q_1 + q_2 = 2ku$:

$$ku = A_c\dot{x} + \frac{V}{2\beta}\dot{\Delta p} + C_l\Delta p$$

Equation (3.416) becomes

$$M\ddot{x} = f + A_c\Delta p - C_f\dot{x}$$

and one has the state equations:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{M}(f + A_cx_3 - C_fx_2)$$
$$\dot{x}_3 = \frac{2\beta}{V}(ku - A_cx_2 - C_lx_3)$$

or:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{C_f}{M} & \frac{A_c}{M} \\ 0 & -\frac{2\beta A_c}{V} & -\frac{2BC_l}{V} \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ \frac{2\beta k}{V} \end{bmatrix}u + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \end{bmatrix}f$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 4.6

The output equation is: $y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x}$

Now inserting the numerical value using the units from example 3.26:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.2 & 300 \\ 0 & -700 & -4.667 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 93.33 \end{bmatrix}, \quad \mathbf{B}_v = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

The eigenvalues for **A** are:

$$\lambda_{\mathbf{A}} = \begin{cases} 0 \\ -2.434 \pm j458.3 \end{cases} \qquad \text{(see for example page 193)}$$

b. Using now Matlab and Ackermann's formula, the eigenvalues for the closed loop system are:

$$\text{>> evr} = [-20 \quad -12 \pm j*12 \quad -12 - j*12]$$

Using

$$\text{>> K} = \text{acker(A,B,evr);}$$

one finds:

$$\mathbf{K} = [0.2057 \quad -7.473 \quad 0.4193]$$

c. The closed loop system is:

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}u + \mathbf{B}_v v, \qquad y = \mathbf{C}x$$
$$u = -\mathbf{Kx} + r$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 4.6

or:

$$\begin{cases} \dot{\mathbf{x}} = (\mathbf{A}-\mathbf{BK})\mathbf{x}+\mathbf{B}r+\mathbf{B}_v v = \mathbf{A_K}x+\mathbf{B}r+\mathbf{B}_v\mathbf{V} \\ y = \mathbf{C}x \end{cases}$$

Stationary state for $r = 0$ is:

$$\dot{\mathbf{x}} = 0 \Rightarrow \mathbf{A_K}\mathbf{x}_0+\mathbf{B}_v v_0 = 0$$
$$\Rightarrow \mathbf{x}_0 = -\mathbf{A_K}^{-1}\mathbf{B}_v v_0$$

for $v_0 = 50$ (note that the force unit is 10 Newton) we find

$$x_0 = \begin{bmatrix} 0.7604 \\ 0 \\ -0.3333 \end{bmatrix}$$

and thus

$$y_0 = Cx_0 = 0.7604 \text{ cm}$$

d. The augmented state vector is: $\mathbf{x}_a = \begin{bmatrix} x \\ x_i \end{bmatrix}$

$$\mathbf{A}_1 = \begin{bmatrix} \mathbf{A} & 0 \\ -\mathbf{C} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.2 & 300 & 0 \\ 0 & -700 & -4.667 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

Problem 4.6

$$\mathbf{B}_1 = \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 93.33 \\ 0 \end{bmatrix}, \qquad \mathbf{B}_{v1} = \begin{bmatrix} \mathbf{B}_v \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} C & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

again using use MATLAB and Ackermann's formula:

```
>> A1 = [A  zeros(3,1);–C  0];
>> B1 = [B;0];   Bv1 = [Bv;0];   C1 = [C  0];
>> evri = [evr  -16];
>> K1 = acker(A1,B1,evri);
```

One obtains:

$$\mathrm{K1} = [0.6446 \quad -7.448 \quad 0.5907 \quad -3.292]$$

and therefore

$$\mathbf{K} = [0.6446 \quad -7.448 \quad 0.5907] \quad \text{and} \quad K_i = -3.292$$

e. The overall closed loop system is then:

$$\dot{\mathbf{x}}_a = \mathbf{A}_{\mathbf{K}1}\mathbf{x}_a + \mathbf{B}_r r + \mathbf{B}_{v1} v \quad \text{where} \quad \mathbf{A}_{\mathbf{K}1} = \mathbf{A}_{\mathbf{K}1} = \mathbf{A}_1 - \mathbf{B}_1\mathbf{K}_1$$

Stationary state for $r = 0$:

$$\mathbf{A}_{K1}\mathbf{x}_{a0} + \mathbf{B}_{v1}v_0 = 0 \Rightarrow \mathbf{x}_{a0} = -\mathbf{A}_{\mathbf{K}1}^{-1}\mathbf{B}_{v1}v_0$$

For $v_0 = 50$ one finds:

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 4.6

$$\mathbf{x}_{a0} = \begin{bmatrix} 0 \\ 0 \\ -0.3333 \\ -0.06489 \end{bmatrix} \quad \text{and} \quad y_0 = 0$$

f. In MATLAB one can generate a time vector:

$$t = 0{:}0.01{:}1;$$

Unit step response for the system without integration:

$$\gg \ [y, x] = \text{step(AK,B,C,0,1,t);}$$

and with integration

$$\gg \ [\text{y1, x1}] = \text{step(AK1,Br,C1,0,1,t);}$$

Note: here $\mathbf{B}_r = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$.

The responses may be plotted with the command (see plots on following page).

$$\gg \ \text{plot(t,y,t,y1), grid on}$$

The responses for a 500N force disturbance $(v_0 = 50)$, can be calculated by the commands:

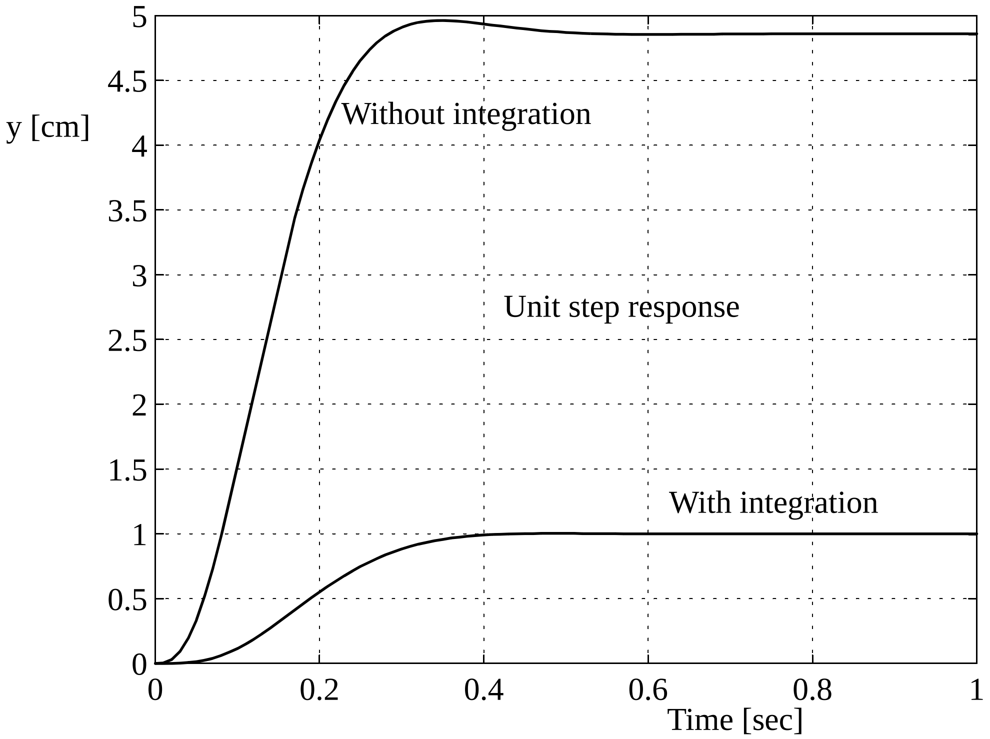
```
>> [yf,xf] = step(AK,BV,C,0,1,t);
>> [yf1, xf1] = step(AK1,BV1,C1,0,1,t);
>> plot(t,50*yf,t,50*yf1),grid on
```

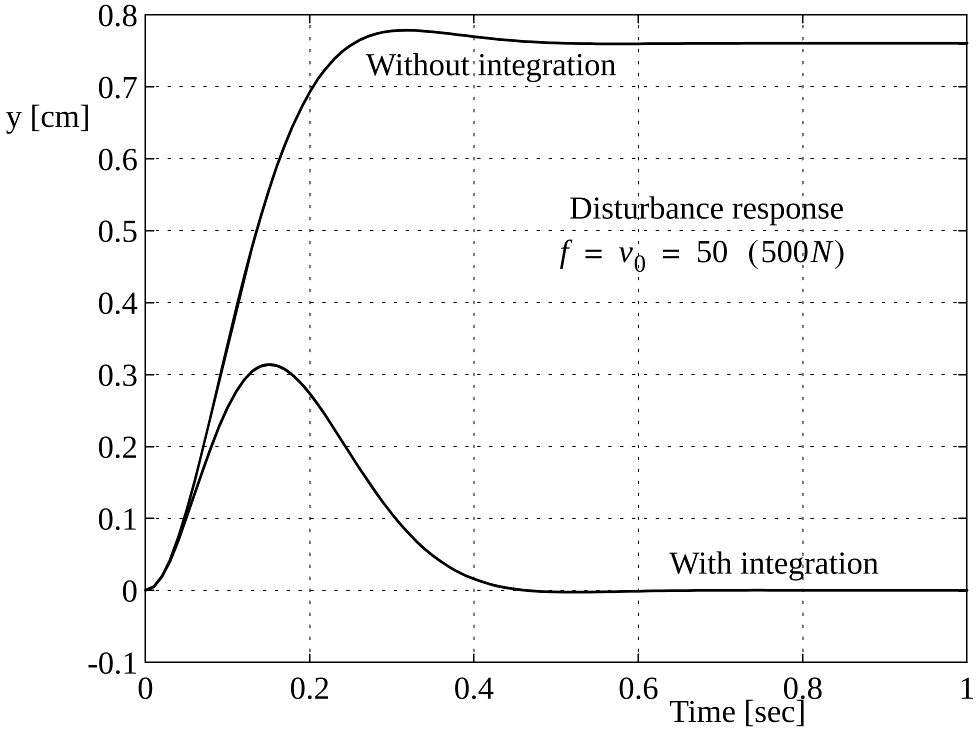
The plots of the responses are on the last page.

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 4.6





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LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 4.8

a. The question here is whether or not it is possible to make a full order observer for the system in Problem 4.3.

$$\mathbf{M}_0 = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -6 & 6 \end{bmatrix}, \ det(\mathbf{M}_0) = 18$$

$$\Rightarrow rank(\mathbf{M}_0) = 2 \Rightarrow \text{the system is observable}$$

The answer is that: yes it is possible.

b. To make and observer for the system use the method from section 3.9.2:

$$P_{Ch,\mathbf{A}}(\lambda) = \det[\lambda\mathbf{I} - \mathbf{A}] = \begin{bmatrix} \lambda & -1 \\ 2 & \lambda - 1 \end{bmatrix} = \lambda^2 - 2\lambda + 2$$

$$\Rightarrow a_1 = -2, \qquad a_0 = 2$$

$$q_1^T = \mathbf{C} = \begin{bmatrix} 0 & 3 \end{bmatrix}$$

$$q_2^T = \mathbf{CA} + q_{n-1}\mathbf{C} = [-6 \quad 6] - 2[0 \quad 3] = [-6 \quad 0]$$

$$\mathbf{Q} = \begin{bmatrix} q_2^T \\ q_1^T \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{Q}^{-1} = \begin{bmatrix} -\frac{1}{6} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

$$\mathbf{A}_t = \mathbf{A}_{0c} = \mathbf{QAQ}^{-1} = \begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{C}_t = \mathbf{C}_{0c} = \mathbf{CQ}^{-1} = [0 \quad 1] \qquad \text{since } \mathbf{L}_t = \begin{bmatrix} l_{1t} \\ l_{2t} \end{bmatrix}$$

$$\mathbf{A}_{Lt} = \mathbf{A}_{0c} - \mathbf{L}_t\mathbf{C}_{0c} = \begin{bmatrix} 0 & -2 - l_{1t} \\ 1 & 2 - l_{2t} \end{bmatrix}$$

Desired eigenvalues for observer:

$$\lambda_{\mathbf{A_L}} = -4 \pm j4$$

$$P_{Ch,\mathbf{A_L}} = (\lambda + 4 + j4)(\lambda + 4 - j4) = \lambda^2 + 8\lambda + 32$$

Problem 4.8

$$\mathbf{A}_{Lt} = \begin{bmatrix} 0 & -32 \\ 1 & -8 \end{bmatrix} \text{(observer canonical form)}$$

$$\mathbf{A}_{Lt} = \begin{bmatrix} 0 & -32 \\ 1 & -8 \end{bmatrix} = \begin{bmatrix} 0 & -2-l_{1t} \\ 0 & 2-l_{2t} \end{bmatrix}$$

$$\Rightarrow \begin{cases} l_{1t} = 30 \\ l_{2t} = 10 \end{cases}$$

For the original system one has that:

$$\mathbf{A_L} = \mathbf{A} - \mathbf{LC} = \mathbf{Q}^{-1}\mathbf{A}_{0c}\mathbf{Q} - \mathbf{LC}_{0c}\mathbf{Q} = \mathbf{Q}^{-1}\mathbf{A}_{0c}\mathbf{Q} - \mathbf{Q}^{-1}\mathbf{QLC}_{0c}\mathbf{Q}$$
$$= \mathbf{Q}^{-1}[\mathbf{A}_{0c} - \mathbf{QLC}_{0c}]\mathbf{Q} = \mathbf{Q}^{-1}\mathbf{AL}_t\mathbf{Q}$$
$$\Rightarrow \mathbf{L}_t = \mathbf{QL} \quad \Rightarrow \quad \mathbf{L} = \mathbf{Q}^{-1}\mathbf{L}_t = \begin{bmatrix} -5 \\ 3.333 \end{bmatrix}$$

c. The design is now to be repeated using Ackermann's formula:

$$P_{Ch,\,\mathbf{A_L}}(\mathbf{A}) = \mathbf{A}^2 + 8\mathbf{A} + 32\mathbf{I} = \begin{bmatrix} 30 & 10 \\ -20 & 50 \end{bmatrix}$$

$$\mathbf{L} = P_{Ch,\,\mathbf{A_L}}(\mathbf{A})\mathbf{M}_0^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3.333 \end{bmatrix}$$

d. A block diagram is now to be drawn of the observer and state controller of Problem 4.3.

The observer equation is:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}(y - \hat{y})$$
$$= \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}\hat{\mathbf{x}} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}(y - \hat{y})$$

LINEAR SYSTEMS CONTROL

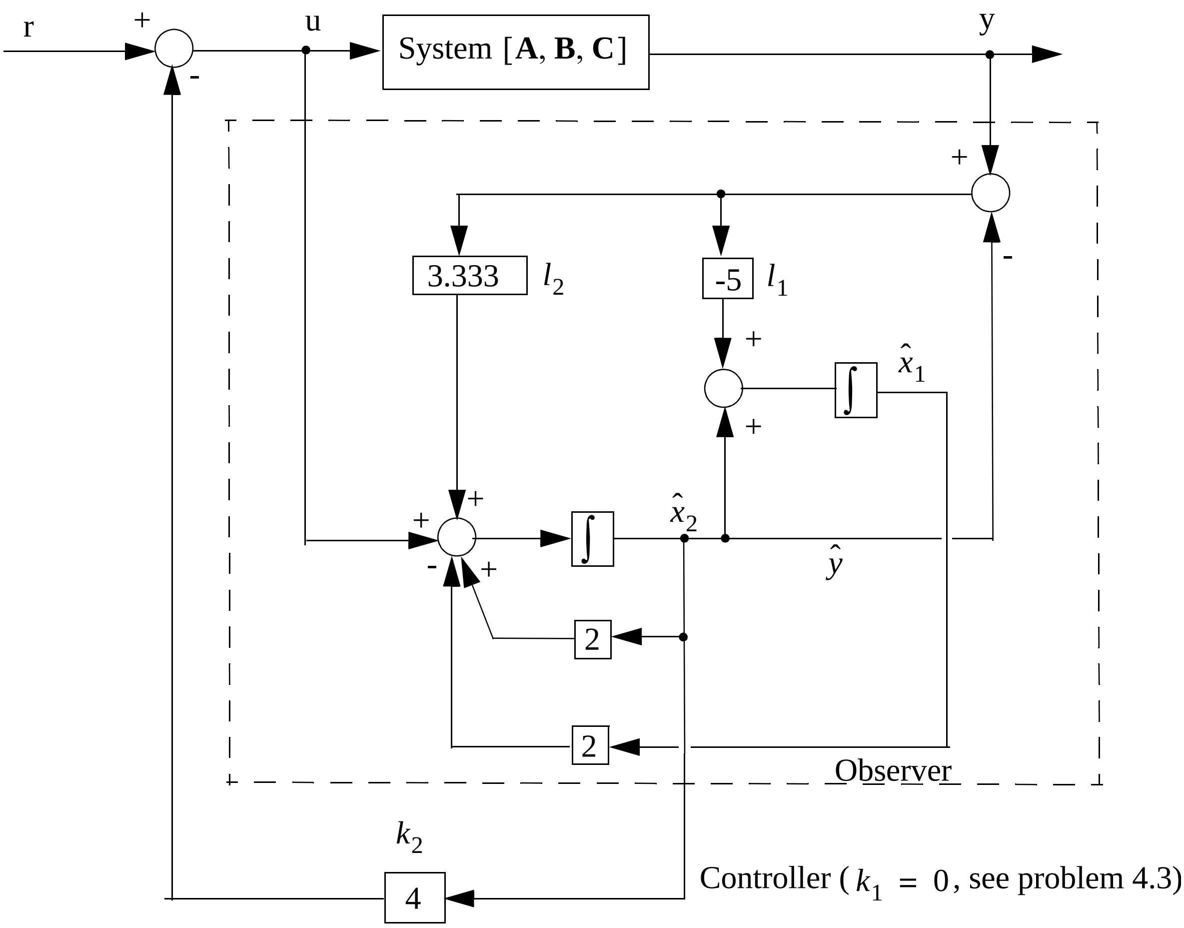
Solutions to Problems

Problem 4.8

$$\Rightarrow \begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + l_1(y - \hat{y}) \\ \dot{\hat{x}}_2 = -2\hat{x}_1 + 2\hat{x}_2 + u + l_2(y - \hat{y}) \end{cases}$$

The controller is:

$$u = -K\hat{x} + r = -k_1\hat{x}_1 - k_2\hat{x}_2 + r$$



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LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 5.10

This problem concerns a D.C. motor which is to be used as an electronic throttle control for a spark ignition engine. The control object is basically a D.C. motor which has an increased moment of inertia due to the throttle plate (see example 2.3).

a. The system matrices are (given that position control is desired):

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ \beta \end{pmatrix}, \quad \mathbf{R}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{R}_2 = r_2$$

The Riccati equation for this system is

$$0 = \mathbf{PA} + \mathbf{A}^T\mathbf{P} + \mathbf{R}_1 - \mathbf{PBR}_2^{-1}\mathbf{B}^T\mathbf{P}$$

$$0 = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & -\alpha \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}$$

$$+ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}\begin{pmatrix} 0 \\ \beta \end{pmatrix}\frac{1}{r_2}\begin{pmatrix} 0 & \beta \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}$$

$$0 = 1 - \frac{\beta^2}{r_2}p_{12}^2$$

$$\Rightarrow \qquad 0 = p_{11} - \alpha\, p_{12} - \frac{\beta^2}{r_2}p_{12}p_{22}$$

$$0 = 2(p_{12} - \alpha p_{22}) - \frac{\beta^2}{r_2}p_{22}^2$$

These equations can be solved simultaneously to find that:

$$p_{12} = \frac{\sqrt{r_2}}{\beta}$$

$$p_{22} = \frac{r_2}{\beta^2}\left(-\alpha + \sqrt{\alpha^2 + \frac{2\beta}{\sqrt{r_2}}}\right)$$

$$p_{11} = \frac{\sqrt{r_2}}{\beta}\sqrt{\alpha^2 + \frac{2\beta}{\sqrt{r_2}}}$$

LINEAR SYSTEM CONTROL

Solutions to Problems

Problem 5.10

The LQR gain is then given by

$$\mathbf{K} = \mathbf{R}_2^{-1}\mathbf{B}^T\mathbf{P} = -\frac{1}{r_2}\begin{pmatrix} 0 & \beta \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}$$

$$= -\frac{\beta}{r_2}\begin{pmatrix} p_{12} & p_{22} \end{pmatrix}$$

$$= -\left[\frac{1}{\sqrt{r_2}} \quad \frac{1}{\beta}\left(-\alpha + \sqrt{\alpha^2 + \frac{2\beta}{\sqrt{r_2}}}\right)\right]$$

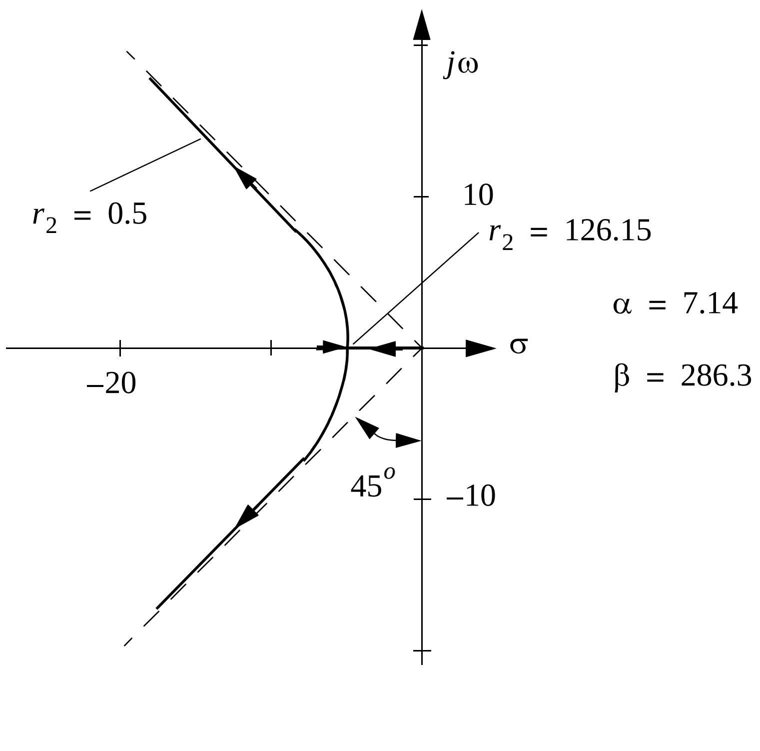
b. The characteristic equation for the system is

$$\det[s\mathbf{I} - (\mathbf{A} - \mathbf{BK})] = s^2 + s\sqrt{\alpha^2 + \frac{2\beta}{\sqrt{r_2}}} + \frac{\beta}{\sqrt{r_2}} = 0$$

which has the solutions:

$$s_0 = -\frac{1}{2}\sqrt{\alpha^2 + \frac{2\beta}{\sqrt{r_2}}} \pm \frac{1}{2}\sqrt{\alpha^2 - \frac{2\beta}{\sqrt{r_2}}}.$$

The plot below shows the root curve as a function of the weight paramter r_2.



LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 5.10

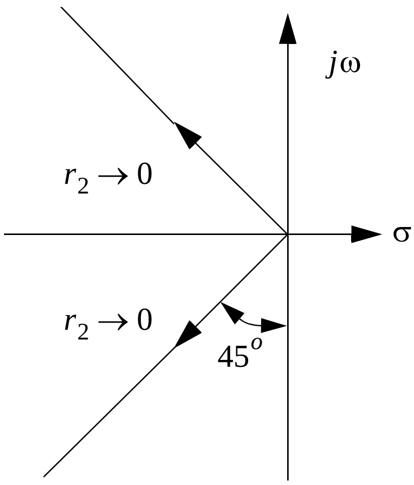
The asymptotes for the root curve are shown dashed on the figure: they correspond to the order of the system which is 2. Using the parameter values, $\alpha = 7.14$, $\beta = 286.3$, and the input weighting, $r_2 = 0.5$, the specified system response time (time constant) of 50 msec can be achieved with the feedback matrix:

$$\mathbf{K} = [1.414 \quad 0.0775].$$

The time constant is reasonable for a real automotive throttle control.

The branch point is at $s_{bp} = -5.049$. Notice that as $r_2 \to 0$ more and more power is used to control the system as it becomes faster and faster.

c. If the motor damping, α, is zero then the root curve look like the figure below



d. The value of r_2 which gives a response time (time constant) of 50 msec is 0.5. The system response to a given set of initial conditions can be calculated by finding the systems transition matrix.

$$\bar{x}(t) = \bar{\Phi}(t, 0)\ \bar{x}(0) \qquad \text{since} \quad u(t) = 0$$

LINEAR SYSTEM CONTROL

Solutions to Problems

Problem 5.10

$$\mathbf{A_K} = \mathbf{A} - \mathbf{BK} = \begin{pmatrix} 0 & 1 \\ -404.8 & -29.33 \end{pmatrix}$$

$$\Rightarrow [s\mathbf{I} - \mathbf{A_K}]^{-1} = \begin{pmatrix} \dfrac{s+29.33}{\Delta} & \dfrac{1}{\Delta} \\ \dfrac{-404.8}{\Delta} & \dfrac{s}{\Delta} \end{pmatrix},$$

$$\Delta = s^2 + 29.33s + 404.8$$

The roots of the characteristic equation are $s_0 = -14.67 \pm j13.77$.

Note now that:

$$L^{-1}\left\{\frac{1}{(s+a)^2+b^2}\right\} = \frac{e^{-at}\sin bt}{b}$$

$$L^{-1}\left\{\frac{s+a}{(s+a)^2+b^2}\right\} = e^{-at}\cos bt$$

$$s^2 + 2sa + a^2 + b^2 : = s^2 + 29.33s + 404.8$$

$$\begin{cases} \Rightarrow 2a = 29.33 \Rightarrow a = 14.665 \\ \Rightarrow a^2 + b^2 = 404.8 \quad \Rightarrow b = 13.775 \end{cases}$$

$$\Phi(t,0) = L^{-1}[s\mathbf{I} - \mathbf{A_K}]^{-1} = \begin{pmatrix} \Phi_{11}(t,0) & \Phi_{12}(t,0) \\ \Phi_{21}(t,0) & \Phi_{22}(t,0) \end{pmatrix}$$

$$x(0) = \begin{pmatrix} 0.3 \\ 0 \end{pmatrix} \quad \Rightarrow \quad x(t) = \Phi(t,0)x(0) = 0.3\begin{bmatrix} \Phi_{11}(t,0) \\ \Phi_{21}(t,0) \end{bmatrix}$$

Thus

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 5.10

$$x(t) = 0.3L^{-1}\begin{bmatrix} \dfrac{s + 29.33}{(s + 14.665)^2 + 13.775^2} \\ \dfrac{-404.8}{(s + 14.665)^2 + 13.775^2} \end{bmatrix}$$

$$= \begin{bmatrix} 0.3e^{-14.665t}\cos(1.89.74t) \\ -0.64e^{-14.665t}\sin(189.74t) \end{bmatrix}$$

❒

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 5.11

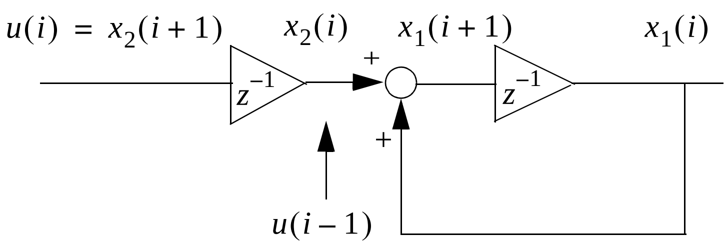
a. The state equation of the stock resupply problem can be found by letting $x(i)$ = stock supply on day i and $x_2(i)$ = delayed order on day i. This means that

$$x_1(i) = x(i), \qquad x_2(i) = u(i-1)$$
$$x_1(i+1) = x_1(i) + x_2(i)$$
$$\Rightarrow \; x_2(i+1) = u(i)$$

$$\Rightarrow \begin{pmatrix} x_1(i+1) \\ x_2(i+1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(i) \\ x_2(i) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(i)$$

$$y(i) = (1 \quad 0) \begin{pmatrix} x_1(i) \\ x_2(i) \end{pmatrix}$$

Block diagram:



b. What is desired is a regulator for which can achieve a steady state output given a step in put in at most 2 days. The system is of second order. This implies that a deadbeat regulator is to be designed.

To design such a regulator let

$$|\lambda \mathbf{I} - \mathbf{F} + \mathbf{GK}| = \lambda^2$$

this implies that

Problem 5.11

$$\Rightarrow \quad |\lambda\mathbf{I} - \mathbf{F} + \mathbf{GK}| = \left| \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (k_1 k_2) \right|$$

$$= (\lambda - 1)(\lambda + k_2) + k_1$$

$$= \lambda^2 + (k_2 - 1)\lambda - k_2 + k_1$$

$$= \lambda^2$$

$$\Rightarrow \quad k_2 - 1 = 0, \quad -k_2 + k_1 = 0 \quad \Rightarrow \quad k_1 = k_2 = 1$$

c From the specification what is required is a steady state LQR regulator.The Riccato equation which must be solved is

$$\mathbf{P}_d = \mathbf{R}_{1d} + \mathbf{F}^T \mathbf{P}_d [\mathbf{I} + \mathbf{G}\mathbf{R}_2^{-1}\mathbf{G}\mathbf{P}]^{-1} \mathbf{F}$$

where

$$\mathbf{K} = (k_1 \quad k_2), \quad \mathbf{P} = \begin{pmatrix} p_{11} & p_{12} \\ p_{11} & p_{22} \end{pmatrix}$$

$$R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad R_2 = \rho$$

One finds that

$$\mathbf{K}_d = [\mathbf{R}_{2d} + \mathbf{G}^T \mathbf{P}_d \mathbf{G}]^{-1} \mathbf{G}\mathbf{P}_d \mathbf{F} = \frac{1}{\rho + p_{22}} (\, p_{12} \quad p_{22} \,)$$

and

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 5.11

$$p_{11} = p_{11} + 1 - \frac{\rho_{12}}{\rho + p_{22}} p_{12}$$

$$p_{12} = p_{11} + 1 - \frac{\rho_{12}}{\rho + p_{22}} p_{12}$$

$$p_{22} = p_{11} + 1 - \frac{\rho_{12}}{\rho + p_{22}} p_{12}$$

$$\Rightarrow p_{11} = p_{12} = p_{22} = p$$

$$p^2 - p - \rho = 0 \quad \Rightarrow \quad p = \frac{1 \pm \sqrt{1 + 4\rho}}{2}$$

$$\Rightarrow \quad k_1 = k_2 = \frac{1 + \sqrt{1 + 4\rho}}{2\rho + 1 + \sqrt{1 + 4\rho}}$$

d. Note:

$$\rho \gg 1 \Rightarrow k = \frac{1}{1 + \sqrt{\rho}}$$

$$\rho \ll 1 \quad \Rightarrow \quad k = 1 - \rho \quad \Rightarrow \quad \rho \to 0$$

gives a deadbeat regulator.

With this type of regulator the input level will be very high as the state power level is not weighted (punished) at all.

❐

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 5.8

a. The index which must be minimized implies that what is required is a steady state LQR regulator. This means that a Riccati equation must be solved which is

$$-\dot{\mathbf{P}} = \mathbf{0} = \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{R}_1 - \mathbf{P}\mathbf{B}\mathbf{R}_2^{-1}\mathbf{B}^T\mathbf{P}$$

$$0 = -\alpha p - p\alpha + 1 - p\ \beta\ \frac{1}{\rho}\ \beta\ p$$

$$= -2\alpha p + 1 - \frac{\beta^2}{\rho}\ p^2$$

$$\Rightarrow\ \left(\frac{\beta^2}{\rho}\ p^2 + 2\alpha p - 1\right) = 0$$

$$p = \frac{-\alpha \pm \sqrt{\alpha^2 + \frac{\beta^2}{\rho}}}{\frac{\beta^2}{\rho}} = -\frac{\alpha\rho}{\beta^2} + \frac{\rho}{\beta^2}\sqrt{\alpha^2 + \frac{\beta^2}{\rho}}$$

$$k = -\mathbf{R}_2^{-1}\mathbf{B}^T\mathbf{P} = -\frac{1}{\rho}\ \beta\ p = -\frac{\alpha}{\beta} + \sqrt{\frac{\alpha^2}{\beta^2} + \frac{1}{\rho}}$$

❐

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 5.9

In the exercise text the performance index should be written as:

$$J = \frac{1}{2}\mathbf{x}^T(t_1)\mathbf{x}(t_1) + \int_{t_0}^{t_1} (\mathbf{x}^T(t)\mathbf{x}(t) + r_2 u^2(t))\ dt \ ,$$

where r_2 is a constant. There is an error in the book text.

a. The solution to this exercise is given in the book text in examples 5.3 and 5.5. It is a good idea go throught these examples in detail and to try to integrate the differential equations using Matlab different input weights and various initial conditions.

b. The stationary solution of the three differential equations in example 5.3 can be found by solving the following algebraic equations simultaneously (see example 5.5):

$$0 = r_p - \frac{1}{r_2}p_{12}^2$$

$$0 = p_{11} - \frac{1}{r_2}p_{12}p_{22}$$

$$0 = r_v + 2p_{12} - \frac{1}{r_2}p_{22}^2$$

This can be done by solving the first equation for p_{12} and then using this expression in the third equation to find p_{12}. Then it is easy to use the second equation to find p_{11}. The results are (see example 5.5):

$$p_{12} = \sqrt{r_p r_2}$$
$$p_{22} = \sqrt{2r_2 p_{12} + r_v r_2}$$
$$p_{11} = \frac{1}{r_2}p_{12}p_{22}$$

Using the values given in example 5.3 for r_p, r_v and r_2, one finds that $p_{12} = 1.732$, $p_{22} = 2.732$ and $p_{11} = 4.732$. These values are those which can be observed at $t = 0$ in figure 5.9. Remember the three differential equations are solve backwards in time.

The corresponding LQR gain is given by:

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 5.9

$$\mathbf{K} = \frac{1}{r_2}\mathbf{B}^T\mathbf{P}_\infty = \frac{1}{r_2}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \frac{1}{r_2}\begin{bmatrix} p_{11} & p_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{\frac{r_p}{r_2}} & \sqrt{\frac{r_p}{r_2}}\sqrt{2\sqrt{r_p r_2} + r_v} \end{bmatrix}$$

❒

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 6.1

Experiment with a coin: the coin is flipped two times.

In order to assign numbers to the sample space, assign: T (Tails) = 0 , H (Heads) = 1.

Sample	HH	HT	TH	TT
x	2	1	1	0

a. Construction of the probability density function:

$$Pr(HH) = \frac{1}{4}, \quad Pr(HT) = Pr(TH) = \frac{1}{4}, \quad Pr(TT) = \frac{1}{4}$$

$$Pr(X = 0) = Pr(TT) = \frac{1}{4}$$

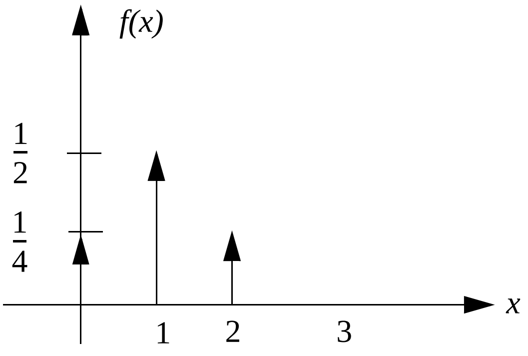
$$Pr(X = 1) = Pr(HT \ v \ TH) = Pr(HT) + Pr(TH)$$

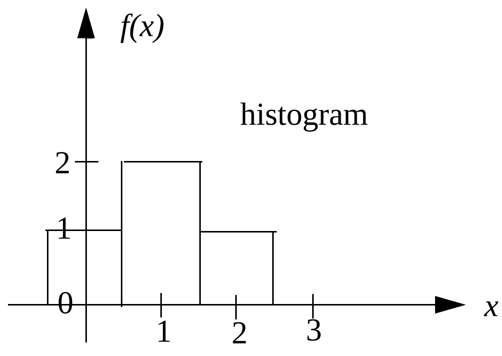
$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$Pr(X = 2) = Pr(HH) = \frac{1}{4}$$

The probability density function table is sketched on the figures below:

x	0	1	2
$f(x)$	1/4	1/2	1/4





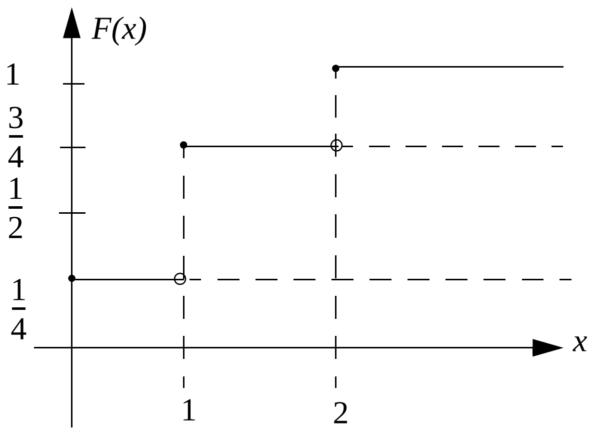
LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 6.1

b. Probability distribution function is sketched on the figure below:

$$F(x) = \begin{cases} 0, \quad -\infty < x < 0 \\ Pr(X = 0)U(x) = \frac{1}{4}, 0 \le x < 1 \\ Pr(X = 0)U(x) + Pr(X = 1)U(X-1) \\ \quad = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}, \quad 1 \le x < 2 \\ Pr(X = 0)U(x) + Pr(X = 1)U(X-1) \\ + Pr(X = 2)U(X-2) = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1, \\ \quad 2 \le x < \infty \end{cases}$$



c. The probability of finding *X* during the experiment:

$$E\{X\} = x_1 Pr(X = x_1) + \ldots x_n Pr(X = x_n)$$

$$= \sum_{j=1}^{n} x_j \, Pr(X = x_1)$$

$$= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 6.1

d. The probability of finding X^2:

$$E\{X^2\} = x_1^2 Pr(x = x_1) + \dots + x_n^2 Pr(x = x_0)$$
$$= 0 \cdot \frac{1}{4} + 1^2 \cdot \frac{1}{2} + 2^2 \ \frac{1}{4} = 1 \cdot \frac{1}{2}$$

e. Given the probability of finding X and X^2, the standard deviation is easily found:

$$\sigma_x^2 = \{E(X-m)^2\} = E\{X^2\} - [E\{X\}]^2$$
$$= \left(\frac{3}{2}\right)^2 - 1^2 = \frac{5}{4}$$
$$\sigma_x = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$$

❐

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 6.11

a. The spectral density matrix is found by Fourier transforming the covariance function

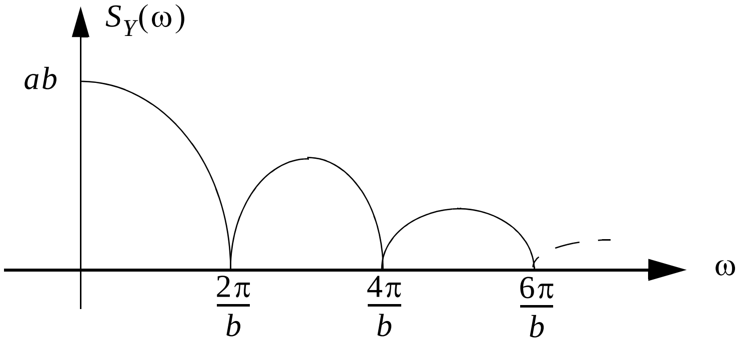
$$R_\gamma(\tau) = \begin{bmatrix} a\left(1 - b\dfrac{|\tau|}{b}\right) & |\tau| < b \\ 0 & \text{otherwise} \end{bmatrix}$$

$$\begin{aligned}
S_Y(\omega) &= \int_{-\infty}^{\infty} a\left(1 - \frac{|\tau|}{b}\right) e^{-j\omega\tau} d\omega \\
&= \int_{-b}^{b} a e^{-j\omega} d\tau + \frac{a}{b}\left[\int_{-b}^{0} \tau e^{-j\omega\tau} d\tau + \int_{b}^{0} \tau e^{-j\omega\tau} d\tau\right] \\
&= \frac{a}{j\omega}(e^{j\omega b} - e^{-j\omega b}) + \frac{a}{b}\left[\frac{2}{\omega^2} - \frac{b}{j\omega}e^{j\omega b} - \frac{1}{\omega^2}e^{j\omega b} + \frac{b}{j\omega}e^{-j\omega b} - \frac{1}{\omega^2}e^{-j\omega b}\right] \\
&= \frac{2a\ \sin\omega b}{\omega} + \frac{2a}{\omega^2 b}(1 - \cos\omega b) - \frac{2a\ \sin\omega b}{\omega} = \frac{2a}{\omega^2 b}(1 - \cos\omega b)
\end{aligned}$$

As $\int \tau e^{-j\omega\tau} d\tau = -\frac{\tau}{j\omega}e^{-j\omega\tau} + \frac{1}{\omega^2}\ e^{-j\omega\tau}$, which is found by partial integration.

The D.C. value of the spectrum is found from

$$S_Y(0) = \lim_{\omega \to 0} \frac{2a}{\omega^2 b}(1 - \cos\omega b) = \lim_{\omega \to 0} \frac{2a}{\omega^2 b}\left(1 - 1 + \frac{1}{2}\omega^2 b^2\right) = ab$$



Problem 6.11

b. No because $S_Y(\omega)$ has infinitely many zeroes.

c.

$$H(s) = \frac{p}{s+q}, \quad V = 1$$

$$S_Y(\omega) = H(j\omega)\, V\, H^T(-j\omega) = \frac{p}{q+j\omega} \cdot \frac{p}{q-j\omega} = \frac{p^2}{q^2+\omega^2}$$

For the power in Y one finds:

$$P_Y = \int_{-\infty}^{\infty} S_Y(\omega)df = \frac{p^2}{2\pi q^2} \int_{-\infty}^{\infty} \frac{1}{1+\left(\frac{\omega}{q}\right)^2} d\omega$$

$$= \frac{p^2}{2\pi q} \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \frac{p^2}{2\pi q}[\operatorname{atan} x]\big|_{-\infty}^{\infty} = \frac{p^2}{2q}, \quad \left(x = \frac{\omega}{q}\right)$$

The D.C. power is

$$P_\gamma = R_Y(0) = a$$

It is required that:

$$(1)\ P_\gamma = P_Y \quad \Leftrightarrow \quad \frac{p^2}{2q} = a$$

$$(2)\ S_\gamma(0) = S_Y(0) \quad \Leftrightarrow \quad \frac{p^2}{q^2} = ab$$

which must be solved for p and q.

$$\frac{p^2}{q^2} = ab = \frac{p^2}{2q} b \Leftrightarrow q = \frac{2}{b}$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

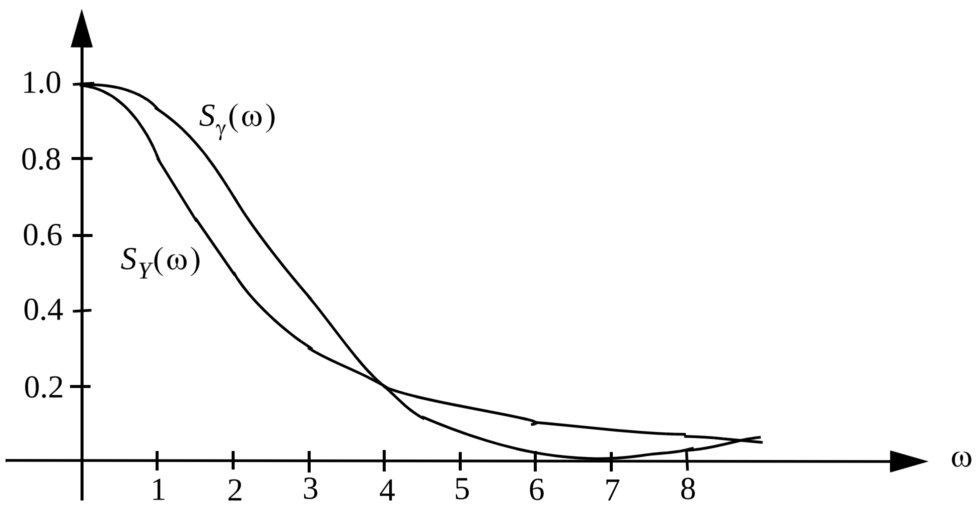
Problem 6.11

$$\frac{p^2}{q^2} = ab \quad \Leftrightarrow \quad p = q\sqrt{ab} = 2\sqrt{\frac{a}{b}}$$

$$H(s) = \frac{2\sqrt{\frac{a}{b}}}{s + \frac{2}{b}}$$

d.

$$a = 1, \quad b = 1, \quad \left\{ \begin{array}{l} S_\gamma(\omega) = \frac{2}{\omega^2}(1 - \cos\omega) \\ S_Y(\omega) = \frac{4}{\omega^2 + 4} \end{array} \right\}$$



The power bandwidth for Y can be found from:

$$P_Y^{\omega_0} = \int_{-\frac{\omega_0}{2\pi}}^{\frac{\omega_0}{2\pi}} S_Y(\omega)df = \frac{p^2}{\pi q}\left[\operatorname{atan}\frac{\omega}{q}\right]$$

Problem 6.11

ω_0 must be found so that

$$P_Y^{\omega_0} = \frac{1}{2}P_Y \quad \Leftrightarrow \quad \operatorname{atan}\frac{\omega_0}{q} = \frac{\pi}{4}$$
$$\Leftrightarrow \omega_0 = \frac{2}{b}$$

which is the power bandwidth for *Y*.

❐

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 6.2

a. In order that *f(x)* is a proper p.d.f. it is necessary that

$$\int_{-\infty}^{\infty} f(\chi)d\chi = 1 = \int_{-\infty}^{\infty} \frac{a}{\chi^2 + 1}\, d\chi$$

$$= a \ \mathrm{atan}(\chi)\Big|_{-\infty}^{\infty}$$

$$= a\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = a\pi$$

$$\Rightarrow a = \frac{1}{\pi}$$

b. The distribution function corresponding to the p.d.f. is given by

$$F(x) = \int_{-\infty}^{x} f(\chi)d\chi = \frac{1}{\pi} = \int_{-\infty}^{x} \frac{1}{1+\chi^2}\, d\chi$$

$$= \frac{1}{\pi}[\mathrm{atan}\ \chi]\Big|_{-\infty}^{x} = \frac{1}{\pi}[\mathrm{atan}\ x - \mathrm{atan}\ (-\infty)]$$

$$= \frac{1}{\pi}\left[\mathrm{atan}\ x + \frac{\pi}{2}\right] = \frac{1}{\pi}\mathrm{atan}\ x + \frac{1}{2}$$

c. It is given that the range of X is:

$$\frac{1}{4} \le X \le 1$$

Thus the probability which must be found is:

$$Pr\left\{\frac{1}{4} \le X \le 1\right\} = \int_{\frac{1}{2}}^{1} f(\chi)d\chi = \frac{1}{\pi}\left[\mathrm{atan}(1) - \mathrm{atan}\left(\frac{1}{2}\right)\right] = \frac{1}{\pi}\left(\frac{\pi}{4} - 0.148\pi\right) = 0.102$$

d. It is given that

$$\frac{1}{4} \le X^2 \le 1 \Rightarrow \frac{1}{2} \le X \le 1 \quad \text{or} \quad -1 \le X \le -\frac{1}{2}$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 6.2

$$\Rightarrow Pr\left\{\frac{1}{4} \le X^2 \le 1\right\} = \int_{-1}^{-\frac{1}{2}} f(\chi)d\chi + \int_{\frac{1}{2}}^{1} f(\chi)d\chi$$

so that

$$Pr\left\{\frac{1}{4} \le X^2 \le 1\right\} = \frac{2}{\pi}\int_{\frac{1}{2}}^{1} \frac{1}{1+\chi^2}d\chi = \frac{2}{\pi}\left[\operatorname{atan}(1) - \operatorname{atan}\left(\frac{1}{2}\right)\right]$$

$$= \frac{2}{\pi}\left(\frac{\pi}{4} - 0.148\pi\right) = 0.204$$

❐

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 6.5

The differential equations desired are found with direct differentiation of the definitions using the Leibniz's integral differentiation formula:

$$\frac{dv_{avg}}{dt} = \frac{d}{dt}\left(\frac{1}{t}\int_0^t v\ d\tau\right) = -\frac{1}{t^2}\int_0^t v\ d\tau + \frac{1}{t}\ v$$

$$= \frac{1}{t}\left(v - \frac{1}{t}\int_0^t v\ d\tau\right)$$

$$\Rightarrow \frac{dv_{avg}}{dt} = \frac{1}{t}(v - v_{avg}), \qquad t > 0$$

$$\frac{dv_{rms}}{dt} = \frac{d}{dt}\left(\sqrt{\frac{1}{t}\int_0^t v^2\ d\tau}\right)$$

$$= \frac{1}{2}\left(\frac{1}{t}\int_0^t v^2\ d\tau\right)^{-\frac{1}{2}}\left[-\frac{1}{t^2}\int_0^t v^2\ d\tau + \frac{1}{t}\ v^2\right]$$

$$= \frac{1}{2t}\frac{\left[v^2 - \frac{1}{t}\int_0^t v^2\ d\tau\right]}{\left(\frac{1}{t}\int_0^t v^2\ d\tau\right)^{\frac{1}{2}}} = \frac{1}{2t}\ \frac{(v^2 - v_{rms}^2)}{v_{rms}}$$

$$\frac{dv_{rms}}{dt} = \frac{1}{2t}\left(\frac{v^2}{v_{rms}} - v_{rms}\right), \quad t > 0,\ \ v_{rms} > 0$$

a. If one uses these equations with a stationary input signal, one will for large times find an approximate (iterative) answer for its average and root mean square values over long integration times.

LINEAR SYSTEM CONTROL

Solutions to Problems

Problem 6.5

b. The equations can be used for a practical determination of the mean and RMS values of the input signal if the averaging time is very large compared to period of signal. Practically one could use a constant integration time instead of an (effectively) infinite time.

c. The answer to b. is yes.

The advantage of using the differential equations is that they give an answer which is as exact as possible given the definition of average and RMS value.

The main disadvantage is that the answers found are dependent on the averaging times used and that the signals measured must be stationary. This is the natural result of the form of the definitions themselves.

d. The differential equations above may be used on a random or noise signal with proper choice of the integration algorithm and the sample time used. The integration time should be 20 - 100 times the noise generator sample time.

e. The differential equations can be linearized immediately with differentiation:

$$\mathbf{A} = \nabla_x f\Big|_{\substack{x = x_n \\ u = u_n}} = \begin{bmatrix} -\frac{1}{t} & 0 \\ 0 & -\frac{1}{2t}\left(1 + \frac{v_0^2}{v_{orms}^2}\right) \end{bmatrix}$$

$$\mathbf{B} = \nabla_u f\Big|_{\substack{x = x_n \\ u = u_n}} = \begin{bmatrix} \frac{1}{t} \\ \frac{v_0}{t\, v_{orms}} \end{bmatrix}$$

e. The equations above are decoupled because the definition equations for v_{avg} and v_{rms} are. The eigen frequencies for the linearized differential equations are negative and thus they represent stable filters for $t > 0$. As the eigenfrequencies are proportional with $1/t$ the filters' bandwidths become smaller and smaller as time increases. This filters out high fre-

Problem 6.5

quency changes in the input signals. For large times the filters are very insensitive to their inputs.

f. The differential equations thus have the same limitiation as analog instruments: an increasingly accurate exact answer can be found for longer integration times assuming a stationary input signal.

❐

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 7.1

a. A reasonable measurement model is one with additive white noise

$$y = \omega(t) + \omega_2(t)$$

b. A Kalman filter can be constructed for the D.C. motor by appending the state equation with an innovation term using the measurement of the shaft speed

$$\dot{\hat{\omega}}(t) = -\alpha\hat{\omega}(t) + \beta V_a(t) + l(\omega_m - \hat{\omega})$$

where ω_m is the measurement.

In order to calculate the Kalman gain, *l*, the Riccati equation for the system must be solved. This Riccati equation is that for the stationary case.

$$0 = \mathbf{AQ} + \mathbf{QA}^T + \mathbf{V}_1 - \mathbf{QC}^T\mathbf{V}_2^{-1}\mathbf{CQ}$$

Here this equastion becomes

$$0 = -\alpha q - q\alpha + V_1 - q\frac{1}{V_2}q$$

$$\Rightarrow q^2 + 2\alpha V_2 q - V_1 V_2 = 0$$

where the bars indicate that the steady state solution is being found. The solution is:

$$q = \frac{-2\alpha V_2 \pm \sqrt{4\alpha^2 V_2^{\ 2} + 4V_1 V_2}}{2}$$

$$= \alpha V_2\left(\sqrt{1 + \frac{V_1}{\alpha^2 V_2}} - 1\right)$$

The corresponding Kalman gain is

$$l = QC^T V_2^{-1} = \alpha\left(\sqrt{1 + \frac{V_1}{\alpha^2 V_2}} - 1\right)$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 7.1

c. If the signal to noise ratio is poor then $\frac{V_1}{V_2} \ll 1$ and $l \cong 0$. There is no feedback in the observer and thus the observer is a primative model of the system itself. Such a construct is called a feedforward or degenerate observer.

d. If the signal to noise ratio is good then $\frac{V_1}{V_2} \gg 1$ and l will be large. The filter then uses the measurement strongly.

❐

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 7.11

a. Newton's second law gives the expression

$$m\frac{d^2X}{dt^2} = F(V) + F(X, I) + mA - mg$$

This equation can be written in state space form as

$$\begin{aligned}\dot{X} &= V \\ \dot{V} &= \frac{1}{m}F(V) + \frac{1}{m}F(X, I) + A - g\end{aligned}$$

b. The system above can be linearized using differentiation

$$\begin{aligned}\dot{x} &= v \\ \dot{v} &= \frac{1}{m}\left.\frac{\partial F(X, I)}{\partial X}\right|_{\substack{\mathbf{X} = \mathbf{X}_n \\ u = u_n}} x + \frac{1}{m}\left.\frac{\partial F(V)}{\partial V}\right|_{\substack{\mathbf{X} = \mathbf{X}_n \\ u = u_n}} v\end{aligned}$$

where $\mathbf{X}_n = (x_n \quad v_n)^T,\ u_n = i_n,\ x = \Delta X,\ v = \Delta V,\ i = \Delta I,\ a = \Delta A$

The stationary bias current can be found by letting

$$F(X, I) = mg$$

remembering that

$$\begin{aligned}V_n &= 0 \Rightarrow F(V_n) = 0, A_n = 0 \\ X_n &= 2.5 \cdot 10^{-3} \Rightarrow \\ F(2.5 \cdot 10^{-3}, I) &= 0.01128 + 0.0361I - 0.01625 \\ &\quad + 0.02888I^2 + 0.3740I - 0.1742 \\ &= 0.02888I^2 + 0.4101I - 0.1792 = 0.0981 \\ \Rightarrow \quad I^2 + 14.20I - 9.601 &= 0 \Rightarrow I_n = 0.6467\text{amps}\end{aligned}$$

which is the requried bias current.

LINEAR SYSTEM CONTROL

Solutions to Problems

Problem 7.11

Assuming this operating current the coefficients in the linearized equation can be found.

$$\frac{1}{m}\left.\frac{\partial F(X,I)}{\partial X}\right|_{\substack{X = 2.5\cdot 10^{-3}\\ I = 0.6467}} = \frac{1}{0.01}(3.61\cdot 10^{3}X + 14.44I - 6.488)\Big|_{\substack{X = 2.5\cdot 10^{-3}\\ I = 0.6467}}$$

$$= 11.94\ m/\sec^2 = a_{21}$$

$$\frac{1}{m}\left.\frac{\partial F(X,I)}{\partial X}\right|_{\substack{X = 2.5\cdot 10^{-3}\\ I = 0.6467}} = \frac{1}{0.01}(14.44X + 0.05776I + 0.3740)\Big|_{\substack{X = 2.5\cdot 10^{-3}\\ I = 0.6467}}$$

$$= 44.75\ \frac{m/\sec^2}{amp} = b$$

One can also find a graphic solution from the figure given. In this case one finds that $I_n = 0.650$ amps and

$$\left.\frac{\partial F}{\partial X}\right|_{\substack{V = X_n\\ I = I_n}} = 12.3\ N/m \quad and \quad \left.\frac{\partial F}{\partial I}\right|_{\substack{X = X_n\\ I = I_n}} = 0.42\ N/amp$$

$$\frac{1}{m}\left.\frac{\partial F(V)}{\partial V}\right|_{V_n = 0} = \frac{1}{m}\frac{\partial}{\partial V}\begin{Bmatrix}(-V(c_1 + c_2V), & V > 0\\ (-V(c_1 + c_2V), & V < 0\end{Bmatrix} = -\frac{c_1}{m}$$

Thus

$$a_{22} = -\frac{c_1}{m} = -1.55\cdot 10^{-4}$$

c. The numerical values of a_{21}, a_{22} and b have been found above. The equations $\dot{X} = V$ and $\dot{x} = v$ are linear and have the same form. Thus $a_{12} = 1$.

d. The transfer function for the system can be found from

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 7.11

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\begin{pmatrix} \rho - a_{22} & 1 \\ a_{21} & \rho \end{pmatrix}}{\Delta} \begin{bmatrix} 0 \\ b \end{bmatrix} = \frac{b}{\rho^2 - a_{22}\rho - a_{2:}}$$

since $\mathbf{C} = (\; 1 \;\; 0 \;)$, $\quad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ a_{21} & a_{22} \end{pmatrix}$, $\quad \mathbf{B} = \begin{pmatrix} 0 \\ b \end{pmatrix}$.

where $a_{22} < 0$ and $a_{21} > 0$. The system is thus unstable in itself with roots in the right half plan. The roots are actually in $s = \pm 34.6$ rad/sec. Feedback is thus necessary to keep the ball in its nominal quiescent position.

e. In order to design a LQ regulator for the accelerometer the following specifications are necessary

$$\mathbf{R}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad R_2 = \rho$$

The Riccati equation for the system is

$$0 = \mathbf{PA} + \mathbf{A}^T\mathbf{P} + \mathbf{R}_1 - \mathbf{PBR}_2^{-1}\mathbf{B}^T\mathbf{P}$$

$$0 = \mathbf{P}\begin{pmatrix} 0 & 1 \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} 0 & a_{21} \\ 1 & a_{22} \end{pmatrix}\mathbf{P} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$-\mathbf{P}\begin{bmatrix} 0 \\ b \end{bmatrix} \frac{1}{\rho} \begin{bmatrix} 0 & b \end{bmatrix}\mathbf{P}$$

where $\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}$

This is to say that the following system of equations must be solved:

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Solutions to Problems

Problem 7.11

$$2a_{11}p_{12} + 1 - \frac{p_{12}^2}{\rho}b^2 = 0$$

$$a_{22}p_{12} + a_{21}p_{22} + p_{11} - \frac{p_{12}p_{22}}{\rho}b^2 = 0$$

$$2(p_{12} + a_{22}p_{22}) - \frac{p_{22}^2}{\rho}b^2 = 0$$

One finds that

$$p_{12} = -a_{21}\frac{\rho}{b^2} \pm \sqrt{a_{21}^2\frac{\rho^2}{b^4} + \frac{\rho}{b^2}}$$

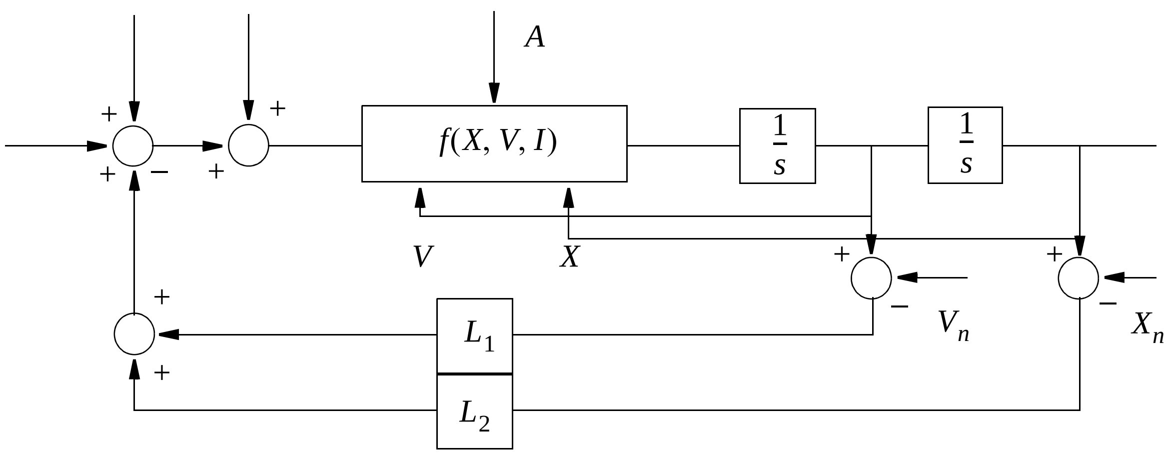
$$p_{22} = -a_{22}\frac{\rho}{b^2} \pm \sqrt{a_{22}^2\frac{\rho^2}{b^4} + \frac{\rho}{b^2}p_{12}}$$

$$p_{11} = p_{12}p_{22}\frac{\rho}{b^2} - a_{22}p_{12} - a_{21}p_{22}$$

From this the LQR gain can be found

$$K = R_2^{-1}BP = -\frac{1}{\rho}(0 \quad b)\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} = -\frac{b}{\rho}(p_{12} \quad p_{22})$$

f. The system block diagram is thus:



LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 7.11

g. It is possible to measure the unknown acceleration by measuring the current, *i*, which it calls forth with respect to I_n . When the system is in equilibrium one can find a from the equation

$$\frac{1}{m}F(X_n, I_n + i) = a - g$$

or from a linear approximation to this equation. The accuracy of the system is dependent on the size of the weighting factor ρ as there is no integration in the feedback loop. It is possible to make the system more accurate by including an integration in the feedback loop for the ball position.

h. In order to estimate a as a slowly varying constant the state vector must be augmented with the acceleration as an extra state.

The observer is then

$$\begin{aligned}
\dot{\hat{x}} &= \hat{v} + k_1(y - \hat{x}) \\
\dot{\hat{v}} &= a_{21}\hat{x} + a_{22}\hat{v} + \hat{a} + bi + k_2(y - \hat{x}) \\
\dot{\hat{a}} &= k_3(y - \hat{x})
\end{aligned}$$

The corresponding Riccati equation is

$$\mathbf{AQ} + \mathbf{QA}^T + \mathbf{V}_1 - \mathbf{QCV}_2^{-1}\mathbf{CQ} = 0$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ a_{21} & a_{22} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

and $\mathbf{V}_2^{-1} = \frac{1}{V_2}$, $\quad V_1 = diag(0, 0, V_1)$. From this the following system of equations can be found:

Problem 7.11

$$-\frac{q_1^2}{V_2} + 2q_2 = 0$$

$$-\frac{q_1 q_2}{V_2} + a_{21}q_1 + a_{22}q_2 + q_3 + q_4 = 0$$

$$-\frac{q_1 q_3}{V_2} + q_5 = 0$$

$$-\frac{q_2^2}{V_2} + 2a_{21}q_2 + 2a_{22}q_4 + 2q_5 = 0$$

$$-\frac{q_2 q_3}{V_2} + a_{21}q_3 + a_{22}q_5 + q_6 = 0$$

$$-\frac{q_3^2}{V_2} + V_1 = 0$$

where $q_1 = q_{11}$, $q_2 = q_{12}$, $q_3 = q_{13}$, $q_4 = q_{22}$, $q_5 = q_{23}$, $q_6 = q_{33}$. No solution to these equations was requested.

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LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 7.3

a. A Kalman filter is to be designed for the system which consists of the D.C. motor of problem 7.1 with the provision of a state describing the angular position of the system.

The state equation for the D.C. motor is

$$\dot{\omega}(t) = -\alpha\omega(t) + \beta V_a(t)$$

The angular position of the system can be found by integrating the angular velocity:

$$\theta(t) = \int_0^t \omega(t)dt \Rightarrow \dot{\theta}(t) = \omega(t)$$

Thus the state equation which describes the overall system is

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= -\alpha\omega + \beta V_a(t) + T_d(t) \end{aligned}$$

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= -\alpha\omega + \beta V_a(t) + T_d(t) \end{aligned}$$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \beta \end{pmatrix}, \quad T_d\text{: torque dist.}$$

b. If one attempts to design a Kalman filter based on a velocity measurement alone then one discovers that the system is not observable.

$$C = (0 \quad 1), \quad A = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix}$$

$$\begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix}: \quad \text{rank } 1$$

An observer for the system can thus not be constructed.

c. Now a Kalman filter is to be constructed which uses the position measurement. This makes the system observable.

LINEAR SYSTEM CONTROL

Solutions to Problems

Problem 7.3

$$C = (0 \quad 1), \quad A = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix}$$

$$\begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}: \quad \text{rank } 2$$

A steady state filter is to be designed, thus the algebraic Riccati equation has to be solved.

$$0 = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix}\begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix} + \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & -\alpha \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 \\ 0 & V \end{pmatrix} - \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\frac{1}{V_2}(\ 1\ \ 0\ )\begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix}$$

This results in a set of quadratic equations which must be solved simultaneously for the q_{ij}‘s, $\quad i = j = 1, 2$.

$$0 = 2q_{12} - \frac{1}{V_2}\, q_{11}$$

$$0 = q_{22} - \alpha q_{12} - \frac{1}{V_2} q_{11} q_{12}$$

$$0 = -2\alpha q_{22} + V_1 - \frac{1}{V_2}\, q_{12}^2$$

The solution of these equations can be found with some difficulty to be

$$q_{11} = V_2\left(-\alpha + \sqrt{\alpha^2 + 2\sqrt{\frac{V_1}{V_2}}}\right)$$

$$q_{12} = \alpha^2 + \sqrt{\frac{V_1}{V_2}} \; - \; \alpha\sqrt{\alpha^2 + \sqrt{\frac{V_1}{V_2}}}$$

$$q_{22} = -\alpha^3 - 2\alpha\sqrt{\frac{V_1}{V_2}} + \left(\alpha^2 + \sqrt{\frac{V_1}{V_2}}\right)\sqrt{\alpha^2 + 2\sqrt{\frac{V_1}{V_2}}}$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 7.3

The Kalman gain is thus

$$
\begin{aligned}
\mathbf{L} &= \mathbf{QC}^T\mathbf{V}_2^{-1} \\
&= \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\frac{1}{V_2} \\
&= \frac{1}{V_2}\begin{bmatrix} q_{11} & q_{12} \end{bmatrix}
\end{aligned}
$$

d. It is clear that this is the only Kalman filter which can be built for the system: this it is the “best” of the two.

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LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 7.8

a. The state equations for the ballon must be written so that they only involve the given input of the system and its state variables.

The first state equation for the ballon is

$$\dot{T} = -a(T - T_a) + cq$$

The second state equation is

$$M\frac{dV}{dt} = Vg(\rho_a - \rho) - Mg - bv$$

But it is given that $\rho T = \rho_a T_a \Rightarrow \rho_a = \frac{\rho T}{T_a}$. Thus the second state equation becomes

$$\begin{aligned}\dot{v} &= \frac{Vg}{M}\left(\frac{\rho T}{T_a} - \rho\right) - g - \frac{b}{M}v \\ &= \frac{Vg}{M}\frac{\rho}{T_a}(T - T_a) - g - \frac{b}{M}v\end{aligned}$$

b. In the stationary state

$$0 = -a(T - T_a) + cq$$

which implies that

$$T_0 = \frac{c}{a}q_0 + T_a$$

In the second state equation in the steady state $\dot{v} = 0$ and $v = 0$ thus

$$\frac{\rho_0}{T_a}(T_0 - T_a) - \frac{M}{V} = 0$$

Using the expression for T_0 above in this equation

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 7.8

$$\rho_0 = \frac{T_a}{\rho_0}\frac{a}{C}\frac{M}{V}$$

also

$$\frac{\rho_0 T_0}{T_a} - \rho_0 = \frac{\rho_a T_a}{T_a} - \rho_0 = \frac{M}{V}$$

$$\rho_0 = \rho_a - \frac{M}{V}$$

Finally

$$q_0 = \frac{T_a}{\rho_0}\frac{a}{c}\frac{M}{V} = \frac{a}{c}\frac{T_a}{\rho_a - \frac{M}{V}}\frac{M}{V}$$

$$= \frac{a}{c}T_a \frac{\frac{M}{\rho_a V}}{1 - \frac{M}{\rho_a V}}$$

Notice that the term $r = \frac{M}{\rho_a V}$ is the ratio of the effective density of the air in the ballon to the ambient density.

c. Linearization can be carried out using differentiation or power series development. One finds

$$\Delta\dot{T} = -\frac{1}{\tau}\Delta T + c\Delta q + \frac{1}{\tau}\Delta T_a \quad , \quad \frac{1}{\tau} = a$$

$$\Delta\dot{v} = -\frac{b}{M}\Delta v + \frac{Vg}{M}\rho_a \frac{T_a}{T_0^2}\Delta T - \frac{Vg}{M}\frac{\rho_0 T_0}{T_a^2}\Delta T_a$$

or

$$\begin{pmatrix} \Delta\dot{T} \\ \Delta\dot{V} \end{pmatrix} = \begin{pmatrix} -a & 0 \\ \frac{Vg}{M}\frac{\rho_a T_a}{T_0^2} & -\frac{b}{M} \end{pmatrix} \begin{pmatrix} \Delta T \\ \Delta V \end{pmatrix} + \begin{pmatrix} 0 & a \\ 0 & \frac{Vg}{M}\frac{\rho_0 T_0}{T_a} \end{pmatrix} \begin{pmatrix} \Delta q \\ \Delta T_a \end{pmatrix}$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 7.8

d. This system is in itself stable as both of its eigenfrequencies are in the left half plan,

$$\lambda = -a \quad , \qquad \lambda = -\frac{b}{m} \ .$$

In order to find out if the system is controllable, the controllability matrix has to be found.

Let $d = \dfrac{Vg\rho_0 T_0}{M}$

$$M_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} c & a & -ac & a^2 \\ 0 & -\dfrac{d}{T_a^2} & \dfrac{cd}{T_0^2} & \dfrac{cd}{T_0^2} + \dfrac{bd}{MT_0^2} \end{bmatrix}$$

The matrix has full rank so that the underlying system is controllable.

e. The state equation of the simplified system is:

$$\begin{aligned} \Delta T &= A\Delta T + B\Delta q + B_v \Delta T_a \\ &= -a\Delta T + c\Delta q + a\Delta T \quad \Rightarrow \\ A &= -a, \quad B = c, \quad B_v = a \end{aligned}$$

f. The optimization index

$$J = \int_0^a (\Delta T^2 + Q\Delta q^2)dt$$

The corresponding Riccati equation is

$$\mathbf{PA} + \mathbf{PA}^T + \mathbf{R}_1 - \mathbf{PBR}_2^{-1}\mathbf{B^TP} = 0$$

$$-2ap + 1 - \frac{c^2}{Q}p^2 = 0$$

$$p^2 + \frac{2aQ}{c^2}p - \frac{Q}{c^2} = 0$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 7.8

$$p = -\frac{aQ}{c} + \sqrt{\frac{a^2Q^2}{c^4} + \frac{Q}{c^2}} = \frac{aQ}{c}\left(\sqrt{1 + \frac{c^2}{a^2Q}} - 1\right),$$

where the positive definite solution has been selected.

The state feedback is then

$$\Delta q = -K\Delta T \quad \text{where} \quad k = Q^{-1}B^Tp$$

Thus

$$k = \frac{a}{c}\left(\sqrt{1 + \frac{c^2}{a^2Q}} - 1\right)$$

g. The system to be modelled is

$$\dot{\Delta T} = -a\Delta T + c\Delta q + a\Delta T_a$$

where $\Delta T_a \in N(0,V)$.

The measurement model is

$$y = \Delta T + w$$

where $E\{\Delta T_a^2\} \in N(0, V)$, $w \in N(0,W)$ and ΔT_a and w are independent: $E\{\Delta T \cdot w\} = 0$.

The Riccati equation for the system is:

$$-aq - aq + V + qW^{-1}q = 0$$

$$q^2 + 2aWq + VW = 0 \quad \text{or}$$

$$q = -aW + \sqrt{a^2W^2 + VW} = aW\left(\sqrt{1 + \frac{V}{a^2W}} + 1\right)$$

LINEAR SYSTEMS CONTROL

Solutions to Problems

Problem 7.8

The Kalman gain is then

$$L = qW^{-1} = aW\left(\sqrt{1+\frac{V}{a^2W}}+1\right)\frac{1}{W}$$

$$= a\left(\sqrt{1+\frac{V}{a^2W}}-1\right)$$

The corresponding Kalman filter is

$$\dot{\hat{\Delta T}} = -a\hat{\Delta T} + c\Delta q + L(\Delta T_{meas} - \hat{\Delta T})$$

where ΔT_{meas} is the temperature measured with respect to the linearization point.

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Linear Systems Control

Solutions to Problems

Problem 2.9

a) Volume conservation law:

$$\left.\begin{aligned} A_1\dot{x}_1 &= ku_1 - q_1 \\ A_2\dot{x}_2 &= q_1 - q_0 + q_2 \\ A_3\dot{x}_3 &= -q_2 + ku_2 \end{aligned}\right\} \qquad (1)$$

Flow equations

$$\left.\begin{aligned} q_1 &= c_1\sqrt{x_1 - x_2} \\ q_2 &= c_2\sqrt{x_3 - x_2} \\ q_0 &= c_0\sqrt{x_2} \end{aligned}\right\} \qquad (2)$$

(2) is inserted into (1):

$$\begin{aligned} \dot{x}_1 &= \frac{1}{A_1}(-c_1\sqrt{x_1 - x_2} + ku_1) = f_1(x, u) \\ \dot{x}_2 &= \frac{1}{A_2}(c_1\sqrt{x_1 - x_2} + c_2\sqrt{x_3 - x_2} - c_0\sqrt{x_2}) = f_2(x, u) \\ \dot{x}_3 &= \frac{1}{A_3}(ku_2 - c_2\sqrt{x_3 - x_2}) = f_3(x, u) \end{aligned}$$

$$y = q_0 = c_0\sqrt{x_2} = g(x, u)$$

b) Stationary states:

$$\dot{x}_1 = 0 \qquad ; \qquad \dot{x}_2 = 0 \qquad ; \qquad \dot{x}_3 = 0$$

$$\begin{aligned} ku_{10} &= c_1\sqrt{x_{10} - x_{20}} && 3) \\ c_1\sqrt{x_{10} - x_{20}} + c_2\sqrt{x_{30} - x_{20}} &= c_0\sqrt{x_{20}} && 4) \\ ku_{20} &= c_2\sqrt{x_{30} - x_{20}} && 5) \end{aligned}$$

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3) and 5) are inserted into 4):

$$\begin{aligned} ku_{10} + ku_{20} &= c_0\sqrt{x_{20}} \\ \Rightarrow x_{20} &= \left(\frac{k}{c_0}(u_{10} + u_{20})\right)^2 \end{aligned}$$

From 3) and 5) is obtained:

$$\begin{aligned} x_{10} &= \left(\frac{k}{c_1}u_{10}\right)^2 + x_{20} \\ x_{30} &= \left(\frac{k}{c_2}u_{20}\right)^2 + x_{20} \end{aligned}$$

Output:

$$y_0 = c_0\sqrt{x_{20}}$$

We define the deviation variables:

$$\begin{aligned} x_1 &= x_{10} + \Delta x_1 & \qquad u_1 &= u_{10} + \Delta u_1 \\ x_2 &= x_{20} + \Delta x_2 & \qquad u_2 &= u_{20} + \Delta u_2 \\ x_3 &= x_{30} + \Delta x_3 & \qquad y &= y_0 + \Delta y \end{aligned}$$

$$A = \left\{\frac{\partial f_i}{\partial x_j}\right\}_0$$

$$= \begin{bmatrix} \dfrac{-c_1}{2A_1\sqrt{x_{10}-x_{20}}} & \dfrac{c_1}{2A_1\sqrt{x_{10}-x_{20}}} & 0 \\ \dfrac{c_1}{2A_2\sqrt{x_{10}-x_{20}}} & -\dfrac{c_1}{2A_2\sqrt{x_{10}-x_{20}}} - \dfrac{c_0}{2A_2\sqrt{x_{20}}} - \dfrac{c_2}{2A_2\sqrt{x_{30}-x_{20}}} & \dfrac{c_2}{2A_2\sqrt{x_{30}-x_{20}}} \\ 0 & \dfrac{c_2}{2A_3\sqrt{x_{30}-x_{20}}} & \dfrac{c_2}{2A_3\sqrt{x_{30}-x_{20}}} \end{bmatrix}$$

Problem 2.9

$$B = \left\{ \frac{\partial f_i}{\partial u_j} \right\}_0 =$$

$$\begin{bmatrix} \frac{k}{A_1} & 0 \\ 0 & 0 \\ 0 & \frac{k}{A_3} \end{bmatrix}$$

$$C = \left\{ \frac{\partial g_i}{\partial x_j} \right\}_0 = \left\{ 0 \quad \frac{c_0}{2\sqrt{x_{20}}} \quad 0 \right\}$$

$$D = \left\{ \frac{\partial g_i}{\partial u_j} \right\}_0 = 0$$