Technical University of Denmark

Written examination, date 14/12/2013 Page 1 of 17 pages

Course name: Linear Control Design II

Course number: 31310

Aids allowed: All

Exam duration: 2 hours

Weighting: For each of the 12 questions there is only one correct answer, 4 wrong answers and one "I do not know" answer. Each correct answer gives 5 points; each wrong answer gives -1 point; the "I do not know" answer gives 0 points.

Student number: ____

Name: ____

Table number: ____

Signature: ____

The passing level is to be set after the correction of the exams.

Use large capital letters in the scheme

Question	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Answer												

Remember to include your Student number, Name, Table number, and Signature.

Linear Control Design II   Fall 2013
Final Exam

1. Consider the block diagram of a magnetic suspension shown in Fig. 1 where g is the gravity constant. The inductance of the electromagnet is given by

$$L\left(y\right)=L_1+\frac{L_0}{1+\frac{y}{a}}$$

where L_0, L_1, and a are positive constants. Which of the following state space models is associated with the block diagram? (In the diagram the black dots address intersection of signal flows.)

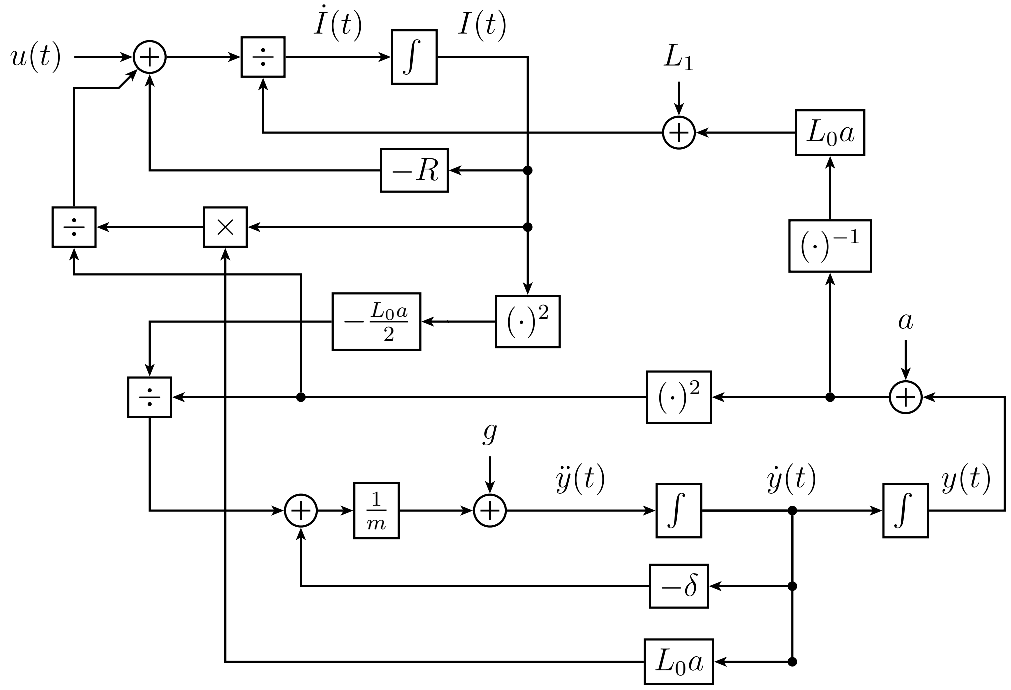


Figure 1: Block diagram of a magnetic suspension.

(a)

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= g - \frac{\delta}{m}x_2 - \frac{L_0 a x_3^2}{2m\left(a+x_1\right)^2} \\ \dot{x}_3 &= \frac{1}{L\left(x_1\right)}\left(-Rx_3 + \frac{L_0 x_2 x_3}{\left(a+x_1\right)^2} + u\right)\end{aligned}$$

(b)

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= g - \frac{\delta}{m}x_2 - \frac{\kappa}{m}x_1 - \frac{L_0 a x_3^2}{2m\left(a+x_1\right)^2} \\ \dot{x}_3 &= \frac{1}{L\left(x_1\right)}\left(-Rx_3 + \frac{L_0 x_2 x_3}{\left(a+x_1\right)^2} + u\right)\end{aligned}$$

(c)

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= g - \frac{\delta}{m}x_2 - \frac{\kappa}{m}x_1 - \frac{L_0 a x_3^2}{2m\left(a+x_1\right)^2} \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{1}{L\left(x_1\right)}\left(-Rx_3 + \frac{L_0 x_2 x_4}{\left(a+x_1\right)^2} + u\right)\end{aligned}$$

(d)

$$\begin{aligned}\dot{x}_1 &= g - \frac{\delta}{m}x_1 - \frac{L_0 a x_2^2}{2m\left(a+x_1\right)^2} \\ \dot{x}_2 &= \frac{1}{L\left(x_1\right)}\left(-Rx_2 + \frac{L_0 x_1 x_2}{\left(a+x_1\right)^2} + u\right)\end{aligned}$$

(e)

$$\begin{aligned}\dot{x}_1 &= x_2 - \frac{R}{L\left(x_1\right)}x_3 \\ \dot{x}_2 &= g - \frac{\delta}{m}x_2 - \frac{L_0 a x_3^2}{2m\left(a+x_1\right)^2} \\ \dot{x}_3 &= \frac{1}{L\left(x_1\right)}\left(-Rx_3 + \frac{L_0 x_2 x_3}{\left(a+x_1\right)^2} + u\right)\end{aligned}$$

(f) I do not know.

2. Consider the 2$^{\text{nd}}$ order LTI continuous time system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -\alpha & \beta \\ -\frac{\beta}{2} & \alpha \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \gamma & \gamma \\ \gamma & \gamma \end{bmatrix} \mathbf{u}(t), \quad \mathbf{x} \in \mathbb{R}^2, \mathbf{u} \in \mathbb{R}^2$$

$$\mathbf{y}(t) = \begin{bmatrix} -\delta & 2\delta \\ \delta & -2\delta \end{bmatrix} \mathbf{x}(t), \quad \mathbf{y} \in \mathbb{R}^2$$

where α, β, γ, and δ are real positive constants, and

$$\frac{\alpha}{\beta} = \frac{3}{4}.$$

Which of the following statements is correct?

(a) The system is controllable but not observable.

(b) The system is not controllable but observable.

(c) The system is neither controllable nor observable.

(d) The system is controllable and observable.

(e) The system is asymptotically stable.

(f) I do not know.

3. Consider the 4^{th} order LTI discrete time system

$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{F}\mathbf{x}(k) + \mathbf{G}\mathbf{u}(k), \quad \mathbf{x} \in \mathbb{R}^4, \mathbf{u} \in \mathbb{R}^2 \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k), \quad \mathbf{y} \in \mathbb{R}^2\end{aligned}$$

Assume that the reachable subspace has dimension 3

$$\dim(\mathcal{R}) = 3,$$

the observable subspace has dimension 2

$$\dim(\mathcal{O}) = 2,$$

and that the intersection of the two subspaces has dimension 1

$$\dim(\mathcal{R} \cap \mathcal{O}) = 1.$$

Which of the following are the state and output responses? ($\mathbf{x}_0$ is the system initial condition at time $k = 0$)

(a)

$$\begin{aligned}\mathbf{x}(k) = {} & \left(\mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0\lambda_1^k\right)\mathbf{v}_1 + \left(\mathbf{w}_2^{\mathrm{T}}\mathbf{x}_0\lambda_2^k\right)\mathbf{v}_2 + \left(\mathbf{w}_3^{\mathrm{T}}\mathbf{x}_0\lambda_3^k\right)\mathbf{v}_3 + \left(\mathbf{w}_4^{\mathrm{T}}\mathbf{x}_0\lambda_4^k\right)\mathbf{v}_4 \\ & + \left(\sum_{i=0}^{k-1}\lambda_1^{k-1+i}\mathbf{w}_1^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_1 + \left(\sum_{i=0}^{k-1}\lambda_2^{k-1+i}\mathbf{w}_2^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_2 \\ & + \left(\sum_{i=0}^{k-1}\lambda_3^{k-1+i}\mathbf{w}_3^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_3 + \left(\sum_{i=0}^{k-1}\lambda_4^{k-1+i}\mathbf{w}_4^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_4 \\ \mathbf{y}(k) = {} & \left(\mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0\lambda_1^k\right)\mathbf{C}\mathbf{v}_1 + \left(\mathbf{w}_2^{\mathrm{T}}\mathbf{x}_0\lambda_2^k\right)\mathbf{C}\mathbf{v}_2 + \left(\mathbf{w}_3^{\mathrm{T}}\mathbf{x}_0\lambda_3^k\right)\mathbf{C}\mathbf{v}_3 + \left(\mathbf{w}_4^{\mathrm{T}}\mathbf{x}_0\lambda_4^k\right)\mathbf{C}\mathbf{v}_4 \\ & + \left(\sum_{i=0}^{k-1}\lambda_1^{k-1+i}\mathbf{w}_1^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{C}\mathbf{v}_1 + \left(\sum_{i=0}^{k-1}\lambda_2^{k-1+i}\mathbf{w}_2^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{C}\mathbf{v}_2 \\ & + \left(\sum_{i=0}^{k-1}\lambda_3^{k-1+i}\mathbf{w}_3^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{C}\mathbf{v}_3 + \left(\sum_{i=0}^{k-1}\lambda_4^{k-1+i}\mathbf{w}_4^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{C}\mathbf{v}_4\end{aligned}$$

(b)

$$\begin{aligned}\mathbf{x}(k) = {} & \left(\mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0\lambda_1^k\right)\mathbf{v}_1 + \left(\mathbf{w}_2^{\mathrm{T}}\mathbf{x}_0\lambda_2^k\right)\mathbf{v}_2 + \left(\mathbf{w}_3^{\mathrm{T}}\mathbf{x}_0\lambda_3^k\right)\mathbf{v}_3 + \left(\mathbf{w}_4^{\mathrm{T}}\mathbf{x}_0\lambda_4^k\right)\mathbf{v}_4 \\ & + \left(\sum_{i=0}^{k-1}\lambda_1^{k-1+i}\mathbf{w}_1^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_1 + \left(\sum_{i=0}^{k-1}\lambda_3^{k-1+i}\mathbf{w}_3^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_3 \\ & + \left(\sum_{i=0}^{k-1}\lambda_4^{k-1+i}\mathbf{w}_4^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_4 \\ \mathbf{y}(k) = {} & \left(\mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0\lambda_1^k\right)\mathbf{C}\mathbf{v}_1 + \left(\mathbf{w}_2^{\mathrm{T}}\mathbf{x}_0\lambda_2^k\right)\mathbf{C}\mathbf{v}_2 + \left(\mathbf{w}_3^{\mathrm{T}}\mathbf{x}_0\lambda_3^k\right)\mathbf{C}\mathbf{v}_3 \\ & + \left(\sum_{i=0}^{k-1}\lambda_1^{k-1+i}\mathbf{w}_1^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{C}\mathbf{v}_1 + \left(\sum_{i=0}^{k-1}\lambda_3^{k-1+i}\mathbf{w}_3^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_3\end{aligned}$$

(c)

$$\begin{aligned}
\mathbf{x}(k) &= \left(\mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0\lambda_1^k\right)\mathbf{v}_1 + \left(\mathbf{w}_2^{\mathrm{T}}\mathbf{x}_0\lambda_2^k\right)\mathbf{v}_2 + \left(\mathbf{w}_3^{\mathrm{T}}\mathbf{x}_0\lambda_3^k\right)\mathbf{v}_3 + \left(\mathbf{w}_4^{\mathrm{T}}\mathbf{x}_0\lambda_4^k\right)\mathbf{v}_4 \\
&\quad + \left(\sum_{i=0}^{k-1}\lambda_2^{k-1+i}\mathbf{w}_2^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_2 + \left(\sum_{i=0}^{k-1}\lambda_3^{k-1+i}\mathbf{w}_3^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_3 \\
&\quad + \left(\sum_{i=0}^{k-1}\lambda_4^{k-1+i}\mathbf{w}_4^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_4 \\
\mathbf{y}(k) &= \left(\mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0\lambda_1^k\right)\mathbf{C}\mathbf{v}_1 + \left(\mathbf{w}_2^{\mathrm{T}}\mathbf{x}_0\lambda_2^k\right)\mathbf{C}\mathbf{v}_2 + \left(\sum_{i=0}^{k-1}\lambda_2^{k-1+i}\mathbf{w}_2^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{C}\mathbf{v}_2
\end{aligned}$$

(d)

$$\begin{aligned}
\mathbf{x}(k) &= \left(\mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0\lambda_1^k\right)\mathbf{v}_1 + \left(\mathbf{w}_2^{\mathrm{T}}\mathbf{x}_0\lambda_2^k\right)\mathbf{v}_2 + \left(\mathbf{w}_3^{\mathrm{T}}\mathbf{x}_0\lambda_3^k\right)\mathbf{v}_3 + \left(\mathbf{w}_4^{\mathrm{T}}\mathbf{x}_0\lambda_4^k\right)\mathbf{v}_4 \\
\mathbf{y}(k) &= \mathbf{0}
\end{aligned}$$

(e)

$$\begin{aligned}
\mathbf{x}(k) &= \left(\mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0\lambda_1^k\right)\mathbf{v}_1 + \left(\mathbf{w}_2^{\mathrm{T}}\mathbf{x}_0\lambda_2^k\right)\mathbf{v}_2 + \left(\mathbf{w}_3^{\mathrm{T}}\mathbf{x}_0\lambda_3^k\right)\mathbf{v}_3 + \left(\mathbf{w}_4^{\mathrm{T}}\mathbf{x}_0\lambda_4^k\right)\mathbf{v}_4 \\
&\quad + \left(\sum_{i=0}^{k-1}\lambda_1^{k-1+i}\mathbf{w}_1^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_1 + \left(\sum_{i=0}^{k-1}\lambda_3^{k-1+i}\mathbf{w}_3^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_3 \\
&\quad + \left(\sum_{i=0}^{k-1}\lambda_4^{k-1+i}\mathbf{w}_4^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_4 \\
\mathbf{y}(k) &= \left(\mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0\lambda_1^k\right)\mathbf{C}\mathbf{v}_1 + \left(\mathbf{w}_3^{\mathrm{T}}\mathbf{x}_0\lambda_3^k\right)\mathbf{C}\mathbf{v}_3 + \left(\mathbf{w}_4^{\mathrm{T}}\mathbf{x}_0\lambda_4^k\right)\mathbf{C}\mathbf{v}_4 \\
&\quad + \left(\sum_{i=0}^{k-1}\lambda_1^{k-1+i}\mathbf{w}_1^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{C}\mathbf{v}_1 + \left(\sum_{i=0}^{k-1}\lambda_3^{k-1+i}\mathbf{w}_3^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{C}\mathbf{v}_3 \\
&\quad + \left(\sum_{i=0}^{k-1}\lambda_4^{k-1+i}\mathbf{w}_4^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{C}\mathbf{v}_4
\end{aligned}$$

(f) I do not know

4. Consider a n-th order controllable but not observable LTI SISO system

$$\sum_x : \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, & \mathbf{x} \in \mathbb{R}^n, u \in \mathbb{R} \\ y = \mathbf{C}\mathbf{x}, & y \in \mathbb{R} \end{cases}$$

The full state feedback controller

$$u(t) = -\mathbf{K}_x \mathbf{x}(t)$$
$$\mathbf{K}_x = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix} \mathbf{M}_{c,x}^{-1} P_{ch,\mathbf{A_K}}(\mathbf{A})$$

is designed to regulate the system. What is the observer for the dual system $\sum_z$ associated with $\sum_x$?

(a)

$$\dot{\hat{\mathbf{z}}}(t) = \mathbf{A}^{\mathrm{T}}\hat{\mathbf{z}}(t) + \mathbf{C}^{\mathrm{T}}u(t) + \mathbf{L}_z(y(t) - \hat{y}(t))$$
$$\hat{y}(t) = \mathbf{B}^{\mathrm{T}}\hat{\mathbf{z}}(t)$$
$$\mathbf{L}_z = P_{ch,\mathbf{A_K}}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}})\left(\mathbf{M}_{c,x}^{-1}\right)^{\mathrm{T}} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

(b)

$$\dot{\hat{\mathbf{z}}}(t) = \mathbf{A}\hat{\mathbf{z}}(t) + \mathbf{B}u(t) + \mathbf{L}_z(y(t) - \hat{y}(t))$$
$$\hat{y}(t) = \mathbf{C}\hat{\mathbf{z}}(t)$$
$$\mathbf{L}_z = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix} \mathbf{M}_{c,x}^{-1} P_{ch,\mathbf{A_K}}(\mathbf{A})$$

(c)

$$\dot{\hat{\mathbf{z}}}(t) = \mathbf{A}^{\mathrm{T}}\hat{\mathbf{z}}(t) + \mathbf{C}^{\mathrm{T}}u(t) + \mathbf{L}_z(y(t) - \hat{y}(t))$$
$$\hat{y}(t) = \mathbf{B}^{\mathrm{T}}\hat{\mathbf{z}}(t)$$
$$\mathbf{L}_z = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix} \mathbf{M}_{c,x}^{-1} P_{ch,\mathbf{A_K}}(\mathbf{A})$$

(d)

$$\dot{\hat{\mathbf{z}}}(t) = \mathbf{A}^{\mathrm{T}}\hat{\mathbf{z}}(t) + \mathbf{C}^{\mathrm{T}}u(t) + \mathbf{L}_z(y(t) - \hat{y}(t))$$
$$\hat{y}(t) = \mathbf{B}^{\mathrm{T}}\hat{\mathbf{z}}(t)$$
$$\mathbf{L}_z = P_{ch,\mathbf{A_K}}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}})\left(\mathbf{M}_{o,x}^{-1}\right)^{\mathrm{T}} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

(e) It is not possible to design an observer for the dual system $\sum_z$ because the original system $\sum_x$ is not observable.

(f) I do not know.

5. Consider the 2nd order LTI continuous time system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -\alpha & \beta \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ \gamma \end{bmatrix} u(t), \quad \mathbf{x} \in \mathbb{R}^2, u \in \mathbb{R}$$
$$y(t) = \begin{bmatrix} 0 & \delta \end{bmatrix} \mathbf{x}(t), \quad y \in \mathbb{R}$$

Assume that

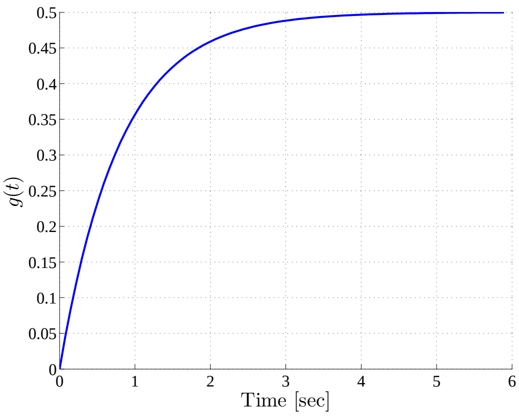
$$\alpha > 0,\ \gamma > 0,\ \delta < 0$$

and that

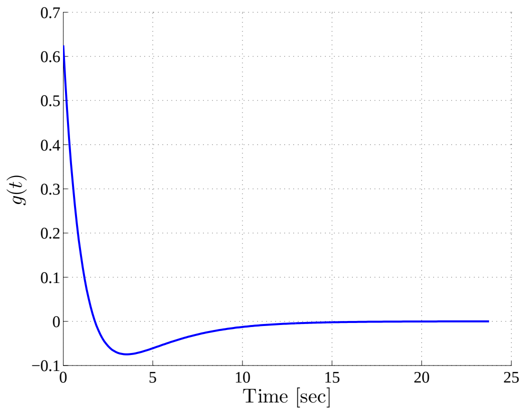
$$\beta < -2\alpha \ \vee \ \beta > 2\alpha.$$

Which of the following plots shows the impulse response associated with the system?

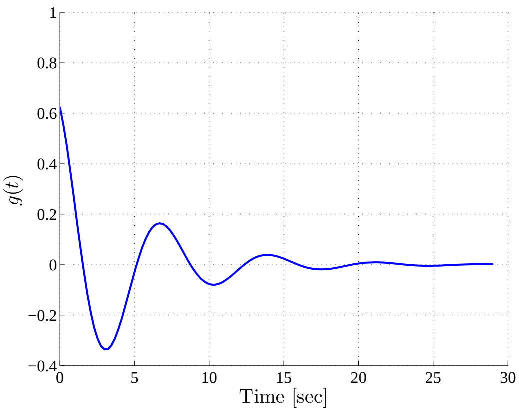
(f) I do not know.



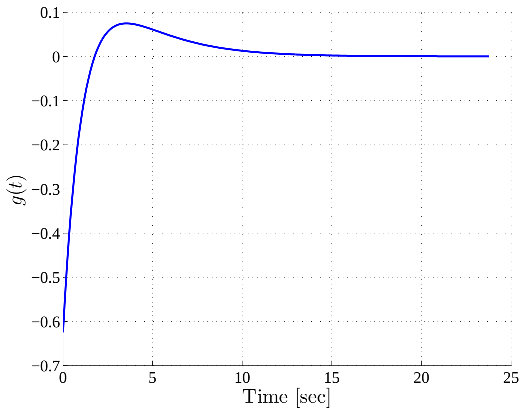
(a)



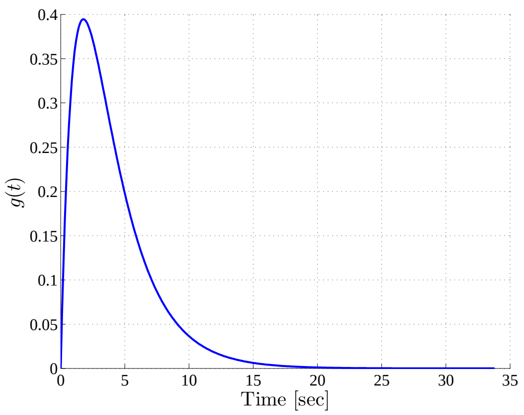
(b)



(c)



(d)



(e)

6. Consider the 3rd order LTI asymptotically stable continuous time system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -\alpha & \beta & -\beta \\ \gamma & -2\alpha & 0 \\ 0 & \delta & -\varepsilon \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ -\rho \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ \eta \\ 0 \end{bmatrix} d(t), \quad \mathbf{x} \in \mathbb{R}^3, u \in \mathbb{R}, d \in \mathbb{R}$$
$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t), \quad y \in \mathbb{R}$$

with initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, where u is the control input and d is an unknown disturbance. The constants α, β, γ, δ, ε, ρ, and η are positive. A continuous time reduced order observer is designed to estimate the unmeasured states x_2 and x_3. Assuming that

$$\mathbf{x} = \begin{bmatrix} x_1 & \mathbf{x}_2 \end{bmatrix}^{\mathrm{T}}$$

then the dynamics of the reduced order observer is given by

$$\begin{aligned} \dot{\mathbf{z}}(t) &= \mathbf{M}\mathbf{z}(t) + \mathbf{N}u(t) + \mathbf{P}y(t) \\ \hat{\mathbf{x}}_2(t) &= \mathbf{z}(t) + \mathbf{L}y(t) \end{aligned}$$

with $\mathbf{z}(t_0)$ such that $\hat{\mathbf{x}}_2(t_0) = \mathbf{x}_2(t_0)$. Let $\mathbf{e}(t) = \mathbf{x}_2(t) - \hat{\mathbf{x}}_2(t)$ be the estimation error. If

$$d(t) = \begin{cases} 0, & 0 \le t < t_1 \\ d_0, & t \ge t_1 \end{cases}$$

is the disturbance acting on the system, what is the behavior of the estimation error $\mathbf{e}(t)$?

(a)
$$\mathbf{e}(t) = \mathbf{0} \text{ for } 0 \le t < t_1 \text{ and } \lim_{t\to+\infty} \mathbf{e}(t) = 0$$

(b)
$$\mathbf{e}(t) = \bar{\mathbf{e}} < \infty \text{ for } 0 \le t < t_1 \text{ and } \lim_{t\to+\infty} \mathbf{e}(t) = 0$$

(c)
$$\mathbf{e}(t) = \mathbf{0} \text{ for } 0 \le t < t_1 \text{ and } \lim_{t\to+\infty} \mathbf{e}(t) = +\infty$$

(d)
$$\mathbf{e}(t) = \mathbf{0} \text{ for } 0 \le t < t_1 \text{ and } \lim_{t\to+\infty} \mathbf{e}(t) = \bar{\mathbf{e}} < \infty$$

(e)
$$\mathbf{e}(t) = \bar{\mathbf{e}} < \infty \text{ for } 0 \le t < t_1 \text{ and } \lim_{t\to+\infty} \mathbf{e}(t) = \bar{\mathbf{e}} < \infty$$

(f) I do not know

7. Consider a 3rd order LTI discrete time SISO system

$$\sum_k : \begin{cases} \mathbf{x}(k+1) = \begin{bmatrix} \mathbf{F}_1 & \mathbf{0} \\ \mathbf{0} & F_2 \end{bmatrix} \mathbf{x}(k) + \mathbf{G}u(k), & \mathbf{x} \in \mathbb{R}^3,\ u \in \mathbb{R} \\ y(k) = \mathbf{C}\mathbf{x}(k), \quad y \in \mathbb{R} \end{cases}$$

where $\mathbf{F}_1$ is a 2×2 matrix and F_2 is a scalar. The zero input solution of the system to the initial condition $\mathbf{x}_0 = [x_{10}, x_{20}, x_{30}]^T$ with $x_{10} > 0$, $x_{20} > 0$, and $x_{30} < 0$ is shown in Fig. 2 (the sampling time is $T_s = 0.1\,\text{s}$).

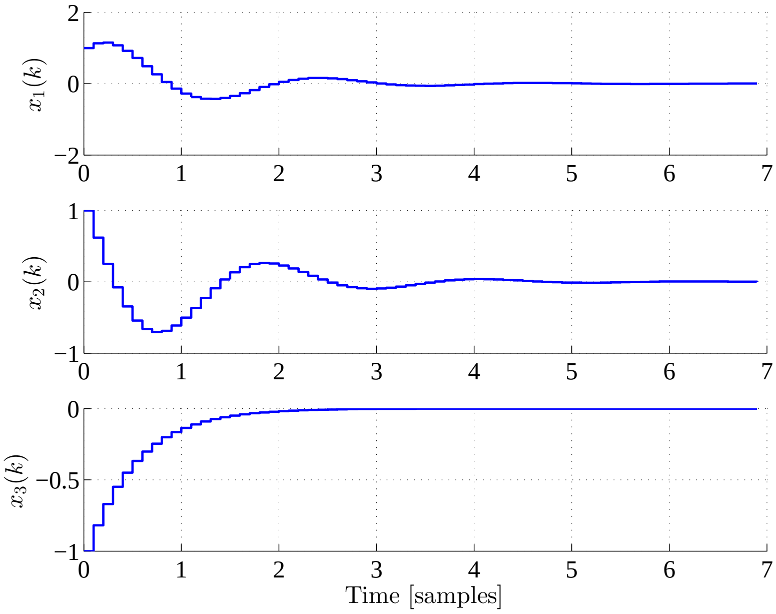


Figure 2: Zero input response

Which of the following statements is **not correct**?

(a) The system is asymptotically stable.

(b) The system has one real eigenvalue whose magnitude is less than one.

(c) The continuous time system associated with $\sum_k$ has a pair of natural modes which oscillate with frequency

$$\omega_n = \frac{|\ln(\lambda_1(\mathbf{F}_1))|}{T_s}$$

and asymptotically decay to zero ($\lambda_1(\mathbf{F}_1)$ is one of the eigenvalues of the submatrix $\mathbf{F}_1$).

(d) The natural mode associated with the dynamics of the state x_3 is unstable.

(e) The eigenvalues of the system have magnitude less than one.

(f) I do not know.

8. Consider the 4^{th} order LTI continuous time system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & -2\zeta_1\omega_1 & \gamma & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_2^2 & -2\zeta_2\omega_2 \end{bmatrix} \mathbf{x}(t), \quad \mathbf{x} \in \mathbb{R}^4$$

where ω_1 and ω_2 are real positive constants; ζ_1, ζ_2, and γ are real constants. Which of the following statement is **not correct**?

(a) If $0 < \zeta_1 < 1$ and $0 < \zeta_2 < 1$ then the system has two pairs of asymptotically stable complex eigenvalues.

(b) If $\zeta_1 = 0$, $\zeta_2 = 0$, and $\omega_1 \neq \omega_2$ then the system has two pairs of stable imaginary eigenvalues.

(c) If $\zeta_1 = 0$, $\zeta_2 = 0$, $\omega_1 = \omega_2$ and $\gamma \neq 0$ then the system has one pair of unstable imaginary eigenvalues with algebraic multiplicity equal to two.

(d) If $\zeta_1 = 0$, $\zeta_2 = 0$, $\omega_1 = \omega_2$, and $\gamma = 0$ then the system has two pairs of stable imaginary eigenvalues.

(e) If $-1 < \zeta_1 < 0$ and $-1 < \zeta_2 < 0$ then the system has two pairs of asymptotically stable real eigenvalues.

(f) I do not know.

9. Consider the 3rd order LTI continuous time system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -\beta & \gamma & 0 \\ -\gamma & -\beta & \varepsilon_1 \\ 0 & \varepsilon_2 & \alpha \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \delta_1 & 0 \\ 0 & 0 \\ 0 & \delta_2 \end{bmatrix} \mathbf{u}(t), \quad \mathbf{x} \in \mathbb{R}^3, \mathbf{u} \in \mathbb{R}^2$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t), \quad y \in \mathbb{R}$$

where $\beta > 0$ and $\gamma > 0$. A steady state optimal controller is designed using the performance index

$$J(\mathbf{u}) = \int_0^{+\infty} \mathbf{x}^{\mathrm{T}}(t)\,\mathbf{R}_1\mathbf{x}(t) + \mathbf{u}^{\mathrm{T}}(t)\,\mathbf{R}_2\mathbf{u}(t)\ \mathrm{d}t$$

where

$$\mathbf{R}_1 = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \quad , \quad \mathbf{R}_2 = \rho \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

with $r_1 > 0$, $r_2 > 0$, and $\rho > 0$. Which of the following statements is **not correct**? ($\mathbf{P}_\infty$ is the solution of the steady state Riccati equation)

(a) The optimal control law

$$\mathbf{u}(t) = -\mathbf{R}_2^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}_\infty\mathbf{x}(t)$$

gives an asymptotically stable closed-loop system if

$$\begin{array}{lll} \alpha > 0 & \varepsilon_1 = 0 & \varepsilon_2 \neq 0 \\ r_3 = 0 & \delta_1 \neq 0 & \delta_2 = 0 \end{array}$$

(b) The optimal control law

$$\mathbf{u}(t) = -\mathbf{R}_2^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}_\infty\mathbf{x}(t)$$

gives an asymptotically stable closed-loop system if

$$\begin{array}{lll} \alpha > 0 & \varepsilon_1 \neq 0 & \varepsilon_2 = 0 \\ r_3 > 0 & \delta_1 = 0 & \delta_2 \neq 0 \end{array}$$

(c) The optimal control law

$$\mathbf{u}(t) = -\mathbf{R}_2^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}_\infty\mathbf{x}(t)$$

gives an asymptotically stable closed-loop system if

$$\begin{array}{lll} \alpha > 0 & \varepsilon_1 = 0 & \varepsilon_2 = 0 \\ r_3 = 0 & \delta_1 \neq 0 & \delta_2 \neq 0 \end{array}$$

(d) The optimal control law

$$\mathbf{u}(t) = -\mathbf{R}_2^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}_\infty\mathbf{x}(t)$$

gives an asymptotically stable closed-loop system if

$$\begin{array}{lll} \alpha > 0 & \varepsilon_1 \neq 0 & \varepsilon_2 = 0 \\ r_3 = 0 & \delta_1 \neq 0 & \delta_2 = 0 \end{array}$$

(e) The optimal control law

$$\mathbf{u}(t) = -\mathbf{R}_2^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}_\infty\mathbf{x}(t)$$

gives an asymptotically stable closed-loop system if

$$\begin{array}{lll} \alpha < 0 & \varepsilon_1 = 0 & \varepsilon_2 = 0 \\ r_3 = 0 & \delta_1 \neq 0 & \delta_2 = 0 \end{array}$$

(f) I do not know.

10. Consider an asymptotically stable, minimum phase LTI SISO system with transfer function $G(s)$

$$Y(s) = G(s)\, U(s)$$

where $U(s)$ and $Y(s)$ are the Laplace transforms of the input signal $u(t)$ and the output signal $y(t)$, respectively. The input $u(t)$ is a wide sense stationary white noise process with noise intensity equal to σ_u^2. The power spectral density $S_y(\omega)$ of the output $y(t)$ is

$$S_y(\omega) = \frac{\omega^2 + \alpha}{\omega^4 + \beta}\sigma_u^2$$

where α and β are real positive constants. Which of the following is the transfer function that shapes the input noise $u(t)$ into the output spectral density $S_y(\omega)$?

(a)

$$G(s) = \frac{1}{s + \sqrt{\beta}}$$

(b)

$$G(s) = \frac{s + \sqrt{\alpha}}{s^2 + \sqrt[4]{\beta}\sqrt{2}s + \sqrt{\beta}}$$

(c)

$$G(s) = \frac{s + \sqrt{\alpha}}{s^2 + \sqrt{\beta}}$$

(d)

$$G(s) = \frac{s - \sqrt{\alpha}}{s^2 + \sqrt[4]{\beta}\sqrt{2}s + \sqrt{\beta}}$$

(e)

$$G(s) = 1$$

(f) I do not know.

11. Consider the 1st order continuous time LTI system

$$\dot{x}(t) = \alpha x(t) + v_1(t)$$
$$y(t) = \beta x(t) + v_2(t)$$

where α and β are real constants. The process noise $v_1(t)$ and the measurement noise $v_2(t)$ are uncorrelated white noise sources with noise intensities σ_1^2 and σ_2^2, respectively. The steady state Kalman filter

$$\dot{\hat{x}}(t) = (\alpha - l\beta)\,\hat{x}(t) + l y(t)$$

is designed in order to reconstruct the state variable $x(t)$ based on the measurement $y(t)$. Which of the following statements is correct?

(a) The steady state Kalman gain is given by

$$l = \frac{1}{\beta}\left(\alpha + \sqrt{\alpha^2 + \beta^2 \frac{\sigma_2^2}{\sigma_1^2}}\right)$$

(b) If the process noise is zero and $\alpha > 0$ then the eigenvalue of the Kalman filter is the reflection of the eigenvalue of the system in the left-half plane.

(c) If $(\sigma_1^2/\sigma_2^2) \gg 1$ then the magnitude of the steady state Kalman gain decreases and the filter relies heavily on the measurement.

(d) If $\alpha = 0$ and the process noise is zero the steady state Kalman gain goes to infinity.

(e) If $(\sigma_1^2/\sigma_2^2) \ll 1$ then the magnitude of the steady state Kalman gain increases and the filter relies heavily on the system model.

(f) I do not know.

12. Consider the n-th order LTI system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{B}_n\mathbf{n}_1(t), \quad \mathbf{x} \in \mathbb{R}^n,\ \mathbf{u} \in \mathbb{R}^m,\ \mathbf{n}_1 \in \mathbb{R}^q$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{n}_2(t), \quad \mathbf{y} \in \mathbb{R}^p,\ \mathbf{n}_2 \in \mathbb{R}^p$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{B}_n$, and $\mathbf{C}$ are matrices with constant coefficients; $\mathbf{n}_1$ is the process noise characterized as white noise with zero mean and covariance matrix $\mathbf{\Sigma}_1$; $\mathbf{n}_2$ is the measurement noise characterized as white noise with zero mean and covariance matrix $\mathbf{\Sigma}_2$. The two noise sources are uncorrelated, that is

$$E\left\{\mathbf{n}_1(t)\,\mathbf{n}_2^{\mathrm{T}}(t)\right\} = 0.$$

The steady state Linear Quadratic Gaussian (LQG) regulator

$$\mathbf{u}(t) = -\mathbf{K}_\infty\hat{\mathbf{x}}(t),$$

where $\hat{\mathbf{x}}$ is the state estimate provided by a continuous time Kalman filter, is the *optimal linear solution* associated with the minimization of the performance index

$$J(\mathbf{u}) = E\left\{\mathbf{x}^{\mathrm{T}}(t)\,\mathbf{R}_1\mathbf{x}(\mathbf{t}) + \mathbf{u}^{\mathrm{T}}(t)\,\mathbf{R}_2\mathbf{u}(t)\right\}.$$

Which of the following statements is **not correct**?

(a) The steady state Linear Quadratic regulator and the Kalman filter, which constitute the LQG regulator, can be designed independently of each other with guaranteed overall optimality and asymptotic stability of the closed-loop system.

(b) If both $\mathbf{n}_1(t)$ and $\mathbf{n}_2(t)$ are Gaussian white noise processes, and the initial condition $\mathbf{x}_0 = \mathbf{x}(t_0)$ is Gaussian then the optimal linear solution is the optimal solution without qualification.

(c) If the control input $\mathbf{u}(t)$ is not weighted at all then the minimum of the performance index is zero, that is

$$\lim_{\mathbf{R}_2\to 0} J(\mathbf{u}) = 0$$

(d) The position of the closed-loop poles is strongly influenced by the choice of the weighting matrix $\mathbf{R}_2$ and by the covariance matrix $\mathbf{\Sigma}_2$ of the measurement noise.

(e) Although there is no measurement noise affecting the output of the system, the minimum of the performance index is still larger than zero, that is

$$\lim_{\mathbf{\Sigma}_2\to 0} J(\mathbf{u}) \geq tr\left(\mathbf{P}_\infty\mathbf{\Sigma}_1\right)$$

where $\mathbf{P}_\infty$ is the solution of the algebraic Riccati equation associated with the Linear Quadratic regulation problem.

(f) I do not know.

Linear Control Design II Fall 2013
Final Exam Solutions

The correct answers are listed in the following table:

Question	1	2	3	4	5	6	7	8	9	10	11	12
Answer	A	C	C	A	D	D	D	E	D	B	B	C

Detailed solutions of each question are given below.

1. In order to find the state space model associated with the block diagram in Fig. 1 we first need to identify the state variables. Since the block diagram has three integrators then the state of the system is a three dimensional vector whose entries are the output of the integrators, that is

$$\mathbf{x} = \begin{bmatrix} y(t) \\ \dot{y}(t) \\ I(t) \end{bmatrix}$$

Since $\mathbf{x} \in \mathbb{R}^3$ then the state space model must have 3 differential equations describing the dynamics of the magnetic suspension. Therefore answers **(c)** and **(d)** are not correct since they include 4 and 2 differential equations respectively. To choose among the remaining three options two approaches can be followed:

- to derive the differential equations and check which solution matches
- to compare the given solutions with the block diagram and find the mistakes in the solutions

Using the second approach the following can be concluded. The first differential equation must read

$$\dot{x}_1 = x_2$$

since the link between the position $y(t)$ and the velocity $\dot{y}(t)$ is just a differentiation. Based on this answer **(e)** can be discarded. By comparing answer **(a)** with answer **(b)** it is possible to notice that the only difference is represented by the term $-(\kappa/m)\,x_1$ in the differential equation for $\dot{x}_2$. However by inspecting the block diagram in Fig. 1 it is clear that there is no linear feedback of the position $y(t)$ into the acceleration $\ddot{y}(t)$. Therefore the correct answer is **(a)**.

2. To determine which statement is correct we need to check three properties: stability, controllability, and observability. To assess if the system is asymptotically stable the eigenvalues of the system matrix are to be found. The characteristic polynomial is

$$\begin{aligned} P_{ch}(\lambda) &= (\lambda+\alpha)(\lambda-\alpha) + \frac{\beta^2}{2} \\ &= \lambda^2 - \alpha^2 + \frac{\beta^2}{2} \end{aligned}$$

whose roots are

$$\lambda_{1,2} = \pm\sqrt{\alpha^2 - \frac{\beta^2}{2}}$$

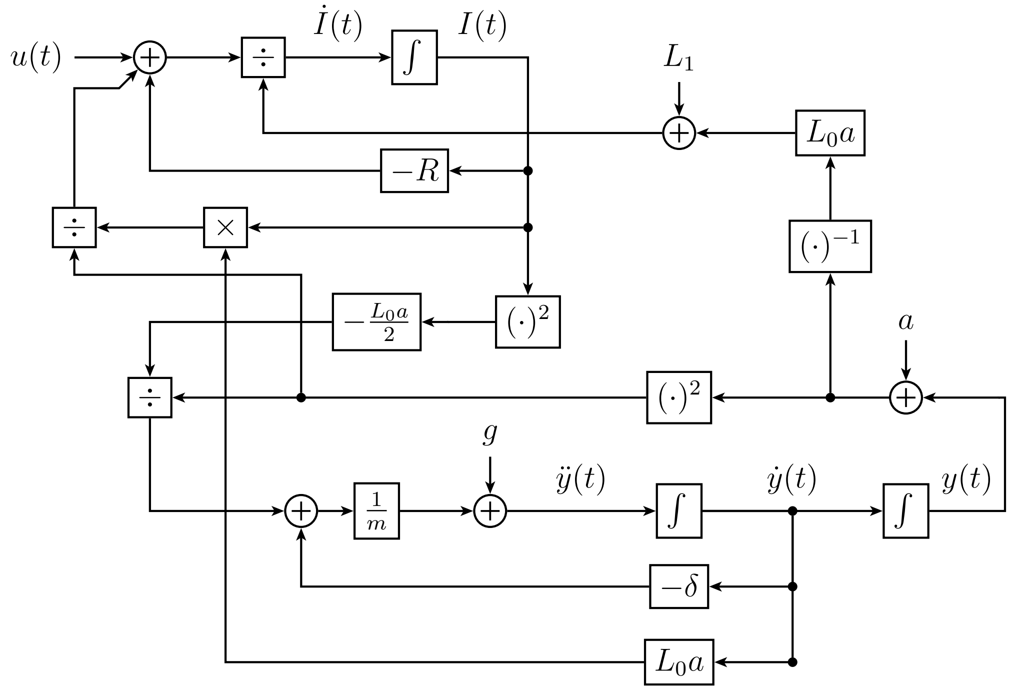


Figure 1: Block diagram of a magnetic suspension.

If $(\alpha^2 - \beta^2/2) > 0$ then both eigenvalues are real, but only one would be negative; hence the system would be unstable. Instead if $(\alpha^2 - \beta^2/2) < 0$ then both eigenvalues are imaginary; hence the system would be (marginally) stable but not asymptotically stable. Therefore answer **(e)** is not correct. For what concerns controllability and observability the controllability matrix and the observability matrix are to be computed and the respective rank conditions to be checked. The controllability matrix reads

$$\begin{aligned}\mathbf{M}_c &= \begin{bmatrix}\mathbf{B} & \mathbf{AB}\end{bmatrix}\\ &= \begin{bmatrix}\gamma & \gamma & (\beta-\alpha)\,\gamma & (\beta-\alpha)\,\gamma\\ \gamma & \gamma & \left(\alpha-\frac{\beta}{2}\right)\gamma & \left(\alpha-\frac{\beta}{2}\right)\gamma\end{bmatrix}\end{aligned}$$

The input matrix $\mathbf{B}$ provides only one linearly independent vector; hence for the system to be controllable we need to check if the third column of $\mathbf{M}_c$ is linearly independent from the vector $\mathbf{b}_1 = [\gamma, \gamma]^{\mathrm{T}}$. If the two vectors are linearly dependent then there exists $\alpha > 0$ and $\beta > 0$ such that

$$c_1 \begin{bmatrix}(\beta-\alpha)\,\gamma\\ \left(\alpha-\frac{\beta}{2}\right)\gamma\end{bmatrix} + c_2 \begin{bmatrix}\gamma\\ \gamma\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

for c_1 and c_2 different from zero. The equations read

$$\begin{aligned}c_1\,(\beta-\alpha) + c_2 &= 0\\ c_1\left(\alpha-\frac{\beta}{2}\right) + c_2 &= 0\end{aligned}$$

By adding the second equation to the first one we obtain

$$c_1\frac{\beta}{2} = -2c_2 \Rightarrow \beta = -\frac{4c_2}{c_1}$$

Inserting this value of β into the first equation we obtain

$$-c_1\alpha = 3c_2 \Rightarrow \alpha = -\frac{3c_2}{c_1}$$

Hence if $\alpha/\beta = 3/4$ then the third column of the controllability matrix $\mathbf{M}_c$ is linearly dependent on $\mathbf{b}_1 = [\gamma, \gamma]^{\mathrm{T}}$, and therefore the system is not controllable. The same procedure is applied to the observability matrix

$$\begin{aligned}\mathbf{M}_o &= \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} \\ &= \begin{bmatrix} -\delta & 2\delta \\ \delta & -2\delta \\ \alpha\delta - \beta\delta & 2\alpha\delta - \beta\delta \\ \beta\delta - \alpha\delta & \beta\delta - 2\alpha\delta \end{bmatrix}\end{aligned}$$

and by testing the linear independency of the third row to the first row the exact same conclusion about α and β is reached. Hence the system is not observable. The correct answer is then **(c)**.

3. The general forms of the state and output responses for a LTI discrete time system are (*Linear Systems Control* - Section 3.8.10)

$$\mathbf{x}(k) = \sum_{j=1}^{4} \left[\left(\mathbf{w}_j^{\mathrm{T}} \mathbf{x}_0 \lambda_j^k\right) \mathbf{v}_j + \left(\sum_{i=0}^{k-1} \lambda_j^{k-1+i} \mathbf{w}_j^{\mathrm{T}} \mathbf{G} \mathbf{u}(i) \right) \mathbf{v}_j \right]$$
$$\mathbf{y}(k) = \sum_{j=1}^{4} \left[\left(\mathbf{w}_j^{\mathrm{T}} \mathbf{x}_0 \lambda_j^k\right) \mathbf{C}\mathbf{v}_j + \left(\sum_{i=0}^{k-1} \lambda_j^{k-1+i} \mathbf{w}_j^{\mathrm{T}} \mathbf{G} \mathbf{u}(i) \right) \mathbf{C}\mathbf{v}_j \right]$$

Since the reachable subspace has dimension 3 ($\dim(\mathcal{R}) = 3$) then it means that there exists one left eigenvector $\mathbf{w}_j$ such that

$$\mathbf{w}_j^{\mathrm{T}} \mathbf{G} = \mathbf{0}$$

Moreover since the observable subspace has dimension 2 ($\dim(\mathcal{O}) = 2$) then it means that there exist two right eigenvectors $\mathbf{v}_i$ and $\mathbf{v}_j$ such that

$$\begin{aligned}\mathbf{C}\mathbf{v}_i &= \mathbf{0} \\ \mathbf{C}\mathbf{v}_j &= \mathbf{0}\end{aligned}$$

Last since the reachable and observable subspaces have one element in common ($\dim(\mathcal{R} \cap \mathcal{O}) = 1$) then it means that there exist one left eigenvector $\mathbf{w}_j$ and the corresponding right eigenvector $\mathbf{v}_j$ such that

$$\begin{aligned}\mathbf{w}_j^{\mathrm{T}} \mathbf{G} &\neq \mathbf{0} \\ \mathbf{C}\mathbf{v}_j &\neq \mathbf{0}\end{aligned}$$

The given assumptions can also be read as the system has two eigenmodes that are reachable but not observable; one eigenmode that is observable but not reachable; one eigenmode that is both reachable and observable. Based on this analysis the following can be concluded:

- answer **(a)** is not correct because all four eigenmodes are reachable and observable

$$\begin{aligned} \dim(\mathcal{R}) &= 4 \\ \dim(\mathcal{O}) &= 4 \\ \dim(\mathcal{R} \cap \mathcal{O}) &= 4 \end{aligned}$$

- answer **(b)** is not correct because three eigenmodes are reachable, three eigenmodes are observable, and two eigenmodes are both reachable and observable

$$\begin{aligned} \dim(\mathcal{R}) &= 3 \\ \dim(\mathcal{O}) &= 3 \\ \dim(\mathcal{R} \cap \mathcal{O}) &= 2 \end{aligned}$$

- answer **(c)** is correct because three eigenmodes are reachable, two eigenmodes are observable, and one eigenmode (namely λ_2^k) is both reachable and observable

$$\begin{aligned} \dim(\mathcal{R}) &= 3 \\ \dim(\mathcal{O}) &= 2 \\ \dim(\mathcal{R} \cap \mathcal{O}) &= 1 \end{aligned}$$

- answer **(d)** is not correct because no eigenmode is reachable and no eigenmmode is observable

$$\begin{aligned} \dim(\mathcal{R}) &= 0 \\ \dim(\mathcal{O}) &= 0 \\ \dim(\mathcal{R} \cap \mathcal{O}) &= 0 \end{aligned}$$

- answer **(e)** is not correct because three eigenmodes are both reachable and observable

$$\begin{aligned} \dim(\mathcal{R}) &= 3 \\ \dim(\mathcal{O}) &= 3 \\ \dim(\mathcal{R} \cap \mathcal{O}) &= 3 \end{aligned}$$

4. The dual system associated with the given system is

$$\sum\nolimits_z : \begin{cases} \dot{\mathbf{z}} = \mathbf{A}^{\mathrm{T}}\mathbf{z} + \mathbf{C}^{\mathrm{T}}u, & \mathbf{z} \in \mathbb{R}^n, u \in \mathbb{R} \\ y = \mathbf{B}^{\mathrm{T}}\mathbf{z}, & y \in \mathbb{R} \end{cases}$$

In order to identify which is the observer for the dual system we need to exploit the relationships existing between the original and the dual system (*Linear Systems Control* - Sections 3.8.9 and 4.7.2). Answer **(e)** can immediately be excluded because if the controller $\mathbf{K}_x$ can be designed for the system $\sum_x$ it means that the system is controllable; hence the dual system $\sum_z$ is observable. Answer **(b)** can also be excluded because a full order observer for an LTI system is based on the copy of the dynamics of the system to be observed, that is $\sum_z$. However, the observer given in answer **(b)** uses the dynamics of the system $\sum_x$, hence it is not correct. Answer **(c)** is not correct because the product $\mathbf{L}_z\mathbf{B}^{\mathrm{T}}$ is not defined since both the observer gain $\mathbf{L}_z$ and the output vector $\mathbf{B}^{\mathrm{T}}$ are row vectors, and hence the injection term $\mathbf{L}_z\mathbf{B}^{\mathrm{T}}(\mathbf{z}-\hat{\mathbf{z}})$ cannot be created. Answer **(d)** is also wrong because the observer gain $\mathbf{L}_z$ is computed using the transpose of the observability matrix of the original system $\sum_x$, which is the controllability matrix for the dual system $\sum_z$. So the correct answer is **(a)**, that is the observer for the dual system is built by using a copy of the dynamics of the dual system itself, and the observer gain $\mathbf{L}_z$ is found by transposing the controller gain $\mathbf{K}_x$ of the original system.

5. In order to determine which plot represent the impulse response associated with the given system first we need to compute the impulse response of the system. The impulse response is the inverse Laplace transform of the transfer function

$$
\begin{aligned}
G(s) &= \begin{bmatrix} 0 & \delta \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\alpha & \beta \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ \gamma \end{bmatrix} \\
&= \frac{\gamma\delta s}{s^2-\beta s+\alpha}
\end{aligned}
$$

hence

$$
\begin{aligned}
g(t) &= \mathcal{L}^{-1}\left(\frac{\gamma\delta s}{s^2-\beta s+\alpha}\right) \\
&= ae^{-s_1 t} - be^{-s_2 t}
\end{aligned}
$$

where

$$
\begin{aligned}
s_1 &= \frac{\beta}{2} + \sqrt{\frac{\beta^2}{4}-\alpha} \\
s_2 &= \frac{\beta}{2} - \sqrt{\frac{\beta^2}{4}-\alpha} \\
a &= \frac{s_1}{s_1-s_2}\gamma\delta \\
b &= -\frac{s_2}{s_1-s_2}\gamma\delta
\end{aligned}
$$

By assumption $\alpha > 0$ and

$$
\beta < -2\alpha \ \vee \ \beta > 2\alpha
$$

this means that the poles of $G(s)$ are both real and therefore the impulse response will not show any oscillatory behavior. If $\beta > 2\alpha$ then both poles are positive and the system will be unstable; however none of the impulse responses show a growing unbounded behavior so $\beta < -2\alpha$. Moreover

$$
\begin{aligned}
&g(t=0) = a-b = \gamma\delta < 0 \\
&\lim_{t\to\infty} g(t) = 0
\end{aligned}
$$

so the only impulse response matching these conditions is that one shown in answer **(d)**.

6. The estimation error dynamics associated with the given reduced order observer is (*Linear Systems Control* - Sections 4.9, Lecture Notes 11)

$$
\dot{\mathbf{e}}(t) = \mathbf{M}\mathbf{e}(t) + \begin{bmatrix} \eta \\ 0 \end{bmatrix} d(t)
$$

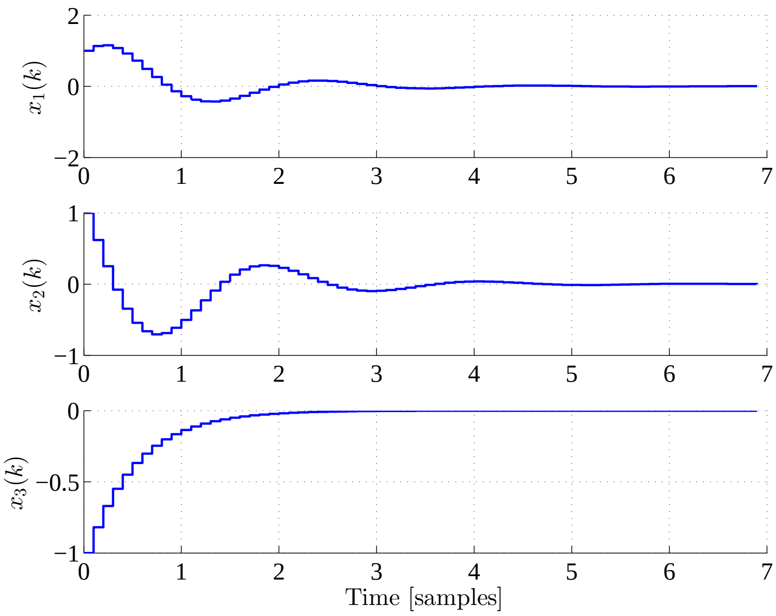


Figure 2: Zero input response

where the additional term $[\eta, 0]^{\mathrm{T}} d(t)$ is due to the fact that the disturbance is unknown and hence it cannot be included in the observer dynamics. To analytically assess the effect of a step change in $d(t)$ on the estimation error $\mathbf{e}(t)$ the transfer functions from disturbance to error can be computed, that is

$$\begin{aligned}
\mathbf{G}_{\mathbf{ed}}(s) &= \left(s\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \mathbf{M}\right)^{-1}\begin{bmatrix}\eta\\ 0\end{bmatrix}\\
&= \begin{bmatrix}\frac{\eta(s+\varepsilon-l_2\beta)}{(s+2\alpha+l_1\beta)(s+\varepsilon-l_2\beta)+l_1\beta(-\delta+l_2\beta)}\\ \frac{\eta(\delta-l_2\beta)}{(s+2\alpha+l_1\beta)(s+\varepsilon-l_2\beta)+l_1\beta(-\delta+l_2\beta)}\end{bmatrix}
\end{aligned}$$

When $d(t) = 0$, for $0 \le t < t_1$, the estimation error is obviously zero since its dynamics is asymptotically stable by design and the reduced order observer is initialized such that $\hat{\mathbf{x}}_2(t_0) = \mathbf{x}_2(t_0)$. When $d(t) = d_0$, for $t \ge t_1$, we can apply the Final Value Theorem and find that

$$\begin{aligned}
\lim_{t\to+\infty} \mathbf{e}(t) &= \lim_{s\to 0} s\mathbf{G}_{\mathbf{ed}}(s)\frac{d_0}{s} = \begin{bmatrix}\frac{d_0\eta(\varepsilon-l_2\beta)}{2\alpha\varepsilon-2\alpha\beta l_2-\beta\delta l_1+\beta\varepsilon l_1}\\ \frac{d_0\eta(\delta-l_2\beta)}{2\alpha\varepsilon-2\alpha\beta l_2-\beta\delta l_1+\beta\varepsilon l_1}\end{bmatrix}\\
&= \bar{\mathbf{e}} < \infty
\end{aligned}$$

Therefore the correct answer is **(d)**. The same conclusion could have been derived in a more straightforward way by simply considering that the disturbance is not modeled into the dynamics of the observer, hence its effect cannot be completely compensated for in the estimates (in Lecture 11 a similar situation was addressed for the 2 tanks system). This can also be seen as the problem of counteracting the action of a constant disturbance by a simple full state feedback controller (no integral action) for the dual system.

7. Based on the zero input response shown in Fig. 2 the following can be concluded:

- The system is certainly asymptotically stable since all three state variables converge to the zero value; hence answer **(a)** is correct

- Differently from $x_1(t)$ and $x_2(t)$ which converge to zero oscillating, $x_3(t)$ converges to zero as a discrete time decaying exponential, that is

$$x_3(t) = c\lambda_3^k, \quad |\lambda_3| < 1, c \in \mathbb{R}$$

  hence answer **(b)** is correct

- $x_1(t)$ and $x_2(t)$ converge to zero oscillating, that is the system dynamical matrix has a pair of complex eigenvalues, which in continuous time are characterized by the frequency

$$\omega_n = \frac{|\ln(\lambda_1(\mathbf{F}_1))|}{T_s}$$

  therefore answer **(c)** is correct

- All three eigenmodes are asymptotically stable since all states converges to zero if initialized away from zero; hence answer **(d)** is not correct

- Since all states are converging to zero, then the magnitude of the eigenvalues is certainly less than one; hence answer **(e)** is correct

8. The system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & -2\zeta_1\omega_1 & \gamma & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_2^2 & -2\zeta_2\omega_2 \end{bmatrix} \mathbf{x}(t)$$

is given by the interconnection of two second order subsystems (the upper left 2×2 block, and the bottom right 2×2 block). The stability properties of the whole system are related to the values of the damping coefficients ζ_1 and ζ_2, and to the presence/absence of the interconnection constant γ. The not correct statement is answer **(e)** since if the damping coefficients are negative ($-1 < \zeta_1 < 0$ and $-1 < \zeta_2 < 0$) then the system has two pairs of unstable complex eigenvalues.

9. The steady state optimal controller

$$\mathbf{u}(t) = -\mathbf{R}_2^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}_\infty \mathbf{x}(t)$$

is guaranteed to give an asymptotically stable closed loop system if the system dynamics

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -\beta & \gamma & 0 \\ -\gamma & -\beta & \varepsilon_1 \\ 0 & \varepsilon_2 & \alpha \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \delta_1 & 0 \\ 0 & 0 \\ 0 & \delta_2 \end{bmatrix} \mathbf{u}(t)$$

is stabilizable and the pair $\left(\mathbf{A}, \sqrt{\mathbf{R}_1}\right)$ is detectable (*Linear Systems Control* - Sections 5.5). These are very broad conditions, which give the weakest requirements for obtaining an optimal controller that stabilizes the system. If the system fulfills stronger conditions such as controllability, then asymptotic stabilization is obviously guaranteed. Based on this the following can be concluded about the given statements:

- If

$$\begin{array}{lll} \alpha > 0 & \varepsilon_1 = 0 & \varepsilon_2 \neq 0 \\ r_3 = 0 & \delta_1 \neq 0 & \delta_2 = 0 \end{array}$$

then the system has an unstable mode associated with the eigenvalue $\lambda_3 = \alpha$ which is controllable through the state $x_2(t)$. Hence asymptotic stabilization can be achieved through the optimal controller.

- If
$$\begin{array}{lll} \alpha > 0 & \varepsilon_1 \neq 0 & \varepsilon_2 = 0 \\ r_3 > 0 & \delta_1 = 0 & \delta_2 \neq 0 \end{array}$$
then the system has an unstable mode that is directly controllable (the input gain $\delta_2 \neq 0$) and observable by the performance index. Hence the optimal controller asymptotically stabilizes the system dynamics.

- If
$$\begin{array}{lll} \alpha > 0 & \varepsilon_1 = 0 & \varepsilon_2 = 0 \\ r_3 = 0 & \delta_1 \neq 0 & \delta_2 \neq 0 \end{array}$$
then the system has an unstable mode that is directly controllable; hence the system can be asymptotically stabilized by the optimal controller.

- If
$$\begin{array}{lll} \alpha > 0 & \varepsilon_1 \neq 0 & \varepsilon_2 = 0 \\ r_3 = 0 & \delta_1 \neq 0 & \delta_2 = 0 \end{array}$$
then the system has an unstable mode that is not controllable (the eigenmode associated with $\lambda_3 = \alpha$ cannot be controlled neither directly, $\delta_2 = 0$, nor indirectly, $\varepsilon_2 = 0$). Hence there is no optimal control law that can asymptotically stabilizes the system.

- If
$$\begin{array}{lll} \alpha < 0 & \varepsilon_1 = 0 & \varepsilon_2 = 0 \\ r_3 = 0 & \delta_1 \neq 0 & \delta_2 = 0 \end{array}$$
then the system is stabilizable and the pair $\left(\mathbf{A}, \sqrt{\mathbf{R}_1}\right)$ is detectable; hence the optimal control law asymptotically stabilizes the system.

10. In order to determine which transfer function $G(s)$ shapes the input noise $u(t)$ into the output spectral density $S_y(\omega)$ two approaches can be taken based on the same fundamental relation (*Linear Systems Control* - Sections 6.4.2)
$$S_y(\omega) = G(j\omega)\, S_u(\omega)\, G(-j\omega)$$
where $S_u(\omega) = 1$ is the spectral density of the input $u(t)$. The first approach is based on the decomposition of $S_y(\omega)$ into the product of the two complex numbers $G(j\omega)$ and $G(-j\omega)$, that is
$$S_y(\omega) = \frac{j\omega + \sqrt{\alpha}}{-\omega^2 + \sqrt{2}\sqrt[4]{\beta}j\omega + \sqrt{\beta}} \frac{-j\omega + \sqrt{\alpha}}{-\omega^2 - \sqrt{2}\sqrt[4]{\beta}j\omega + \sqrt{\beta}}$$
hence
$$G(s) = \frac{s + \sqrt{\alpha}}{s^2 + \sqrt{2}\sqrt[4]{\beta}s + \sqrt{\beta}}$$
and answer **(b)** is correct. The second approach is instead based onto the analysis of each possible given solution. Answer **(a)** can be discarded because the transfer function has no zero in $s = -\sqrt{a}$; answer **(d)** can be discarded because $G(s)$ has a zero in $s = \sqrt{\alpha}$;

answer **(e)** can be discarded because if $G(s) = 1$ then the output signal $y(t)$ will have the same spectral density of $u(t)$. To chose which between answer **(b)** and **(c)** is the correct one the output spectrum $S_y(\omega)$ is to be computed for both cases. Starting with the easier transfer function, namely

$$G(s) = \frac{s + \sqrt{\alpha}}{s^2 + \sqrt{\beta}}$$

we find that

$$\begin{aligned} S_y(\omega) &= \frac{j\omega + \sqrt{\alpha}}{-\omega^2 + \sqrt{\beta}} \frac{-j\omega + \sqrt{\alpha}}{-\omega^2 + \sqrt{\beta}} \\ &= \frac{\omega^2 + \alpha}{\omega^4 - 2\sqrt{\beta}\omega^2 + \beta} \end{aligned}$$

which is certainly not the matching the given output spectrum. Therefore answer **(c)** is wrong and answer **(b)** is correct.

11. The steady state Kalman gain of the Kalman filter associated with the given system is

$$l = \frac{1}{\beta}\left(\alpha + \sqrt{\alpha^2 + \beta^2 \frac{\sigma_1^2}{\sigma_2^2}}\right)$$

therefore answer **(a)** is not correct. Moreover the ratio between the covariance of the process noise σ_1^2 and the covariance of the measurement noise σ_2^2 specifies how the filter works. $(\sigma_1^2/\sigma_2^2) \gg 1$ means that the measurement noise is small compared to the process noise; hence the Kalman filter will definitely use the information contained in the measurements in order to reconstruct the state. This is reflected in a large gain l. Conversely $(\sigma_1^2/\sigma_2^2) \ll 1$ means that the measurement noise is very large; therefore the filter relies much more on the model of the system in order to compute the state estimate. This is reflected in a small gain l. Answers **(c)** and **(e)** are then wrong. If $\alpha = 0$ and the process noise is zero $(\sigma_1^2 = 0)$ the Kalman gain is equal to zero, hence answer **(d)** is not correct too. The correct answer is hence **(b)** and it can be easily shown by using the steady state Kalman gain into the equation of the estimation error dynamics.

12. The correct answer is **(c)**. For details about the other answers please read Section 7.6 (pages 466 - 472) of *Linear Systems Control*.

Linear Control Design 2 E18 - Theoretical Questionnaire

Page 1

◉ Show correct answers
○ Hide correct answers

Analysis of open loop systems (Part 1)

Question 1

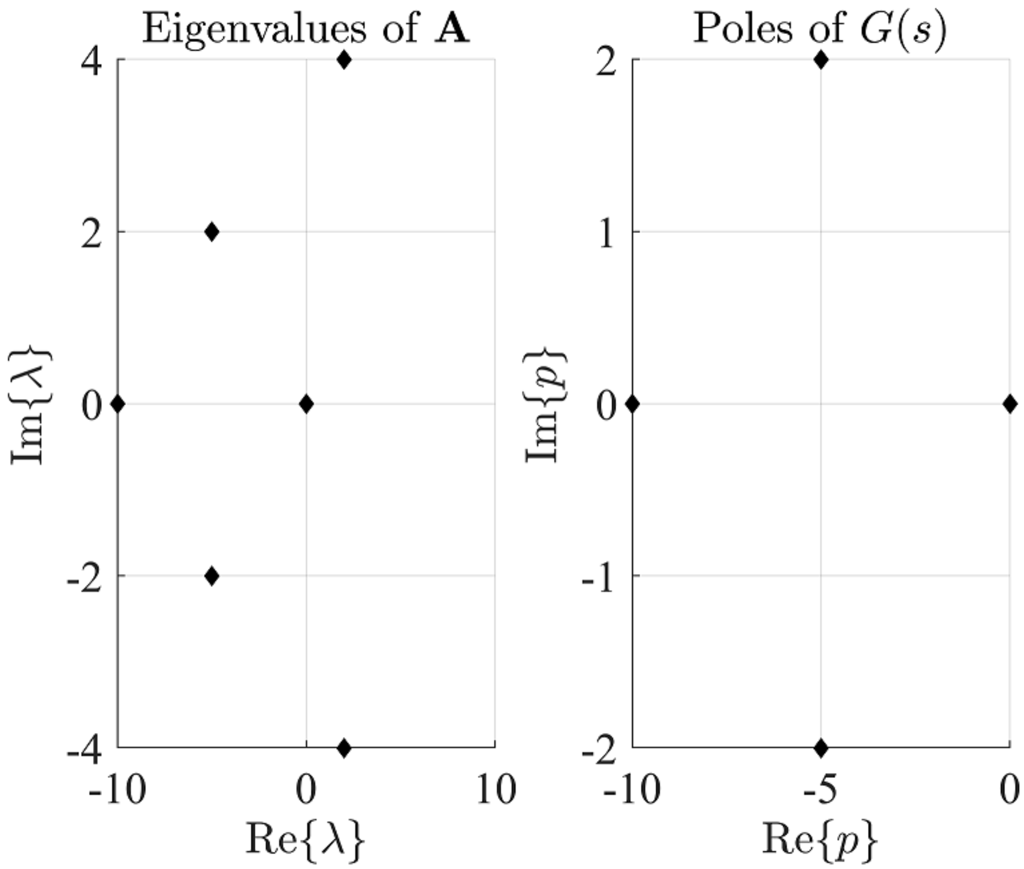
Consider the sixth order continuous time LTI SISO system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \qquad \mathbf{x} \in \mathbb{R}^n, u \in \mathbb{R}$$
$$y = \mathbf{C}\mathbf{x} \qquad y \in \mathbb{R}$$

with the associated transfer function

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

The following figure shows in the complex plane the position of the eigenvalues of the system dynamical matrix $\mathbf{A}$ as well as the position of the poles of the transfer function $G(s)$.



Based on the eigenvalues and poles maps, which of the following statements is correct?

- ☐ The system is internally unstable and BIBO stable.
- ☐ The system is internally marginally stable and not BIBO stable.
- ☐ The system is internally asymptotically stable and BIBO stable.
- ☑ The system is internally unstable and not BIBO stable.
- ☐ The system is internally marginally stable and BIBO stable.

Question 2

Consider the third order continuous time LTI system

$$\dot{\mathbf{x}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \neq \mathbf{0}$$

where λ_i are the system eigenvalues. Said c_i a constant dependent on the initial condition and $\mathbf{v}_i$ the right eigenvector associated with the eigenvalue λ_i, what is the zero-input response of the system to the given initial condition?

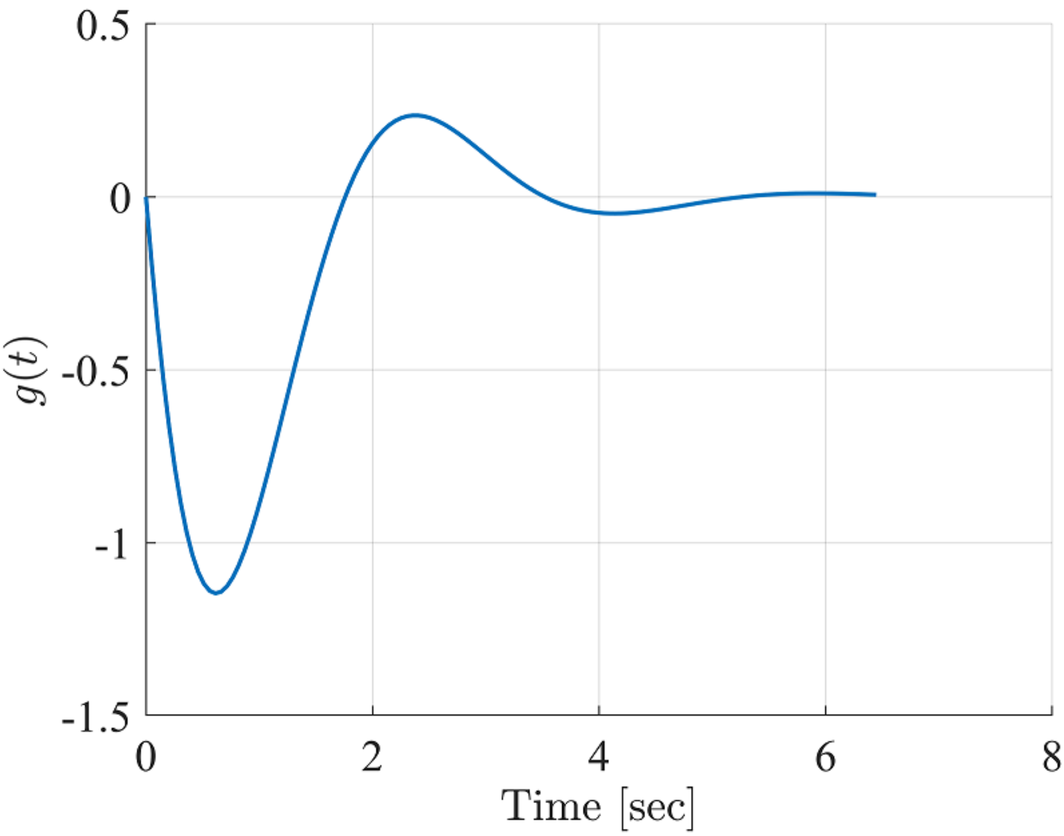
- [ ] $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_2 t}$
- [ ] $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + 2c_2\mathbf{v}_2 e^{\lambda_2 t}$
- [x] $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3(\mathbf{v}_2 t e^{\lambda_2 t} + \mathbf{v}_3 e^{\lambda_2 t})$
- [ ] $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_2 t e^{\lambda_2 t}$
- [ ] $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + (c_2\mathbf{v}_2 e^{\lambda_2 t})^2$

Question 3

The second order continuous time LTI SISO system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \begin{bmatrix} 0 \\ b \end{bmatrix} u, \qquad \mathbf{x} \in \mathbb{R}^2, u \in \mathbb{R}$$
$$y = \begin{bmatrix} c & 0 \end{bmatrix} \mathbf{x}, \qquad y \in \mathbb{R}$$

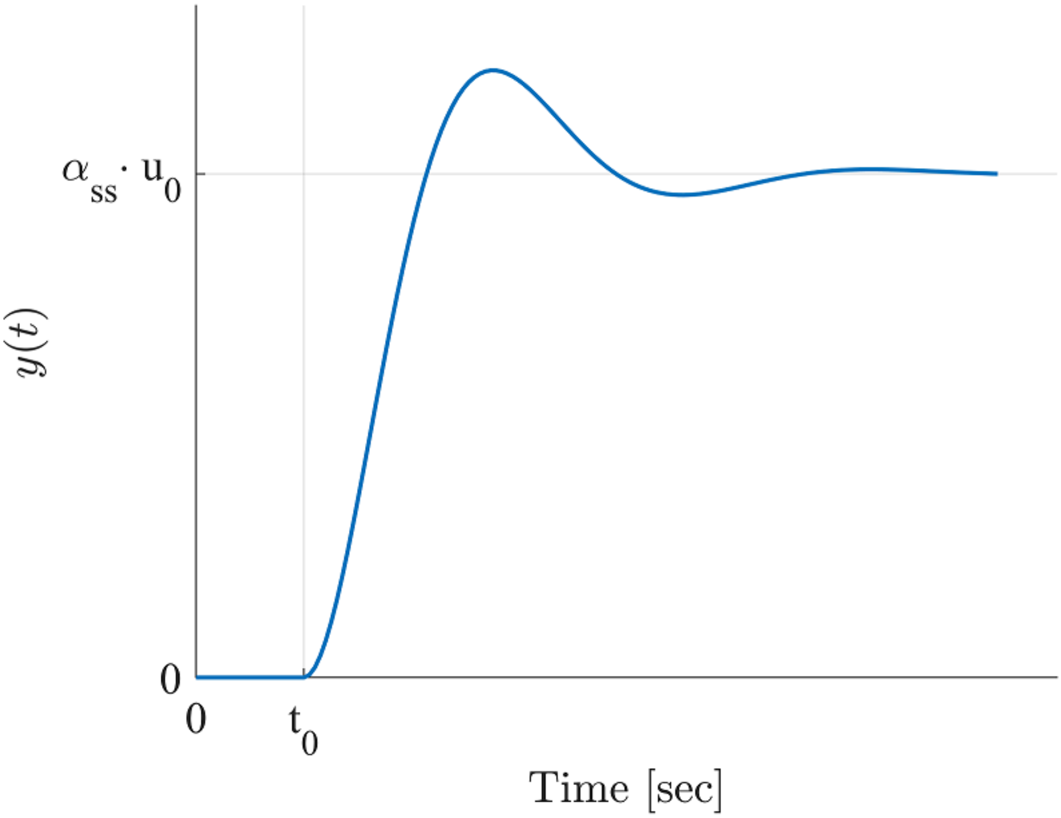
has the following impulse response function in response to a positive impulse (the coefficients b and c are real numbers different from zero)



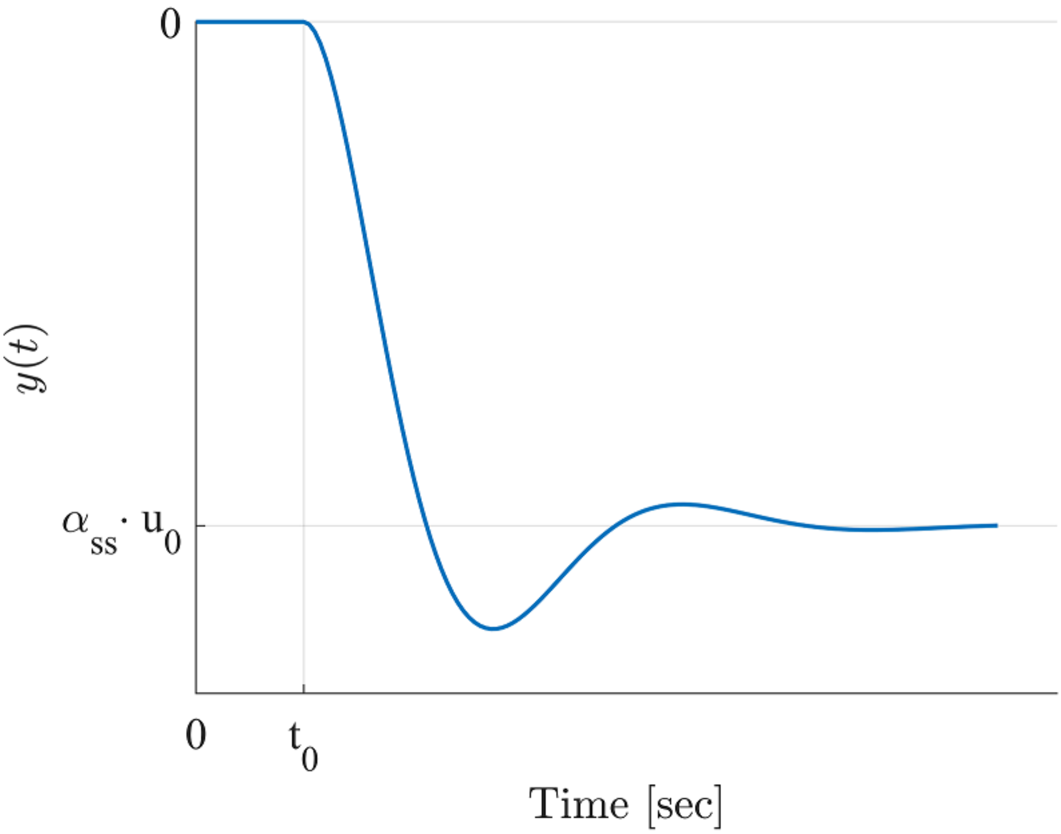
Said α_{ss} the steady state gain of the system, what is the zero-state output response of the system if the input is

$$u(t) = \begin{cases} 0 & 0 \leq t < t_0 \\ u_0 & t \geq t_0 \end{cases}$$

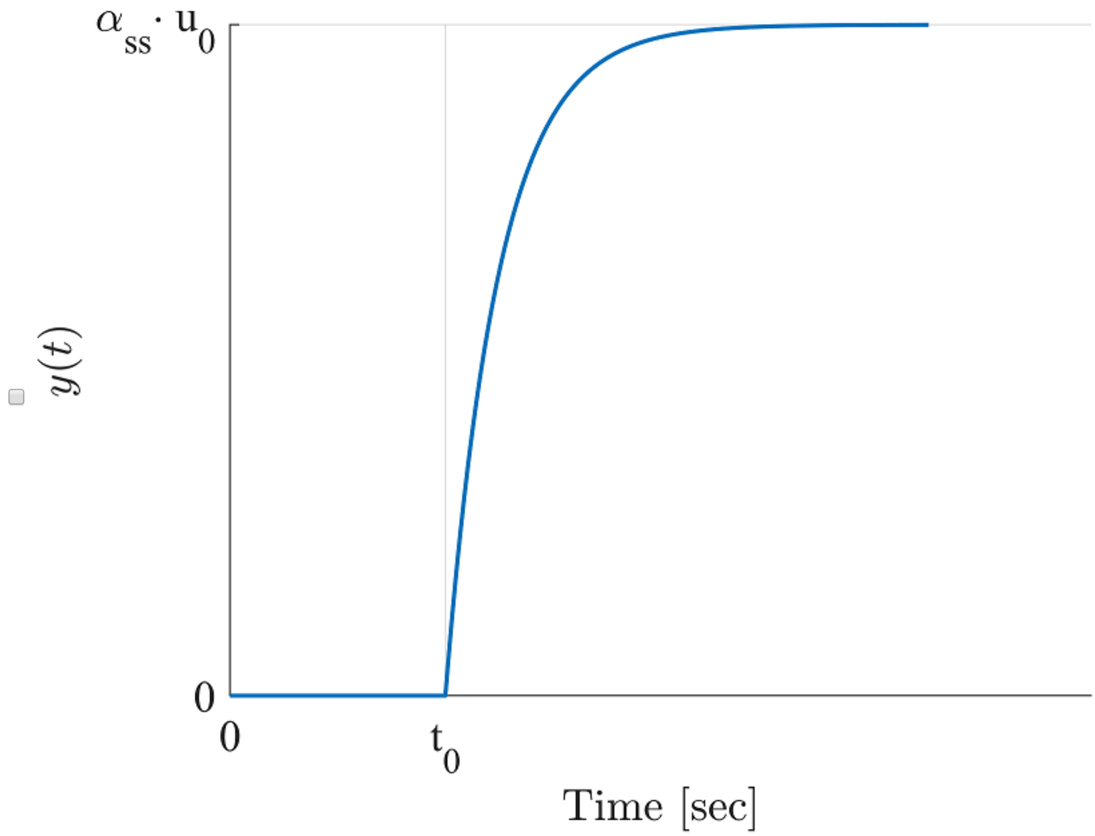
with $u_0 < 0$?

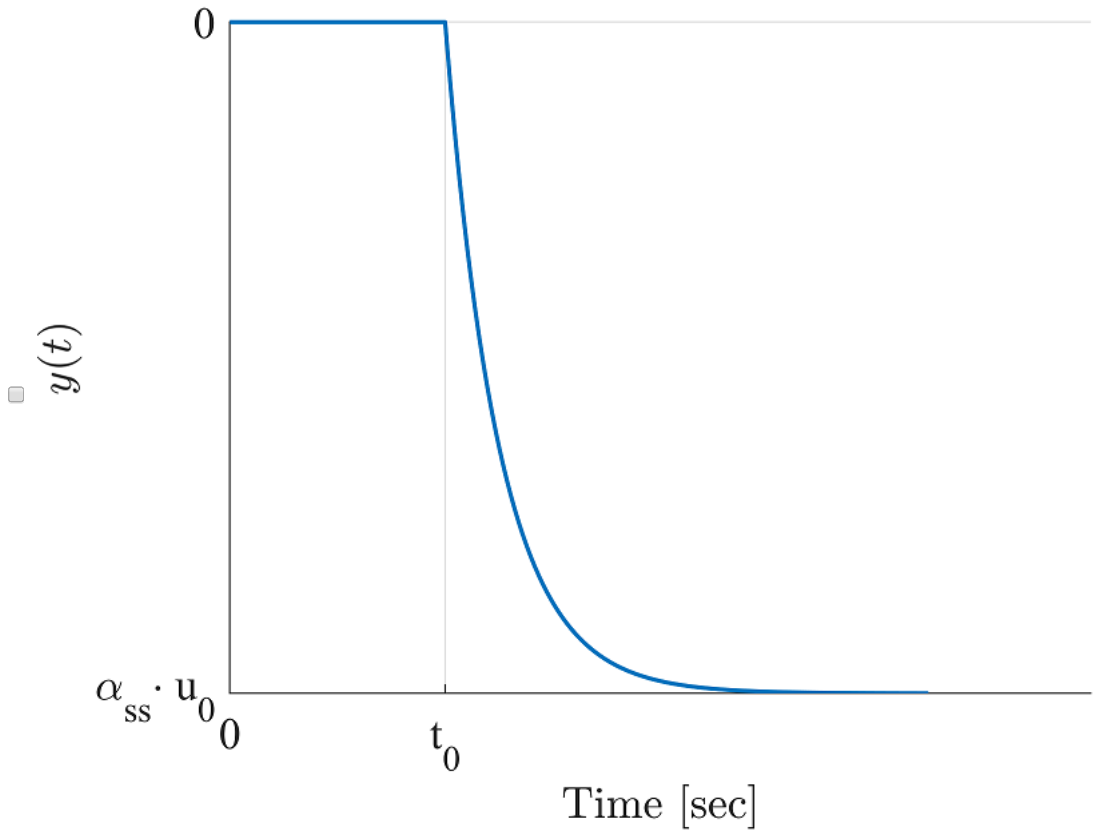
- [x] 

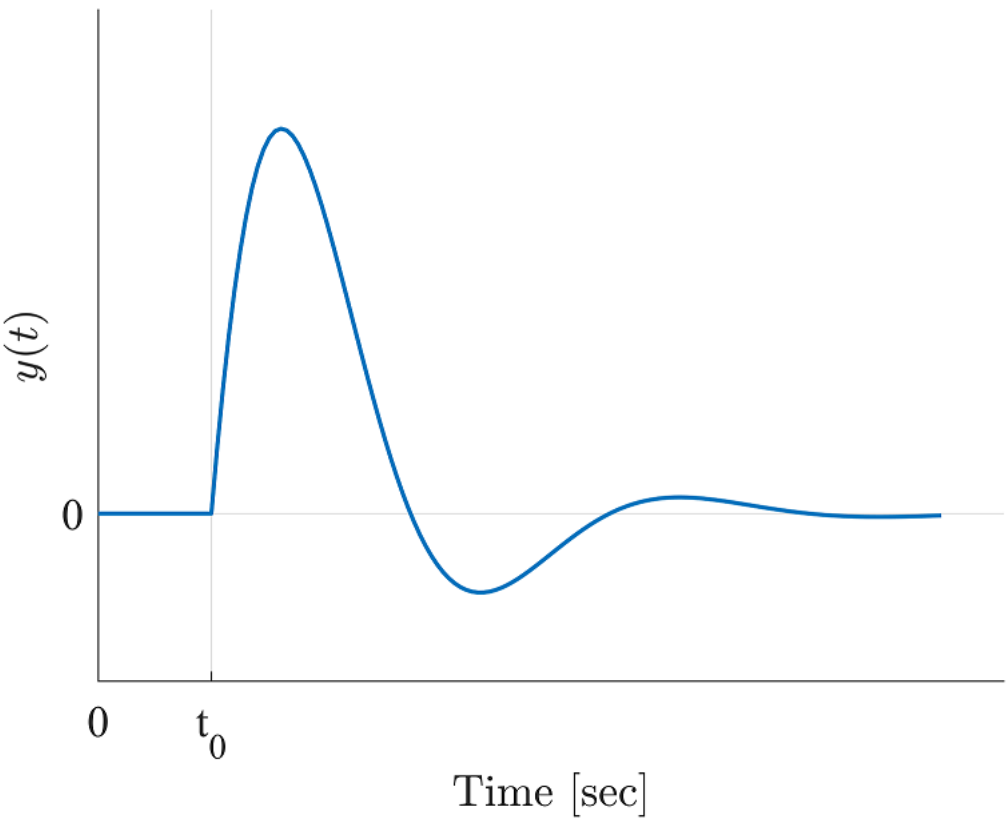
$\alpha_{ss} \cdot u_0$
$y(t)$
0
0
t_0
Time [sec]

- [ ] 

0
$y(t)$
$\alpha_{ss} \cdot u_0$
0
t_0
Time [sec]


$\alpha_{ss} \cdot u_0$
0
$y(t)$
0
t_0
Time [sec]


0
$y(t)$
$\alpha_{ss} \cdot u_0$
0
t_0
Time [sec]



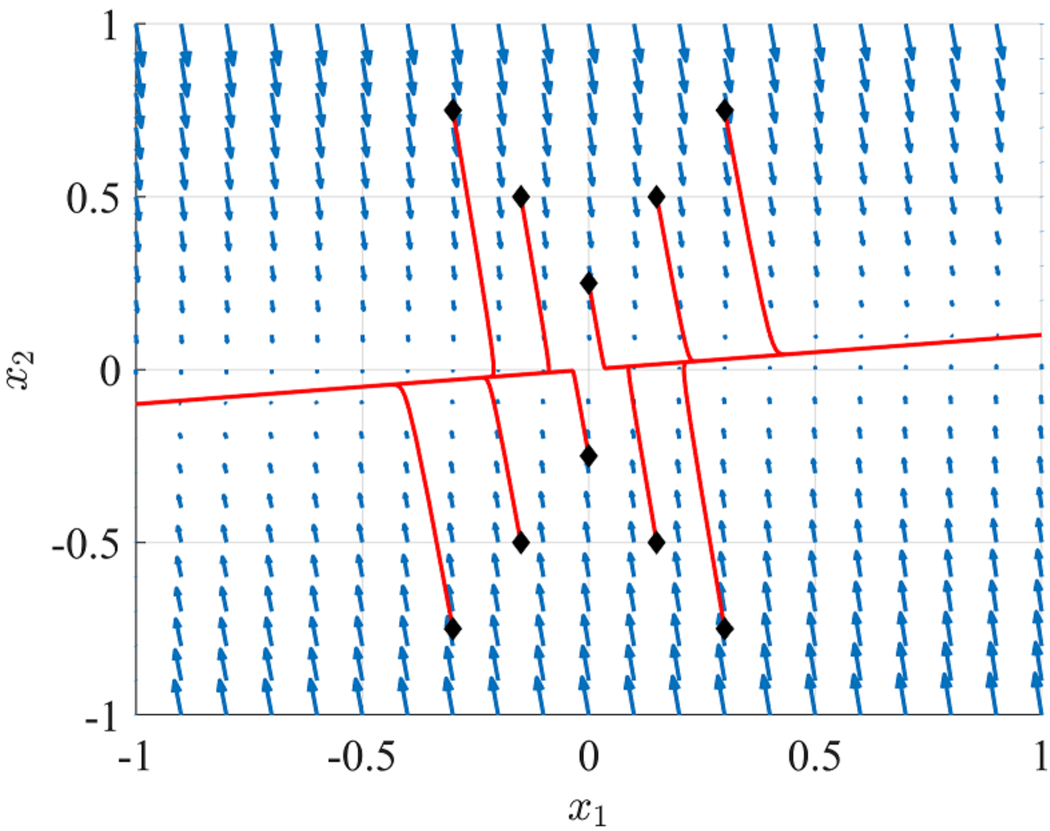
y(t)
0
0
t0
Time [sec]

Page 2

Analysis of open loop systems (Part 2)

Question 4

The phase portrait of a second order continuous time LTI system is shown in the following figure (in the given phase portraits each black diamond represents an initial condition $\mathbf{x}(0) = [x_{10}, x_{20}]^{\mathrm{T}}$ for the system; each red line is a trajectory of the system originated from the initial condition; the blue arrows represent the direction of the vector field in the neighborhood of the origin.)



Which of the following statement is correct?

- [ ] The equilibrium point is a stable node.
- [ ] The equilibrium point is an unstable focus.
- [x] The equilibrium point is a saddle point.
- [ ] The equilibrium point is an unstable node.
- [ ] The equilibrium point is a centre.

Question 5

Consider the third order discrete time LTI system

$$\mathbf{x}(k+1) = \begin{bmatrix} \alpha & \varepsilon_1 & \varepsilon_2 \\ 0 & \gamma & \beta \\ 0 & -\beta & \gamma \end{bmatrix} \mathbf{x}(k)$$

where

$$|\alpha| < 1 \wedge |\gamma \pm j\beta| = 1$$

and $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$.

Which of the following statements is **not correct**?

- [ ] Given the initial condition $\mathbf{x}(0) = [x_{10}, 0, 0]^{\mathrm{T}}$, then for $k \to +\infty$ the zero-input response converges to $\mathbf{x} = [0, 0, 0]^{\mathrm{T}}$ as α^k.
- [ ] Given the initial condition $\mathbf{x}(0) = [0, x_{20}, x_{30}]^{\mathrm{T}}$ such that $x_{20}^2 + x_{30}^2 = 1$, then for $k \to +\infty$ the zero-input response converges to $\mathbf{x} = [c_1 \cos(\beta k T_s) + c_2 \sin(\beta k T_s), \cos(\beta k T_s), \sin(\beta k T_s)]^{\mathrm{T}}$.
- [ ] Given the initial condition $\mathbf{x}(0) = [0, 0, 0]^{\mathrm{T}}$, then the zero-input response stays at $\mathbf{x} = \mathbf{x}(0)$ for all future times.
- [x] Given the initial condition $\mathbf{x}(0) = [0, x_{20}, x_{30}]^{\mathrm{T}}$ such that $x_{20}^2 + x_{30}^2 = 1$, then for $k \to +\infty$ the zero-input response converges to $\mathbf{x} = [0, \cos(\beta k T_s), \sin(\beta k T_s)]^{\mathrm{T}}$.
- [ ] Given the initial condition $\mathbf{x}(0) = [x_{10}, x_{20}, x_{30}]^{\mathrm{T}}$ such that $x_{20}^2 + x_{30}^2 = 1$, then for $k \to +\infty$ the zero-input response converges to $\mathbf{x} = [c_1 \cos(\beta k T_s) + c_2 \sin(\beta k T_s), \cos(\beta k T_s), \sin(\beta k T_s)]^{\mathrm{T}}$ as α^k.

Question 6

Consider the third order continuous time LTI SISO system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & \alpha \\ \beta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ \gamma \end{bmatrix} u, \qquad \mathbf{x} \in \mathbb{R}^3, u \in \mathbb{R}$$
$$y = \begin{bmatrix} 0 & \delta & 0 \end{bmatrix} \mathbf{x} \qquad y \in \mathbb{R}$$

where $\alpha,\ \beta,\ \gamma$ and δ belong to the set of real numbers. Which of the following statements is correct?

- [x] The system is controllable if $\alpha \neq 0\ \wedge\ \beta \neq 0\ \wedge\ \gamma \neq 0$.
- [ ] The system is controllable if $\gamma \neq 0$ and $\forall\, \alpha, \beta \in \mathbb{R}$.
- [ ] The system is controllable if $\alpha \neq 0\ \wedge\ \beta \neq 0$ and $\forall \gamma \in \mathbb{R}$.
- [ ] The system is controllable $\forall \alpha, \beta, \gamma \in \mathbb{R}$.
- [ ] The system is controllable if $\delta \neq 0$ and $\forall \alpha, \beta, \gamma \in \mathbb{R}$.

Question 7

Consider the first order continuous time LTI system

$$\dot{x} = \alpha x + \beta v$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$. The system is driven by the stochastic process v that is white noise with zero mean and noise intensity V_1. Which of the following statements is correct?

- [ ] The variance q_x of the state x is found as the solution of the steady state Lyapunov equation $2q_x\alpha + \beta^2 V_1 = 0$.
- [x] The variance q_x of the state x is found as the solution of the time-varying Lyapunov equation $\dot{q}_x = 2q_x\alpha + \beta^2 V_1$.
- [ ] The variance q_x of the state x cannot be computed because the system is unstable.
- [ ] The variance q_x of the state x equals the noise intensity V_1 since the dynamic system has no effect onto the stochastic process v.
- [ ] The variance q_x of the state x is zero because the system is unstable.

Page 3

Control system design and closed loop system analysis

Question 8

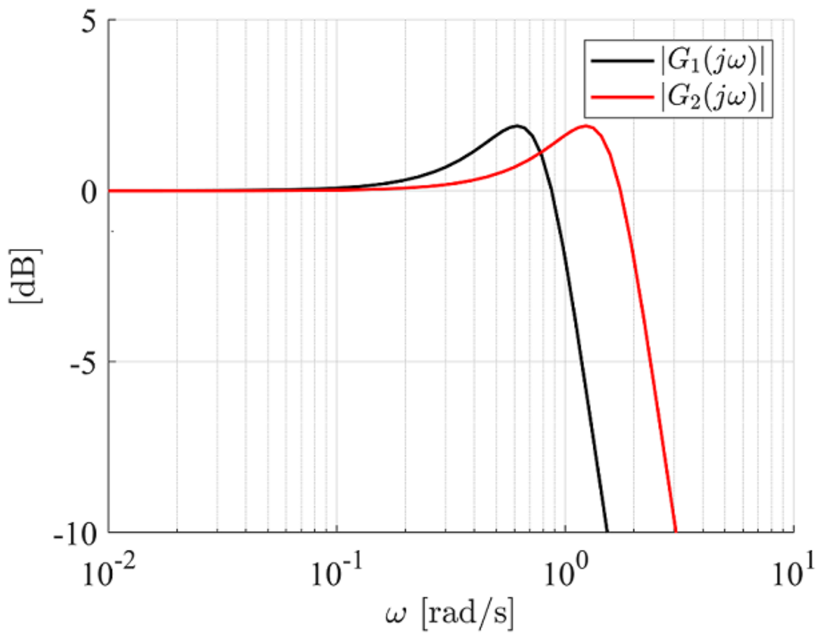
Consider the second order LTI continuous time system

$$\Sigma_x : \begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ \omega_0^2 \end{bmatrix} u(t), \quad \mathbf{x} \in \mathbb{R}^2, u \in \mathbb{R} \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t), \quad y \in \mathbb{R} \end{cases}$$

where $\omega_0 > 0$ and $0 < \zeta < 1$. The transfer function associated with open loop system reads

$$G_1(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

The figure below shows the Bode diagrams of the open loop transfer function $G_1(s)$ and the transfer function $G_2(s)$ associated with the closed-loop system.



Which of the following control architectures has been applied to the open loop system Σ_x in order to obtained the closed-loop system $G_2(s)$?

- [ ] $u(t) = -\frac{1}{\omega_0^2}\left[(\omega_1^2 - \omega_0^2) \;\; 2\zeta(\omega_1 - \omega_0)\right] \mathbf{x}$

- [x] $u(t) = -\frac{1}{\omega_0^2}\left[(\omega_1^2 - \omega_0^2) \;\; 2\zeta(\omega_1 - \omega_0)\right] \mathbf{x} + \frac{\omega_1^2}{\omega_0^2} r$

  where r is a reference signal.

- [ ] $u(t) = -\frac{1}{\omega_0^2}\left[0 \;\; 2\omega_0(\zeta_1 - \zeta)\right] \mathbf{x} + r$

  where r is a reference signal.

- [ ] $u(t) = -\frac{1}{\omega_0^2}\left[(\omega_1^2 - \omega_0^2) \;\; 2(\zeta_1\omega_1 - \zeta\omega_0)\right] \mathbf{x} + \frac{\omega_1^2}{\omega_0^2} r$

  where r is a reference signal.

- [ ] $u(t) = -\frac{1}{\omega_0^2}\left[(\omega_1^2 - \omega_0^2) \;\; 2(\zeta_1\omega_1 - \zeta\omega_0)\right] \mathbf{x} + r$

  where r is a reference signal.

Question 9

Consider the second order continuous time LTI SISO system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\alpha_1 & -\alpha_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \beta \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} + \gamma d$$

where $\alpha_1, \alpha_2, \beta$ and γ are real and positive coefficients. The signal $d(t)$ acting on the system output is an unknown constant disturbance.

Let T_s be a properly chosen sampling time and $\mathbf{e}(k) = \mathbf{x}(k) - \hat{\mathbf{x}}(k)$ be the estimation error associated with a discrete time observer. Which of the following discrete time observers achieves

$$\lim_{k \to +\infty} \mathbf{e}(k) = \mathbf{0}$$

for a step change in the disturbance $d(t)$?

- [ ] $$\hat{\mathbf{x}}(k+1) = \begin{bmatrix} 1 & T_s \\ -\alpha_1 T_s & 1 - \alpha_2 T_s \end{bmatrix} \hat{\mathbf{x}}(k) + \begin{bmatrix} \beta \frac{T_s^2}{2} \\ \beta T_s - \alpha_2 \beta \frac{T_s^2}{2} \end{bmatrix} u(k) + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} (y(k) - \hat{y}(k))$$
$$\hat{y}(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{\mathbf{x}}(k)$$

- [ ] $$\hat{\mathbf{x}}(k+1) = \begin{bmatrix} 1 & T_s \\ -\alpha_1 T_s & 1 - \alpha_2 T_s \end{bmatrix} \hat{\mathbf{x}}(k) + \begin{bmatrix} \beta \frac{T_s^2}{2} \\ \beta T_s - \alpha_2 \beta \frac{T_s^2}{2} \end{bmatrix} u(k) + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} (y(k) - \hat{y}(k))$$
$$\hat{y}(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{\mathbf{x}}(k) + \gamma d(k)$$

- [x] $$\hat{\mathbf{x}}_a(k+1) = \begin{bmatrix} 1 & T_s & 0 \\ -\alpha_1 T_s & 1 - \alpha_2 T_s & 0 \\ 0 & 0 & 1 \end{bmatrix} \hat{\mathbf{x}}_a(k) + \begin{bmatrix} \beta \frac{T_s^2}{2} \\ \beta T_s - \alpha_2 \beta \frac{T_s^2}{2} \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} (y(k) - \hat{y}(k))$$
$$\hat{y}(k) = \begin{bmatrix} 1 & 0 & \gamma \end{bmatrix} \hat{\mathbf{x}}_a(k)$$
where $\hat{\mathbf{x}}_a = [\hat{\mathbf{x}}^{\mathrm{T}}, \hat{d}]^{\mathrm{T}}$.

- [ ] $$\hat{\mathbf{x}}_a(k+1) = \begin{bmatrix} 1 & T_s & \gamma \frac{T_s^2}{2} \\ -\alpha_1 T_s & 1 - \alpha_2 T_s & \gamma T_s - \alpha_2 \gamma \frac{T_s^2}{2} \\ 0 & 0 & 1 \end{bmatrix} \hat{\mathbf{x}}_a(k) + \begin{bmatrix} \beta \frac{T_s^2}{2} \\ \beta T_s - \alpha_2 \beta \frac{T_s^2}{2} \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} (y(k) - \hat{y}$$
$$\hat{y}(k) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \hat{\mathbf{x}}_a(k)$$
where $\hat{\mathbf{x}}_a = [\hat{\mathbf{x}}^{\mathrm{T}}, \hat{d}]^{\mathrm{T}}$.

- [ ] $$\hat{\mathbf{x}}_a(k+1) = \begin{bmatrix} 1 & T_s & 0 \\ -\alpha_1 T_s & 1 - \alpha_2 T_s & 0 \\ 0 & 0 & 0 \end{bmatrix} \hat{\mathbf{x}}_a(k) + \begin{bmatrix} \beta \frac{T_s^2}{2} \\ \beta T_s - \alpha_2 \beta \frac{T_s^2}{2} \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} (y(k) - \hat{y}(k))$$
$$\hat{y}(k) = \begin{bmatrix} 1 & 0 & \gamma \end{bmatrix} \hat{\mathbf{x}}_a(k)$$
where $\hat{\mathbf{x}}_a = [\hat{\mathbf{x}}^{\mathrm{T}}, \hat{d}]^{\mathrm{T}}$.

Question 10

Consider the second order continuous time LTI SISO system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} u + \begin{bmatrix} 0 \\ \alpha \end{bmatrix} d$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}$$

where $\omega_n > 0, 0 < \zeta < 1$ and $\alpha \in \mathbb{R}$. The disturbance $d(t)$ acting on the state equation is time-varying and unknown, i.e. $d(t) = d_0(t - t_0)$ where d_0 is the unknown slope and t_0 is the time when the disturbance enters the system.

Which of the following control architectures (CAs) guarantees perfect tracking of the constant reference $r(t) = r_0$ in the presence of the given disturbance?

- [x] $$\text{CA} : \begin{cases} u = -\mathbf{K}\mathbf{x} + \mathbf{K}_i\mathbf{x}_i \\ \dot{\mathbf{x}}_i = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{x}_i + \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r \end{cases}$$
  where $\mathbf{K} = [K_1, K_2], \mathbf{K}_i = [K_{i,1}, K_{i,2}]$ and $\mathbf{x}_i = [x_{i,1}, x_{i,2}]^{\mathrm{T}}$.

- [ ] $$\text{CA} : \begin{cases} u = -\mathbf{K}\mathbf{x} + K_i x_i \\ \dot{x}_i = \begin{bmatrix} -1 & 0 \end{bmatrix} \mathbf{x} + r \end{cases}$$
  where $\mathbf{K} = [K_1, K_2]$.

- [ ] $$\text{CA} : \left\{ u = -\mathbf{K}\mathbf{x} + Nr \right.$$
  where $\mathbf{K} = [K_1, K_2], N = (\omega_n^2 + K_1)/\omega_n^2$.

- [ ] $$\text{CA} : \left\{ u = -\mathbf{K}\mathbf{x} + r \right.$$
  where $\mathbf{K} = [K_1, K_2]$.

- [ ] $$\text{CA} : \begin{cases} u = -\mathbf{K}\hat{\mathbf{x}} - \mathbf{K}_d\hat{d} + Nr \\ \dot{\mathbf{x}}_o = \begin{bmatrix} 0 & 1 & 0 \\ -\omega_n^2 & -2\zeta\omega_n^2 & \alpha \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x}_o + \begin{bmatrix} 0 \\ \omega_n^2 \\ 0 \end{bmatrix} u + \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} (y - \hat{y}) \\ \hat{y} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}_o \end{cases}$$
  where $\mathbf{K} = [K_1, K_2], N = (\omega_n^2 + K_1)/\omega_n^2$ and $\mathbf{x}_o = [\hat{\mathbf{x}}^{\mathrm{T}}, \hat{d}]^{\mathrm{T}}$.

Question 11

Consider the first order LTI continuous time system

$$\dot{x} = ax + b_v v$$
$$y_1 = x + w_1$$
$$y_2 = x + w_2$$

where $a \in \mathbb{R} \setminus \{0\}$, $b_v \in \mathbb{R} \setminus \{0\}$, v is white Gaussian noise with zero mean and noise intensity $\sigma_v^2 \geq 0$, w_1 and w_2 are uncorrelated white Gaussian noise sources with zero mean and noise intensity matrix

$$\mathbf{V} = \begin{bmatrix} \sigma_{w_1}^2 & 0 \\ 0 & \sigma_{w_2}^2 \end{bmatrix} > 0\,.$$

Which of the following statements is correct? (Given a matrix $\mathbf{M}$ the symbol $\|\mathbf{M}\|_\infty$ indicates the infinity norm of the matrix, which is defined as $\|\mathbf{M}\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |m_{ij}|$, where m_{ij} is the i-th row and j-th column entry of the matrix. The symbol $\ll$ means "much smaller than" and the symbol $\gg$ means "much larger than").

- [ ] If $\|\mathbf{V}\|_\infty \gg \sigma_v^2$ then the Kalman filter strongly relies on the measurements to estimate x.
- [ ] If $\sigma_{w_1}^2 \ll \sigma_{w_2}^2$ then the Kalman gain associated with the measurement y_1 is smaller than the Kalman gain associated with the measurement y_2.
- [ ] If the plant dynamics is unstable ($a > 0$) then the dynamics of the estimation error $e = x - \hat{x}$ is also unstable.
- [x] If the plant dynamics is asymptotically stable ($a < 0$) and the intensity of the process noise is zero ($\sigma_v^2 = 0$) then the solution of the Riccati equation associated with the design of the Kalman gain is zero.
- [ ] If the plant dynamics is unstable ($a > 0$) and the intensity of the process noise is zero ($\sigma_v^2 = 0$) then the solution of the Riccati equation associated with the design of the Kalman gain is zero.

Question 12

Consider the n-th order continuous time LTI system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{x} \in \mathbb{R}^n,\ \mathbf{u} \in \mathbb{R}^m$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}, \quad \mathbf{y} \in \mathbb{R}^p$$

where $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ are matrices with constant coefficients. A finite-time optimal regulator problem is set-up for the given system using the performance index

$$J(\mathbf{u}) = \int_{t_0}^{t_1} \mathbf{x}^{\mathrm{T}}\mathbf{Q}(t)\,\mathbf{x} + \mathbf{u}^{\mathrm{T}}\mathbf{R}(t)\,\mathbf{u}\;\mathrm{d}t + \mathbf{x}^{\mathrm{T}}(t_1)\,\mathbf{S}(t_1)\,\mathbf{x}(t_1)$$

where the weighting matrices fulfill the following inequalities

$$\mathbf{Q}(t) \geq 0, \quad \forall t \geq t_0$$
$$\mathbf{R}(t) > 0, \quad \forall t \geq t_0$$
$$\mathbf{S}(t_1) \geq 0, \quad \forall t_1$$

Which of the following statements is correct?

- [ ] The optimal controller $\mathbf{K}(t)$ is time independent when $\mathbf{Q}(t)$ and $\mathbf{R}(t)$ are constant weighting matrices.
- [ ] If the pair $(\mathbf{A}, \mathbf{B})$ is not stabilizable there is no finite solution to the finite-time optimal control problem.
- [ ] If $\mathbf{S}(t_1) = 0$ then it is possible to achieve a constant non zero control law $\mathbf{K}(t) = \bar{\mathbf{K}} \neq \mathbf{0}$.
- [x] If $\mathbf{S}(t_1) = \bar{\mathbf{P}}$ ($\bar{\mathbf{P}}$ being the solution of the algebraic Riccati equation), and $\mathbf{Q}(t)$ and $\mathbf{R}(t)$ are constant weighting matrices then it is possible to achieve a constant non zero control law $\mathbf{K}(t) = \bar{\mathbf{K}} \neq \mathbf{0}$.
- [ ] The performance index $J(\mathbf{u})$ reaches its minimum at the end of the optimization time horizon $[t_0, t_1]$.

Final exam E19 - Questionnaire

Side 1

Vis rigtige svar
Skjul rigtige svar

Open-loop system analysis (Part I)

Spørgsmål 1

The wheel dynamics of a car travelling with linear velocity v is given by

$$\Sigma : \begin{cases} m\dot{x}_1 = -mg\mu(\lambda) \\ I\dot{x}_2 = -Bx_2 + mgR\mu(\lambda) - u \end{cases}$$

where x_1 is the linear velocity v, x_2 is the angular velocity Ω of the wheel and u is the input breaking torque. The parameters in the above equation are: *m* is the wheel mass, *I* is the wheel moment of inertia, *g* is the gravity constant, *R* is the wheel radius, *B* is the bearing friction coefficient, λ is the wheel slip and $\mu(\lambda)$ is the friction force coefficient.

The wheel slip is defined as $\lambda = (x_1 - Rx_2)/x_1$.

Let $\mathbf{x}_0 = [x_{10}, x_{20}]^\mathrm{T}$ be the stationary point related to the stationary input u_0. Which of the following systems is the linearized system around the point of operation?

- [x] $\mathbf{A} = \begin{bmatrix} -g\frac{\partial\mu}{\partial\lambda}\frac{Rx_{20}}{x_{10}^2} & g\frac{\partial\mu}{\partial\lambda}\frac{R}{x_{10}} \\ \frac{mgR}{I}\frac{\partial\mu}{\partial\lambda}\frac{Rx_{20}}{x_{10}^2} & -\frac{B}{I} - \frac{mgR}{I}\frac{\partial\mu}{\partial\lambda}\frac{R}{x_{10}} \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} 0 \\ -\frac{1}{I} \end{bmatrix}$

- [ ] $\mathbf{A} = \begin{bmatrix} -g\frac{Rx_{20}}{x_{10}^2} & g\frac{R}{x_{10}} \\ \frac{mgR}{I}\frac{Rx_{20}}{x_{10}^2} & -\frac{B}{I} - \frac{mgR}{I}\frac{R}{x_{10}} \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} 0 \\ -\frac{1}{I} \end{bmatrix}$

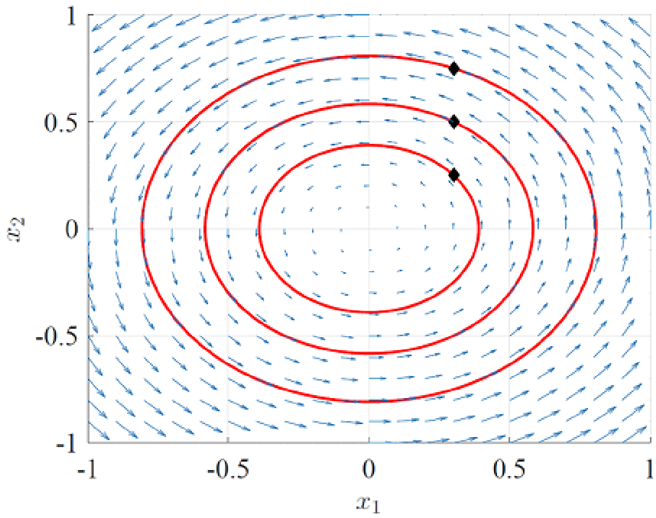
- [ ] $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ \frac{mgR}{I}\frac{Rx_{20}}{x_{10}^2} & -\frac{B}{I} - \frac{mgR}{I}\frac{R}{x_{10}} \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} 0 \\ -\frac{1}{I} \end{bmatrix}$

- [ ] $\mathbf{A} = \begin{bmatrix} -g\frac{\partial\mu}{\partial\lambda} & g\frac{\partial\mu}{\partial\lambda} \\ \frac{mgR}{I}\frac{\partial\mu}{\partial\lambda} & -\frac{B}{I} - \frac{mgR}{I}\frac{\partial\mu}{\partial\lambda} \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} 0 \\ -\frac{1}{I} \end{bmatrix}$

- [ ] $\mathbf{A} = \begin{bmatrix} -g\frac{\partial\mu}{\partial\lambda} & g\frac{\partial\mu}{\partial\lambda} \\ \frac{mgR}{I}\frac{\partial\mu}{\partial\lambda} & -\frac{B}{I} - \frac{mgR}{I}\frac{\partial\mu}{\partial\lambda} \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

Spørgsmål 2

Consider the phase portrait shown in the following figure (in the given phase portrait each black diamond represents an initial condition $\mathbf{x} = \begin{bmatrix} x_{10} & x_{20} \end{bmatrix}^{\mathrm{T}}$ for the system; each red line is a trajectory of the system originated from the initial condition; the blue arrows represent the direction of the vector field in the neighborhood of the origin).



Which of the following linear systems describes the dynamical behavior shown in the figure when initialized at $\mathbf{x}_0 \neq 0$?

- [ ] $\dot{\mathbf{x}} = \begin{bmatrix} -\alpha & 0 \\ 0 & -\beta \end{bmatrix} \mathbf{x}, \qquad \alpha, \beta \in \mathbb{R}_+$
- [ ] $\dot{\mathbf{x}} = \begin{bmatrix} -\alpha & \gamma \\ -\gamma & -\alpha \end{bmatrix} \mathbf{x}, \qquad \alpha, \gamma \in \mathbb{R}_+$
- [ ] $\dot{\mathbf{x}} = \begin{bmatrix} -\alpha & 0 \\ 0 & 0 \end{bmatrix} \mathbf{x}, \qquad \alpha \in \mathbb{R}_+$
- [x] $\dot{\mathbf{x}} = \begin{bmatrix} 0 & \gamma \\ -\gamma & 0 \end{bmatrix} \mathbf{x}, \qquad \gamma \in \mathbb{R}_+$
- [ ] $\dot{\mathbf{x}} = \begin{bmatrix} -\alpha & 1 \\ 0 & -\alpha \end{bmatrix} \mathbf{x}, \qquad \alpha \in \mathbb{R}_+$

Spørgsmål 3

Consider the n-th order continuous time linear system

$$\Sigma : \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} & \mathbf{x} \in \mathbb{R}^n,\ \mathbf{u} \in \mathbb{R}^m \\ \mathbf{y} = \mathbf{C}\mathbf{x} & \mathbf{y} \in \mathbb{R}^p \end{cases}$$

where $m > 1$. Which of the following statements is **not correct**?

- [ ] Σ is controllable if and only if the controllability gramian $$\mathbf{W}_c = \int_0^{+\infty} e^{-\mathbf{A}t}\mathbf{B}\mathbf{B}^{\mathrm{T}}e^{-\mathbf{A}^{\mathrm{T}}t}\,\mathrm{d}t$$ has determinant different from zero.
- [ ] Σ is controllable if and only the controllability matrix $$\mathbf{M}_c = \begin{bmatrix}\mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix}$$ is full rank.
- [x] Σ is controllable if and only the controllability matrix $$\mathbf{M}_c = \begin{bmatrix}\mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix}$$ has determinant different from zero.
- [ ] Σ is controllable if and only if no left eigenvector $\mathbf{w}_i$ of $\mathbf{A}$ exists such that $\mathbf{w}_i^{\mathrm{T}}\mathbf{B} = 0$.
- [ ] Let $\mathbf{M}$ be the modal matrix associated with $\mathbf{A}$. Σ is controllable if and only if the diagonalized system $$\dot{\mathbf{z}} = \mathbf{\Lambda}\mathbf{z} + \mathbf{\Gamma}\mathbf{u}$$ has the input matrix $\mathbf{\Gamma}$ with no zero rows; where $\mathbf{\Lambda} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}$, $\mathbf{\Gamma} = \mathbf{M}^{-1}\mathbf{B}$ and $\mathbf{z} = \mathbf{M}^{-1}\mathbf{x}$.

Spørgsmål 4

Consider the n-th order continuous time linear system

$$\Sigma : \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} & \mathbf{x} \in \mathbb{R}^n,\ \mathbf{u} \in \mathbb{R}^m \\ \mathbf{y} = \mathbf{C}\mathbf{x} & \mathbf{y} \in \mathbb{R}^p \end{cases}$$

where $p > 1$. Which of the following statements is correct?

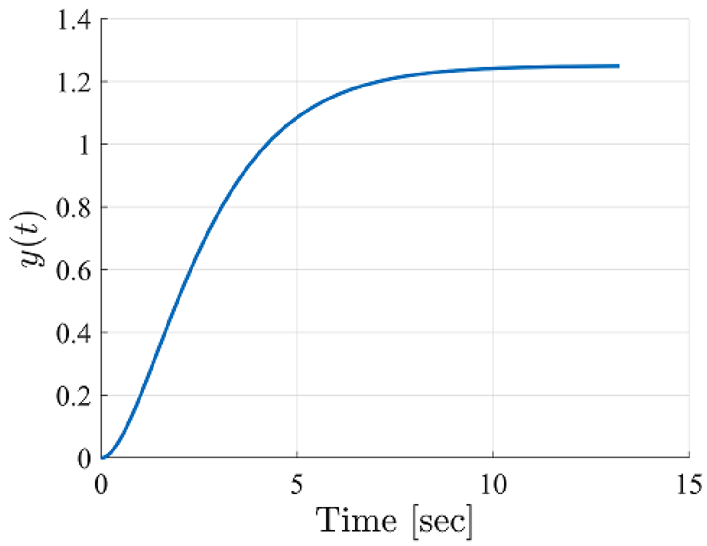
- [ ] Σ is observable if and only if the observability gramian $\mathbf{W}_o = \int_0^{+\infty} e^{\mathbf{A}^{\mathrm{T}}t}\mathbf{C}^{\mathrm{T}}\mathbf{C}e^{\mathbf{A}t}\,\mathrm{d}t$ has determinant equal to zero.
- [ ] Σ is observable if and only if the observability matrix $\mathbf{M}_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \mathbf{C}\mathbf{A}^2 \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix}$ is rank deficient, i.e. $\operatorname{rank}\{\mathbf{M}_o\} < n$.
- [ ] Σ is observable if and only if no left eigenvector $\mathbf{w}_i$ of $\mathbf{A}$ exists such that $\mathbf{w}_i^{\mathrm{T}}\mathbf{B} = 0$.
- [x] Σ is observable if and only if no right eigenvector $\mathbf{v}_i$ of $\mathbf{A}$ exists such that $\mathbf{C}\mathbf{v}_i = 0$.
- [ ] Let $\mathbf{M}$ be the modal matrix associated with $\mathbf{A}$. Σ is observable if and only if the diagonalized system $\begin{cases} \dot{\mathbf{z}} = \mathbf{\Lambda}\mathbf{z} + \mathbf{\Gamma}\mathbf{u} \\ \mathbf{y} = \mathbf{\Xi}\mathbf{z} \end{cases}$ has the output matrix $\mathbf{\Xi}$ with at least one zero column; where $\mathbf{\Lambda} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M},\ \mathbf{\Gamma} = \mathbf{M}^{-1}\mathbf{B},\ \mathbf{\Xi} = \mathbf{C}\mathbf{M}$ and $\mathbf{z} = \mathbf{M}^{-1}\mathbf{x}$.

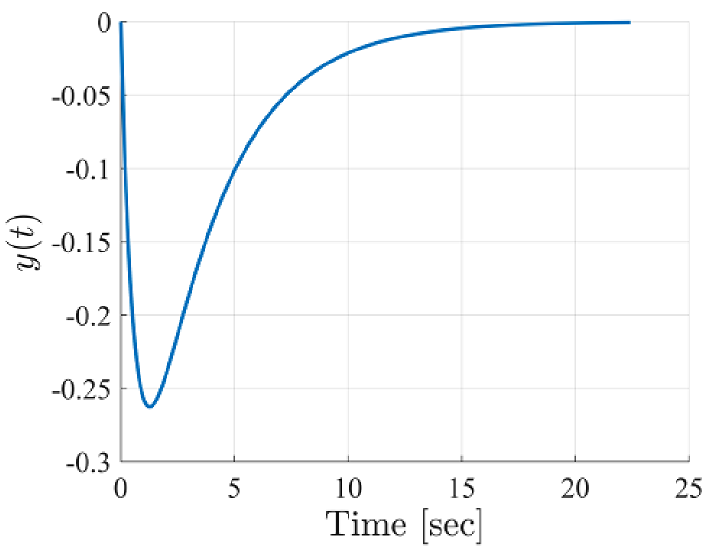
Spørgsmål 5

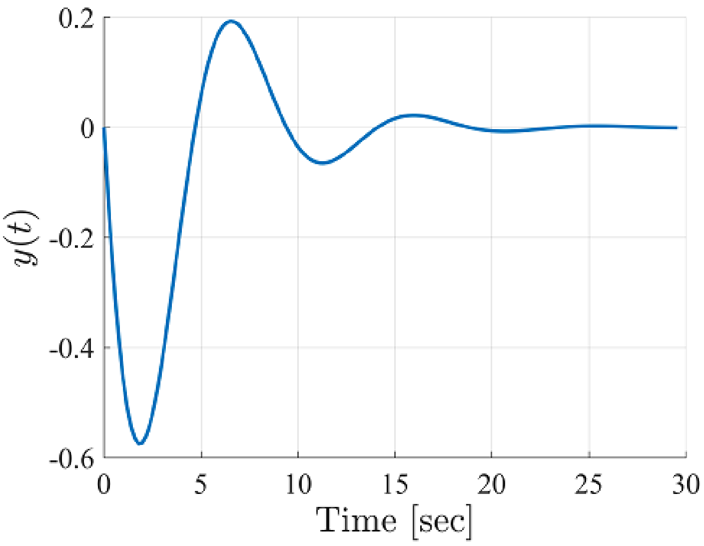
Consider the 2nd order LTI continuous time system

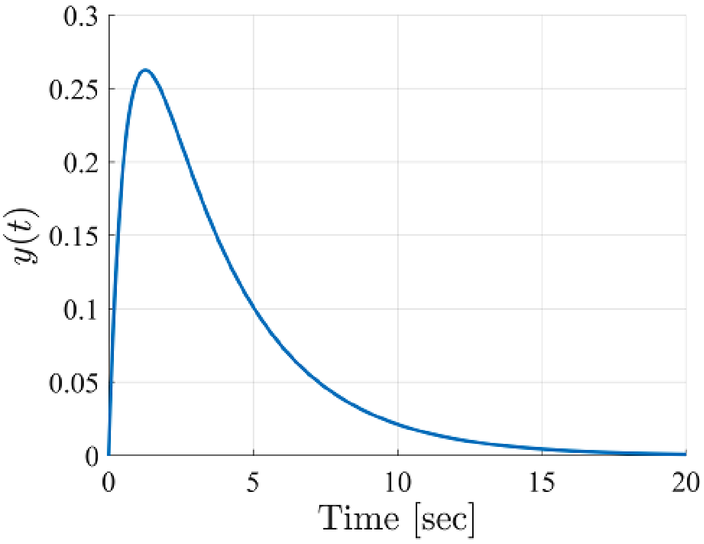
$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -\alpha & \beta \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ \gamma \end{bmatrix} u(t), \quad \mathbf{x} \in \mathbb{R}^2, u \in \mathbb{R}$$
$$y(t) = \begin{bmatrix} 0 & \delta \end{bmatrix} \mathbf{x}(t), \quad y \in \mathbb{R}$$

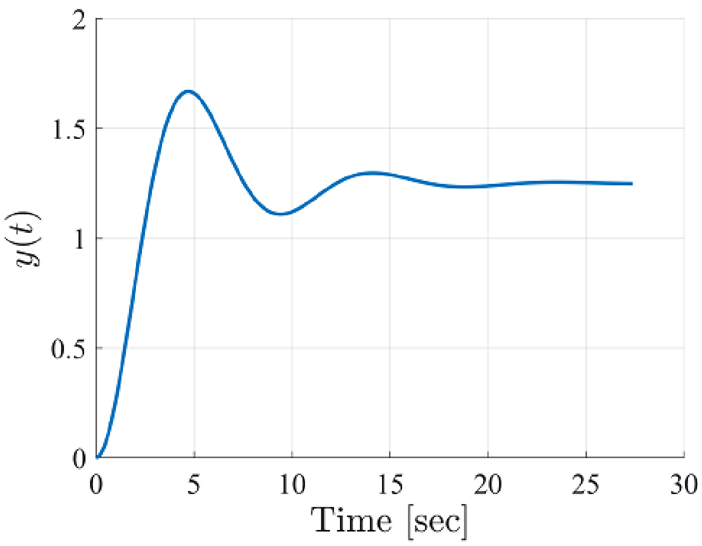
Assume that $\alpha > 0,\ \gamma > 0,\ \delta < 0$ and that $\beta < -2\sqrt{\alpha}\ \vee\ \beta > 2\sqrt{\alpha}$. Which of the following plots shows the step response associated with the system?

☐ 

☑ 

- [ ] 
0.2
0
-0.2
-0.4
-0.6
$y(t)$
0
5
10
15
20
25
30
Time [sec]

- [ ] 
0.3
0.25
0.2
0.15
0.1
0.05
0
$y(t)$
0
5
10
15
20
Time [sec]

- [ ] 
2
1.5
1
0.5
0
$y(t)$
0
5
10
15
20
25
30
Time [sec]

Side 2

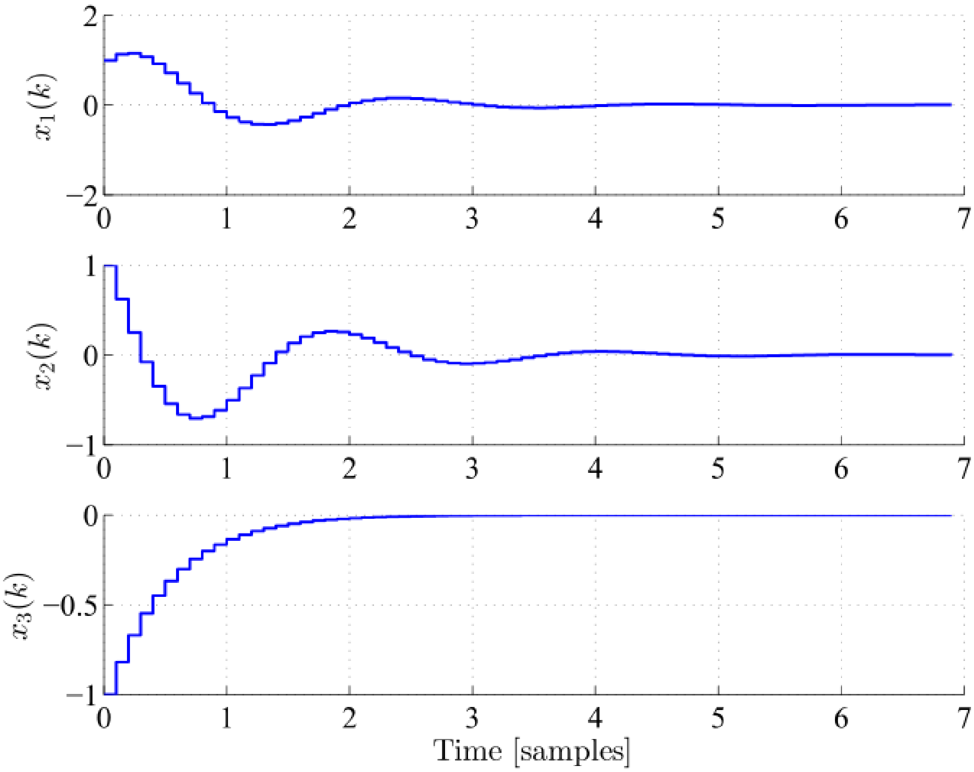
Open-loop analysis (Part II)

Spørgsmål 6

Consider a 3rd order LTI discrete time SISO system

$$\Sigma : \begin{cases} \mathbf{x}(k+1) = \begin{bmatrix} \mathbf{F}_1 & \mathbf{0} \\ \mathbf{0} & F_2 \end{bmatrix} \mathbf{x}(k) + \mathbf{G}u(k), & \mathbf{x} \in \mathbb{R}^3,\ u \in \mathbb{R} \\ y(k) = \mathbf{C}\mathbf{x}(k), & y \in \mathbb{R} \end{cases}$$

where $\mathbf{F}_1$ is a 2x2 matrix and F_2 is a scalar. The zero input response of the system to the initial condition $\mathbf{x}_0 = [x_{10}, x_{20}, x_{30}]^T$ with $x_{10} > 0,\ x_{20} > 0$, and $x_{30} < 0$ is shown in the following figure.



Which of the following statements is **not correct** ?

- ☐ The system is asymptotically stable.
- ☐ The system has one real eigenvalue whose magnitude is less than one.
- ☐ The continuous time system associated with Σ has a pair of natural modes which oscillate with frequency $$\omega_n = \frac{|\ln(\lambda_1(\mathbf{F}_1))|}{T_s}$$ and asymptotically decay to zero ($\lambda_1(\mathbf{F}_1)$ is one of the eigenvalues of the submatrix $\mathbf{F}_1$ and T_s is the sampling time).
- ☑ The natural mode associated with the dynamics of the state variable x_3 is unstable.
- ☐ The eigenvalues of the system have magnitude less than one.

Spørgsmål 7

Consider the 4th order LTI continuous time system

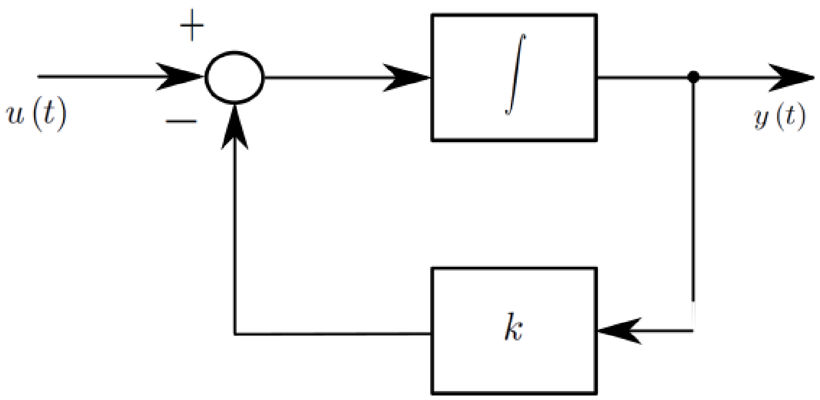
$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & -2\zeta_1\omega_1 & \gamma & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_2^2 & -2\zeta_2\omega_2 \end{bmatrix} \mathbf{x}$$

where ω_1 and ω_2 are real positive constants; ζ_1, ζ_2, and γ are real constants. Which of the following statement is **not correct**?

- [ ] If $0 < \zeta_1 < 1$ and $0 < \zeta_2 < 1$ then the system has two pairs of asymptotically stable complex eigenvalues.
- [ ] If $\zeta_1 = 0$, $\zeta_2 = 0$, and $\omega_1 \neq \omega_2$ then the system has two pairs of stable imaginary eigenvalues for any $\gamma \in \mathbb{R}$.
- [ ] If $\zeta_1 = 0$, $\zeta_2 = 0$, and $\omega_1 = \omega_2$ and $\gamma \neq 0$ then the system has one pair of unstable imaginary eigenvalues with algebraic multiplicity equal to two.
- [ ] If $\zeta_1 = \zeta_2 = 0$, $\omega_1 = \omega_2$, and $\gamma = 0$ then the system has two pairs of stable imaginary eigenvalues.
- [x] If $-1 < \zeta_1 < 0$ and $-1 < \zeta_2 < 0$ then the system has two pairs of asymptotically stable real eigenvalues.

Spørgsmål 8

Consider the system shown in the following block diagram



where *u(t)* is a stochastic process whose autocorrelation function is given by

$$R_u(\tau) = \sigma_u^2 e^{-\beta|\tau|}$$

with σ_u^2 being the variance of the input signal, and $\beta > 0$. What is the variance σ_y^2 of the output signal *y(t)*?

- [ ] $\sigma_y^2 = \sigma_u^2$
- [x] $\sigma_y^2 = \dfrac{1}{k(\beta + k)}\sigma_u^2$
- [ ] $\sigma_y^2 = \dfrac{\beta - k}{\beta + k}\sigma_u^2$
- [ ] $\sigma_y^2 = 0$
- [ ] $\sigma_y^2 = \dfrac{k}{\beta}\sigma_u^2$

Side 3

Closed-loop analysis and synthesis

Spørgsmål 9

Consider the 3rd order LTI asymptotically stable continuous time system

$$\dot{\mathbf{x}} = \begin{bmatrix} -\alpha & \beta & -\beta \\ \gamma & -2\alpha & 0 \\ 0 & \delta & -\varepsilon \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ -\rho \end{bmatrix} u + \begin{bmatrix} 0 \\ \eta \\ 0 \end{bmatrix} d, \quad \mathbf{x} \in \mathbb{R}^3, u \in \mathbb{R}, d \in \mathbb{R}$$
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}, \quad y \in \mathbb{R}$$

with initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, where *u* is the control input and *d* is an unknown disturbance. The constants $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \rho,\ \text{and}\ \eta$ are positive.

A continuous time reduced order observer is designed to estimate the unmeasured states $x_2 \text{ and } x_3$. Assuming that $\mathbf{x} = \begin{bmatrix} x_1 & \mathbf{x}_2 \end{bmatrix}^{\mathrm{T}}$ then the dynamics of the reduced order observer is given by

$$\dot{\mathbf{z}}(t) = \mathbf{M}\mathbf{z}(t) + \mathbf{N}u(t) + \mathbf{P}y(t)$$
$$\hat{\mathbf{x}}_2(t) = \mathbf{z}(t) + \mathbf{L}y(t)$$

with $\mathbf{z}(t_0)$ such that $\hat{\mathbf{x}}_2(t_0) = \mathbf{x}_2(t_0)$. Let $\mathbf{e}(t) = \mathbf{x}_2(t) - \hat{\mathbf{x}}_2(t)$ be the estimation error. If

$$d(t) = \begin{cases} 0, & 0 \le t < t_1 \\ d_0, & t \ge t_1 \end{cases}$$

is the disturbance acting on the system, what is the behavior of the estimation error?

- [ ] $\mathbf{e}(t) = \mathbf{0} \text{ for } 0 \le t < t_1 \text{ and } \lim_{t \to +\infty} \mathbf{e}(t) = 0$
- [ ] $\mathbf{e}(t) = \bar{\mathbf{e}} < \infty \text{ for } 0 \le t < t_1 \text{ and } \lim_{t \to +\infty} \mathbf{e}(t) = 0$
- [ ] $\mathbf{e}(t) = \mathbf{0} \text{ for } 0 \le t < t_1 \text{ and } \lim_{t \to +\infty} \mathbf{e}(t) = +\infty$
- [x] $\mathbf{e}(t) = \mathbf{0} \text{ for } 0 \le t < t_1 \text{ and } \lim_{t \to +\infty} \mathbf{e}(t) = \bar{\mathbf{e}} < \infty$
- [ ] $\mathbf{e}(t) = \bar{\mathbf{e}}_1 < \infty \text{ for } 0 \le t < t_1 \text{ and } \lim_{t \to +\infty} \mathbf{e}(t) = \bar{\mathbf{e}}_2 < \infty$

Spørgsmål 10

Consider the 2nd order LTI-SISO system

$$\Sigma : \begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} -\alpha & \beta \\ -\beta & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} u + \begin{bmatrix} \delta \\ 0 \end{bmatrix} v, \quad \mathbf{x} \in \mathbb{R}^2, u \in \mathbb{R}, v \in \mathbb{R} \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}, \quad y \in \mathbb{R} \end{cases}$$

where $\alpha, \beta, \gamma_1, \gamma_2, \text{and } \delta$ are real and positive constants; *u* is the control input and *v* is a deterministic disturbance. A full state feedback controller with integral action

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_p & \mathbf{K}_i \end{bmatrix}$$

is designed for the following augmented system

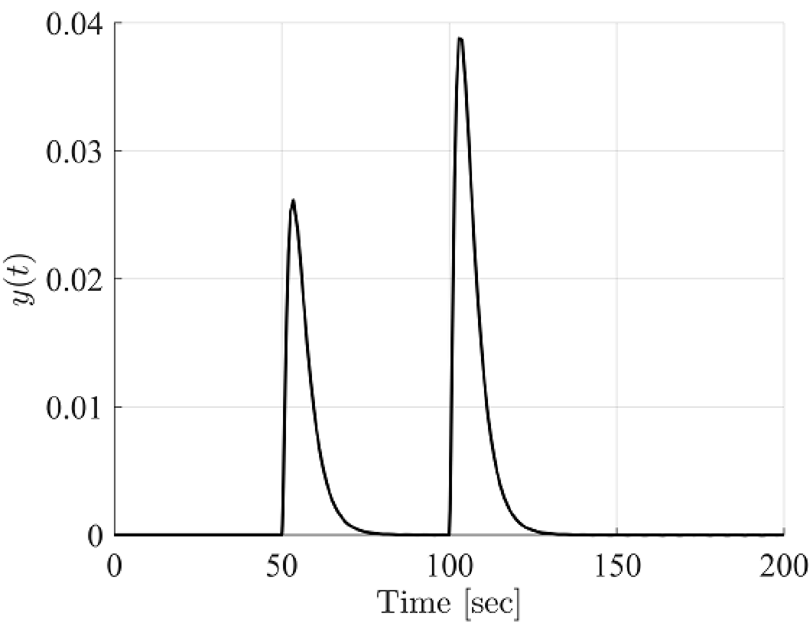
$$\dot{\mathbf{x}}_a = \begin{bmatrix} -\alpha & \beta & 0 & 0 \\ -\beta & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}_a + \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} \delta \\ 0 \\ 0 \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} r, \quad \mathbf{x}_a = \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_i \end{bmatrix} \in \mathbb{R}^4, u \in \mathbb{R}, v \in \mathbb{R}, r \in \mathbb{R}$$

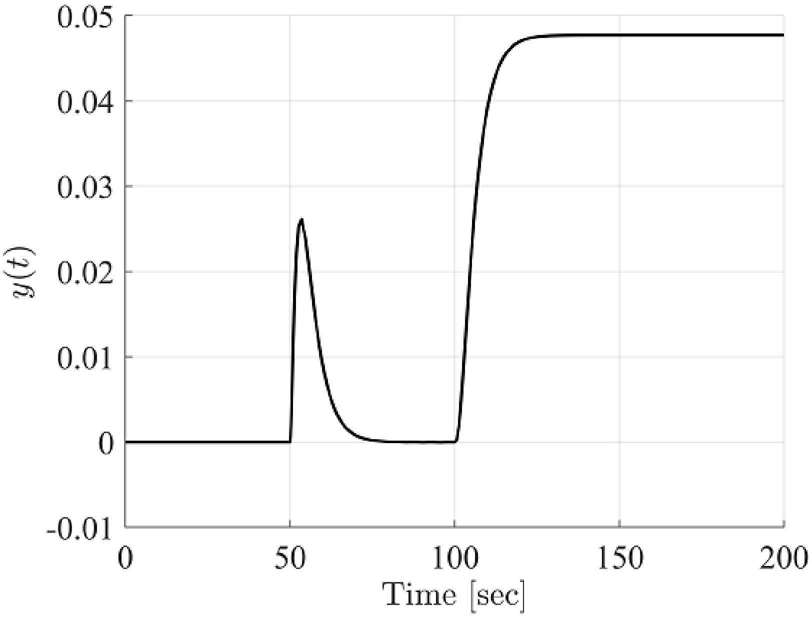
$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \mathbf{x}_a, \quad y \in \mathbb{R}$$

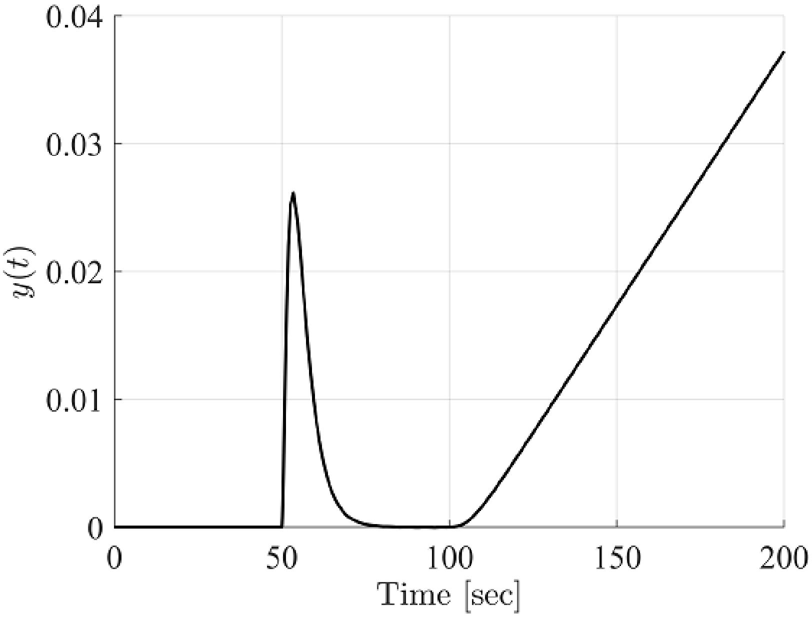
where *r* is reference set-point for the output *y*. Assume that the deterministic disturbance is given by

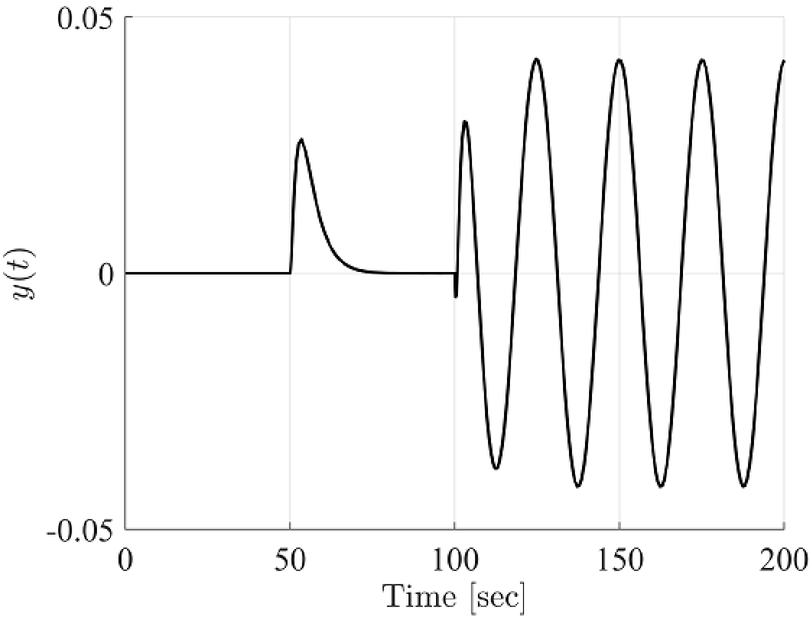
$$v(t) = \begin{cases} v_0 (t - t_0), & t_0 \le t < t_1 \\ v_0 (t - t_0) + v_1 (t - t_1)^2, & t \ge t_1 \end{cases}$$

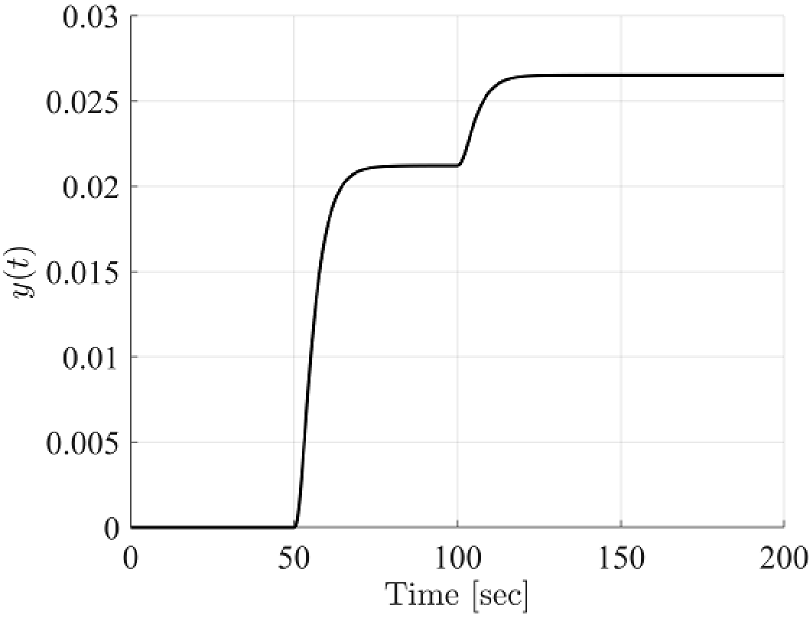
Which of the following plots shows the closed-loop output response when the system is subject to the disturbance *v(t)*?

☐ 

- [x] 
0.05
0.04
0.03
0.02
0.01
0
-0.01
$y(t)$
0
50
100
150
200
Time [sec]

- [ ] 
0.04
0.03
0.02
0.01
0
$y(t)$
0
50
100
150
200
Time [sec]

- [ ] 
0.05
0
-0.05
$y(t)$
0
50
100
150
200
Time [sec]

☐ 
0.03
0.025
0.02
0.015
0.01
0.005
0
y(t)
0
50
100
150
200
Time [sec]

Spørgsmål 11

Consider the three LTI-SISO systems in controllable subspace decomposition form

$$\Sigma^a : \begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} \mathbf{A}_c^a & \mathbf{A}_{12}^a \\ \mathbf{0} & \mathbf{A}_{nc}^a \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{B}_c^a \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} \mathbf{C}_c^a & \mathbf{C}_{nc}^a \end{bmatrix} \mathbf{x} \end{cases}$$

$$\Sigma^b : \begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} \mathbf{A}_c^b & \mathbf{A}_{12}^b \\ \mathbf{0} & \mathbf{A}_{nc}^b \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{B}_c^b \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} \mathbf{C}_c^b & \mathbf{C}_{nc}^b \end{bmatrix} \mathbf{x} \end{cases}$$

$$\Sigma^c : \begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} \mathbf{A}_c^c & \mathbf{A}_{12}^c \\ \mathbf{0} & \mathbf{A}_{nc}^c \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{B}_c^c \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} \mathbf{C}_c^c & \mathbf{C}_{nc}^c \end{bmatrix} \mathbf{x} \end{cases}$$

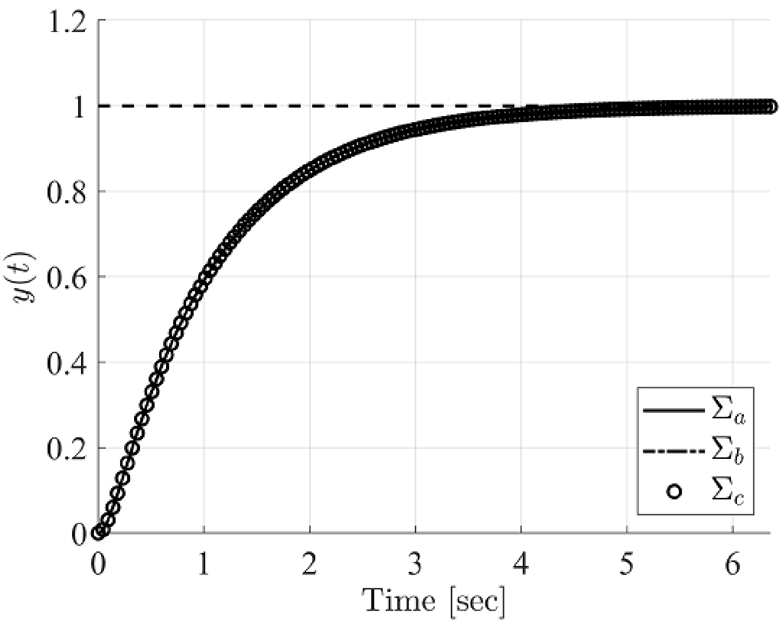
The open loop eigenvalues of the non controllable subsystems are real and they satisfy the following inequality:

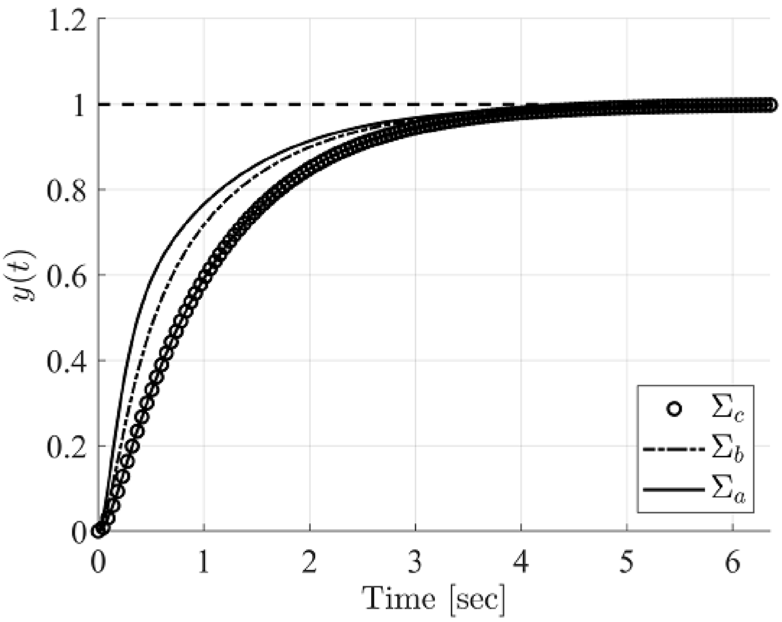
$$\lambda_{ol}\left(\mathbf{A}_{nc}^a\right) < \lambda_{ol}\left(\mathbf{A}_{nc}^b\right) < \lambda_{ol}\left(\mathbf{A}_{nc}^c\right) < 0.$$

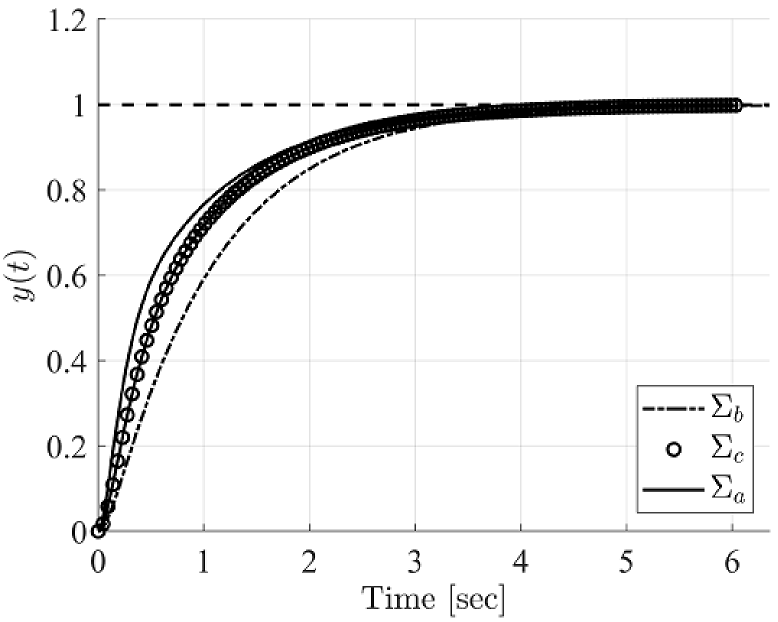
A full state feedback controller with integral action is designed for each of the given systems such that the closed loop eigenvalues of the controllable subsystems are real, negative, and satisfy the following relation:

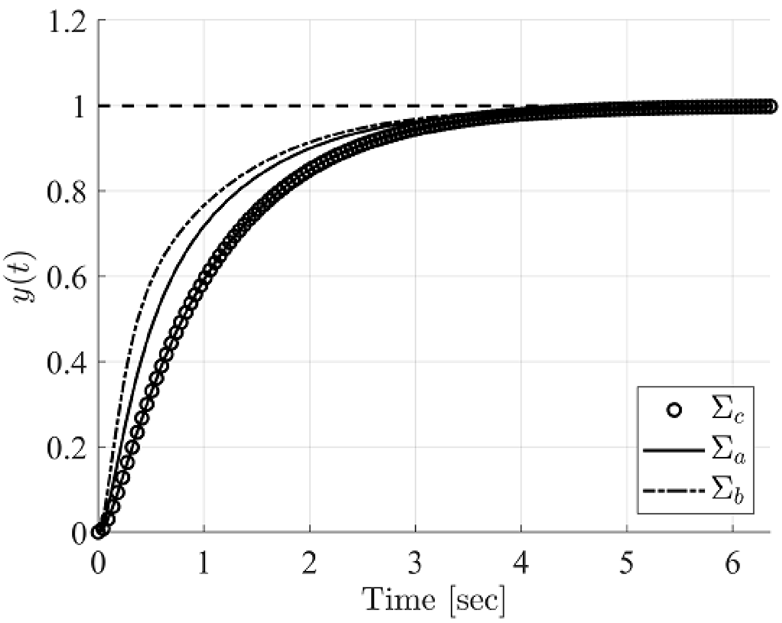
$$\lambda_{cl}\left(\begin{bmatrix} \mathbf{A}_c^a - \mathbf{B}_c^a\mathbf{K}^a & \mathbf{B}_c^a\mathbf{K}_i^a \\ \mathbf{C}_c^a & 0 \end{bmatrix}\right) = \lambda_{cl}\left(\begin{bmatrix} \mathbf{A}_c^b - \mathbf{B}_c^b\mathbf{K}^b & \mathbf{B}_c^b\mathbf{K}_i^b \\ \mathbf{C}_c^b & 0 \end{bmatrix}\right) = \lambda_{cl}\left(\begin{bmatrix} \mathbf{A}_c^c - \mathbf{B}_c^c\mathbf{K}^c & \mathbf{B}_c^c\mathbf{K}_i^c \\ \mathbf{C}_c^c & 0 \end{bmatrix}\right).$$

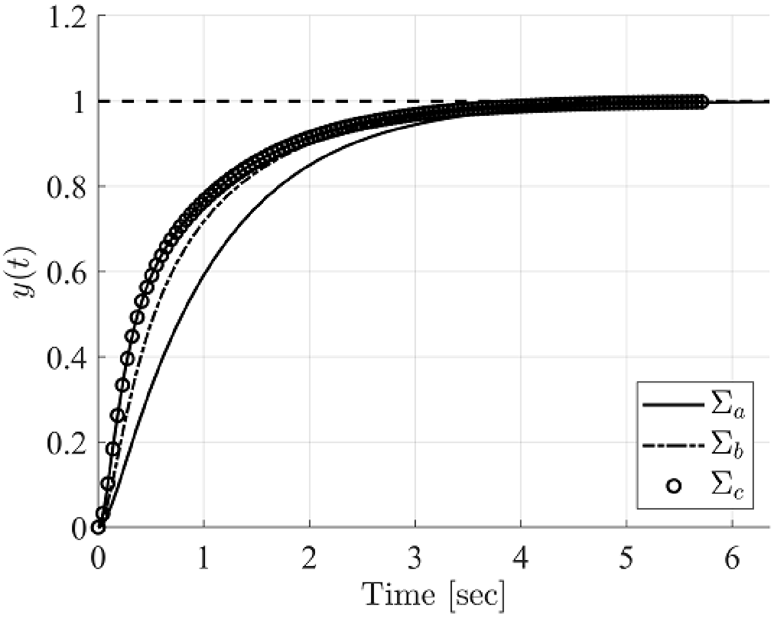
Which of the following plots represents the unit step responses of the closed-loop systems?

- [x] 

☐ 
1.2
1
0.8
0.6
0.4
0.2
0
$y(t)$
0 1 2 3 4 5 6
Time [sec]
Σ_c
Σ_b
Σ_a

☐ 
1.2
1
0.8
0.6
0.4
0.2
0
$y(t)$
0 1 2 3 4 5 6
Time [sec]
Σ_b
Σ_c
Σ_a

☐ 
1.2
1
0.8
0.6
0.4
0.2
0
$y(t)$
0 1 2 3 4 5 6
Time [sec]
Σ_c
Σ_a
Σ_b

▢ 
1.2
1
0.8
0.6
0.4
0.2
0
$y(t)$
0
1
2
3
4
5
6
Time [sec]
Σ_a
Σ_b
Σ_c

Spørgsmål 12

Consider the n-th order LTI continuous time system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{B}_n\mathbf{n}_1(t), \quad \mathbf{x} \in \mathbb{R}^n,\ \mathbf{u} \in \mathbb{R}^m,\ \mathbf{n}_1 \in \mathbb{R}^q$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{n}_2(t), \quad \mathbf{y} \in \mathbb{R}^p,\ \mathbf{n}_2 \in \mathbb{R}^p$$

where $\mathbf{n}_1$ is the process noise characterized as white noise with zero mean and noise intensity matrix $\mathbf{\Sigma}_1$; $\mathbf{n}_2$ is the measurement noise characterized as white noise with zero mean and noise intensity matrix $\mathbf{\Sigma}_2$. The two noise sources are uncorrelated, that is

$$E\left\{\mathbf{n}_1(t)\,\mathbf{n}_2^{\mathrm{T}}(t)\right\} = 0$$

The steady state Linear Quadratic Gaussian (LQG) regulator

$$\mathbf{u}(t) = -\mathbf{K}_\infty\hat{\mathbf{x}}(t),$$

where $\hat{\mathbf{x}}$ is the state estimate provided by a continuous time Kalman filter, is the *optimal linear solution* associated with the minimization of the performance index

$$J(\mathbf{u}) = E\left\{\mathbf{x}^{\mathrm{T}}(t)\,\mathbf{R}_1\mathbf{x}(\mathbf{t}) + \mathbf{u}^{\mathrm{T}}(t)\,\mathbf{R}_2\mathbf{u}(t)\right\}$$

where $\mathbf{R}_1$ and $\mathbf{R}_2$ are constant weighting matrices.Which of the following statements is**not correct**?

- [ ] The steady state Linear Quadratic regulator and the Kalman filter, which constitute the LQG regulator, can be designed independently of each other with guaranteed overall optimality and asymptotic stability of the closed-loop system.

- [ ] If both $\mathbf{n}_1(t)$ and $\mathbf{n}_2(t)$ are Gaussian distributed white noise processes, and the initial condition $\mathbf{x}_0 = \mathbf{x}(t_0)$ is also Gaussian distributed then the optimal linear solution is the optimal solution without qualification.

- [x] If the control input $\mathbf{u}(t)$ is not weighted at all then the minimum of the performance index is zero, that is $\lim_{\mathbf{R}_2 \to 0} J(\mathbf{u}) = 0$

- [ ] The position of the closed-loop poles is strongly influenced by the choice of the weighting matrix $\mathbf{R}_2$ and by the noise intensity matrix $\mathbf{\Sigma}_2$ of the measurement noise.

- [ ] Although there is no measurement noise affecting the output of the system, the minimum of the performance index is still larger than zero, that is
$$\lim_{\mathbf{\Sigma}_2 \to 0} J(\mathbf{u}) \geq tr\left(\mathbf{P}_\infty\mathbf{\Sigma}_1\right)$$
where $\mathbf{P}_\infty$ is the solution of the algebraic Riccati equation associated with the Linear Quadratic regulation problem.

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Roberto Galeazzi

CampusNet / 31310 Linear control design 2 E20 / Assignments

31310 E20 Exam Questionnaire

Page 1

Show correct answers
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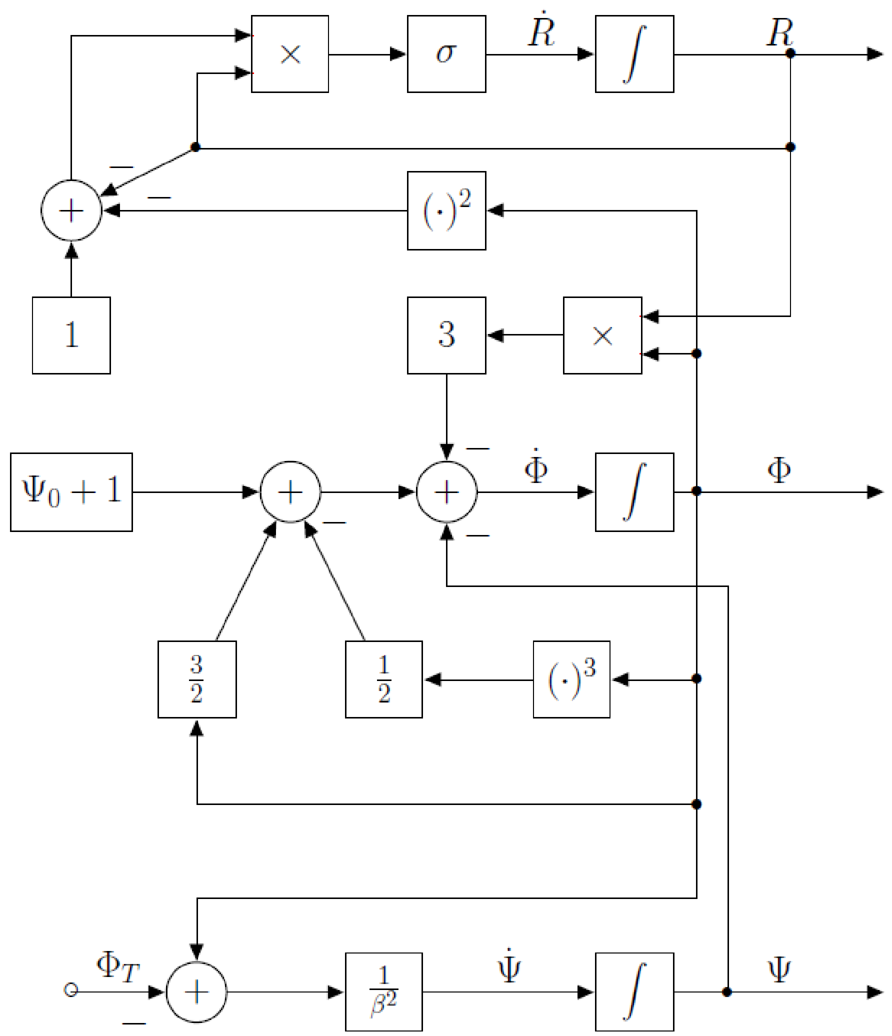
Analysis of open loop systems - Part 1

Question 1

Consider the block diagram of a jet engine compression system shown in the following figure, where Φ is the mass flow, Ψ is the pressure rise and $R \geq 0$ is the normalized stall cell squared amplitude. Φ_T is the mass flow through the throttle, and σ and β are constant positive parameters. Let $\mathbf{x} = [\Phi, \Psi, R]^{\mathrm{T}}$ be the state vector. Which of the following state space models is associated with the block diagram? (In the diagram the black dots address intersection of signal flows.)



The state space model of the system is:

☐ $$\Sigma : \begin{cases} \dot{x}_1 = -x_2 - 3x_1x_3 \\ \dot{x}_2 = \dfrac{1}{\beta^2}(x_2 - \Phi_T) \\ \dot{x}_3 = \sigma x_3(1 - x_1 - x_3) \end{cases}$$

The state space model of the system is:

☐ $$\Sigma : \begin{cases} \dot{x}_1 = -x_2 + \Psi_c(x_1) - 3x_1x_3 \\ \dot{x}_2 = \dfrac{1}{\beta^2}(x_2 - \Phi_T) \\ \dot{x}_3 = \sigma x_3(1 - x_1^2 - x_3) \\ \dot{x}_4 = x_3 \end{cases}$$

where $\Psi_c(x_1) = \Psi_0 + 1 + \frac{3}{2}x_1 - \frac{1}{2}x_1^3$

- [ ] The state space model of the system is:
$$\Sigma : \begin{cases} \dot{x}_1 = -x_2 + \Psi_c(x_1) - 3(x_1 + x_3) \\ \dot{x}_2 = \dfrac{1}{\beta^2}(x_2 - \Phi_T) \\ \dot{x}_3 = \sigma x_3(1 - x_1^2 - x_3) \end{cases}$$
where $\Psi_c(x_1) = \Psi_0 + 1 + \dfrac{3}{2}x_1 - \dfrac{1}{2}x_1^3$

- [x] The state space model of the system is:
$$\Sigma : \begin{cases} \dot{x}_1 = -x_2 + \Psi_c(x_1) - 3x_1x_3 \\ \dot{x}_2 = \dfrac{1}{\beta^2}(x_1 - \Phi_T) \\ \dot{x}_3 = \sigma x_3(1 - x_1^2 - x_3) \end{cases}$$
where $\Psi_c(x_1) = \Psi_0 + 1 + \dfrac{3}{2}x_1 - \dfrac{1}{2}x_1^3$

- [ ] The state space model of the system is:
$$\Sigma : \begin{cases} \dot{x}_1 = -x_2 + \Psi_c(x_1) - 3x_1x_3 \\ \dot{x}_2 = \dfrac{1}{\beta^2}(x_2 - \Phi_T) \end{cases}$$
where $\Psi_c(x_1) = \Psi_0 + 1 + \dfrac{3}{2}x_1 - \dfrac{1}{2}x_1^3$

Question 2

A synchronous generator connected to an infinite bus can be represented by

$$M\ddot{\delta} = P - D\dot{\delta} - \eta_1 E_q \sin\delta$$
$$\tau\dot{E}_q = -\eta_2 E_q + \eta_3 \cos\delta + E_F$$

where δ is the angular position of the generator's shaft, E_q is the voltage, P is the mechanical input power, E_F is the input field voltage, D is a damping coefficient, M is an inertial coefficient, τ is the electrical time constant, and $\eta_1,\ \eta_2,\ \eta_3$ are constant parameters.

Let $\delta = \delta_0 \neq 0$ and $P = P_0 \neq 0$ be the steady state values of the generator shaft position and mechanical input power. What is the steady state solution of the system?

- [ ] The steady state solution of the system is:
$$\cdot\,\text{States}:\quad \delta = \delta_0;\ \dot{\delta} = 0;\ E_q = \frac{P_0}{\eta_1 \sin\delta_0}$$
$$\cdot\,\text{Inputs}:\quad P = P_0;\ E_F = 0$$

- [ ] The steady state solution of the system is:
$$\cdot\,\text{States}:\quad \delta = \delta_0;\ \dot{\delta} = 0;\ E_q = \frac{P_0}{\eta_1 \sin\delta_0}$$
$$\cdot\,\text{Inputs}:\quad P = P_0;\ E_F = \frac{\eta_2 P_0}{\eta_1 \sin\delta_0}$$

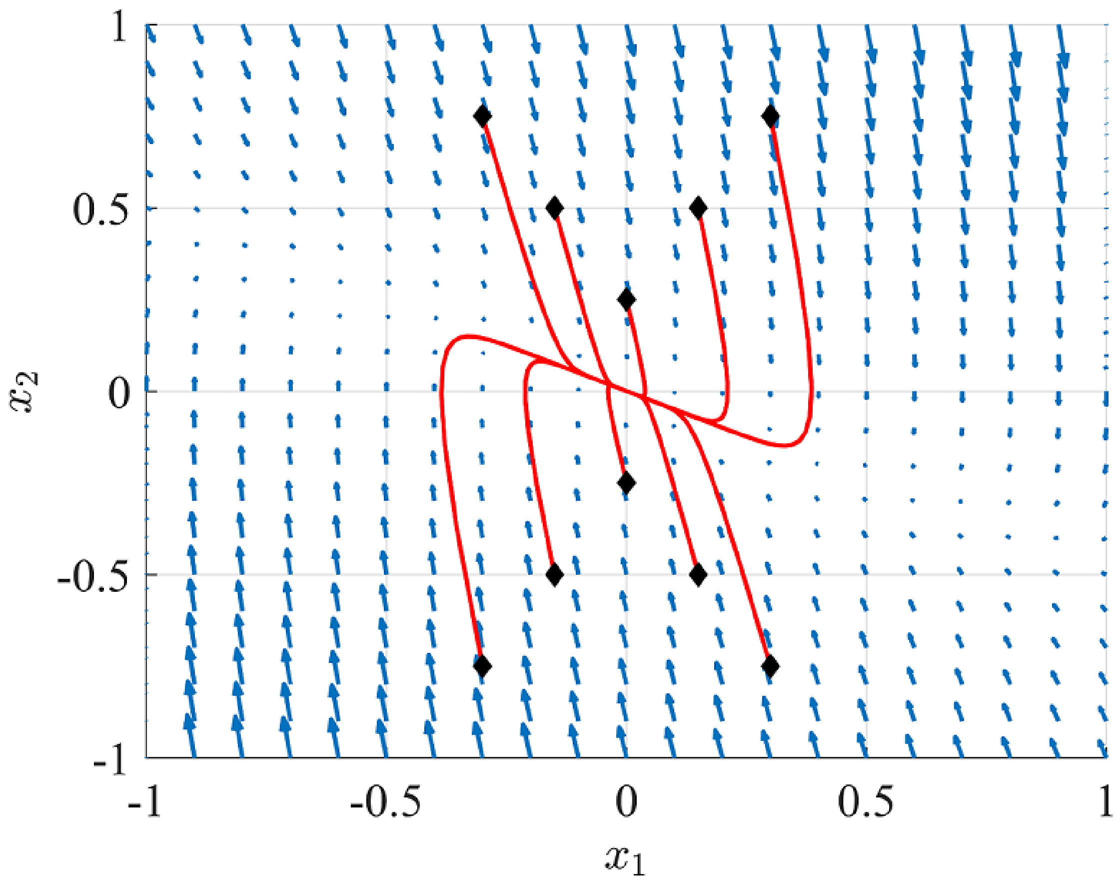
- [ ] The steady state solution of the system is:
$$\cdot\,\text{States}:\quad \delta = \delta_0;\ E_q = \frac{P_0}{\eta_1 \sin\delta_0}$$
$$\cdot\,\text{Inputs}:\quad P = P_0;\ E_F = \frac{\eta_2 P_0}{\eta_1 \sin\delta_0} - \eta_3 \cos\delta_0$$

- [x] The steady state solution of the system is:
$$\cdot\,\text{States}:\quad \delta = \delta_0;\ \dot{\delta} = 0;\ E_q = \frac{P_0}{\eta_1 \sin\delta_0}$$
$$\cdot\,\text{Inputs}:\quad P = P_0;\ E_F = \frac{\eta_2 P_0}{\eta_1 \sin\delta_0} - \eta_3 \cos\delta_0$$

- [ ] The steady state solution of the system is:
$$\cdot\,\text{States}:\quad \delta = \delta_0;\ E_q = \frac{P_0}{\eta_1 \sin\delta_0}$$
$$\cdot\,\text{Inputs}:\quad P = P_0;\ E_F = -\eta_3 \cos\delta_0$$

Question 3

The phase portrait of a second order continuous time LTI system is shown in the following figure (in the given phase portraits each black diamond represents an initial condition $\mathbf{x}(0) = [x_{10}, x_{20}]^{\mathrm{T}}$ for the system; each red line is a trajectory of the system originated from the initial condition; the blue arrows represent the direction of the vector field in the neighborhood of the origin.)



Which of the following statement is correct?

- [x] The equilibrium point is a stable node.
- [ ] The equilibrium point is an unstable focus.
- [ ] The equilibrium point is a saddle point.
- [ ] The equilibrium point is an unstable node.
- [ ] The equilibrium point is a centre.

Question 4

Consider the 4-th order LTI discrete time system

$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{G}\mathbf{u}(k), \quad \mathbf{x} \in \mathbb{R}^4, \mathbf{u} \in \mathbb{R}^2$$
$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k), \quad \mathbf{y} \in \mathbb{R}^2$$

Assume that the reachable subspace has dimension 3

$$\dim(\mathcal{R}) = 3,$$

the observable subspace has dimension 2

$$\dim(\mathcal{O}) = 2,$$

and that the intersection of the two subspaces has dimension 1

$$\dim(\mathcal{R} \cap \mathcal{O}) = 1.$$

Which of the following are the state and output responses? ($\mathbf{x}_0$ is the system initial condition at time $k = 0$)

- [ ] The state and output responses are:

$$\begin{aligned}\mathbf{x}(k) = &\left(\mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0\lambda_1^k\right)\mathbf{v}_1 + \left(\mathbf{w}_2^{\mathrm{T}}\mathbf{x}_0\lambda_2^k\right)\mathbf{v}_2 + \left(\mathbf{w}_3^{\mathrm{T}}\mathbf{x}_0\lambda_3^k\right)\mathbf{v}_3 + \left(\mathbf{w}_4^{\mathrm{T}}\mathbf{x}_0\lambda_4^k\right)\mathbf{v}_4\\
&+ \left(\sum_{i=0}^{k-1}\lambda_1^{k-1+i}\mathbf{w}_1^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_1 + \left(\sum_{i=0}^{k-1}\lambda_2^{k-1+i}\mathbf{w}_2^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_2\\
&+ \left(\sum_{i=0}^{k-1}\lambda_3^{k-1+i}\mathbf{w}_3^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_3 + \left(\sum_{i=0}^{k-1}\lambda_4^{k-1+i}\mathbf{w}_4^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_4\end{aligned}$$

$$\begin{aligned}\mathbf{y}(k) = &\left(\mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0\lambda_1^k\right)\mathbf{C}\mathbf{v}_1 + \left(\mathbf{w}_2^{\mathrm{T}}\mathbf{x}_0\lambda_2^k\right)\mathbf{C}\mathbf{v}_2 + \left(\mathbf{w}_3^{\mathrm{T}}\mathbf{x}_0\lambda_3^k\right)\mathbf{C}\mathbf{v}_3 + \left(\mathbf{w}_4^{\mathrm{T}}\mathbf{x}_0\lambda_4^k\right)\mathbf{C}\mathbf{v}_4\\
&+ \left(\sum_{i=0}^{k-1}\lambda_1^{k-1+i}\mathbf{w}_1^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{C}\mathbf{v}_1 + \left(\sum_{i=0}^{k-1}\lambda_2^{k-1+i}\mathbf{w}_2^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{C}\mathbf{v}_2\\
&+ \left(\sum_{i=0}^{k-1}\lambda_3^{k-1+i}\mathbf{w}_3^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{C}\mathbf{v}_3 + \left(\sum_{i=0}^{k-1}\lambda_4^{k-1+i}\mathbf{w}_4^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{C}\mathbf{v}_4\end{aligned}$$

- [ ] The state and output responses are:

$$\begin{aligned}\mathbf{x}(k) = &\left(\mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0\lambda_1^k\right)\mathbf{v}_1 + \left(\mathbf{w}_2^{\mathrm{T}}\mathbf{x}_0\lambda_2^k\right)\mathbf{v}_2 + \left(\mathbf{w}_3^{\mathrm{T}}\mathbf{x}_0\lambda_3^k\right)\mathbf{v}_3 + \left(\mathbf{w}_4^{\mathrm{T}}\mathbf{x}_0\lambda_4^k\right)\mathbf{v}_4\\
&+ \left(\sum_{i=0}^{k-1}\lambda_1^{k-1+i}\mathbf{w}_1^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_1 + \left(\sum_{i=0}^{k-1}\lambda_3^{k-1+i}\mathbf{w}_3^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_3\\
&+ \left(\sum_{i=0}^{k-1}\lambda_4^{k-1+i}\mathbf{w}_4^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_4\end{aligned}$$

$$\begin{aligned}\mathbf{y}(k) = &\left(\mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0\lambda_1^k\right)\mathbf{C}\mathbf{v}_1 + \left(\mathbf{w}_2^{\mathrm{T}}\mathbf{x}_0\lambda_2^k\right)\mathbf{C}\mathbf{v}_2 + \left(\mathbf{w}_3^{\mathrm{T}}\mathbf{x}_0\lambda_3^k\right)\mathbf{C}\mathbf{v}_3\\
&+ \left(\sum_{i=0}^{k-1}\lambda_1^{k-1+i}\mathbf{w}_1^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{C}\mathbf{v}_1 + \left(\sum_{i=0}^{k-1}\lambda_3^{k-1+i}\mathbf{w}_3^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_3\end{aligned}$$

- [x] The state and output responses are:

$$\mathbf{x}(k) = \left(\mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0\lambda_1^k\right)\mathbf{v}_1 + \left(\mathbf{w}_2^{\mathrm{T}}\mathbf{x}_0\lambda_2^k\right)\mathbf{v}_2 + \left(\mathbf{w}_3^{\mathrm{T}}\mathbf{x}_0\lambda_3^k\right)\mathbf{v}_3 + \left(\mathbf{w}_4^{\mathrm{T}}\mathbf{x}_0\lambda_4^k\right)\mathbf{v}_4 + \left(\sum_{i=0}^{k-1}\lambda_2^{k-1+i}\mathbf{w}_2^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_2 + \left(\sum_{i=0}^{k-1}\lambda_3^{k-1+i}\mathbf{w}_3^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_3 + \left(\sum_{i=0}^{k-1}\lambda_4^{k-1+i}\mathbf{w}_4^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_4$$

$$\mathbf{y}(k) = \left(\mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0\lambda_1^k\right)\mathbf{C}\mathbf{v}_1 + \left(\mathbf{w}_2^{\mathrm{T}}\mathbf{x}_0\lambda_2^k\right)\mathbf{C}\mathbf{v}_2 + \left(\sum_{i=0}^{k-1}\lambda_2^{k-1+i}\mathbf{w}_2^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{C}\mathbf{v}_2$$

- [ ] The state and output responses are:

$$\mathbf{x}(k) = \left(\mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0\lambda_1^k\right)\mathbf{v}_1 + \left(\mathbf{w}_2^{\mathrm{T}}\mathbf{x}_0\lambda_2^k\right)\mathbf{v}_2 + \left(\mathbf{w}_3^{\mathrm{T}}\mathbf{x}_0\lambda_3^k\right)\mathbf{v}_3 + \left(\mathbf{w}_4^{\mathrm{T}}\mathbf{x}_0\lambda_4^k\right)\mathbf{v}_4$$

$$\mathbf{y}(k) = \mathbf{0}$$

- [ ] The state and output responses are:

$$\mathbf{x}(k) = \left(\mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0\lambda_1^k\right)\mathbf{v}_1 + \left(\mathbf{w}_2^{\mathrm{T}}\mathbf{x}_0\lambda_2^k\right)\mathbf{v}_2 + \left(\mathbf{w}_3^{\mathrm{T}}\mathbf{x}_0\lambda_3^k\right)\mathbf{v}_3 + \left(\mathbf{w}_4^{\mathrm{T}}\mathbf{x}_0\lambda_4^k\right)\mathbf{v}_4 + \left(\sum_{i=0}^{k-1}\lambda_1^{k-1+i}\mathbf{w}_1^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_1 + \left(\sum_{i=0}^{k-1}\lambda_3^{k-1+i}\mathbf{w}_3^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_3 + \left(\sum_{i=0}^{k-1}\lambda_4^{k-1+i}\mathbf{w}_4^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{v}_4$$

$$\mathbf{y}(k) = \left(\mathbf{w}_1^{\mathrm{T}}\mathbf{x}_0\lambda_1^k\right)\mathbf{C}\mathbf{v}_1 + \left(\mathbf{w}_3^{\mathrm{T}}\mathbf{x}_0\lambda_3^k\right)\mathbf{C}\mathbf{v}_3 + \left(\mathbf{w}_4^{\mathrm{T}}\mathbf{x}_0\lambda_4^k\right)\mathbf{C}\mathbf{v}_4 + \left(\sum_{i=0}^{k-1}\lambda_1^{k-1+i}\mathbf{w}_1^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{C}\mathbf{v}_1 + \left(\sum_{i=0}^{k-1}\lambda_3^{k-1+i}\mathbf{w}_3^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{C}\mathbf{v}_3 + \left(\sum_{i=0}^{k-1}\lambda_4^{k-1+i}\mathbf{w}_4^{\mathrm{T}}\mathbf{G}\mathbf{u}(i)\right)\mathbf{C}\mathbf{v}_4$$

Page 2

Analysis of open loop systems - Part 2

Question 5

Given the 3rd order LTI discrete time SISO system

$$\mathbf{x}(k+1) = \begin{bmatrix} -\alpha & 0 & 0 \\ 0 & \gamma & \beta \\ 0 & -\beta & \gamma \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(k), \qquad \mathbf{x} \in \mathbb{R}^3,\, u \in \mathbb{R}$$

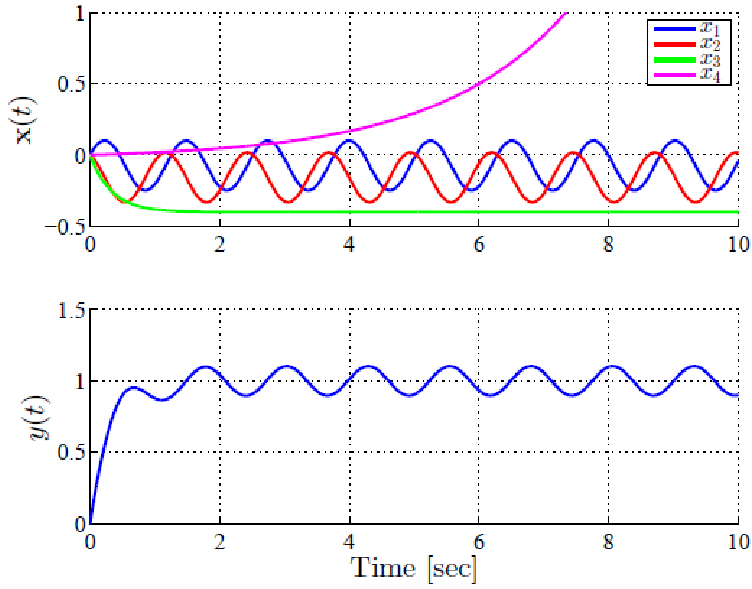
$$y(k) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \mathbf{x}(k), \qquad y \in \mathbb{R}$$

where α, β, γ are real coefficients. Which of the following statements is correct?

- [ ] The system is asymptotically stable for any triple $(\alpha, \beta, \gamma) \in \mathbb{R}^3$.
- [x] The system is asymptotically stable if $|\alpha| < 1 \wedge |\gamma| < 1 \wedge \beta < \sqrt{1-\gamma^2}$
- [ ] The system is asymptotically stable if $\alpha > 0 \wedge \gamma < 0 \wedge \forall \beta \in \mathbb{R}$
- [ ] The system is asymptotically stable if $|\alpha| < 1 \wedge |\gamma| < 1 \wedge \forall \beta \in \mathbb{R}$
- [ ] There is no triple $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ that renders the system asymptotically stable.

Question 6

Consider the step response of a 4th order LTI continuous time SISO system with zero initial conditions shown in the following figure. Which of the following statements is correct?



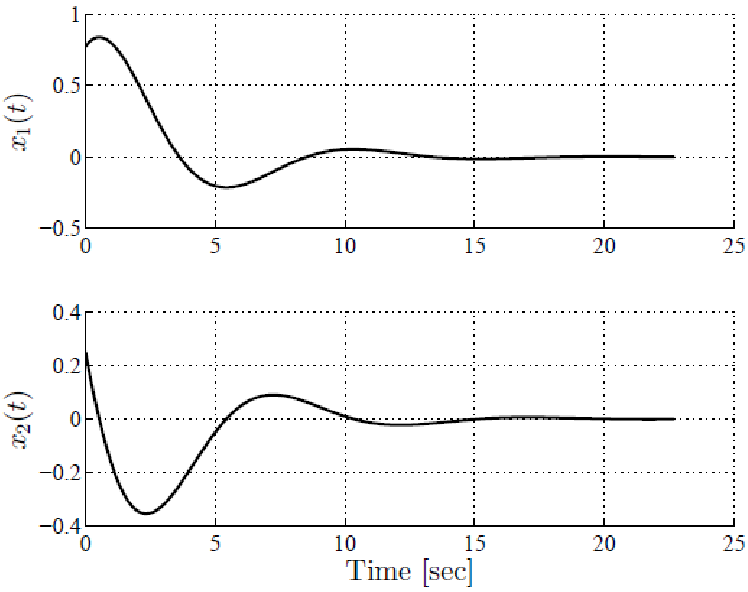
- [ ] The system is internally asymptotically stable and BIBO stable.
- [ ] The system has an unstable eigenmode, which is not controllable.
- [ ] The system is internally asymptotically stable but not observable.
- [x] The system has an unstable eigenmode, which is not observable.
- [ ] The system is internally marginally stable but not BIBO stable.

Question 7

Consider the 2nd order LTI continuous time SISO system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x}$$

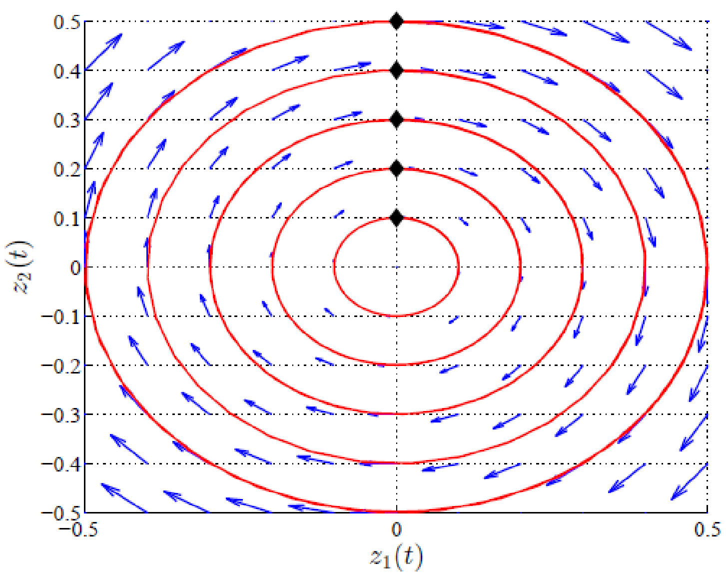
where $\mathbf{x} \in \mathbb{R}^2$, $u \in \mathbb{R}$, and $y \in \mathbb{R}$. The zero-input response of the system to a given initial condition $\mathbf{x} = [x_{10}, x_{20}]^{\mathrm{T}} \neq [0, 0]^{\mathrm{T}}$ is shown in the following figure.

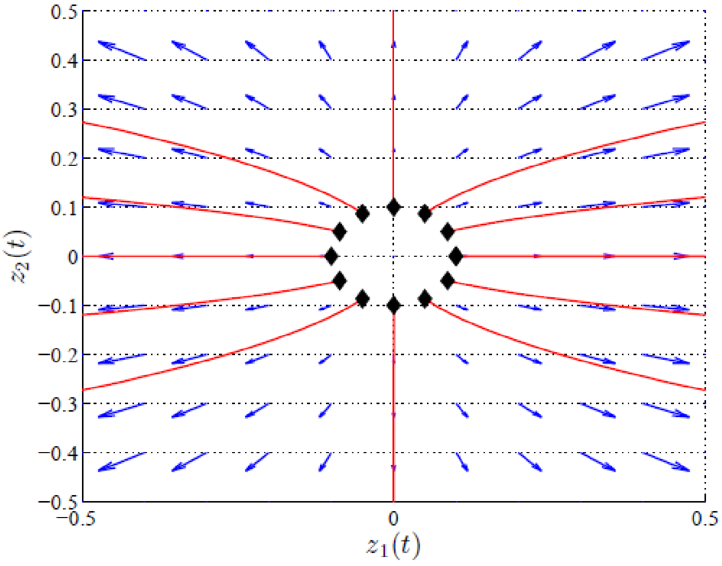


Consider the system Σ_z

$$\Sigma_z : \begin{cases} \dot{\mathbf{z}} = \mathbf{\Lambda}\mathbf{z} + \mathbf{B}_z u \\ y = \mathbf{C}_z \mathbf{z} \end{cases}$$

obtained through the similarity transformation $\mathbf{z} = \mathbf{M}^{-1}\mathbf{x}$, where $\mathbf{M}$ is the modal matrix. Which of the following phase portraits is that one associated with the dynamics of the system Σ_z? (In the given phase portraits each black diamond represents an initial condition $\mathbf{z}(0) = [z_{10}, z_{20}]^{\mathrm{T}}$ for the system Σ_z ; each red line is a trajectory of the system originated from the initial condition; the blue arrows represent the direction of the vector field in the neighbourhood of the origin.)

☐ 

☐ 

0.5
0.4
0.3
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0
−0.1
−0.2
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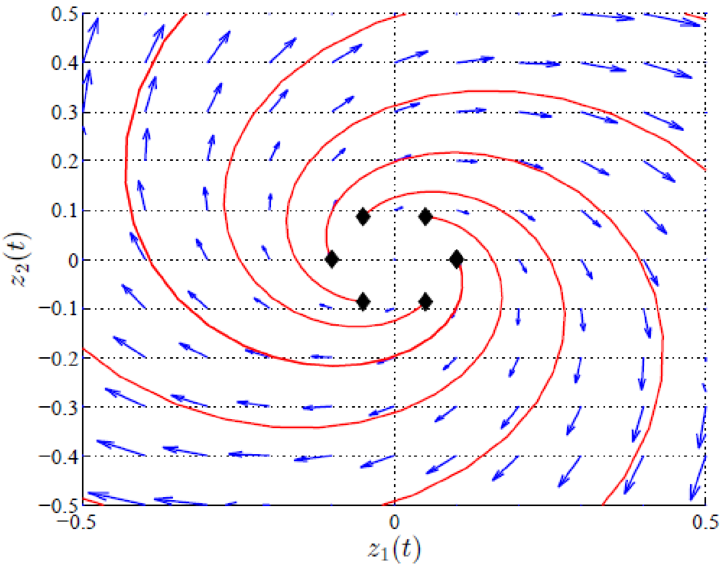
−0.5
−0.5
0
0.5
$z_1(t)$
$z_2(t)$

☐ 

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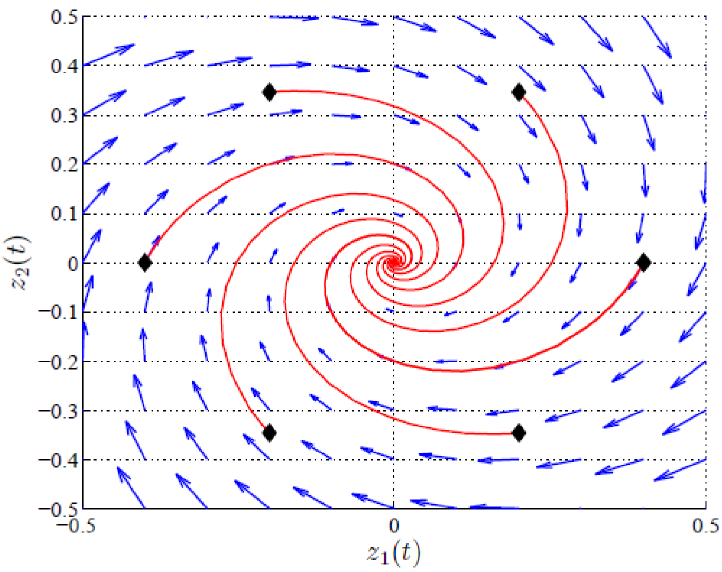
$z_1(t)$
$z_2(t)$

☑ 

0.5
0.4
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0.1
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−0.1
−0.2
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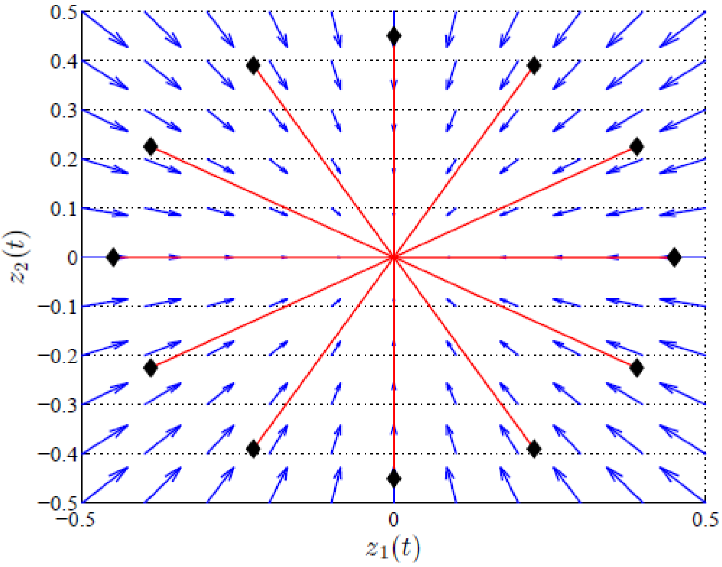
−0.5
0
0.5
$z_1(t)$
$z_2(t)$

☐ 
0.5
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0
−0.1
−0.2
−0.3
−0.4
−0.5
$z_2(t)$
−0.5
0
0.5
$z_1(t)$

Page 3

Analysis of closed-loop systems - Part 1

Question 8

Consider the 2nd order LTI continuous time SISO system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} u + \begin{bmatrix} 0 \\ \alpha \end{bmatrix} d, \qquad \mathbf{x} \in \mathbb{R}^2, u \in \mathbb{R}, d \in \mathbb{R}$$
$$y = \begin{bmatrix} \gamma & 0 \end{bmatrix} \mathbf{x}, \qquad y \in \mathbb{R}$$

where $\omega_n > 0,\ \zeta > 0,\ \alpha > 0,\ \gamma > 0$ are constant parameters. The system is subject to the unknown constant disturbance d.

An output feedback control law is designed as

$$u = -\mathbf{K}\hat{\mathbf{x}} - K_d\hat{d} + Nr$$

where $\mathbf{K} = [K_1\ K_2]$ is the vector of controller gains, $K_d \in \mathbb{R}$ is the disturbance rejection gain, and $N \in \mathbb{R}$ is the reference feedforward gain. $\hat{\mathbf{x}}$ and $\hat{d}$ are the estimates of the state and disturbance provided by the observer. Said y_{des} the desired value of the output y of the closed-loop system, set $r = y_{\text{des}}$.

What are the values of K_d and N that guarantee the fulfillment of the control objective

$$\lim_{t\to\infty} y(t) = r\,?$$

- [ ] $\begin{cases} K_d = \alpha \\ N = \dfrac{\omega_n^2(1+K_1)}{\gamma(1+K_2)} \end{cases}$
- [ ] $\begin{cases} K_d = \alpha \\ N = 1 \end{cases}$
- [x] $\begin{cases} K_d = \dfrac{\alpha}{\omega_n^2} \\ N = \dfrac{(1+K_1)}{\gamma} \end{cases}$
- [ ] $\begin{cases} K_d = \dfrac{\alpha}{\omega_n^2} \\ N = \dfrac{\omega_n^2+K_1}{\gamma\omega_n^2} \end{cases}$
- [ ] There are no values of K_d and N that can fulfil the control objective because the controller does not have integral action.

Question 9

Consider the second order LTI continuous time system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u + \mathbf{B}_v v_1 \quad && \mathbf{x} \in \mathbb{R}^2,\ u \in \mathbb{R},\ v_1 \in \mathbb{R} \\ y &= \mathbf{C}\mathbf{x} + v_2 && y \in \mathbb{R},\ v_2 \in \mathbb{R} \end{aligned}$$

v_1 is a disturbance acting on the system, and it is given by the sum of an unknown constant component $\bar{v}_1$ with a stochastic component $\tilde{v}_1$. $\tilde{v}_1$ is zero mean band-limited noise described by the following autocorrelation function

$$R_{\tilde{v}_1}(\tau) = \sigma_1^2\, e^{-\beta|\tau|}$$

with $\sigma_v^2 > 0$ and $\beta > 0$. v_2 is zero mean white noise with intensity σ_2^2.

Under the assumption that the pair $(\mathbf{A}, \mathbf{C})$ is observable, a Kalman filter is to be designed to estimate the state $\mathbf{x}$ as well as the disturbance v_1. Which of the following state space models provides the correct architecture for the design of the Kalman filter?

- [x] $$\begin{cases} \dot{\mathbf{x}}_{kf} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_v & \mathbf{B}_v \\ \mathbf{0} & 0 & 0 \\ 0 & 0 & -\beta \end{bmatrix} \mathbf{x}_{kf} + \begin{bmatrix} \mathbf{B} \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ 1 & 0 \\ 0 & \sqrt{2\beta}\sigma_1 \end{bmatrix} \boldsymbol{\eta} \\ y_{kf} = \begin{bmatrix} \mathbf{C} & 0 & 0 \end{bmatrix} \mathbf{x}_{kf} + v_2 \end{cases}$$

  where $\mathbf{x}_{kf} = \left[\mathbf{x}^{\mathrm{T}}, \bar{v}_1, \tilde{v}_1\right]^{\mathrm{T}}$ is the state of the Kalman filter and $\boldsymbol{\eta} = [\eta_1, \eta_2]^{\mathrm{T}}$ is the process noise.

- [ ] $$\begin{cases} \dot{\mathbf{x}}_{kf} = \begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 0 \\ 0 & 0 & -\beta \end{bmatrix} \mathbf{x}_{kf} + \begin{bmatrix} \mathbf{B} \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ 1 & 0 \\ 0 & \sqrt{2\beta}\sigma_1 \end{bmatrix} \boldsymbol{\eta} \\ y_{kf} = \begin{bmatrix} \mathbf{C} & 0 & 0 \end{bmatrix} \mathbf{x}_{kf} + v_2 \end{cases}$$

  where $\mathbf{x}_{kf} = \left[\mathbf{x}^{\mathrm{T}}, \bar{v}_1, \tilde{v}_1\right]^{\mathrm{T}}$ is the state of the Kalman filter and $\boldsymbol{\eta} = [\eta_1, \eta_2]^{\mathrm{T}}$ is the process noise.

- [ ] $$\begin{cases} \dot{\mathbf{x}}_{kf} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_v & \mathbf{B}_v \\ \mathbf{0} & 0 & 0 \\ 0 & 0 & -\beta \end{bmatrix} \mathbf{x}_{kf} + \begin{bmatrix} \mathbf{B} \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{\eta} \\ y_{kf} = \begin{bmatrix} \mathbf{C} & 0 & 0 \end{bmatrix} \mathbf{x}_{kf} + v_2 \end{cases}$$

  where $\mathbf{x}_{kf} = \left[\mathbf{x}^{\mathrm{T}}, \bar{v}_1, \tilde{v}_1\right]^{\mathrm{T}}$ is the state of the Kalman filter and $\boldsymbol{\eta} = [\eta_1, \eta_2]^{\mathrm{T}}$ is the process noise.

- [ ] $$\begin{cases} \dot{\mathbf{x}}_{kf} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_v \\ \mathbf{0} & 0 \end{bmatrix} \mathbf{x}_{kf} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} \eta \\ y_{kf} = \begin{bmatrix} \mathbf{C} & 0 \end{bmatrix} \mathbf{x}_{kf} + v_2 \end{cases}$$

  where $\mathbf{x}_{kf} = \left[\mathbf{x}^{\mathrm{T}}, \bar{v}_1\right]^{\mathrm{T}}$ is the state of the Kalman filter and η is the process noise.

☐ $$\begin{cases} \dot{\mathbf{x}}_{kf} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_v \\ \mathbf{0} & -\beta \end{bmatrix} \mathbf{x}_{kf} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u + \begin{bmatrix} \mathbf{0} \\ \sqrt{2\beta}\sigma_1 \end{bmatrix} \eta \\ y_{kf} = \begin{bmatrix} \mathbf{C} & 0 \end{bmatrix} \mathbf{x}_{kf} + v_2 \end{cases}$$

where $\mathbf{x}_{kf} = \left[\mathbf{x}^{\mathrm{T}}, \tilde{v}_1\right]^{\mathrm{T}}$ is the state of the Kalman filter and η is the process noise.

Question 10

Consider the 3rd order LTI discrete time SISO system

$$\mathbf{x}(k+1) = \begin{bmatrix} -\alpha & \beta & 0 \\ -\beta & -\alpha & \gamma \\ 0 & \gamma & \delta \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ \varepsilon \\ 0 \end{bmatrix} u(k), \qquad \mathbf{x} \in \mathbb{R}^3, u \in \mathbb{R}$$
$$y(k) = \begin{bmatrix} 0 & 0 & \nu \end{bmatrix} \mathbf{x}(k), \qquad y \in \mathbb{R}$$

where $\alpha, \beta, \gamma, \delta, \varepsilon,$ and ν are real and positive coefficients.

A full order observer is designed for the given system as

$$\hat{\mathbf{x}}(k+1) = \begin{bmatrix} -\alpha & \beta & 0 \\ -\beta & -\alpha & \gamma \\ 0 & \gamma & \delta \end{bmatrix} \hat{\mathbf{x}}(k) + \begin{bmatrix} 0 \\ \varepsilon \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} (y(k) - \hat{y}(k))$$
$$\hat{y}(k) = \begin{bmatrix} 0 & 0 & \nu \end{bmatrix} \hat{\mathbf{x}}(k)$$

Define the estimation error as $\mathbf{e}_e(k) = \mathbf{x}(k) - \hat{\mathbf{x}}(k)$. Which values of the observer gains render the estimation error exactly equal to zero for $k \geq 3$ regardless of the estimation error initial condition?

☐ The observer gains are:
$l_1 = 0;\ l_2 = 0;\ l_3 = 0$

☐ The observer gains are:
$l_1 = \frac{\alpha}{\nu};\ l_2 = \frac{\alpha}{\nu};\ l_3 = \frac{\delta}{\nu}$

☑ The observer gains are:
$l_1 = \frac{\alpha^3 - 3\alpha\beta^2}{\beta\gamma\nu};\ l_2 = \frac{3\alpha^2 - \beta^2 + \gamma^2}{\gamma\nu};$
$l_3 = \frac{-2\alpha + \delta}{\nu}$

☐ The observer gains are:
$l_1 = \frac{\alpha^3}{\beta\gamma\nu};\ l_2 = \frac{3\alpha^2 - \beta^2}{\gamma\nu};\ l_3 = \frac{-2\alpha}{\nu}$

☐ The observer gains are:
$l_1 = \frac{\alpha(1 - 3\beta^2)}{\gamma\nu};\ l_2 = \frac{3\alpha^2 + \gamma^2}{\gamma\nu};$
$l_3 = \frac{-2\alpha + \delta}{\beta\nu}$

Page 4

Analysis of closed-loop systems - Part 2

Question 11

Consider the n-th order LTI continuous time SISO system

$$\Sigma_x : \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u + \mathbf{B}_d d, & \mathbf{x} \in \mathbb{R}^n, u \in \mathbb{R}, d \in \mathbb{R} \\ y = \mathbf{C}\mathbf{x}, & y \in \mathbb{R} \end{cases}$$

The observability matrix satisfies the following rank condition

$$\text{rank}(\mathbf{M}_o) = n$$

The system is subject to a time-varying not measurable disturbance given by

$$d(t) = d_1 \sin(\beta_1 t) + d_2 \sin(\beta_2 t)$$

where d_1, d_2, β_1 and β_2 are positive real constants. Which of the following state estimators estimates the disturbance $d(t)$?

- [x] The dynamics of the state estimator is
$$\begin{bmatrix} \dot{\hat{\mathbf{x}}} \\ \dot{\hat{\mathbf{w}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_d\mathbf{C}_w \\ \mathbf{0} & \mathbf{A}_w \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} u + \begin{bmatrix} \mathbf{L}_x \\ \mathbf{L}_w \end{bmatrix} (y - \hat{y})$$
$$\hat{y} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix}$$
where
$$\mathbf{A}_w = \begin{bmatrix} 0 & -\beta_1 & 0 & 0 \\ \beta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta_2 \\ 0 & 0 & \beta_2 & 0 \end{bmatrix} \quad , \quad \mathbf{C}_w = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$$

- [ ] The dynamics of the state estimator is
$$\begin{bmatrix} \dot{\hat{\mathbf{x}}} \\ \dot{\hat{\mathbf{w}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_d\mathbf{C}_w \\ \mathbf{0} & \mathbf{A}_w \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} u + \begin{bmatrix} \mathbf{L}_x \\ \mathbf{L}_w \end{bmatrix} (y - \hat{y})$$
$$\hat{y} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix}$$
where
$$\mathbf{A}_w = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad \mathbf{C}_w = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

- [ ] The dynamics of the state estimator is
$$\begin{bmatrix} \dot{\hat{\mathbf{x}}} \\ \dot{\hat{\mathbf{w}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_d\mathbf{C}_w \\ \mathbf{0} & \mathbf{A}_w \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} u + \begin{bmatrix} \mathbf{L}_x \\ \mathbf{L}_w \end{bmatrix} (y - \hat{y})$$
$$\hat{y} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix}$$
where
$$\mathbf{A}_w = \begin{bmatrix} -\beta_1 & 0 \\ 0 & -\beta_2 \end{bmatrix} \quad , \quad \mathbf{C}_w = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

The dynamics of the state estimator is

$$\begin{bmatrix}\dot{\hat{\mathbf{x}}}\\ \dot{\hat{\mathbf{w}}}\end{bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{B}_d\mathbf{C}_w\\ \mathbf{0} & \mathbf{A}_w\end{bmatrix}\begin{bmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{w}}\end{bmatrix} + \begin{bmatrix}\mathbf{B}\\ \mathbf{0}\end{bmatrix}u + \begin{bmatrix}\mathbf{L}_x\\ \mathbf{L}_w\end{bmatrix}(y-\hat{y})$$

- [ ] $$\hat{y} = \begin{bmatrix}\mathbf{C} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{w}}\end{bmatrix}$$

where

$$\mathbf{A}_w = \begin{bmatrix}0 & -\beta_1 & 0\\ \beta_1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \quad , \quad \mathbf{C}_w = \begin{bmatrix}1 & 0 & 1\end{bmatrix}$$

- [ ] It is not possible to design a state estimator to reconstruct the disturbance $d(t)$ because the system Σ_x is not observable.

Question 12

Consider the 2nd order LTI continuous time SISO system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} u, \qquad \mathbf{x} \in \mathbb{R}^2, u \in \mathbb{R}$$
$$y = \begin{bmatrix} \gamma & 0 \end{bmatrix} \mathbf{x}, \qquad y \in \mathbb{R}$$

where $\omega_n > 0,\ \zeta > 0,\ \gamma > 0$ are constant parameters.

Three steady state optimal controllers are designed using the performance index

$$J(\mathbf{u}) = \int_0^{+\infty} \mathbf{x}^{\mathrm{T}}(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^{\mathrm{T}}(t)\mathbf{R}\mathbf{u}(t)\,\mathrm{d}t$$

where the weighting matrices have the following structure

$$\mathbf{Q} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} > 0 \quad , \quad \mathbf{R} = \rho > 0$$

The first controller $\mathbf{K}_1$ is designed choosing (the symbol $\gg$ means "much larger than")

$$\alpha > \beta \gg \rho.$$

The second controller $\mathbf{K}_2$ is designed choosing

$$\beta > \alpha \gg \rho.$$

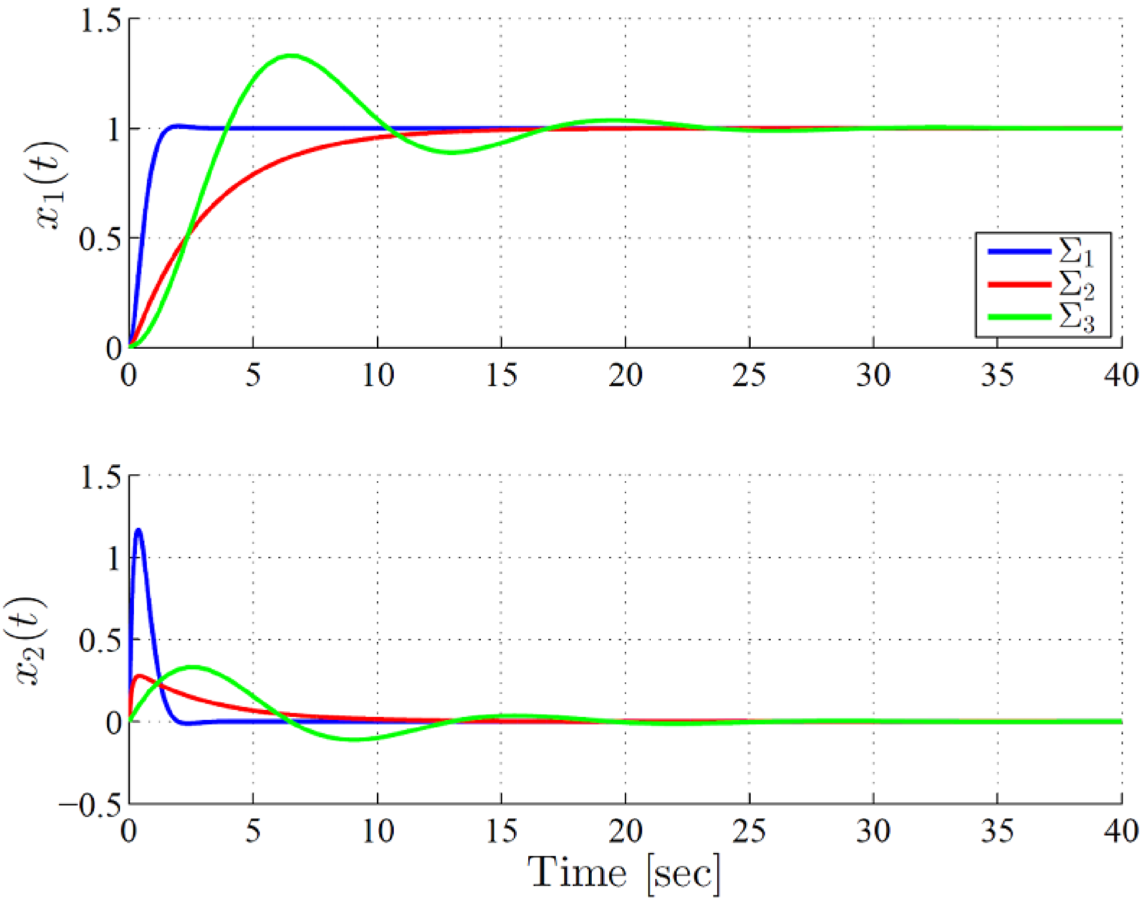
The third controller $\mathbf{K}_3$ is designed choosing

$$\rho \gg \alpha > \beta.$$

Said $u_i = -\mathbf{K}_i x + N_i r$ the control signal with N_i the reference feedforward gain, the following figure shows the state response of the three closed-loop systems ($i = 1, 2, 3$)

$$\Sigma_i : \begin{cases} \dot{\mathbf{x}} = \left(\begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} - \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} \mathbf{K}_i \right) \mathbf{x} + \begin{bmatrix} 0 \\ \omega_n^2 N_i \end{bmatrix} r \\ y = \begin{bmatrix} \gamma & 0 \end{bmatrix} \mathbf{x} \end{cases}$$

when a step of amplitude one is sent through the reference r.



The control effort (CE) of the controller u_i over the time horizon $t \in [0, +\infty)$ is defined as

$$\mathrm{CE}(u_i) = \int_0^{+\infty} u_i^2(t)\,\mathrm{d}t$$

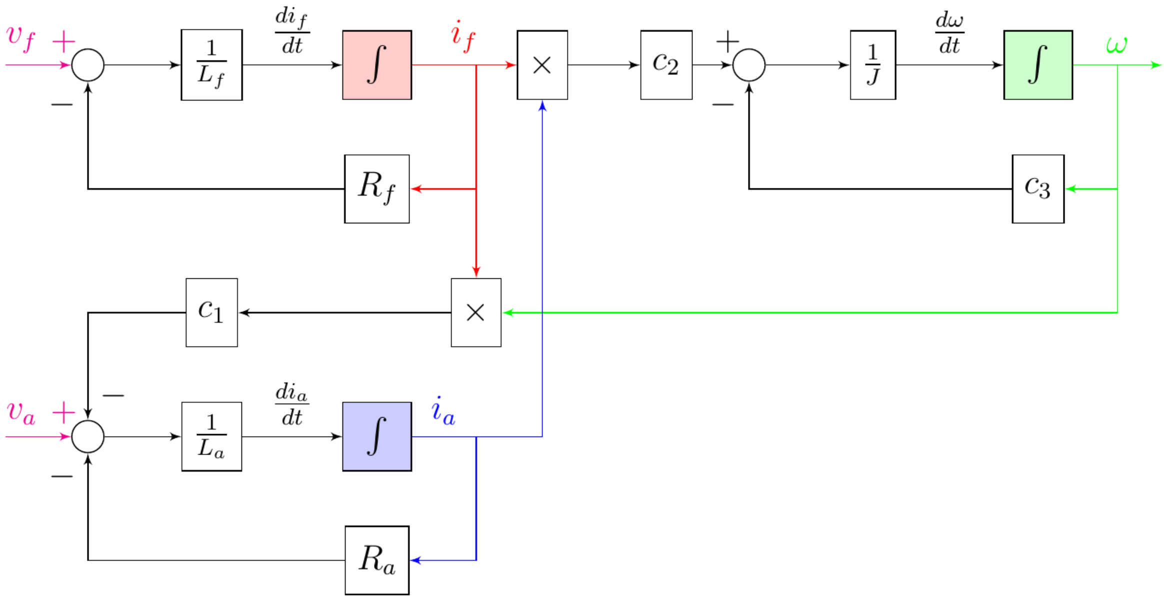
Taking into account the design choices for the controllers $\mathbf{K}_i$, and considering the step responses in the figure above, which of the following statements is correct?

- ☐ $\mathrm{CE}(u_1) > \mathrm{CE}(u_3) > \mathrm{CE}(u_2)$
- ☐ $\mathrm{CE}(u_2) > \mathrm{CE}(u_1) > \mathrm{CE}(u_3)$
- ☑ $\mathrm{CE}(u_1) > \mathrm{CE}(u_2) > \mathrm{CE}(u_3)$
- ☐ $\mathrm{CE}(u_1) = \mathrm{CE}(u_3) = \mathrm{CE}(u_2)$
- ☐ $\mathrm{CE}(u_3) > \mathrm{CE}(u_2) > \mathrm{CE}(u_1)$

Exam 31310 E21 - Multiple Choice Questionnaire

The multiple choice questionnaire includes 12 questions. Each question is provided with 5 answers. There is only one correct choice per question. All questions are equally scored. You can only give one answer for each question.

The block diagram shown in the following figure provides a graphical representation of the dynamics of a DC motor.



The constants L_a, L_f, R_a, R_f, J, c_1, c_2 and c_3 are all positive. The signals v_a and v_f are the armature and field voltages, respectively. The signals i_a and i_f are the armature and field currents, respectively. The signal ω is the motor's shaft angular velocity.

What is the nonlinear state space model associated with this block diagram?

Choose one answer

$$\Sigma : \begin{cases} \dfrac{d\omega}{dt} = \dfrac{1}{J}(-c_3\omega + c_2 i_a i_f) \\ \dfrac{di_f}{dt} = \dfrac{1}{L_f}(-R_f i_f + v_f) \\ \dfrac{di_a}{dt} = \dfrac{1}{L_a}(-R_a i_a - c_1 i_f \omega + v_a) \end{cases}$$

$$\Sigma : \begin{cases} \dfrac{d\omega}{dt} = \dfrac{1}{J}(-c_3\omega + c_2 i_a i_f) \\ \dfrac{di_f}{dt} = -\dfrac{R_f}{L_f} i_f + v_f \\ \dfrac{di_a}{dt} = -\dfrac{R_a}{L_a} - c_1 i_f \omega + v_a \end{cases}$$

$$\Sigma : \begin{cases} \dfrac{di_f}{dt} = \dfrac{1}{L_f}(-R_f i_f + v_f) \\ \dfrac{di_a}{dt} = \dfrac{1}{L_a}(-R_a i_a - c_1 i_f \omega + v_a) \end{cases}$$

$$\Sigma : \begin{cases} \dfrac{d\omega}{dt} = \dfrac{1}{J}(-c_3\omega + c_2 i_a i_f) \\ \dfrac{di_f}{dt} = \dfrac{1}{L_f}(-R_f i_f - c_1 i_a \omega + v_f) \\ \dfrac{di_a}{dt} = \dfrac{1}{L_a}(-R_a i_a + v_a) \end{cases}$$

$$\Sigma : \begin{cases} \dfrac{d\theta}{dt} = \omega \\ \dfrac{d\omega}{dt} = \dfrac{1}{J}(-c_3\omega + c_2 i_a i_f) \\ \dfrac{di_f}{dt} = \dfrac{1}{L_f}(-R_f i_f + v_f) \\ \dfrac{di_a}{dt} = \dfrac{1}{L_a}(-R_a i_a - c_1 i_f \omega + v_a) \end{cases}$$

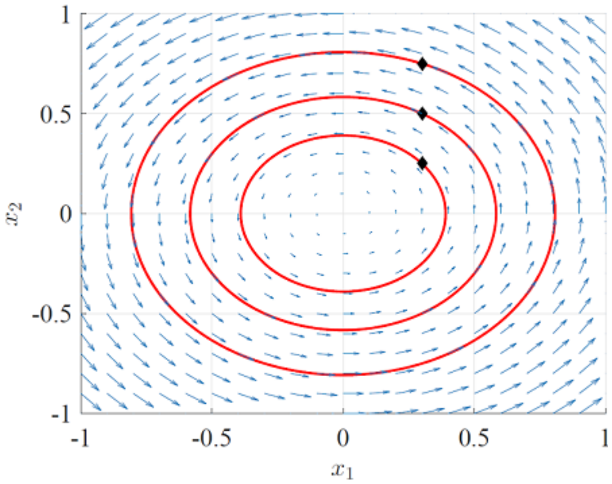
$\boxed{Q1}$ Derive non-linear equations from diagram.

$$\frac{d\omega}{dt} = \frac{1}{J}\left(c_2 \cdot i_f \cdot i_a - c_3 \omega\right)$$

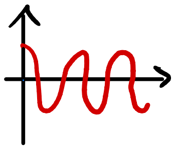
$$\frac{di_f}{dt} = \frac{1}{L_f}\left(v_f - i_f R_f\right)$$

$$\frac{di_a}{dt} = \frac{1}{L_a}\left(v_a - i_a R_a - c_1 \cdot i_f \cdot \omega\right)$$

The phase portrait associated with the dynamics of a second order LTI system Σ_x is shown in the following figure



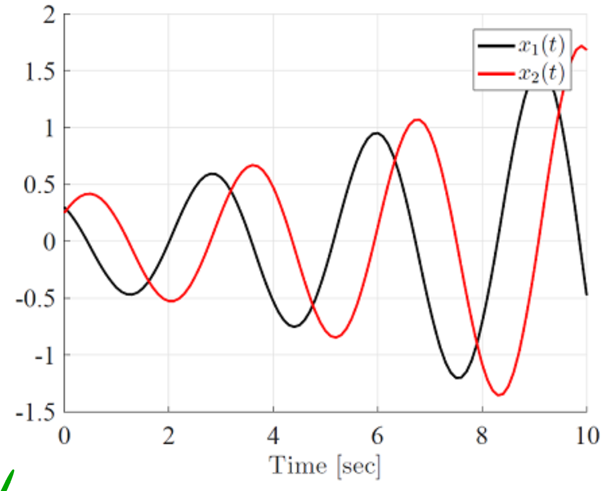
→ Marginally stable system.

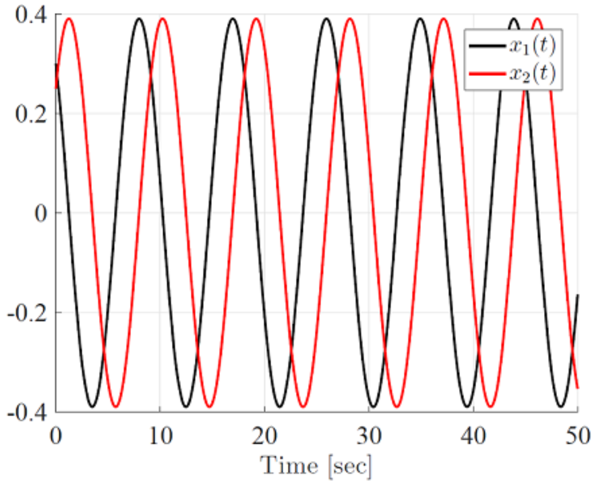


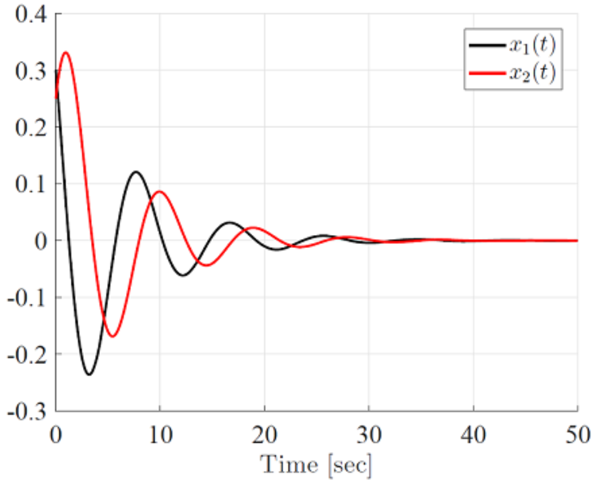
where each black diamond represents an initial condition $\mathbf{x}(0) = [x_{10}, x_{20}]^{\mathrm{T}}$ for the system Σ_x; each red line is a trajectory of the system's solution originated from the initial condition; the blue arrows represent the direction of the vector field.

What is the zero-input response associated with the given phase portrait and that originates from an arbitrary initial condition $\mathbf{x}(0) = [x_{10}, x_{20}]^{\mathrm{T}} \neq [0, 0]^{\mathrm{T}}$?

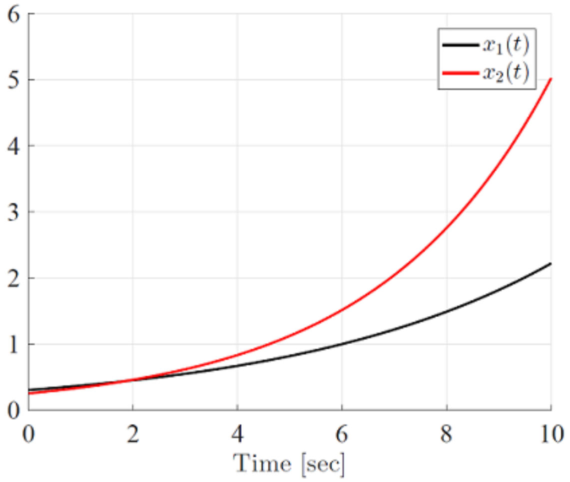
Choose one answer



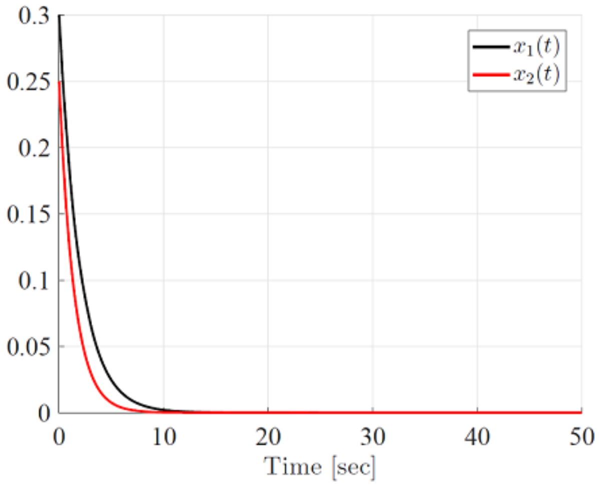



0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0
10
20
30
40
50
Time [sec]
$x_1(t)$
$x_2(t)$

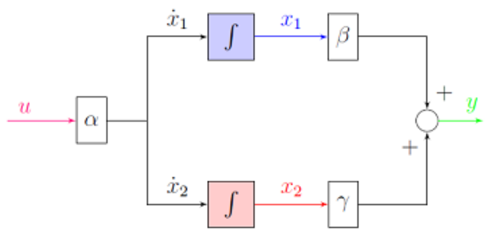
○


6
5
4
3
2
1
0
0
2
4
6
8
10
Time [sec]
$x_1(t)$
$x_2(t)$

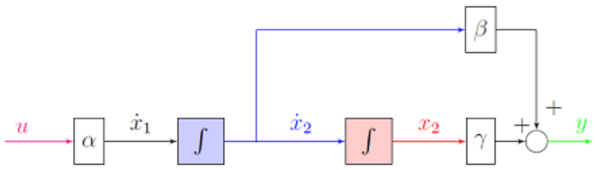
○


0.3
0.25
0.2
0.15
0.1
0.05
0
0
10
20
30
40
50
Time [sec]
$x_1(t)$
$x_2(t)$

Consider the two second order LTI SISO systems shown in the following figure, where the constants α, β and γ are positive.



Block diagram of system Σ_1



Block diagram of system Σ_2

Which of the following statements is <u>**not correct**</u>?

Choose one answer

- ○ The zero-input output responses for the given equal initial condition $\mathbf{x}_0 = [x_{10}, x_{20}]^{\mathrm{T}} \neq [0,0]^{\mathrm{T}}$ are

$$\Sigma_1 : y(t) = \beta x_{10} + \gamma x_{20}$$
$$\Sigma_2 : y(t) = \beta x_{10} + \gamma(x_{10}t + x_{20})$$

- ✓ Σ_1 and Σ_2 are both unstable systems. This statement is wrong.
- ○ Σ_1 and Σ_2 have both two eigenvalues equal to zero.
- ○ Σ_1 is a marginally stable system and Σ_2 is an unstable system.
- ○ The zero-state output responses for $u = u_0$ for all $t \geq 0$ are

$$\Sigma_1 : y(t) = \alpha(\beta + \gamma)u_0 t$$
$$\Sigma_2 : y(t) = \alpha\left(\beta + \frac{1}{2}\gamma t\right) u_0 t$$

$\boxed{Q3}$ Stability analysis.

$$\Sigma_1: \quad \dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} u \qquad \lambda = 0, \quad alg(\lambda) = 2$$

$$y = \begin{bmatrix} \beta & \gamma \end{bmatrix} x \qquad \qquad \| \Rightarrow \text{marginally stable}$$

$$geo(\lambda) = 2$$

$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Sigma_2: \quad \dot{x} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} \alpha \\ 0 \end{bmatrix} u \qquad \lambda = 0, \quad alg(\lambda) = 2$$

$$y = \begin{bmatrix} \beta & \gamma \end{bmatrix} x \qquad geo(\lambda) = \dim(A) - \operatorname{rank}(A - \lambda I)$$

$$= 1 \neq alg(\lambda)$$

$$\Rightarrow \text{unstable.} \qquad v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Q4

Given the third order LTI system

$$\Sigma : \begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} \alpha & 0 & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & -\beta \end{bmatrix} \mathbf{x} + \begin{bmatrix} \gamma_1 & \gamma_2 \\ 0 & 0 \\ \gamma_3 & 0 \end{bmatrix} \mathbf{u}, \quad \mathbf{x} \in \mathbb{R}^3, \mathbf{u} \in \mathbb{R}^2 \\ y = [\alpha \quad \beta \quad \alpha]\,\mathbf{x}, \quad y \in \mathbb{R} \end{cases}$$

where α, β, are real and positive coefficients, while γ_1, γ_2, and γ_3 are real. Which of the following statements is correct?

Choose one answer

- ○ The system is controllable if and only if $\gamma_2 \neq 0 \wedge \forall \gamma_1, \gamma_3 \in \mathbb{R}$.
- ✓ The system is controllable if and only if $(\gamma_1 \neq 0 \wedge \gamma_3 \neq 0) \vee (\gamma_2 \neq 0 \wedge \gamma_3 \neq 0)$.
- ○ There is no triple $(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^3$ that makes the system controllable.
- ○ The system is controllable for any triple $(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^3$.
- ○ The system is controllable if and only if $\gamma_1 \neq 0 \wedge \gamma_2 \neq 0 \wedge \gamma_3 \neq 0$.

$M_c = [B \;\; AB \;\; A^2B]$

$\dot{x}_2$ depends only on x_1, so either γ_1 or γ_2 has to be not zero. γ_1 or $\gamma_2 \neq 0$

$\dot{x}_3$ depends only on x_3. So $\gamma_3 \neq 0$.

Check Matlab for additional calculations.

$\boxed{Q5}$

Consider the third order LTI system

$$\Sigma : \begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} -\alpha & 1 & 0 \\ 0 & -\beta & 0 \\ \gamma & 0 & -\delta \end{bmatrix} \mathbf{x}, & \mathbf{x} \in \mathbb{R}^3 \\ \mathbf{y} = \begin{bmatrix} 0 & \epsilon_1 & 0 \\ \epsilon_2 & 0 & \epsilon_3 \end{bmatrix} \mathbf{x}, & \mathbf{y} \in \mathbb{R}^2 \end{cases} = \begin{matrix} \epsilon_1 x_2 \\ \epsilon_2 x_1 + \epsilon_3 x_3 \end{matrix}$$

where α, β, γ, δ, ϵ_1, ϵ_2 and ϵ_3 are positive constants. Which of the following statements is correct?

Choose one answer

- ○ The system is observable for any triple $(\epsilon_1, \epsilon_2, \epsilon_3) \in \mathbb{R}^3$.
- ○ There exists no triple $(\epsilon_1, \epsilon_2, \epsilon_3) \in \mathbb{R}^3$ such that the system is observable.
- ✓ The system is observable if and only if $\epsilon_3 \neq 0$ and for any $(\epsilon_1, \epsilon_2) \in \mathbb{R}^2$.
- ○ The system is observable if and only if $(\epsilon_1, \epsilon_2, \epsilon_3) \neq (0,0,0)$.
- ○ The system is observable if and only $\epsilon_1 = 0$ and for any pair $(\epsilon_2, \epsilon_3) \neq (0,0)$.

$$M_o = [C \quad CA \quad CA^2]^T$$

If $\epsilon_3 = 0$, the noting is known about $\dot{x}_3$ or x_3, as the othe states are not depending on x_3.

However, $\dot{x}_3$ depends on x_1 and $\dot{x}_1$ depends on x_2, so if $\epsilon_3 \neq 0$, the $(\epsilon_1, \epsilon_2) \in \mathbb{R}^2$ can take any value.

Check Matlab for calculations

Q6

Consider the second order LTI-SISO system

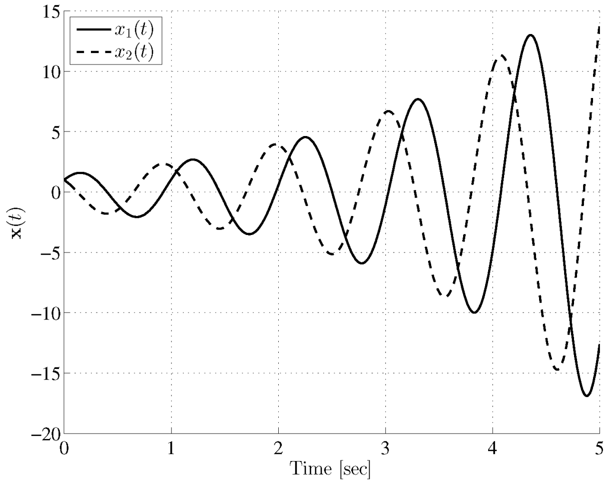
$$\Sigma : \begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} -\beta & \gamma \\ -\gamma & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \delta \\ 0 \end{bmatrix} u, & \mathbf{x} \in \mathbb{R}^2, u \in \mathbb{R} \\ y = [\varepsilon \quad 0]\,\mathbf{x}, & y \in \mathbb{R} \end{cases}$$

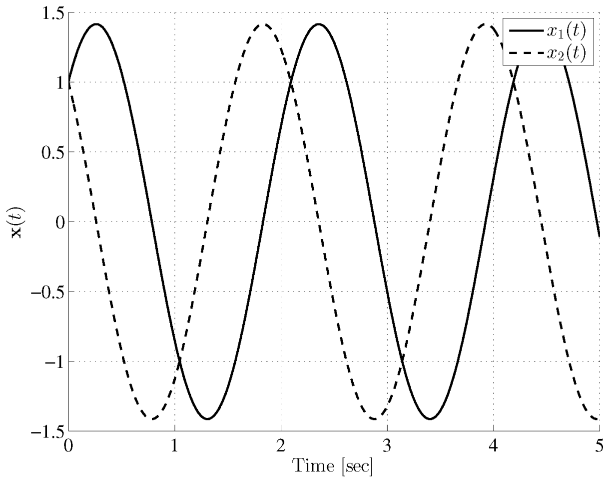
where β, δ, and ε are real and positive coefficients. Further the coefficients β and γ satisfy the following inequality

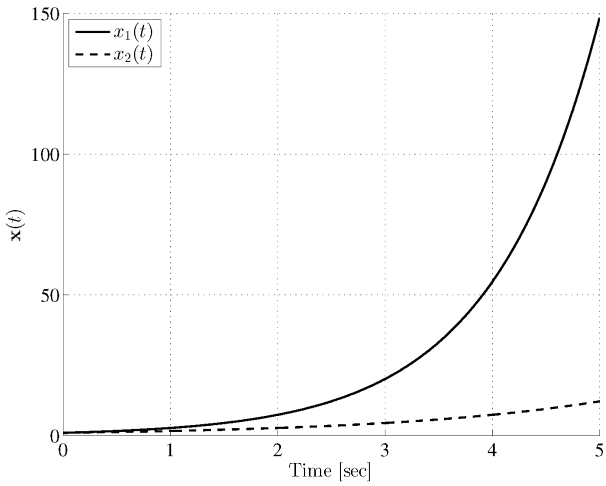
$$\beta > 2|\gamma| > 0, \quad \forall \gamma \in \mathbb{R}, \gamma \neq 0$$

Given an arbitrary initial condition $\mathbf{x}_0 > 0$, which is the zero-input response of the system?

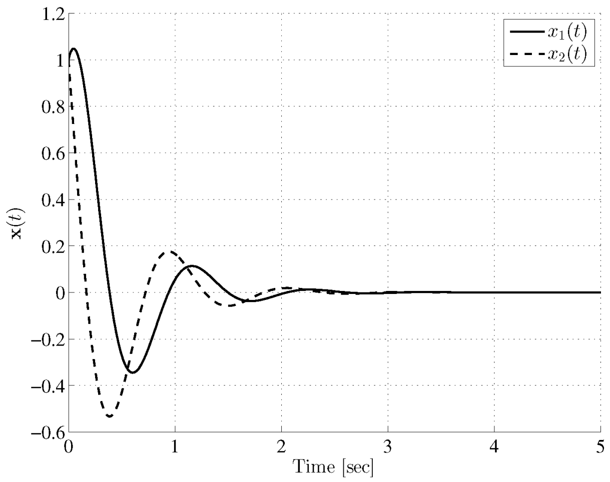
Choose one answer

- ○ 

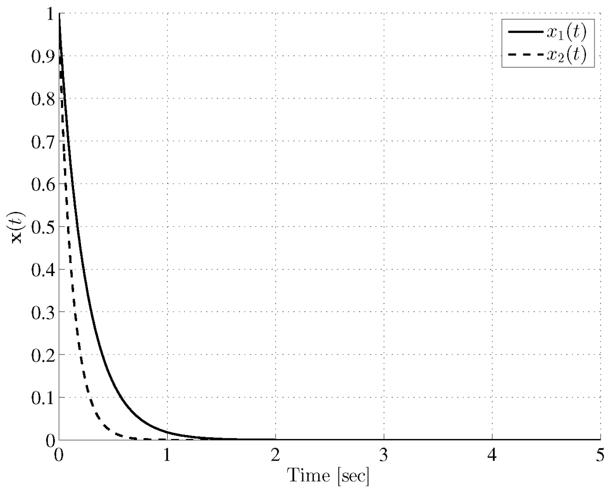
- ○ 

- ○ 

- ○



✓



Stable behaviour from real, negative eigenvalues.

$\boxed{Q6}$ Stability and zero-input response assessment.

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \Rightarrow A = \begin{bmatrix} -\beta & \gamma \\ -\gamma & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \delta \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} \epsilon & 0 \end{bmatrix}$$

$$\lambda = -\frac{\beta}{2} \pm \frac{1}{2} \cdot \sqrt{\underbrace{\beta^2 - 4\gamma^2}_{\beta > \text{ and } > 0}} = -\frac{\beta}{2} \pm \frac{1}{2}\sqrt{\underbrace{(\beta - 2\gamma)}_{>0}\underbrace{(\beta + 2\gamma)}_{>0}}$$

$\beta > 2|\gamma| > 0 \Rightarrow$ $\sqrt{\beta^2 - 4\gamma^2}$ will always be smaller than β, and therefore, the system will always have negative, real eigenvalues.

$$\underline{\lambda \in \mathbb{R} < 0.}$$

See Matlab for simulation

Q7

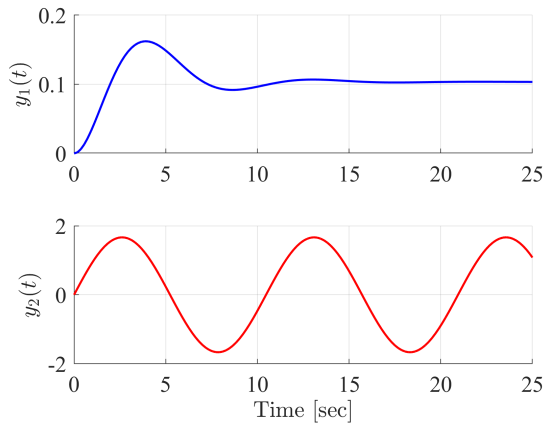
Consider the fifth order LTI continuous time system

$$\Sigma : \begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} -\alpha & \varepsilon & 0 & 0 & 0 \\ 0 & -\beta & -\gamma & 0 & 0 \\ 0 & \gamma & -\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & -\delta \\ 0 & 0 & 0 & \delta & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} u, \quad \mathbf{x} \in \mathbb{R}^5,\, u \in \mathbb{R} \\ \mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}, \quad \mathbf{y} \in \mathbb{R}^2 \end{cases}$$

where $\alpha,\ \beta,\ \gamma$ and δ are real positive constant coefficients. Which plot shows the correct zero-state output response for $u(t) = u_0,\ \forall\, t \geq 0$?

Choose one answer

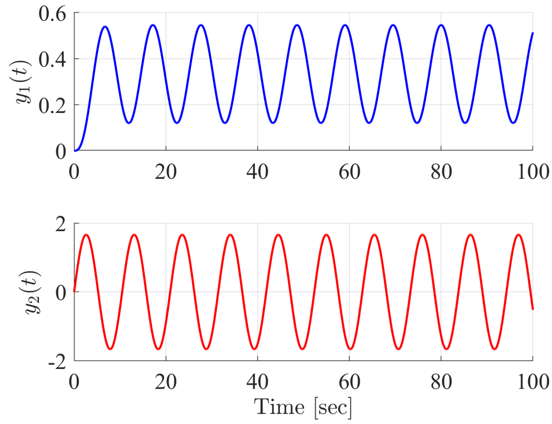
○



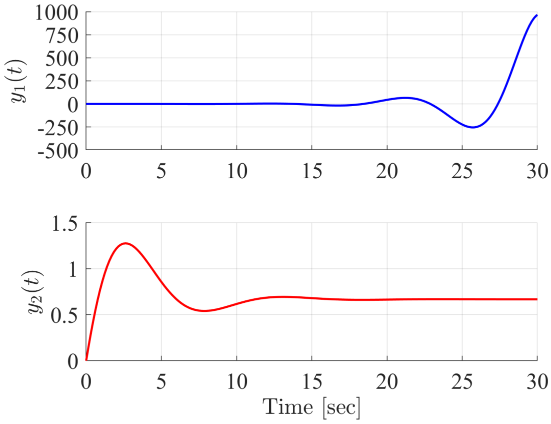
Asymptotic stability

Marginal stability

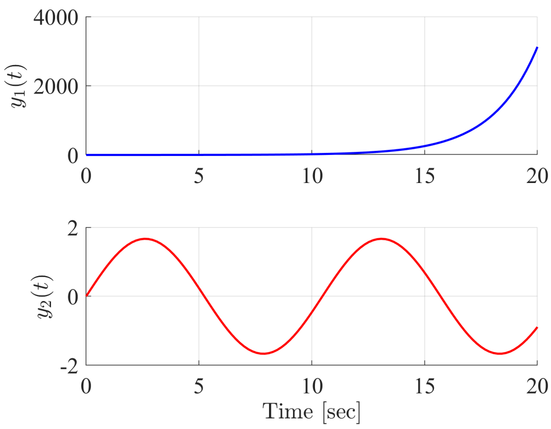
○

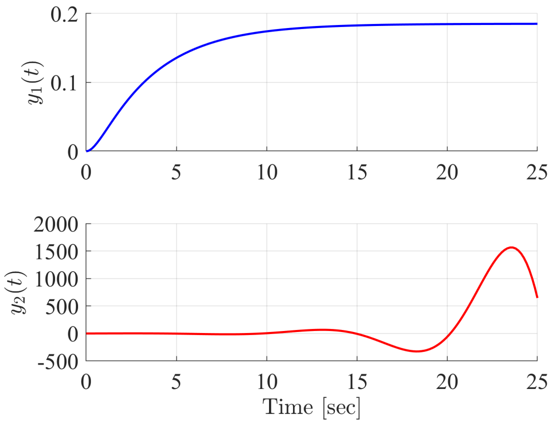


○



○





$$\lambda = \begin{bmatrix} -\alpha \\ -\delta i \\ \delta i \\ -\beta - \gamma i \\ -\beta + \gamma i \end{bmatrix}$$

x_4 and x_5 are uncoupled from the rest of the system with eigenvalues $\delta i, -\delta i \rightarrow$ marginally stable

x_1 is associated with the other states, x_2, x_3, that all contain asymptotically stable eigenvalues.

Q8

OBS! Check with the others - not sure.

Consider the n-th order LTI MIMO system

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{B}_d\mathbf{d}, \quad \mathbf{x} \in \mathbb{R}^n\,,\ \mathbf{u} \in \mathbb{R}^m\,,\ \mathbf{d} \in \mathbb{R}^q \\
\mathbf{y} &= \mathbf{C}\mathbf{x}, \quad \mathbf{y} \in \mathbb{R}^p
\end{aligned}
$$

The system dynamical matrix $\mathbf{A}$ is characterized by $\exists\, \lambda_i \in \lambda(\mathbf{A}) \,|\, \mathrm{Re}\{\lambda_i(\mathbf{A})\} > 0$, where $\lambda(\mathbf{A})$ is the set of eigenvalues of the matrix $\mathbf{A}$. Further the system is fully observable, that is $\mathrm{rank}(\mathbf{M}_o) = n$, where $\mathbf{M}_o$ is the observability matrix.

A full order observer is designed to estimate the state $\mathbf{x}$ and the input disturbance $\mathbf{d}$

$$
\begin{bmatrix} \dot{\hat{\mathbf{x}}} \\ \dot{\hat{\mathbf{w}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_d\mathbf{C}_w \\ \mathbf{0} & \mathbf{A}_w \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{u} + \begin{bmatrix} \mathbf{L}_x \\ \mathbf{L}_w \end{bmatrix} (\mathbf{y} - \hat{\mathbf{y}})
$$

$$
\begin{bmatrix} \hat{\mathbf{y}} \\ \hat{\mathbf{d}} \end{bmatrix} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_w \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix}
$$

where $\mathbf{L} = [\mathbf{L}_x^{\mathrm{T}}, \mathbf{L}_w^{\mathrm{T}}]^{\mathrm{T}}$ is the observer gain matrix, and $(\mathbf{A}_w, \mathbf{C}_w)$ are the dynamical and output matrices of the disturbance model

$$
\begin{aligned}
\dot{\mathbf{w}} &= \mathbf{A}_w\mathbf{w}, \quad \mathbf{w} \in \mathbb{R}^{n_w} \\
\mathbf{d} &= \mathbf{C}_{\mathbf{w}}\mathbf{w}
\end{aligned}
$$

The disturbance model is characterized by $\mathrm{Re}\{\lambda(\mathbf{A}_w)\} = 0$, where $\lambda(\mathbf{A}_w)$ is the set of eigenvalues of the matrix $\mathbf{A}_w$.

Let $\mathbf{e}_x = \mathbf{x} - \hat{\mathbf{x}}$ and $\mathbf{e}_w = \mathbf{w} - \hat{\mathbf{w}}$ the components of the estimation error, the estimation error dynamics is given by

$$
\begin{aligned}
\begin{bmatrix} \dot{\mathbf{e}}_x \\ \dot{\mathbf{e}}_w \end{bmatrix} &= \mathbf{A}_e \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_w \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{A} - \mathbf{L}_x\mathbf{C} & \mathbf{B}_d\mathbf{C}_w \\ -\mathbf{L}_w\mathbf{C} & \mathbf{A}_w \end{bmatrix} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_w \end{bmatrix}
\end{aligned}
$$

Which of the following statements is correct?

Choose one answer

? ✓ The estimation error $\mathbf{e}_x$ converges to zero because $\mathrm{Re}\{\lambda(\mathbf{A} - \mathbf{L}_x\mathbf{C})\} < 0$ by design of the observer; whereas the estimation error $\mathbf{e}_w$ remains bounded away from zero because $\mathrm{Re}\{\lambda(\mathbf{A}_w)\} = 0$.

- The estimation error $\mathbf{e}_x$ diverges to infinity because $\mathrm{Re}\{\lambda(\mathbf{A})\} > 0$; whereas the estimation error $\mathbf{e}_w$ remains bounded away from zero because $\mathrm{Re}\{\lambda(\mathbf{A}_w)\} = 0$.
- Both estimation errors, $\mathbf{e}_x$ and $\mathbf{e}_w$, converge to zero because $\mathrm{Re}\{\lambda(\mathbf{A}_e)\} < 0$ by design of the observer.
- Both estimation errors, $\mathbf{e}_x$ and $\mathbf{e}_w$, diverge to infinity because $\exists\, \lambda_i \in \lambda(\mathbf{A}) \,|\, \mathrm{Re}\{\lambda_i(\mathbf{A})\} > 0$ and $\mathrm{Re}\{\lambda(\mathbf{A}_w)\} = 0$.
- The estimation error $\mathbf{e}_x$ converges to zero because $\mathrm{Re}\{\lambda(\mathbf{A} - \mathbf{L}_x\mathbf{C})\} < 0$ by design of the observer; whereas the estimation error $\mathbf{e}_w$ diverges to infinity because $\mathrm{Re}\{\lambda(\mathbf{A}_w)\} = 0$.

$\boxed{Q8}$

$\exists \lambda_i \in \lambda(A) \mid Re(\lambda_i(A)) > 0 \rightarrow$ unstable A.

$Re(\lambda(A_w)) = 0$, $\lambda(A_w) = $ set $\rightarrow$ marginally stable disturbance.

$\rightarrow$ Noise is a sinusoid!

$A - L_x C$ is asymptotically stable (for observer)

A_w is not asymptotically stable, only marginally.

Q9

Consider the second order LTI SISO system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ \alpha \end{bmatrix} u + \begin{bmatrix} 0 \\ \beta \end{bmatrix} d$$
$$y = [\gamma \quad 0]\,\mathbf{x}$$

where α, β, γ, ω_n, and ζ are all positive constants. Further, consider the three control architectures:

$$\mathrm{CA}_1 : u = -\mathbf{K}\mathbf{x} + Nr$$
$$\mathrm{CA}_2 : u = -\mathbf{K}\mathbf{x} + K_i x_i, \quad x_i = \int_0^t [r(\tau) - y(\tau)]\,\mathrm{d}\tau$$
$$\mathrm{CA}_3 : u = -\mathbf{K}\hat{\mathbf{x}} + Nr - K_d\hat{d}$$
$$\begin{bmatrix} \dot{\hat{\mathbf{x}}} \\ \dot{\hat{d}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\omega_n^2 & -2\zeta\omega_n & \beta \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{\mathbf{x}} \\ \hat{d} \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha \\ 0 \end{bmatrix} u + \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}(y - \hat{y})$$
$$\hat{y} = [\gamma \quad 0 \quad 0]\begin{bmatrix} \hat{\mathbf{x}} \\ \hat{d} \end{bmatrix}$$

where r is a reference signal and $\mathbf{K} = [k_1, k_2] \neq [0, 0]$ is the full state feedback controller gain vector. Which of the following statements is **not correct**?

Choose one answer

- If $d(t) = 0$ and $r(t) = r_0, \forall t \geq 0$, then for $N = (\omega_n^2 + \alpha k_1)/(\alpha\gamma)$ all three control architectures guarantee $\lim_{t\to\infty} y(t) = r_0$.
- ? All three control architectures change the natural frequency ω_n of the open-loop system to the value $\omega_{n,\mathrm{des}}$ if $\mathbf{K} = [(\omega_{n,\mathrm{des}}^2 - \omega_n^2)/\alpha, 2\zeta(\omega_{n,\mathrm{des}} - \omega_n)/\alpha]$.
- ? If $d(t) = d_0$ and $r(t) = 0$, then at steady state the integral action of CA_2 cancels the action of the disturbance, i.e. $K_i x_i = -(\beta/\alpha)d_0$.
- If $d(t) = d_0$ and $r(t) = r_0, \forall t \geq 0$, then for $N = (\omega_n^2 + \alpha k_1)/(\alpha\gamma)$ and $K_d = \beta/\alpha$ all three control architectures guarantee $\lim_{t\to\infty} y(t) = r_0$.
- ? If $d(t) = d_0$ and $r(t) = r_0, \forall t \geq 0$, then CA_1 regulates the output to $y(t) = r_0 + \frac{\beta\gamma}{\omega_n^2 + \alpha k_1} d_0$.

Q9

positive constants
$\alpha, \beta, \gamma, \omega_n, \xi > 0$

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix} x + \begin{bmatrix} 0 \\ \alpha \end{bmatrix} u + \begin{bmatrix} 0 \\ \beta \end{bmatrix} d \\ y = \begin{bmatrix} \gamma & 0 \end{bmatrix} x \end{cases}$$

Setpoint control: $k = C \cdot (A - BK)^{-1} \cdot B$

$$N = (k^T k)^{-1} k^T = \frac{\omega^2 + \alpha K_1}{\alpha \gamma}$$

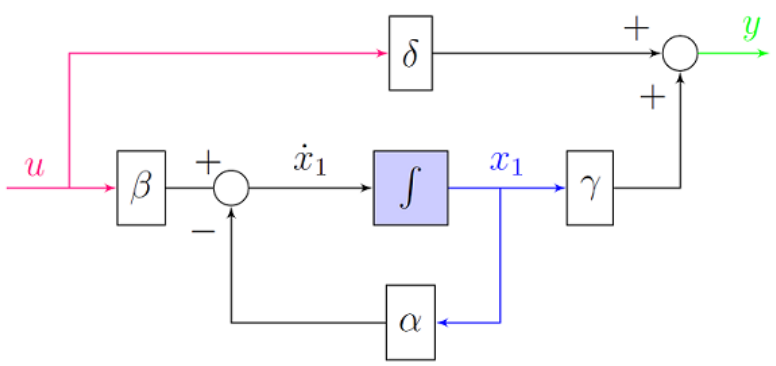
CA_1: $u = -Kx + Nr$

CA_2: $u = -Kx + K_i x_i \Rightarrow$

$$x_a = \begin{bmatrix} x \\ x_i \end{bmatrix}, \quad \dot{x}_a = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} x_a + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ I \end{bmatrix} r + \begin{bmatrix} B_v \\ 0 \end{bmatrix} d$$

Q10 Not sure.

Consider the first order LTI SISO system in the block diagram shown in the following figure



where $\alpha,\ \beta,\ \gamma$ and δ are positive constants. The input signal $u(t)$ is a stochastic process characterized by the autocorrelation function

$$R_u(\tau) = \sigma_u^2 e^{-\lambda|\tau|}$$

with σ_u and λ being both positive and constant. What is the variance of the output $y(t)$?

Choose one answer

- The variance of the output is $\sigma_y^2 = \left(\frac{\gamma^2\beta^2}{\alpha(\alpha+\lambda)} + \frac{2\gamma\delta\beta}{\alpha+\lambda} + \delta^2\right)\sigma_u^2$.
- The variance of the output is $\sigma_y^2 = \sigma_u^2$.
- 2 ✓ The variance of the output is $\sigma_y^2 = \left(\frac{\gamma^2\beta^2}{2\alpha} + \delta^2\right)\sigma_u^2$. Missing the squares.
- The variance of the output is $\sigma_y^2 = \frac{\beta^2}{\alpha(\alpha+\lambda)}\sigma_u^2$
- The variance of the output is $\sigma_y^2 = 0$.

$\boxed{Q10}$ LTI Siso system, Stochastic process, autocorrelation

$$\begin{cases} \dot{x} = -\alpha x + \beta u \\ y = \gamma x + \delta u \end{cases}$$

u is a stochastic process

autocorrelation: $R_u(\tau) = \sigma_u^2 e^{-\lambda|\tau|}$

$\sigma_u, \lambda > 0$

At ss: $\dot{x} = 0 = -\alpha E[x x^T] + \beta E[u u^T]$

$$= -\alpha \sigma_x^2 + \beta \sigma_u^2 \qquad (\tau = 0)$$

$$\Rightarrow \sigma_x^2 = \frac{\beta}{\alpha} \sigma_u^2$$

at ss:

$$E[y(t) y(t)^T] = \sigma_y^2 = \gamma E(x x^T) + \delta E(u u^T)$$

$$\Rightarrow \sigma_y^2 = \gamma \sigma_x^2 + \delta \sigma_u^2 = \frac{\beta\gamma}{\alpha} \sigma_u^2 + \delta \sigma_u^2$$

$$= \left(\frac{\beta\gamma}{\alpha} + \delta\right) \sigma_u^2$$

almost

At ss: $0 = -\alpha x + \beta u \Rightarrow x = \frac{\beta}{\alpha} u$ Insert in y

$$\Rightarrow y = \frac{\gamma\beta}{\alpha} u + \delta u \rightarrow y^2 = \left(\frac{\gamma^2\beta^2}{\alpha^2} + \frac{2\gamma\beta\delta}{\alpha} + \delta^2\right) u^2$$

??

Consider the three LTI-SISO systems in controllable subspace decomposition form

$$\sum{}^{a}: \begin{cases} \dot{\mathbf{x}}^a = \begin{bmatrix} \mathbf{A}_c^a & \mathbf{A}_{12}^a \\ \mathbf{0} & \mathbf{A}_{nc}^a \end{bmatrix} \mathbf{x}^a + \begin{bmatrix} \mathbf{B}_c^a \\ 0 \end{bmatrix} u^a \\ y^a = \begin{bmatrix} \mathbf{C}_c^a & \mathbf{C}_{nc}^a \end{bmatrix} \mathbf{x}^a \end{cases}, \qquad \sum{}^{b}: \begin{cases} \dot{\mathbf{x}}^b = \begin{bmatrix} \mathbf{A}_c^b & \mathbf{A}_{12}^b \\ \mathbf{0} & \mathbf{A}_{nc}^b \end{bmatrix} \mathbf{x}^b + \begin{bmatrix} \mathbf{B}_c^b \\ 0 \end{bmatrix} u^b \\ y^b = \begin{bmatrix} \mathbf{C}_c^b & \mathbf{C}_{nc}^b \end{bmatrix} \mathbf{x}^b \end{cases},$$

$$\sum{}^{c}: \begin{cases} \dot{\mathbf{x}}^c = \begin{bmatrix} \mathbf{A}_c^c & \mathbf{A}_{12}^c \\ \mathbf{0} & \mathbf{A}_{nc}^c \end{bmatrix} \mathbf{x}^c + \begin{bmatrix} \mathbf{B}_c^c \\ 0 \end{bmatrix} u^c \\ y^c = \begin{bmatrix} \mathbf{C}_c^c & \mathbf{C}_{nc}^c \end{bmatrix} \mathbf{x}^c \end{cases}.$$

where $\mathbf{x}^a,\, \mathbf{x}^b,\, \mathbf{x}^c \in \mathbb{R}^n\,; u^a,\, u^b,\, u^c \in \mathbb{R}\,; y^a,\, y^b,\, y^c \in \mathbb{R}\,.$

The open loop eigenvalues of the non controllable subsystems are real and they satisfy the following inequality:

$$\lambda_{ol,\max}\left(\mathbf{A}_{nc}^a\right) < \lambda_{ol,\min}\left(\mathbf{A}_{nc}^b\right) < \lambda_{ol,\max}\left(\mathbf{A}_{nc}^b\right) < \lambda_{ol,\min}\left(\mathbf{A}_{nc}^c\right) < 0.$$

A full state feedback controller with integral action is designed for each of the given systems such that the closed loop eigenvalues of the controllable subsystems are real, negative, and satisfy the following relation:

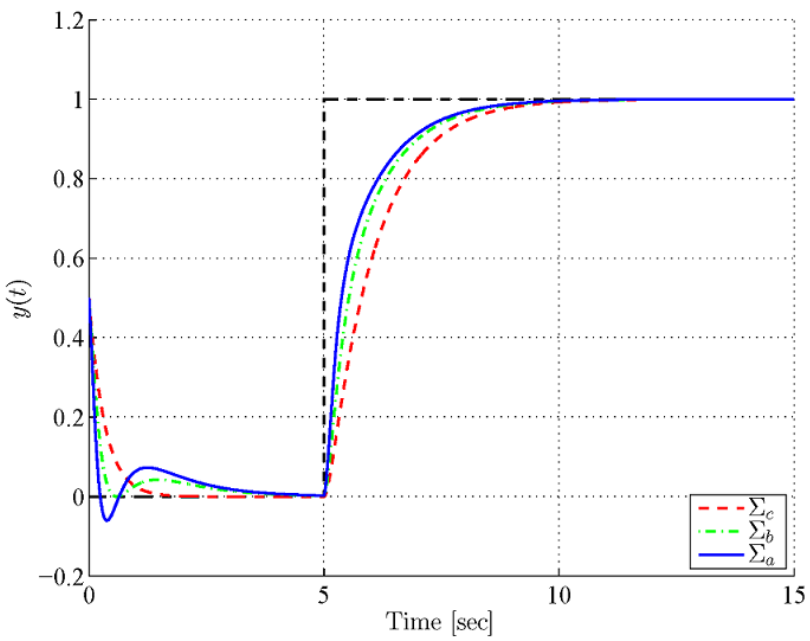
$$\lambda_{cl}\left(\begin{bmatrix} \mathbf{A}_c^a - \mathbf{B}_c^a\mathbf{K}^a & \mathbf{B}_c^a\mathbf{K}_i^a \\ \mathbf{C}_c^a & 0 \end{bmatrix}\right) = \lambda_{cl}\left(\begin{bmatrix} \mathbf{A}_c^b - \mathbf{B}_c^b\mathbf{K}^b & \mathbf{B}_c^b\mathbf{K}_i^b \\ \mathbf{C}_c^b & 0 \end{bmatrix}\right) = \lambda_{cl}\left(\begin{bmatrix} \mathbf{A}_c^c - \mathbf{B}_c^c\mathbf{K}^c & \mathbf{B}_c^c\mathbf{K}_i^c \\ \mathbf{C}_c^c & 0 \end{bmatrix}\right).$$

Let $\mathbf{x}_c^a(t_0) = \mathbf{x}_c^b(t_0) = \mathbf{x}_c^c(t_0) = \mathbf{0}$ be the initial conditions for the part of the state vectors belonging to the controllable subspace; $\mathbf{x}_{nc}^a(t_0) = \mathbf{x}_{nc}^b(t_0) = \mathbf{x}_{nc}^c(t_0) = \mathbf{x}_0 > 0$ be the initial conditions for the part of the state vectors belonging to the non controllable subspace; $\mathbf{x}_i^a(t_0) = \mathbf{x}_i^b(t_0) = \mathbf{x}_i^c(t_0) = \mathbf{0}$ be the initial conditions for the integral state vectors.

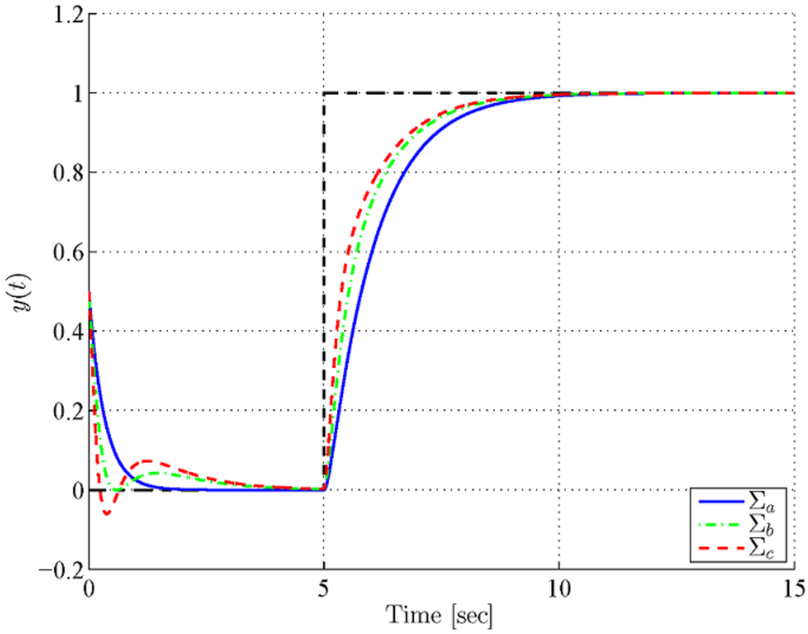
Which of the following plots represents the output response of the closed-loop systems when a reference step change is made at time $t = t_{\text{step}}$? (In the following plots $t_{\text{step}} = 5\ sec$)

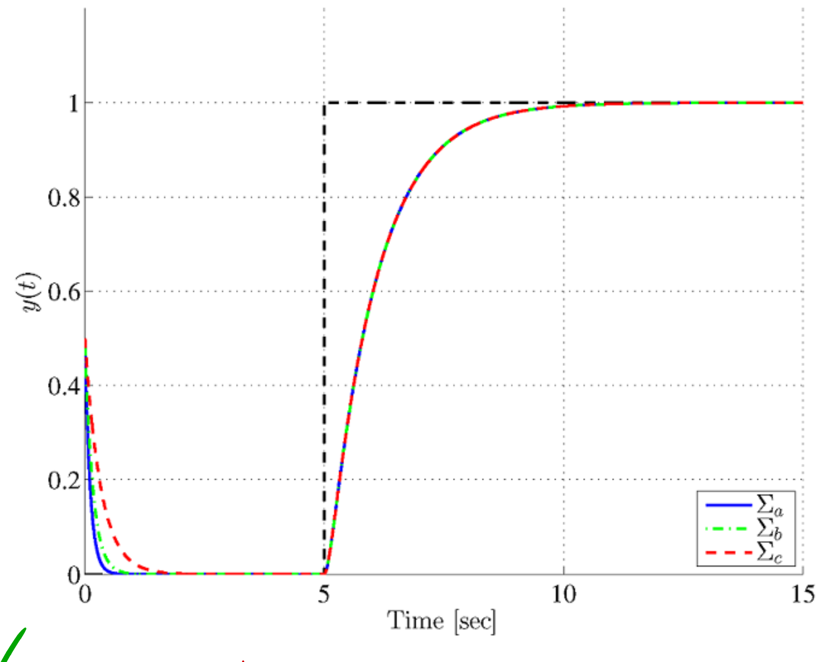
Choose one answer

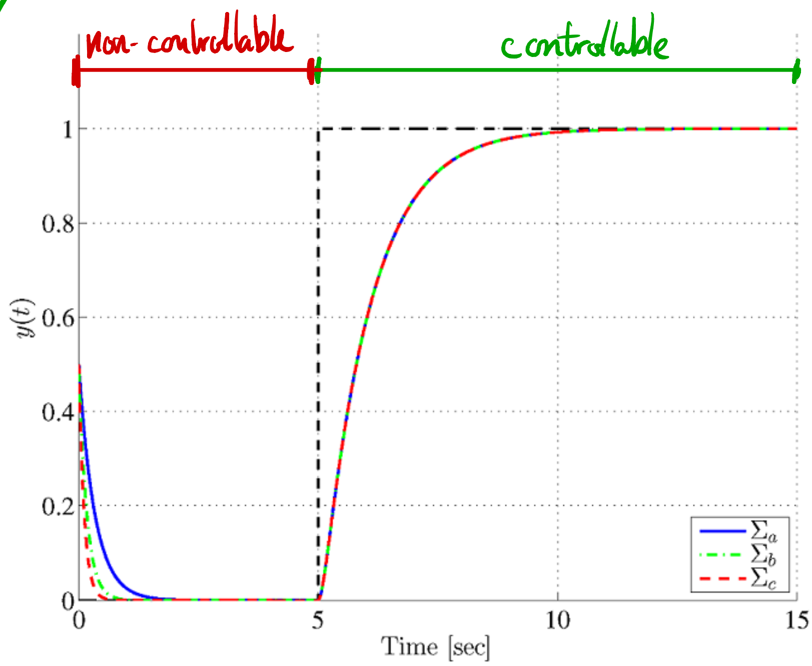
○



○

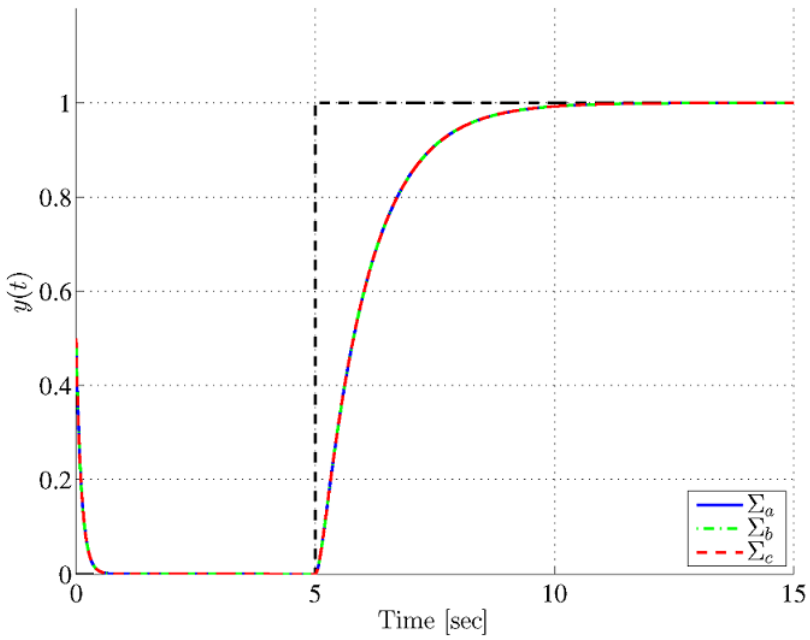






a_{nc} is slowest
c_{nc} is fastest
b_{nc} is in between.

a_c, b_c, c_c are equally fast as they have the same λ's



1
0.8
0.6
0.4
0.2
0
$y(t)$
0
5
10
15
Time [sec]
Σ_a
Σ_b
Σ_c

Q12

Consider the first order continuous time LTI system

$$\dot{x}(t) = \alpha x(t) + v_1(t)$$
$$y(t) = \beta x(t) + v_2(t)$$

where α and β are real constants. The process noise $v_1(t)$ and the measurement noise $v_2(t)$ are uncorrelated white noise sources with noise intensities σ_1^2 and σ_2^2, respectively. The steady state Kalman filter

$$\dot{\hat{x}}(t) = \alpha \hat{x}(t) + l_\infty (y(t) - \hat{y}(t)) = \alpha \hat{x} + l(y - \beta \hat{x}) = \hat{x}(\alpha - l\beta) + ly$$
$$\hat{y}(t) = \beta \hat{x}(t)$$

is designed in order to reconstruct the state variable $x(t)$ based on the measurement $y(t)$. Which of the following statements is **not correct**?

Choose one answer

- ○ The steady state Kalman gain is given by $l_\infty = \frac{1}{\beta}\left(\alpha + \sqrt{\alpha^2 + \beta^2 \frac{\sigma_1^2}{\sigma_2^2}}\right)$
- ○ If the intensity of the process noise is zero and $\alpha > 0$, then the Kalman gain is $l_\infty = 2\alpha$, which ensures the asymptotic stability of the estimation error dynamics.
- ✓ If $(\sigma_1^2/\sigma_2^2) \gg 1$ then the steady state Kalman gain is large and the filter relies heavily on the measurement.
- ✓ If $\alpha = 0$ and the intensity of the process noise is zero, then the steady state Kalman is zero, i.e. $l_\infty = 0$.
- ✗ If $(\sigma_1^2/\sigma_2^2) \ll 1$ then the steady state Kalman gain is large and the filter relies heavily on the system model.

$\boxed{Q12}$ Kalman Filter.

$$l = l_\infty$$

$$\dot{x} = \alpha x + v_1 \qquad \dot{\hat{x}} = \alpha \hat{x} + l(y - \hat{y}) = \hat{x}(\alpha - l\beta) + ly$$

$$y = \beta x + v_2 \qquad \hat{y} = \beta \hat{x}$$

34745 E22 Multiple Choice Questionnaire

Der anvendes en scoringsalgoritme, som er baseret på "One best answer"

Dette betyder følgende:
Der er altid netop ét svar som er mere rigtigt end de andre
Studerende kan kun vælge ét svar per spørgsmål
Hvert rigtigt svar giver 1 point
Hvert forkert svar giver 0 point (der benyttes IKKE negative point)

The following approach to scoring responses is implemented and is based on "One best answer"
There is always only one correct answer – a response that is more correct than the rest
Students are only able to select one answer per question
Every correct answer corresponds to 1 point
Every incorrect answer corresponds to 0 points (incorrect answers do not result in subtraction of points)

The Keynesian model of economic growth is used to describe the dynamics of the expenditure and revenue part of the economy of a nation, and it utilizes the following variables:

- Y is the gross national product,
- G is the government expenditure,
- C is the consumption expenditure,
- I is the investment expenditure.

In dynamic equilibrium at time k the gross national product Y equals the total expenditure E, i.e.

$$Y(k) = E(k)$$

where $E(k) = C(k) + I(k) + G(k)$. Let us assume that the consumption expenditure at time k is given by a fraction of the gross national product at the previous time $k-1$, i.e.

$$C(k) = \alpha Y(k-1)$$

where $0 < \alpha < 1$ is the multiplier factor. Further, let us assume that the investment expenditure at time k is proportional to the rate of change of the gross national product, i.e.

$$I(k) = \beta[Y(k-1) - Y(k-2)]$$

where $\beta > 0$.

Let $\mathbf{x} = [C, I]^T$ be the state vector, $u = G$ the input, and $y = Y$ the output. Which of the following discrete time models describe the Keynesian model of economic growth?

Choose one answer

- ○ $$\Sigma : \begin{cases} \mathbf{x}(k+1) = \begin{bmatrix} \alpha & \alpha \\ \alpha - 1 & \beta \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} \alpha \\ \beta \end{bmatrix} u(k) \\ y(k) = [1 \quad 0]\mathbf{x}(k) + \beta u(k) \end{cases}$$

- ○ $$\Sigma : \begin{cases} \mathbf{x}(k+1) = \begin{bmatrix} \alpha & \alpha \\ \beta & \beta \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} \alpha \\ \beta \end{bmatrix} u(k) \\ y(k) = [1 \quad 1]\mathbf{x}(k) + u(k) \end{cases}$$

- ○ $$\Sigma : \begin{cases} \mathbf{x}(k+1) = \begin{bmatrix} \alpha & \alpha \\ \frac{\beta}{\alpha}(\alpha - 1) & \beta \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} \alpha \\ \beta \end{bmatrix} u(k) \\ y(k) = [1 \quad 1]\mathbf{x}(k) \end{cases}$$

$$\Sigma : \begin{cases} \mathbf{x}(k+1) = \begin{bmatrix} \alpha & \alpha \\ \frac{\beta}{\alpha}(\alpha - 1) & \beta \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} \alpha \\ \beta \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}(k) + u(k) \end{cases}$$

$$\Sigma : \begin{cases} \mathbf{x}(k+1) = \begin{bmatrix} \alpha & \alpha \\ 0 & \beta \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} \alpha \\ \beta \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}(k) + u(k) \end{cases}$$

Consider the 2nd order nonlinear system

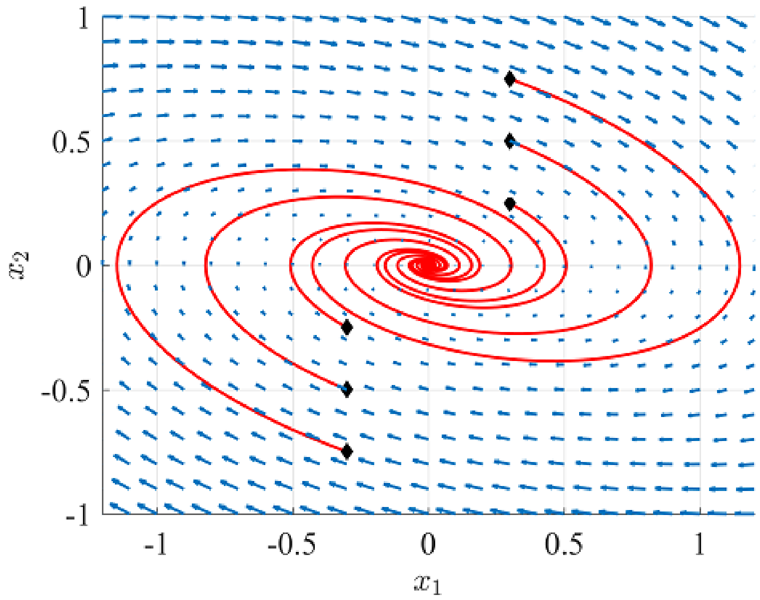
$$\Sigma : \begin{cases} \dot{x}_1 = \dfrac{\alpha x_2^2}{1 + k x_2^2} + \alpha_0 - \gamma x_1 \\ \dot{x}_2 = \beta x_1 - \delta x_2 \end{cases}$$

where the parameters $\alpha_0, \alpha, \beta, \gamma, \delta$ are all real and positive. Let $\mathbf{x}^e = [x_1^e, x_2^e]^T \neq [0, 0]^T$ be an equilibrium point of Σ. What is the linear system obtained by linearizing Σ around the point of equilibrium $\mathbf{x}^e$?

Choose one answer

- $\dot{\Delta\mathbf{x}} = \begin{bmatrix} -\gamma & 2\alpha \\ \beta & -\delta \end{bmatrix} \Delta\mathbf{x}$
- $\dot{\Delta\mathbf{x}} = \begin{bmatrix} -\gamma & 0 \\ \beta & -\delta \end{bmatrix} \Delta\mathbf{x}$
- $\dot{\Delta\mathbf{x}} = \begin{bmatrix} -\gamma & \frac{2\alpha x_2^e (1+k(x_2^e)^2)}{(1+k(x_2^e)^2)^2} \\ \beta & -\delta \end{bmatrix} \Delta\mathbf{x}$
- ✓ $\dot{\Delta\mathbf{x}} = \begin{bmatrix} -\gamma & \frac{2\alpha x_2^e}{(1+k(x_2^e)^2)^2} \\ \beta & -\delta \end{bmatrix} \Delta\mathbf{x}$
- $\dot{\Delta\mathbf{x}} = \begin{bmatrix} -\gamma & 0 \\ 0 & -\delta \end{bmatrix} \Delta\mathbf{x}$

Consider the phase portrait shown in the following figure, where each black diamond is an initial condition of the system $\mathbf{x}(0) = [x_{10}, x_{20}]^T$, each red line is a trajectory of the system originating from the initial condition, and the blue arrows show the direction of the vector field describing the system.



Which of the following statements is correct?

Choose one answer

- ○ The equilibrium point is a centre.
- ○ The equilibrium point is a saddle point.
- ✓ The equilibrium point is a stable focus.
- ○ The equilibrium point is an unstable focus.
- ○ The equilibrium point is a stable node.

Consider the 3rd order LTI system

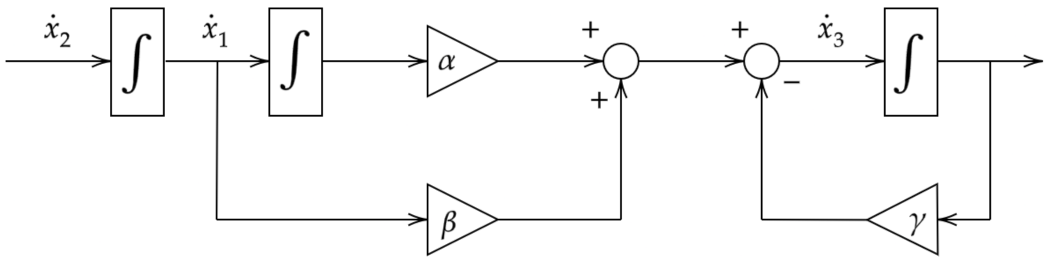
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} = \begin{bmatrix} 0 & 0 & 0 \\ \alpha & -\beta & 0 \\ -\alpha & \beta & 0 \end{bmatrix} \mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^3$$

where $\alpha > 0$ and $\beta > 0$. $\lambda(\mathbf{A})$ is the set of all eigenvalues of the system dynamical matrix $\mathbf{A}$ and λ_i denotes a specific eigenvalue in $\lambda(\mathbf{A})$, which of the following statements is correct?

Choose one answer

- ○ The system is unstable because the eigenvalue $\lambda = 0$ has geometric multiplicity $m_g = 1$.
- ○ The system is asymptotically stable because $\mathrm{Re}\{\lambda_i\} < 0, \forall \lambda_i \in \lambda(\mathbf{A})$
- ○ The system is unstable because the eigenvalue $\lambda = 0$ has algebraic multiplicity $m_a = 2$.
- ○ The system is unstable because $\exists \lambda_i \in \lambda(\mathbf{A})$ such that $\mathrm{Re}\{\lambda_i\} > 0$.
- ✓ The system is marginally stable because the eigenvalue $\lambda = 0$ has geometric multiplicity equal to the algebraic multiplicity, i.e. $m_g = m_a$.

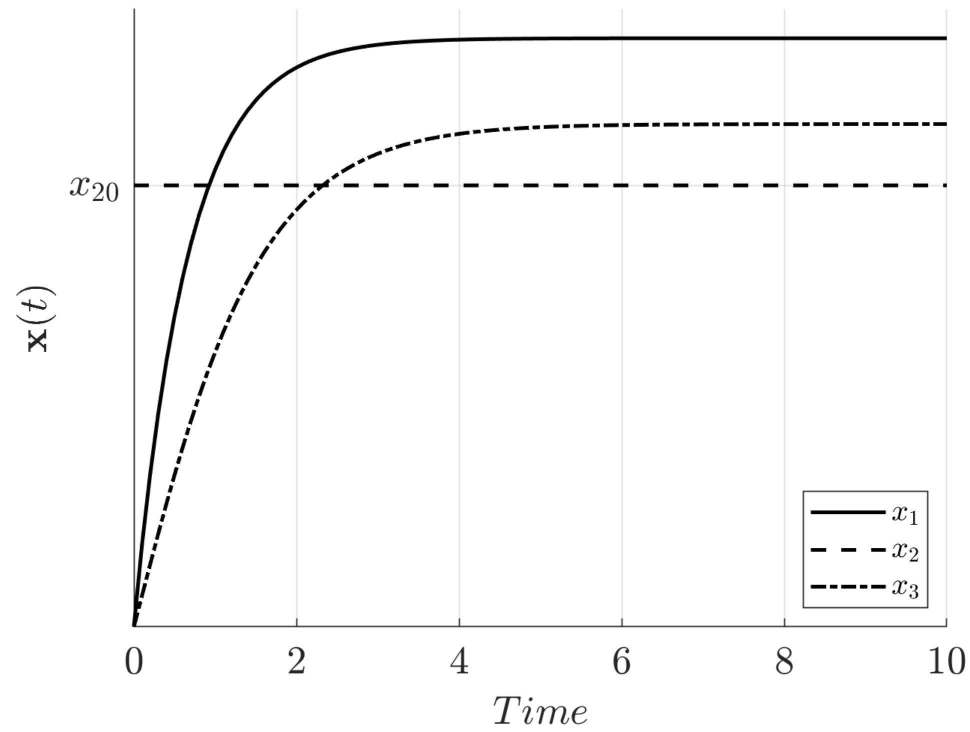
Consider the 3rd order system shown in the block diagram below



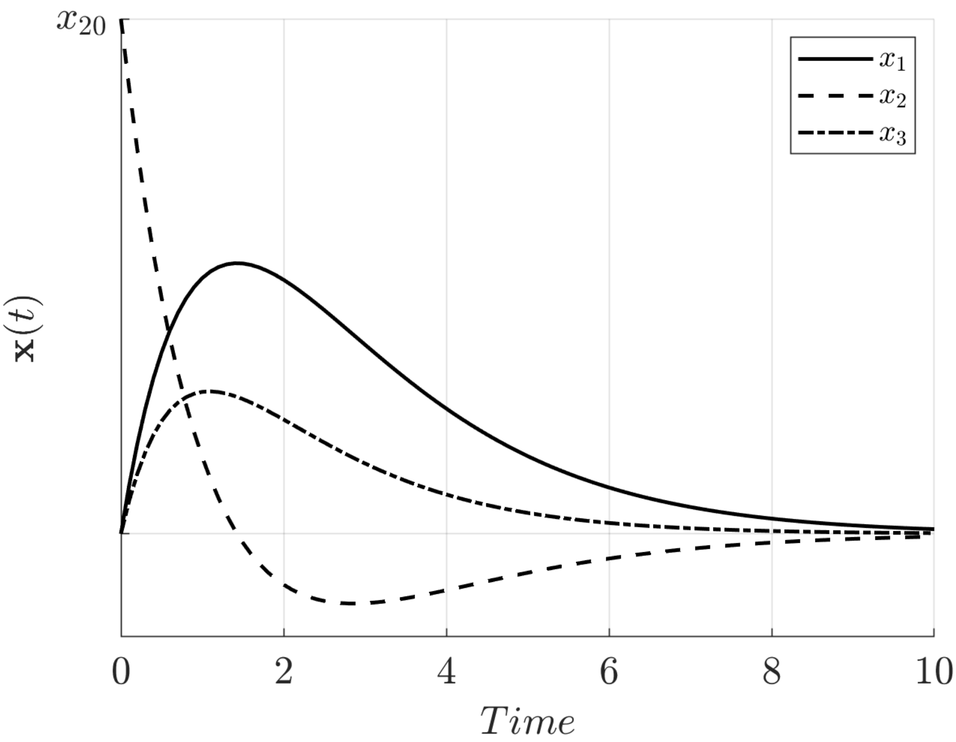
Which of the following zero-input responses is correct when the system is initialized with $\mathbf{x}_0 = [0, x_{20}, 0]^T$?

Choose one answer

○

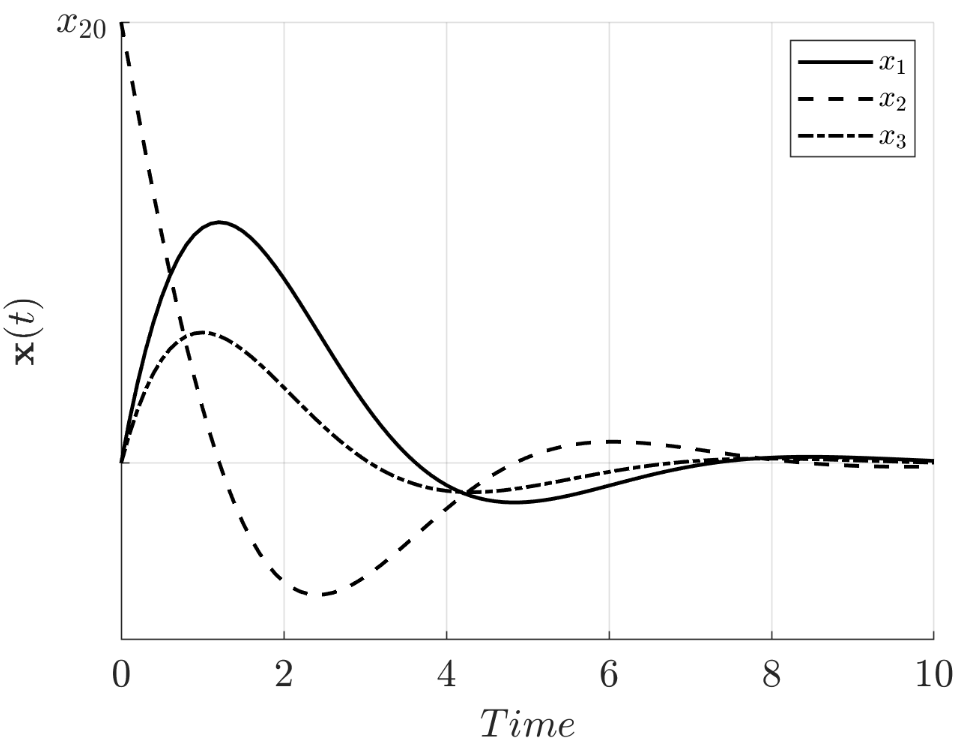


○



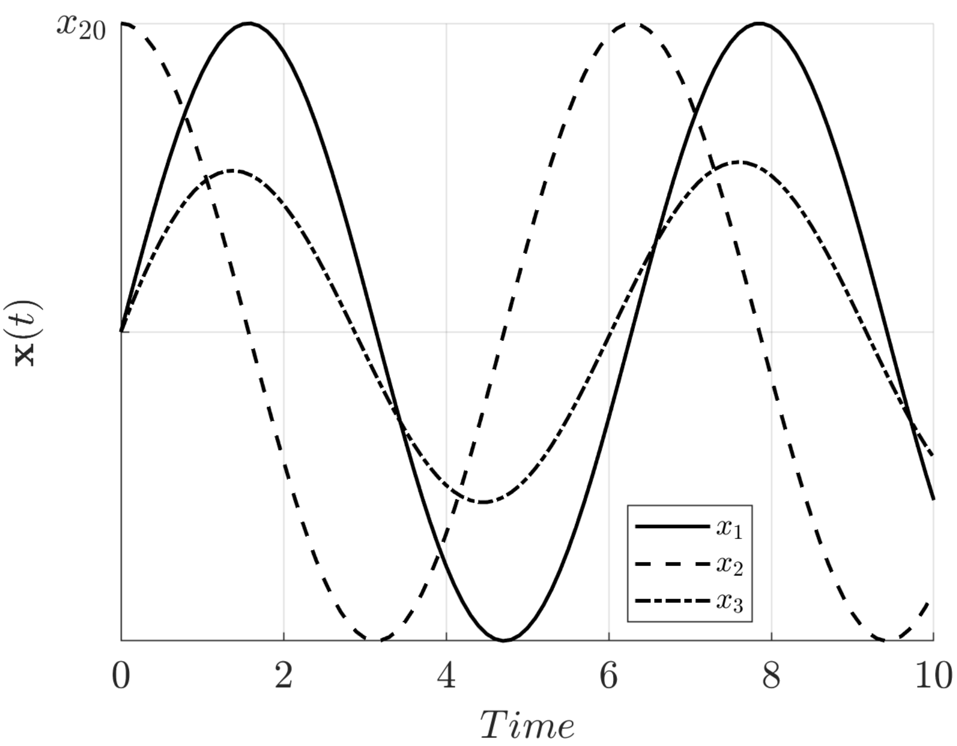
x_{20}
x_1
x_2
x_3
$\mathbf{x}(t)$
0
2
4
6
8
10
$Time$

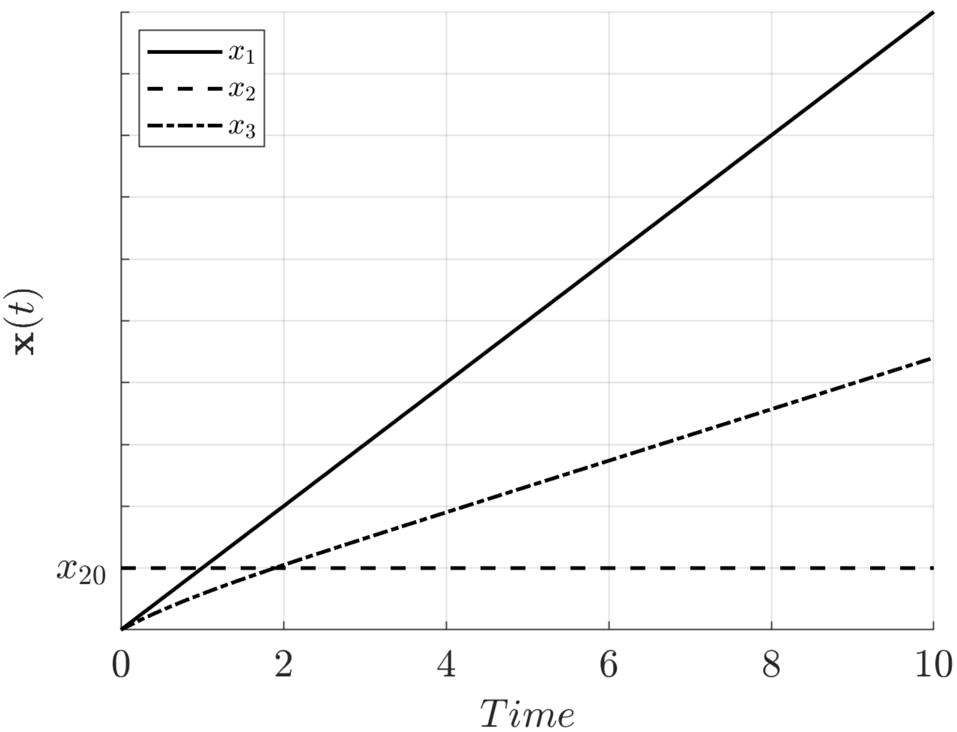
○



x_{20}
x_1
x_2
x_3
$\mathbf{x}(t)$
0
2
4
6
8
10
$Time$

○


x_{20}
$\mathbf{x}(t)$
x_1
x_2
x_3
0
2
4
6
8
10
$Time$


x_1
x_2
x_3
$\mathbf{x}(t)$
x_{20}
0
2
4
6
8
10
$Time$

Consider a 3rd order LTI discrete time SISO system

$$\mathbf{x}(k+1) = \begin{bmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 1 \\ 0 & 0 & -\gamma \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k), \quad \mathbf{x} \in \mathbb{R}^3,\ u \in \mathbb{R}$$
$$\mathbf{y}(k) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}(k), \quad y \in \mathbb{R}$$

where α, β, and γ are real and positive coefficients such that

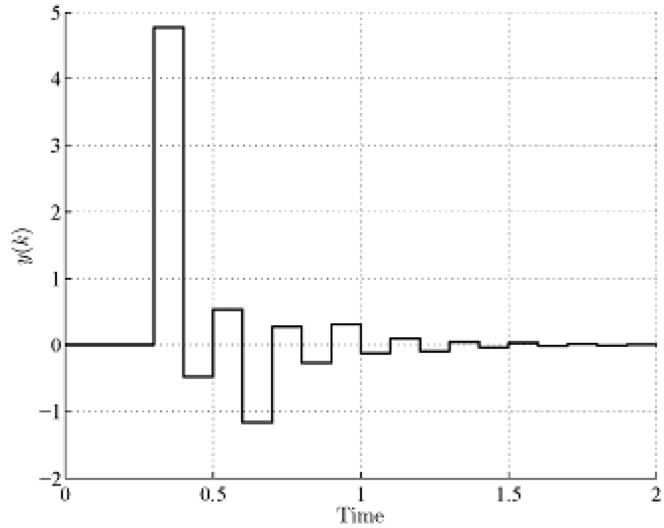
$$|\alpha| < 1 \wedge \beta = \sqrt{1-\alpha^2} \wedge |\gamma| < 1$$

The sampling time is $T_s = 0.1\ \mathrm{s}$. Which of the following plots represents the unit pulse response of the system when a unit pulse

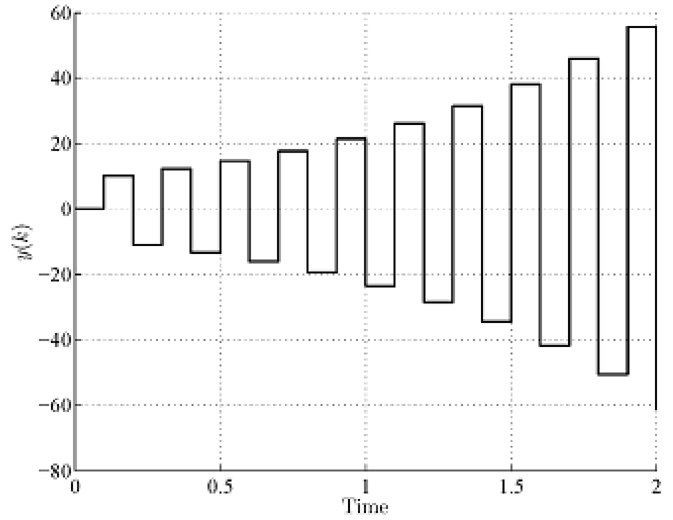
$$u(k) = \begin{cases} 1 & \text{for } k \in [0,1] \subset \mathbb{Z} \\ 0 & \text{for } k > 1 \end{cases}$$

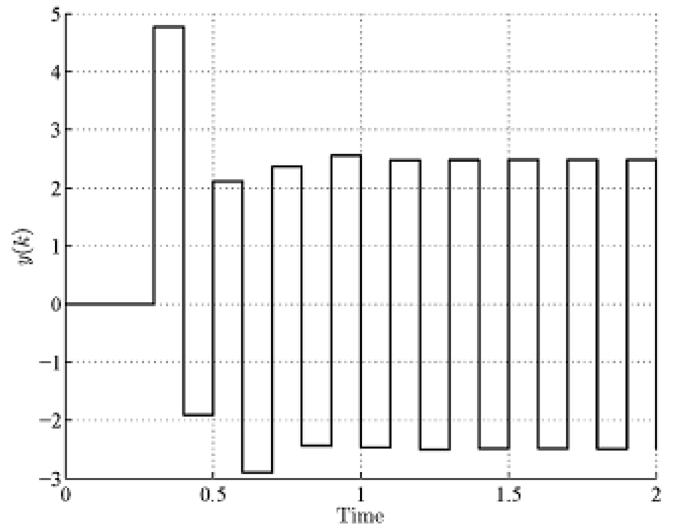
is sent through the input channel?

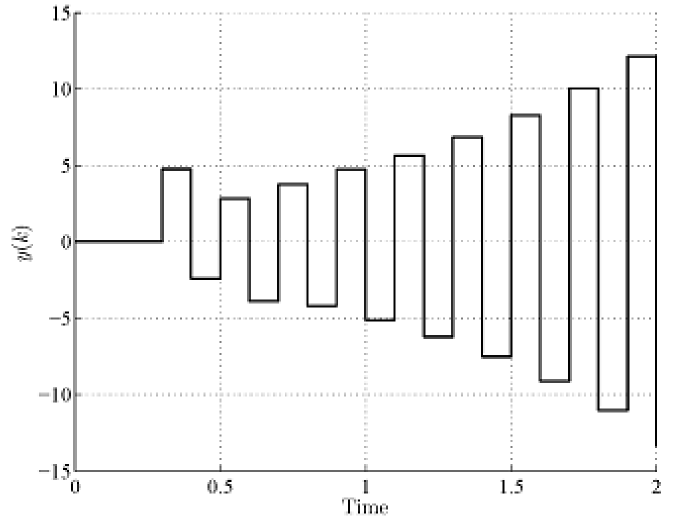
Choose one answer

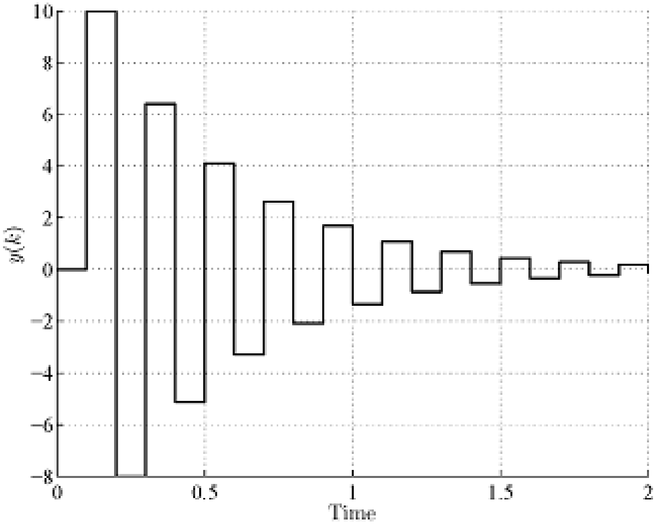
○ 

○


60
40
20
0
−20
−40
−60
−80
0
0.5
1
1.5
2
Time
$y(k)$


5
4
3
2
1
0
−1
−2
−3
0
0.5
1
1.5
2
Time
$y(k)$


15
10
5
0
−5
−10
−15
0
0.5
1
1.5
2
Time
$y(k)$



y(k)
Time
10
8
6
4
2
0
−2
−4
−6
−8
0
0.5
1
1.5
2

Consider the nth order LTI system

$$\Sigma_x : \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u + \mathbf{B}_d d, & \mathbf{x} \in \mathbb{R}^n, u \in \mathbb{R}, d \in \mathbb{R} \\ y = \mathbf{C}\mathbf{x}, \quad y \in \mathbb{R} \end{cases}$$

The observability matrix satisfies the rank condition $\mathrm{rank}(\mathbf{M}_o) = n$. The system is subject to a time-varying not measurable disturbance given by

$$d(t) = d_0 + d_1 \sin(\omega_d t)$$

where d_0, d_1, ω_d are positive real constants. Which of the following observers can estimate the disturbance $d(t)$?

Choose one answer

- ○ The dynamics of the state estimator is

$$\begin{bmatrix} \dot{\hat{\mathbf{x}}} \\ \dot{\hat{\mathbf{w}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_d \mathbf{C}_w \\ \mathbf{0} & \mathbf{A}_w \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} u + \begin{bmatrix} \mathbf{L}_x \\ \mathbf{L}_w \end{bmatrix} (y - \hat{y})$$

$$\hat{y} = [\mathbf{C} \quad \mathbf{0}] \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix}$$

where

$$\mathbf{A}_w = \begin{bmatrix} 0 & 1 \\ -\omega_d^2 & -2\zeta\omega_d \end{bmatrix} \quad , \quad \mathbf{C}_w = [1 \quad 0]$$

- ○ The dynamics of the state estimator is

$$\begin{bmatrix} \dot{\hat{\mathbf{x}}} \\ \dot{\hat{\mathbf{w}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_d \mathbf{C}_w \\ \mathbf{0} & \mathbf{A}_w \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} u + \begin{bmatrix} \mathbf{L}_x \\ \mathbf{L}_w \end{bmatrix} (y - \hat{y})$$

$$\hat{y} = [\mathbf{C} \quad \mathbf{0}] \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix}$$

where

$$\mathbf{A}_w = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad \mathbf{C}_w = [1 \quad 1]$$

- ✓ The dynamics of the state estimator is

$$\begin{bmatrix} \dot{\hat{\mathbf{x}}} \\ \dot{\hat{\mathbf{w}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_d\mathbf{C}_w \\ \mathbf{0} & \mathbf{A}_w \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} u + \begin{bmatrix} \mathbf{L}_x \\ \mathbf{L}_w \end{bmatrix} (y - \hat{y})$$

$$\hat{y} = [\mathbf{C} \quad \mathbf{0}] \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix}$$

where

$$\mathbf{A}_w = \begin{bmatrix} 0 & -\omega_d & 0 \\ \omega_d & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad , \quad \mathbf{C}_w = [1 \quad 0 \quad 1]$$

- ◯ The dynamics of the state estimator is

$$\begin{bmatrix} \dot{\hat{\mathbf{x}}} \\ \dot{\hat{\mathbf{w}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_d\mathbf{C}_w \\ \mathbf{0} & \mathbf{A}_w \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} u + \begin{bmatrix} \mathbf{L}_x \\ \mathbf{L}_w \end{bmatrix} (y - \hat{y})$$

$$\hat{y} = [\mathbf{C} \quad \mathbf{0}] \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix}$$

where

$$\mathbf{A}_w = \begin{bmatrix} -\omega_d & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad \mathbf{C}_w = [1 \quad 1]$$

- ◯ The dynamics of the state estimator is

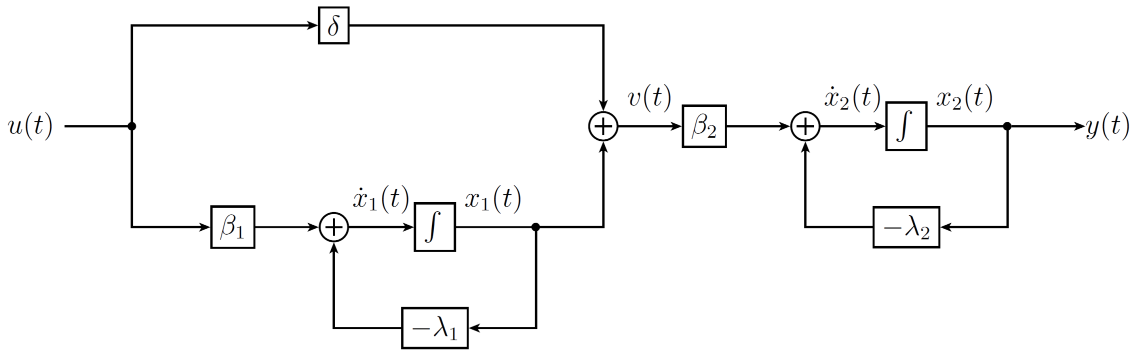
$$\begin{bmatrix} \dot{\hat{\mathbf{x}}} \\ \dot{\hat{\mathbf{w}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_d\mathbf{C}_w \\ \mathbf{0} & \mathbf{A}_w \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} u + \begin{bmatrix} \mathbf{L}_x \\ \mathbf{L}_w \end{bmatrix} (y - \hat{y})$$

$$\hat{y} = [\mathbf{C} \quad \mathbf{0}] \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix}$$

where

$$\mathbf{A}_w = \begin{bmatrix} 0 & -\omega_{d1} & 0 & 0 \\ \omega_{d1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega_{d2} \\ 0 & 0 & \omega_{d2} & 0 \end{bmatrix} \quad , \quad \mathbf{C}_w = [1 \quad 0 \quad 1 \quad 0]$$

Consider the 2nd order system represented in the following block diagram below



where $\beta_1, \beta_2, \delta, \lambda_1$, and λ_2 are real and positive coefficients. Assuming that the input $u(t)$ is Gaussian white noise with zero mean and noise intensity σ_u^2, what is the variance of the output $y(t)$?

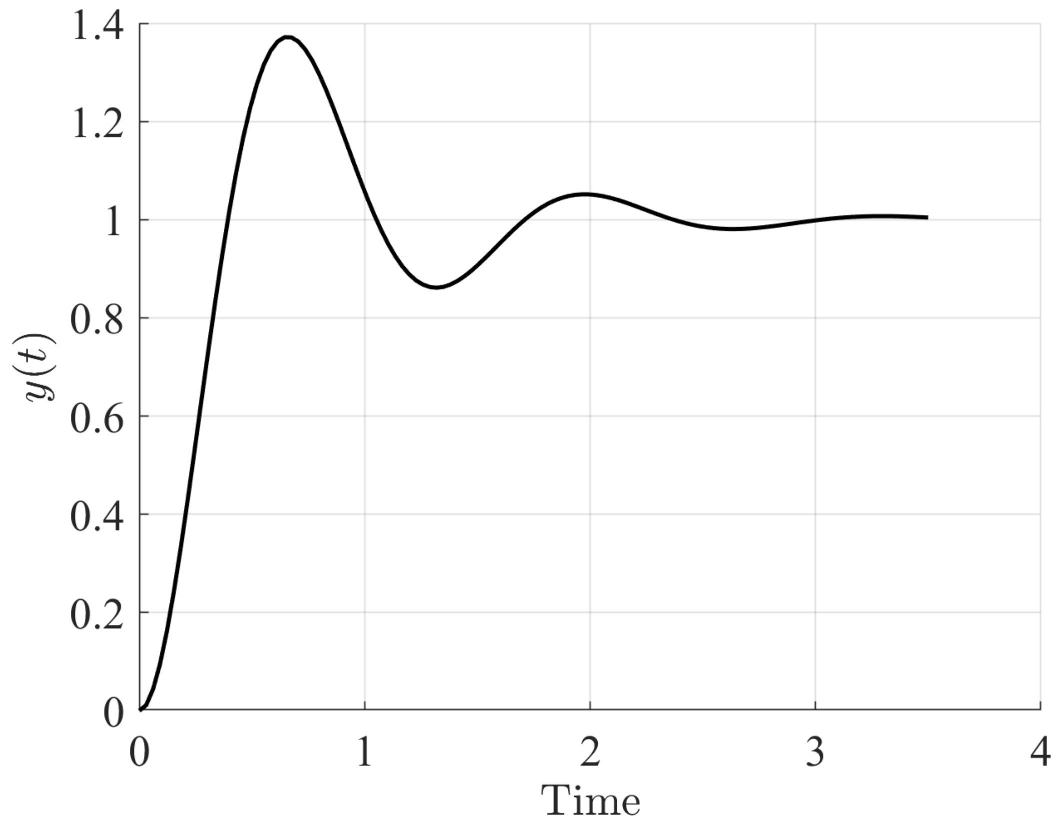
Choose one answer

- [x] $\sigma_y^2 = \frac{\beta_2^2}{2\lambda_2}\left(\delta^2 + \frac{2\beta_1}{\lambda_1+\lambda_2}\left(\delta + \frac{\beta_1}{2\lambda_1}\right)\right)\sigma_u^2$
- [ ] $\sigma_y^2 = \frac{\beta_2^2}{2\lambda_2}\left(\delta^2 + \frac{\beta_1^2}{2\lambda_1}\right)\sigma_u^2.$
- [ ] $\sigma_y^2 = \frac{\beta_2^2}{\lambda_2(\lambda_1+\lambda_2)}\left(\beta_1\delta + \frac{\beta_1^2}{2\lambda_1}\right)\sigma_u^2.$
- [ ] $\sigma_y^2 = 0.$
- [ ] $\sigma_y^2 = \sigma_u^2.$

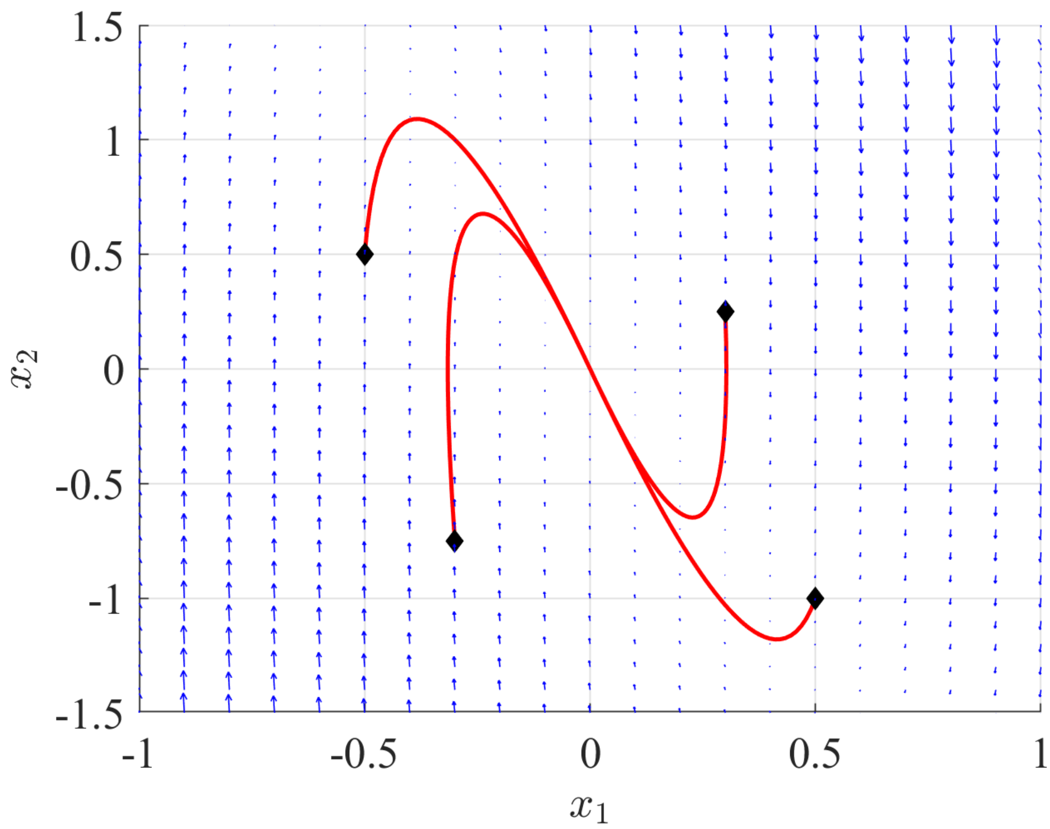
Consider the 2nd order LTI system

$$\Sigma : \begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \omega_0^2 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} \end{cases}$$

where $\omega_0 > 0$ and $0 < \zeta < 1$. The open loop step response of the system to a unit step is shown in the following figure



A full state feedback controller $u = -\mathbf{K}\mathbf{x}$ is designed such that the dynamical behaviour shown in the phase portrait is achieved.



Which of the following full state feedback controller matrix $\mathbf{K} = [\, k_1 \quad k_2 \,]$ achieves the closed-loop behaviour shown in the phase portrait?

Choose one answer

- The closed-loop behaviour can be achieved for $k_1 > -1$ and $-k_2^\star \le k_2 \le k_2^\star$, where

  $k_2^\star = \frac{2}{\omega_0}(\sqrt{1+k_1} - \zeta)$

- The closed-loop behaviour can be achieved for any $k_1, k_2 \neq 0$.

- The closed-loop behaviour can be achieved for $k_1 > 1$ and $k_2 \le -k_2^\star$ where

  $k_2^\star = \frac{2}{\omega_0}(\sqrt{1+k_1} - \zeta)$

- The closed-loop behaviour can be achieved for any $k_1 > 0$ and any $k_2 > 0$.

- The closed-loop behaviour can be achieved for $k_1 > -1$ and $k_2 > k_2^\star$, where

  $k_2^\star = \frac{2}{\omega_0}(\sqrt{1+k_1} - \zeta)$

Consider the 2nd order LTI system

$$\Sigma : \begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix} \mathbf{x} + \begin{bmatrix} \delta \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} \end{cases}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ and $\delta \in \mathbb{R} \setminus \{0\}$. The full state feedback controller $u = \mathbf{K}\mathbf{x}$ with $\mathbf{K} = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, $k_1, k_2 \in \mathbb{R}_+$ is designed to stabilize the system. Which of the following statements is correct?

Choose one answer

- [x] If $\gamma < 0$ then the control law $u = \mathbf{K}\mathbf{x}$ can stabilize the system Σ by arbitrary eigenvalue assignment only to the subsystem $\dot{x}_1 = \alpha x_1 + \beta x_2 + \delta u$.
- [ ] If $\gamma \geq 0$ then the control law $u = \mathbf{K}\mathbf{x}$ can stabilize the system Σ by arbitrary eigenvalue assignment only to the subsystem $\dot{x}_1 = \alpha x_1 + \beta x_2 + \delta u$.
- [ ] The control law $u = \mathbf{K}\mathbf{x}$ can stabilize the system Σ by arbitrary eigenvalue assignment because the open loop system is observable.
- [ ] If $\alpha < 0$ then the control law $u = \mathbf{K}\mathbf{x}$ can stabilize the system Σ by arbitrary eigenvalue assignment only to the subsystem $\dot{x}_2 = \gamma x_2$.
- [ ] The control law $u = \mathbf{K}\mathbf{x}$ can stabilize the system Σ by arbitrary eigenvalue assignment because the open loop system is controllable.

Consider the 2nd order LTI system

$$\Sigma : \begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\alpha & -\beta \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \gamma \end{bmatrix} u + \begin{bmatrix} 0 \\ \delta \end{bmatrix} d \\ y = [1 \quad 0]\, \mathbf{x} \end{cases}$$

where $\alpha, \beta, \in \mathbb{R}_+$ and $\gamma, \delta \in \mathbb{R} \setminus \{0\}$. The system is subject to an unknown disturbance $d(t)$.

The following observer is designed under the assumption that the disturbance acting on the system is constant, i.e.

$$\Sigma_o : \begin{cases} \dot{\hat{\mathbf{x}}}_o = \begin{bmatrix} 0 & 1 & 0 \\ -\alpha & -\beta & \delta \\ 0 & 0 & 0 \end{bmatrix} \hat{\mathbf{x}}_o + \begin{bmatrix} 0 \\ \gamma \\ 0 \end{bmatrix} u + \mathbf{L}(y - \hat{y}_o) \\ \hat{y}_o = [1 \quad 0 \quad 0]\, \hat{\mathbf{x}}_o \end{cases}$$

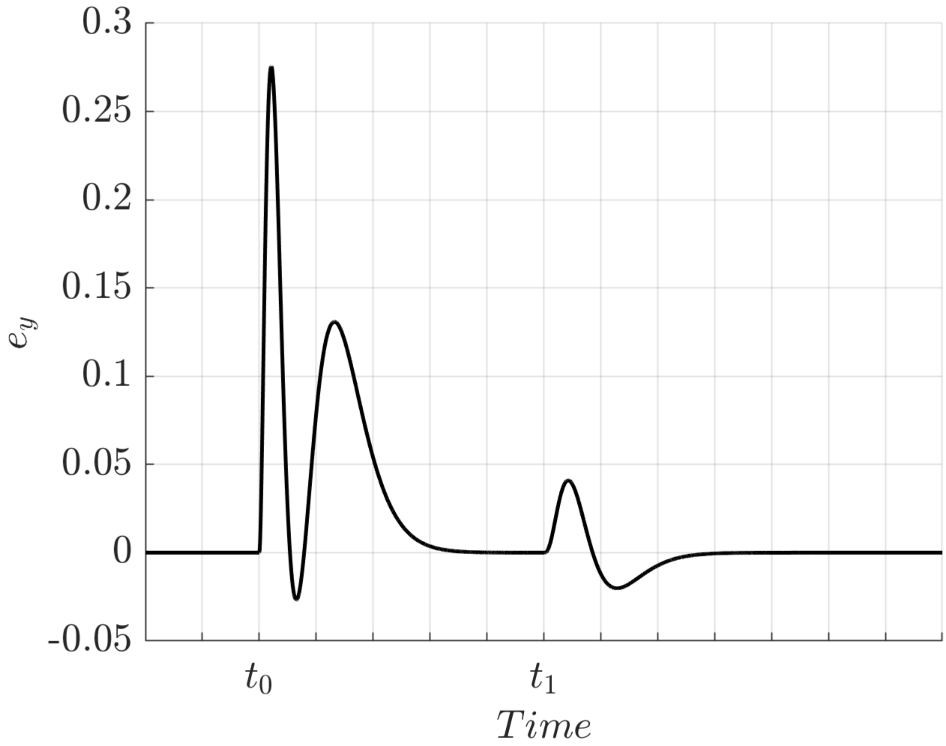
where the state of the observer is $\hat{\mathbf{x}}_o = [\hat{x}_1, \hat{x}_2, \hat{d}\,]^T$. The following disturbance profile acts on the system

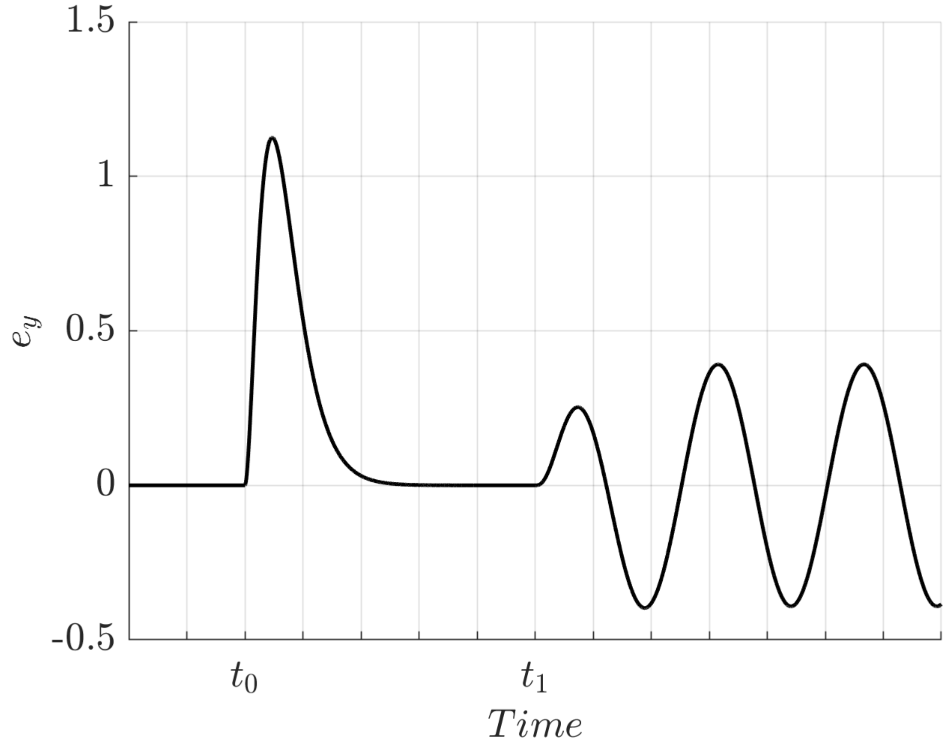
$$d(t) = \begin{cases} 0 & 0 \leq t < t_0 \\ d_0 & t_0 \leq t < t_1 \\ d_0 + A_d \sin(\omega_d t + \varphi_d) & t \geq t_1 \end{cases}$$

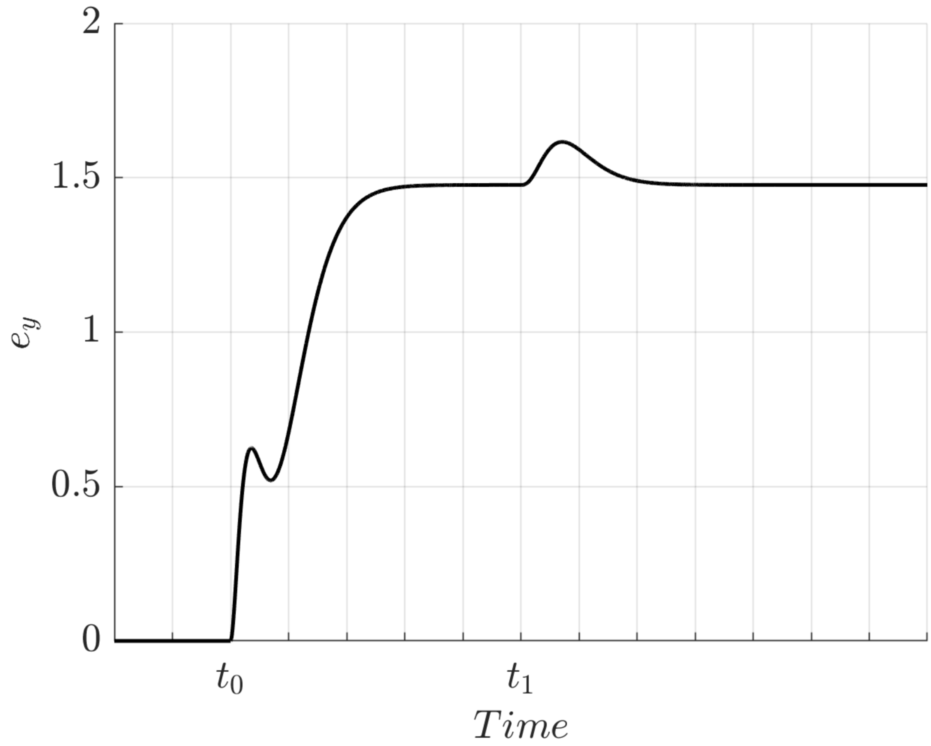
where $d_0, A_d, \omega_d, \varphi_d \in \mathbb{R}_+$. Under the assumption that the observer initial condition matches the system initial condition, i.e. $\hat{\mathbf{x}}_o(0) = [\hat{x}_1(0), \hat{x}_2(0), \hat{d}\,(0)]^T = [x_1(0), x_2(0), 0]^T$, and that $e_y = y - \hat{y}_o$ is the output estimation error, which of the following plots shows the correct behaviour of $e_y(t)$ for all $t \geq 0$?

Choose one answer

○

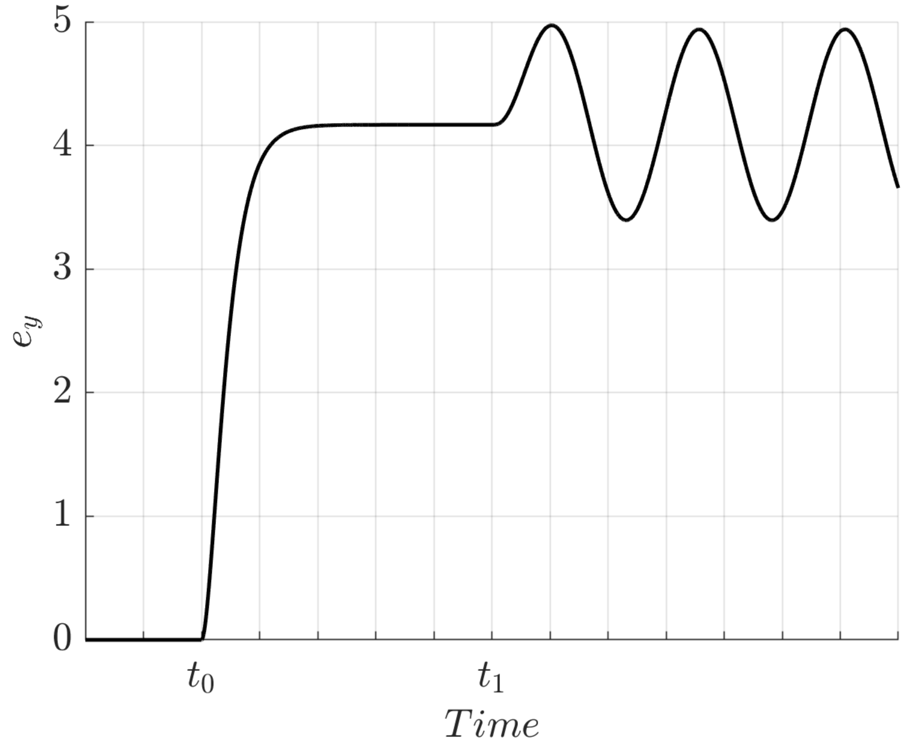

0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
e_y
t_0
t_1
$Time$


1.5
1
0.5
0
-0.5
e_y
t_0
t_1
$Time$



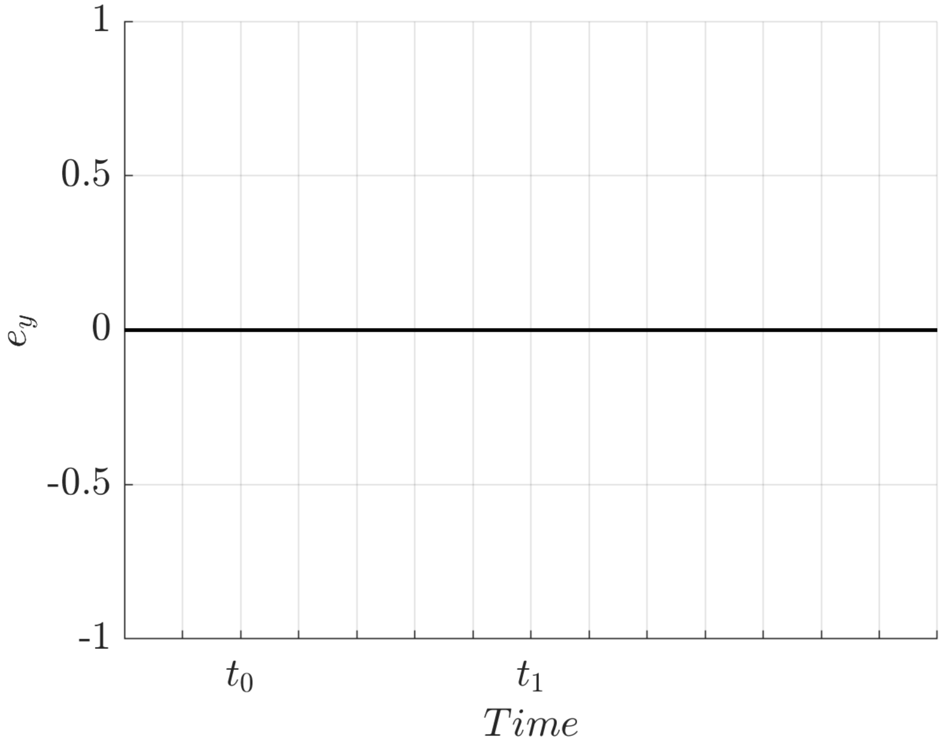
2
1.5
1
0.5
0
e_y
t_0
t_1
$Time$

○



5
4
3
2
1
0
e_y
t_0
t_1
$Time$

○


1
0.5
0
-0.5
-1
e_y
t_0
t_1
$Time$

Consider the 2nd order LTI system

$$\Sigma : \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} g(t) \\ y = \mathbf{C}\mathbf{x} + n \end{cases} \qquad \mathbf{x} \in \mathbb{R}^2, u \in \mathbb{R}, y \in \mathbb{R}$$

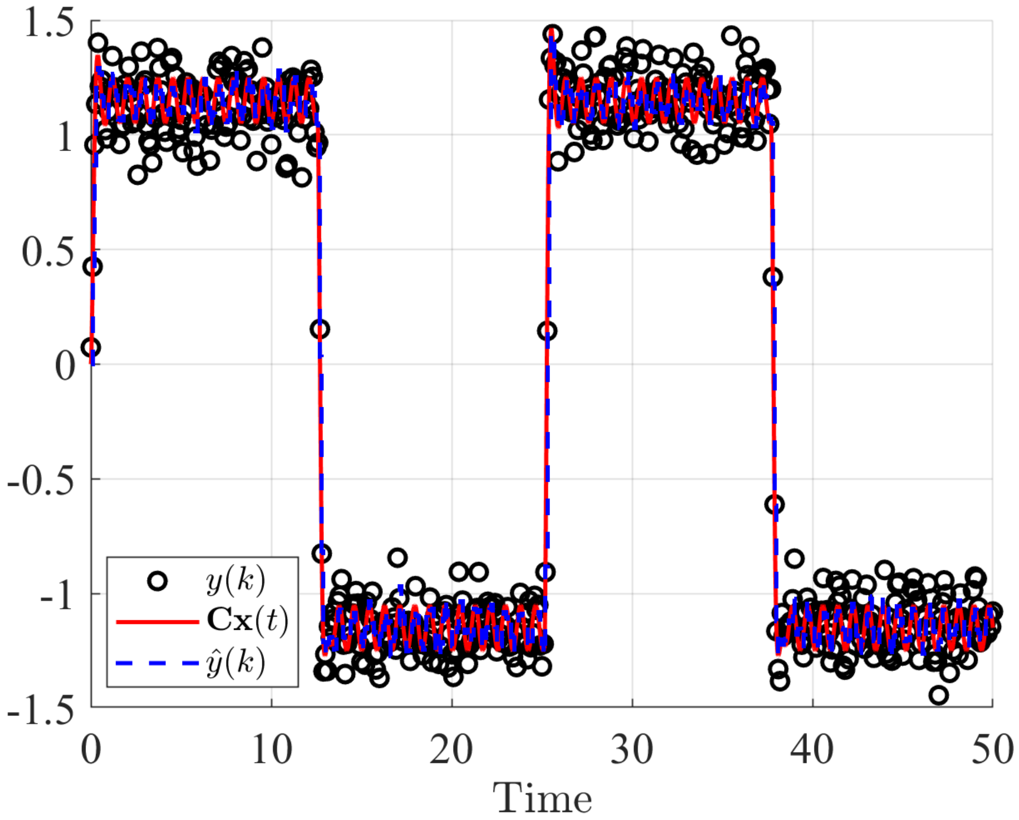
where $g(t)$ is an unknown function of time, and n is zero mean white Gaussian noise with noise intensity σ_n^2.

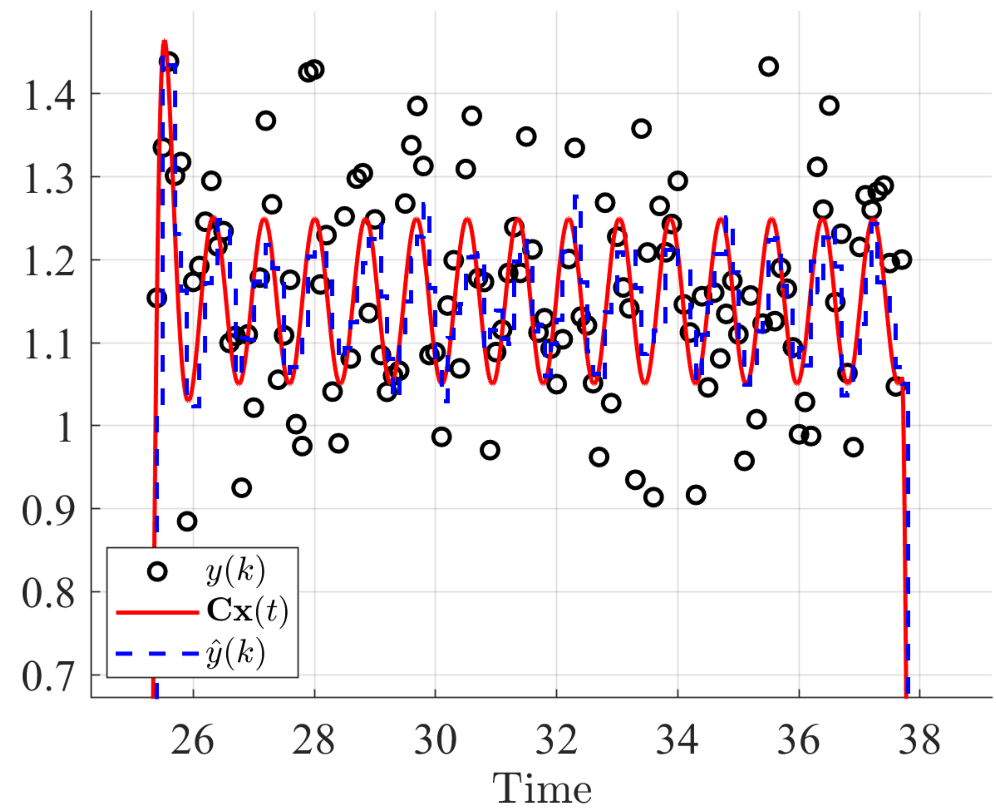
A discrete time Kalman filter is designed to reconstruct the state $\mathbf{x}$ based on noisy measurements y, i.e.

$$\Sigma_{KF} : \begin{cases} \hat{\mathbf{x}}(k+1) = \mathbf{F}\hat{\mathbf{x}}(k) + \mathbf{G}u(k) + \mathbf{L}'(y(k) - \hat{y}(k)) \\ \hat{y}(k) = \mathbf{C}\hat{\mathbf{x}}(k) \end{cases}$$

where $\mathbf{F} = e^{\mathbf{A}T_s}$, $\mathbf{G} = \int_0^{T_s} e^{\mathbf{A}t}\mathbf{B}\,\mathrm{d}t$, $\mathbf{L}' = \mathbf{F}\mathbf{L}$ is the steady state Kalman gain in predictive form, and T_s is the sampling time. The Kalman gain $\mathbf{L}$ is designed based on the variance of the measurement noise n and the variance of the process noise v. The process noise is used to account for the model uncertainty introduced by the function $g(t)$ on the second state equation. The process noise is zero mean white Gaussian noise with noise intensity σ_v^2.

The estimated output $\hat{y}$ is shown in the following figures (second figure is a zoom in), when the system Σ is excited by a square wave and subject to the function $g(t) = A_g \sin \omega_g t$, where both A_g and ω_g are unknown.





Based on the comparison of the estimated output $\hat{y}$ with the true output $\mathbf{Cx}$, which of the following statements is correct?

Choose one answer

- ◯ The Kalman filter estimates the unknown dynamics of the system Σ by tuning the process noise intensity to be equal to the measurement noise intensity, i.e. $\sigma_v^2 = \sigma_n^2$.

- ○ The Kalman filter estimates the unknown dynamics of the system Σ by setting the process noise intensity to zero, i.e. $\sigma_v^2 = 0$.
- ○ The Kalman filter estimates the unknown dynamics of the system Σ by tuning the process noise intensity to be much smaller than the measurement noise intensity, i.e. $\sigma_v^2 \ll \sigma_n^2$.
- ✔ The Kalman filter estimates the unknown dynamics of the system Σ by tuning the process noise intensity to be much larger than the measurement noise intensity, i.e. $\sigma_v^2 \gg \sigma_n^2$.
- ○ The Kalman filter estimates the unknown dynamics of the system Σ regardless of the value of the process and measurement noise intensities.

34745 E22 Multiple Choice Questionnaire

The Keynesian model of economic growth is used to describe the dynamics of the expenditure and revenue part of the economy of a nation, and it utilizes the following variables:

- Y is the gross national product,
- G is the government expenditure,
- C is the consumption expenditure,
- I is the investment expenditure.

In dynamic equilibrium at time k the gross national product Y equals the total expenditure E, i.e.

$$Y(k) = E(k)$$

where $E(k) = C(k) + I(k) + G(k)$. Let us assume that the consumption expenditure at time k is given by a fraction of the gross national product at the previous time $k-1$, i.e.

$$C(k) = \alpha Y(k-1)$$

where $0 < \alpha < 1$ is the multiplier factor. Further, let us assume that the investment expenditure at time k is proportional to the rate of change of the gross national product, i.e.

$$I(k) = \beta[Y(k-1) - Y(k-2)]$$

where $\beta > 0$.

Let $\mathbf{x} = [C, I]^T$ be the state vector, $u = G$ the input, and $y = Y$ the output. Which of the following discrete time models describe the Keynesian model of economic growth?

✔ $$\Sigma : \begin{cases} \mathbf{x}(k+1) = \begin{bmatrix} \alpha & \alpha \\ \frac{\beta}{\alpha}(\alpha-1) & \beta \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} \alpha \\ \beta \end{bmatrix} u(k) \\ y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}(k) + u(k) \end{cases}$$

```
% Y(k) = E(k)
% Y(k) = C(k)+I(k)+G(k)
% C(k) = a*Y(k-1)
% C(k+1) = a*Y(k)
% C(k+1) = a*[C(k)+I(k)+G(k)]
% I(k+1) = b*[Y(k)-Y(k-1)]
% I(k+1) = b*{[C(k)+I(k)+G(k)]-Y(k-1)}
% I(k+1) = b*{[C(k)+I(k)+G(k)]-Y(k-1)}
% I(k+1) = b*{[C(k)-C(k)+I(k)]-Y(k-1)}
```

```
% C(k+1) = a*C(k)+a*I(k)+a*G(k)
% I(k+1) = b*C(k)-b*I(k)+b*G(k)-b*Y(k-1)
% I(k+1) = b*a*Y(k-1)-b*I(k)+b*G(k)-b*Y(k-1)
% sum = Ax+Bu
% sum = [a*C(k)      , a*I(k)] * x + [a*G(k)]
%       [b*a*Y(k-1) , b*I(k)]        [b*G(k)]
% sum = [a        , a] * x + [a]
%       [b*(a-1) , b]        [b]
% u(k) is included
```

Consider the 2nd order nonlinear system

$$\Sigma : \begin{cases} \dot{x}_1 = \dfrac{\alpha x_2^2}{1+kx_2^2} + \alpha_0 - \gamma x_1 \\ \dot{x}_2 = \beta x_1 - \delta x_2 \end{cases}$$

where the parameters $\alpha_0, \alpha, \beta, \gamma, \delta$ are all real and positive. Let $\mathbf{x}^e = [x_1^e, x_2^e]^T \neq [0,0]^T$ be an equilibrium point of Σ. What is the linear system obtained by linearizing Σ around the point of equilibrium $\mathbf{x}^e$?

$$\dot{\Delta\mathbf{x}} = \begin{bmatrix} -\gamma & \frac{2\alpha x_2^e}{(1+k(x_2^e)^2)^2} \\ \beta & -\delta \end{bmatrix} \Delta\mathbf{x}$$

```
syms a x2 k x1 gamma a0
simplify(diff(a*x2^2/(1+k*x2^2)+a0-gamma*x1,x2))
```

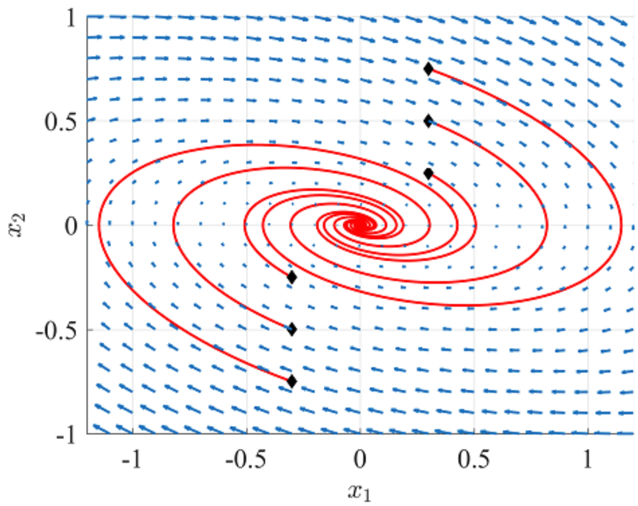
ans =

$$\frac{2\,a\,x_2}{\left(k\,{x_2}^2+1\right)^2}$$

```
simplify(diff(a*x2^2/(1+k*x2^2)+a0-gamma*x1,x1))
```

ans = $-\gamma$

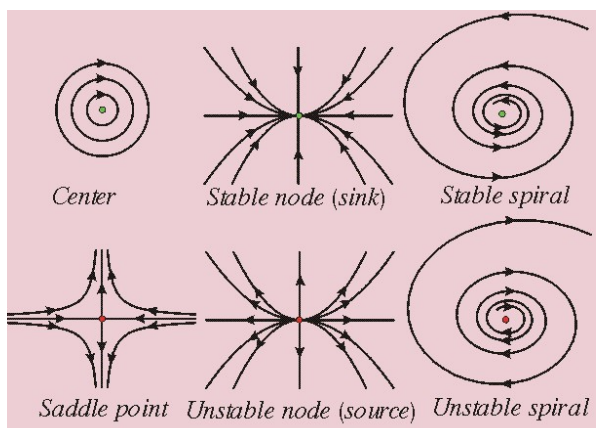
Consider the phase portrait shown in the following figure, where each black diamond is an initial condition of the system $\mathbf{x}(0) = [x_{10}, x_{20}]^T$, each red line is a trajectory of the system originating from the initial condition, and the blue arrows show the direction of the vector field describing the system.



Which of the following statements is correct?

Choose one answer

- The equilibrium point is a centre.
- The equilibrium point is a saddle point.
- The equilibrium point is a stable focus.
- The equilibrium point is an unstable focus.
- The equilibrium point is a stable node.



Consider the 3rd order LTI system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} = \begin{bmatrix} 0 & 0 & 0 \\ \alpha & -\beta & 0 \\ -\alpha & \beta & 0 \end{bmatrix} \mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^3$$

where $\alpha > 0$ and $\beta > 0$. $\lambda(\mathbf{A})$ is the set of all eigenvalues of the system dynamical matrix $\mathbf{A}$ and λ_i denotes a specific eigenvalue in $\lambda(\mathbf{A})$, which of the following statements is correct?

Choose one answer

- The system is unstable because the eigenvalue $\lambda = 0$ has geometric multiplicity $m_g = 1$.
- The system is asymptotically stable because $\text{Re}\{\lambda_i\} < 0, \forall \lambda_i \in \lambda(\mathbf{A})$
- The system is unstable because the eigenvalue $\lambda = 0$ has algebraic multiplicity $m_a = 2$.
- The system is unstable because $\exists \lambda_i \in \lambda(\mathbf{A})$ such that $\text{Re}\{\lambda_i\} > 0$.
- ✓ The system is marginally stable because the eigenvalue $\lambda = 0$ has geometric multiplicity equal to the algebraic multiplicity, i.e. $m_g = m_a$.

eig = [0, -beta, 0]   alg mult = 2 : 0, 0

```
syms alpha beta
A = [0 0 0; alpha -beta 0; -alpha beta 0]
```

A =

$$\begin{pmatrix} 0 & 0 & 0 \\ \alpha & -\beta & 0 \\ -\alpha & \beta & 0 \end{pmatrix}$$

```
e = eig(A)
```

e =

$$\begin{pmatrix} 0 \\ 0 \\ -\beta \end{pmatrix}$$

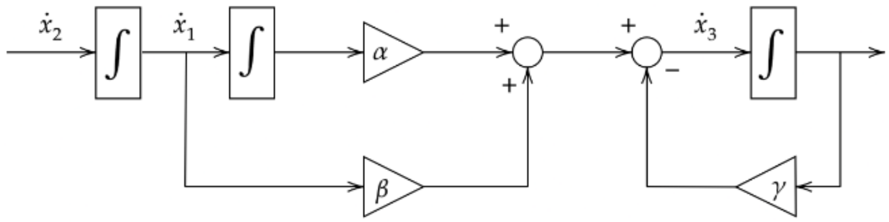
```
[V,D,W] = eig(A);
Ma = 2;
syms v v1 v2 v3

Mg = rank((A-e.*eye(3)).*[v1;v2;v3])
```

Mg = 2

```
if Mg == Ma
    display('system is stable');
elseif Mg < Ma
    display('system is unstable');
end;
```

system is stable

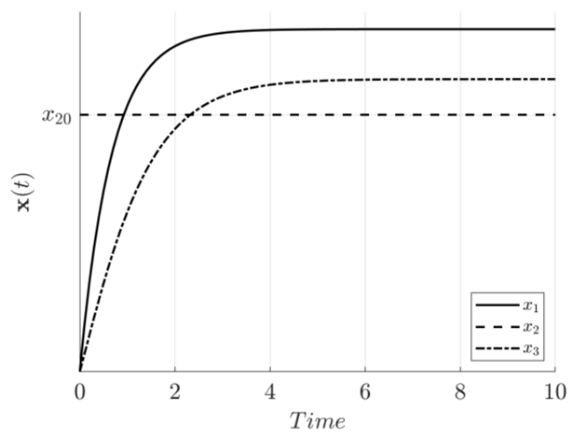
Consider the 3rd order system shown in the block diagram below

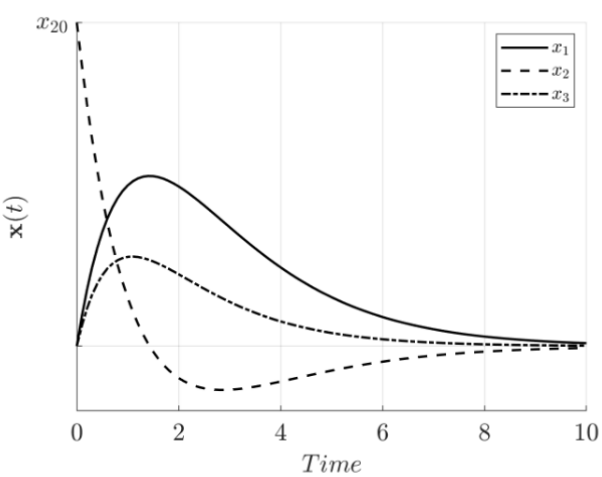


Which of the following zero-input responses is correct when the system is initialized with $\mathbf{x}_0 = [0, x_{20}, 0]^T$?

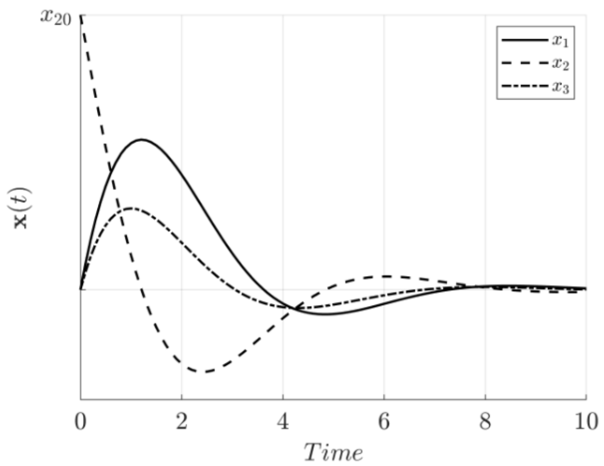
Choose one answer

○

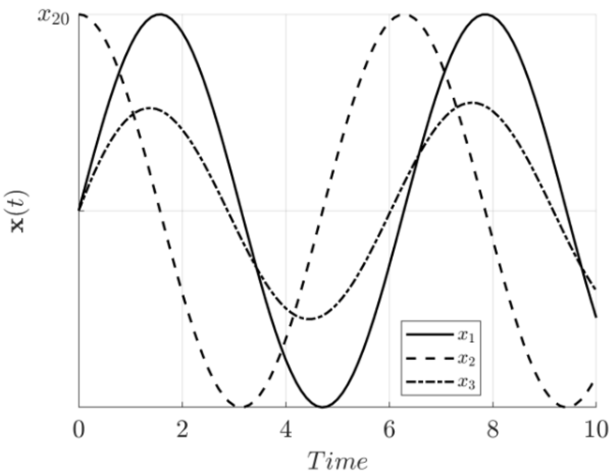




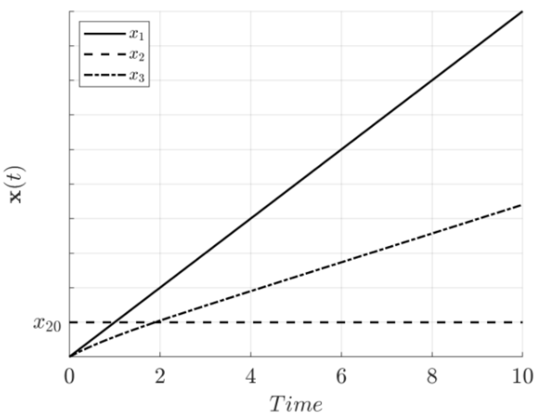
dampened only real since we dont have oscillations



dampened oscillations complex but stable



standing oscillations needs imaginary part = 0




```
syms alpha beta gamma x2
% x1_d = x2
% x2_d = 0
% x3_d = alpha*x1+beta*x2-gamma*x3
A = [0 1 0; 0 0 0; alpha beta -gamma]
```

A =

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \alpha & \beta & -\gamma \end{pmatrix}$$

```
init_cond = [0;x2;0]
```

init_cond =

$$\begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix}$$

```
e = eig(A)
```

e =

$$\begin{pmatrix} 0 \\ 0 \\ -\gamma \end{pmatrix}$$

```
[V,D,W] = eig(A);
Ma = 2;
syms v v1 v2 v3

Mg = rank((A*init_cond-e.*eye(3)).*[v1;v2;v3])
```

Mg = 2

```
if Mg == Ma
    display('system is stable');
elseif Mg < Ma
    display('system is unstable');
end;
```

system is stable

```
% x1(t) = x2*t = linear
```

Consider a 3rd order LTI discrete time SISO system

$$\mathbf{x}(k+1) = \begin{bmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 1 \\ 0 & 0 & -\gamma \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k), \quad \mathbf{x} \in \mathbb{R}^3,\ u \in \mathbb{R}$$
$$\mathbf{y}(k) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}(k), \quad y \in \mathbb{R}$$

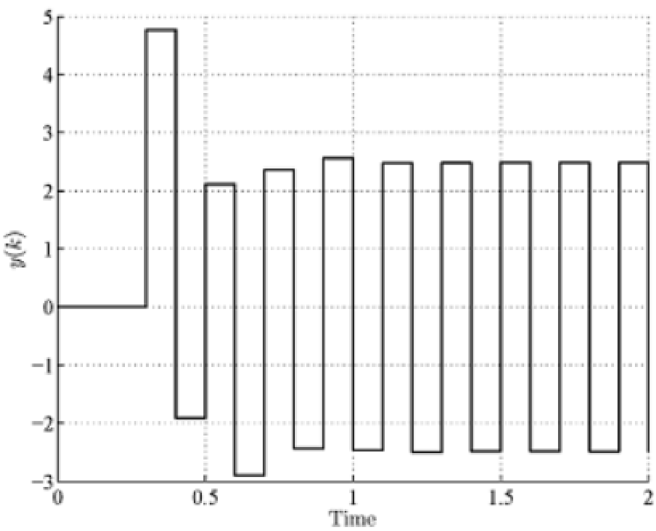
where α, β, and γ are real and positive coefficients such that

$$|\alpha| < 1 \wedge \beta = \sqrt{1 - \alpha^2} \wedge |\gamma| < 1$$

The sampling time is $T_s = 0.1\ \mathrm{s}$. Which of the following plots represents the unit pulse response of the system when a unit pulse

$$u(k) = \begin{cases} 1 & \text{for } k \in [0,1] \subset \mathbb{Z} \\ 0 & \text{for } k > 1 \end{cases}$$

is sent through the input channel?




```
syms alpha beta gamma

A = [alpha beta 0; -beta alpha 1; 0 0 -gamma];
B = [0; 0; 1];
C = [1 0 0];
A_new = subs(A, beta, sqrt(1 - alpha^2))
```

A_new =

$$\begin{pmatrix} \alpha & \sqrt{1-\alpha^2} & 0 \\ -\sqrt{1-\alpha^2} & \alpha & 1 \\ 0 & 0 & -\gamma \end{pmatrix}$$

```
e = eig(A_new)
```

e =

$$\begin{pmatrix} \alpha + \sqrt{(\alpha-1)\ (\alpha+1)} \\ \alpha - \sqrt{(\alpha-1)\ (\alpha+1)} \\ -\gamma \end{pmatrix}$$

```
e_sub = abs(subs(e, [alpha gamma], [0 0]))
```

e_sub =

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

```
% Gamma < 1 : inside circle, alpha < 1 : inside circle, if alpha < 1 beta
% is also < 1 : inside circle

alpha = 0.5, beta =sqrt(1-alpha^2), gamma = 0.5
```

```
alpha = 0.5000
beta = 0.8660
gamma = 0.5000
```

```
A = [alpha beta 0;-beta alpha 1; 0 0 -gamma];
e = eig(A);
ts = 0.1;

norm(e(1))
```

```
ans = 1
```

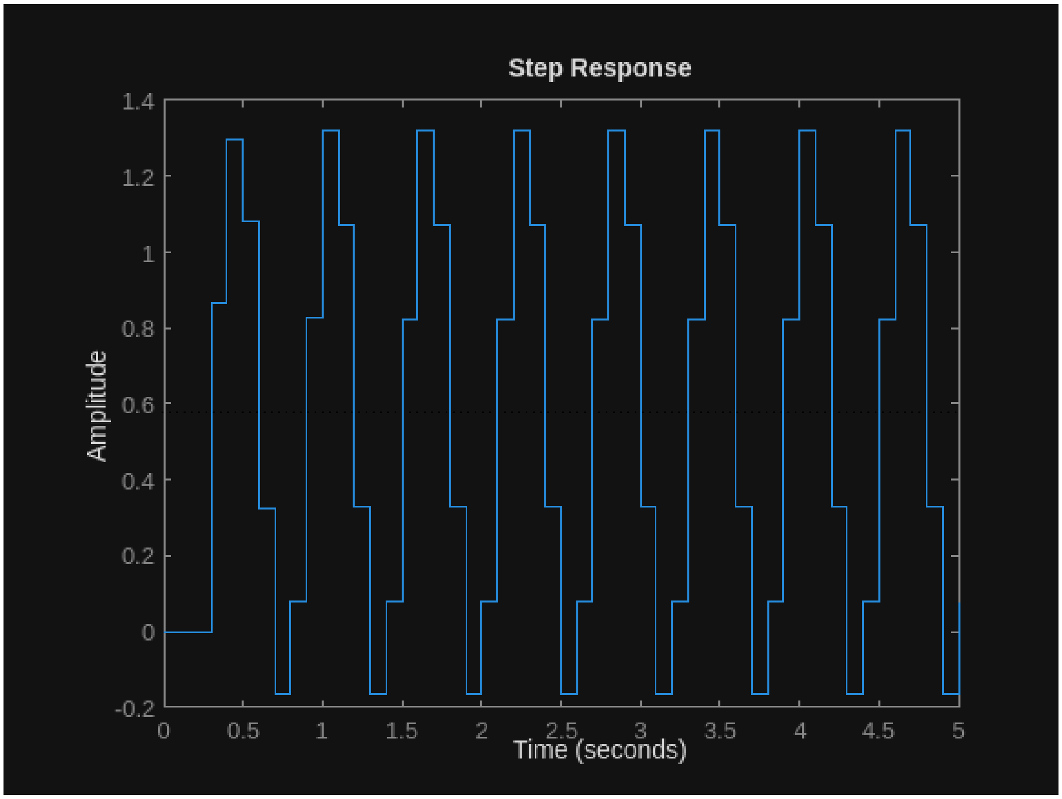
```
norm(e(2))
```

```
ans = 1
```

```
norm(e(3))
```

```
ans = 0.5000
```

```
%marginally stable

sys = ss(A,B,C,D,ts);
step(sys,5)
```



Consider the nth order LTI system

$$\Sigma_x : \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u + \mathbf{B}_d d, & \mathbf{x} \in \mathbb{R}^n, u \in \mathbb{R}, d \in \mathbb{R} \\ y = \mathbf{C}\mathbf{x}, & y \in \mathbb{R} \end{cases}$$

The observability matrix satisfies the rank condition $\text{rank}(\mathbf{M}_o) = n$. The system is subject to a time-varying not measurable disturbance given by

$$d(t) = d_0 + d_1 \sin(\omega_d t)$$

where d_0, d_1, ω_d are positive real constants. Which of the following observers can estimate the disturbance $d(t)$?

$$\mathbf{A}_w = \begin{bmatrix} 0 & 1 \\ -\omega_d^2 & -2\zeta\omega_d \end{bmatrix} \quad , \quad \mathbf{C}_w = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

wrong because it is dampened

$$\mathbf{A}_w = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad \mathbf{C}_w = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

wrong because 0

$$\mathbf{A}_w = \begin{bmatrix} -\omega_d & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad \mathbf{C}_w = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

wrong because it decays

$$\mathbf{A}_w = \begin{bmatrix} 0 & -\omega_{d1} & 0 & 0 \\ \omega_{d1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega_{d2} \\ 0 & 0 & \omega_{d2} & 0 \end{bmatrix} \quad , \quad \mathbf{C}_w = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$$

wrong because we have 2 oscillating factors instead of just 1

The dynamics of the state estimator is

$$\begin{bmatrix} \dot{\hat{\mathbf{x}}} \\ \dot{\hat{\mathbf{w}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_d\mathbf{C}_w \\ \mathbf{0} & \mathbf{A}_w \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} u + \begin{bmatrix} \mathbf{L}_x \\ \mathbf{L}_w \end{bmatrix} (y - \hat{y})$$

$$\hat{y} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{w}} \end{bmatrix}$$

where

$$\mathbf{A}_w = \begin{bmatrix} 0 & -\omega_d & 0 \\ \omega_d & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad , \quad \mathbf{C}_w = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

correct because we have 1 oscillating factor

```
syms wd alpha beta gamma wd1 wd2
% d(t) = d0 + d1 * sin(wd*t)
% d0 is constant d1 is oscillating because of sin
% look at all answers see what decays and what is dampened

eig([0 1; -wd^2 -2*gamma*wd])
```

ans =

$$\begin{pmatrix} \mathrm{wd}\ \sqrt{(\gamma-1)\ (\gamma+1)} - \gamma\ \mathrm{wd} \\ -\mathrm{wd}\ \sqrt{(\gamma-1)\ (\gamma+1)} - \gamma\ \mathrm{wd} \end{pmatrix}$$

```
display("no oscillations")
```

"no oscillations"

```
eig([0 0; 0 0])
```

ans = 2×1

```
0
0
```

```
display("0")
```

```
"0"
```

```
eig([0 wd 0; -wd 0 0; 0 0 0])
```

ans =

$$\begin{pmatrix} 0 \\ -\mathrm{wd}\ \mathrm{i} \\ \mathrm{wd}\ \mathrm{i} \end{pmatrix}$$

```
display("1 oscillating parameter")
```

```
"1 oscillating parameter"
```

```
eig([-wd 0; 0 0])
```

ans =

$$\begin{pmatrix} 0 \\ -\mathrm{wd} \end{pmatrix}$$

```
display("decaying")
```

```
"decaying"
```

```
eig([0 -wd1 0 0; wd1 0 0 0; 0 0 0 -wd2; 0 0 wd2 0])
```

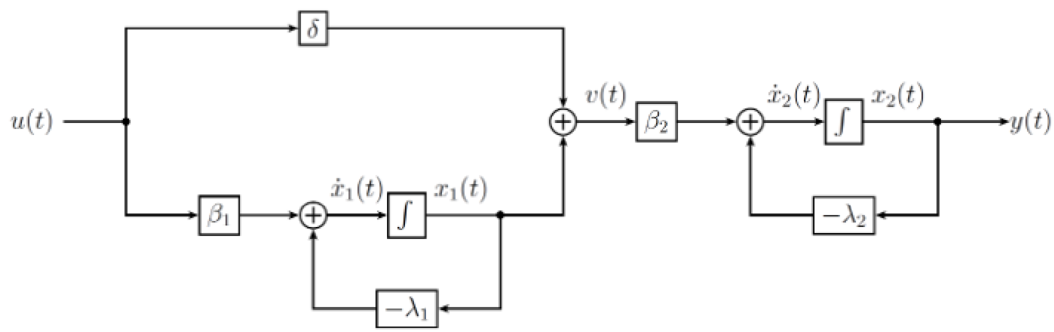
ans =

$$\begin{pmatrix} -\mathrm{wd}_1\ \mathrm{i} \\ \mathrm{wd}_1\ \mathrm{i} \\ -\mathrm{wd}_2\ \mathrm{i} \\ \mathrm{wd}_2\ \mathrm{i} \end{pmatrix}$$

```
display("2 oscillating parameters")
```

```
"2 oscillating parameters"
```

Consider the 2nd order system represented in the following block diagram below



where $\beta_1, \beta_2, \delta, \lambda_1$, and λ_2 are real and positive coefficients. Assuming that the input $u(t)$ is Gaussian white noise with zero mean and noise intensity σ_u^2, what is the variance of the output $y(t)$?

Choose one answer

$$\sigma_y^2 = \frac{\beta_2^2}{2\lambda_2}\left(\delta^2 + \frac{2\beta_1}{\lambda_1+\lambda_2}\left(\delta + \frac{\beta_1}{2\lambda_1}\right)\right)\sigma_u^2$$

$$\sigma_x^2 = \frac{\text{Input Noise Power} \times \text{System Gain}^2}{2 \cdot \text{Eigenvalue of System}}.$$

```
syms lambda1 lambda2 beta1 beta2 sigma_u q11 q12 q21 q22 delta t
% x1_d = -lambda1*x1+beta1*u
% x2_d = -lambda2*x2+beta2*(delta*u+x1)
% u = white_noise(0,sigma^2)
% y = x2
% A*Q+Q*At+Bv*V*Bvt=0

% x1_d + lambda1*x1 = beta1*u

A = [-lambda1 0; beta2 -lambda2]
```

A =

$$\begin{pmatrix} -\lambda_1 & 0 \\ \beta_2 & -\lambda_2 \end{pmatrix}$$

```
Bv = [beta1;beta2*delta]
```

Bv =

$$\begin{pmatrix} \beta_1 \\ \beta_2\,\delta \end{pmatrix}$$

```
V = sigma_u^2
```

V = ${\sigma_u}^2$

```
Q = [q11 q12; q21 q22];
At = transpose(A)
```

At =

$$\begin{pmatrix} -\lambda_1 & \beta_2 \\ 0 & -\lambda_2 \end{pmatrix}$$

```
Bvt = transpose(Bv)
```

Bvt = $\begin{pmatrix} \beta_1 & \beta_2\,\delta \end{pmatrix}$

```
[q11,q12,q21,q22] = solve(A*Q+Q*At+Bv*V*Bvt == 0,[q11,q12,q21,q22]);
simplify(q22)
```

ans =

$$\frac{{\beta_2}^2\,{\sigma_u}^2\,\left({\beta_1}^2+2\,\beta_1\,\delta\,\lambda_1+\delta^2\,{\lambda_1}^2+\lambda_2\,\delta^2\,\lambda_1\right)}{2\,\lambda_1\,\lambda_2\,\left(\lambda_1+\lambda_2\right)}$$

```
expand(2*lambda1*lambda2*(lambda1+lambda2))
```

ans = $2\,{\lambda_1}^2\,\lambda_2+2\,\lambda_1\,{\lambda_2}^2$

```
expand(beta2^2/(2*lambda2)*(delta^2+(2*beta1)/(lambda1+lambda2)*(delta+beta1/
(2*lambda1)))*sigma_u^2)
```

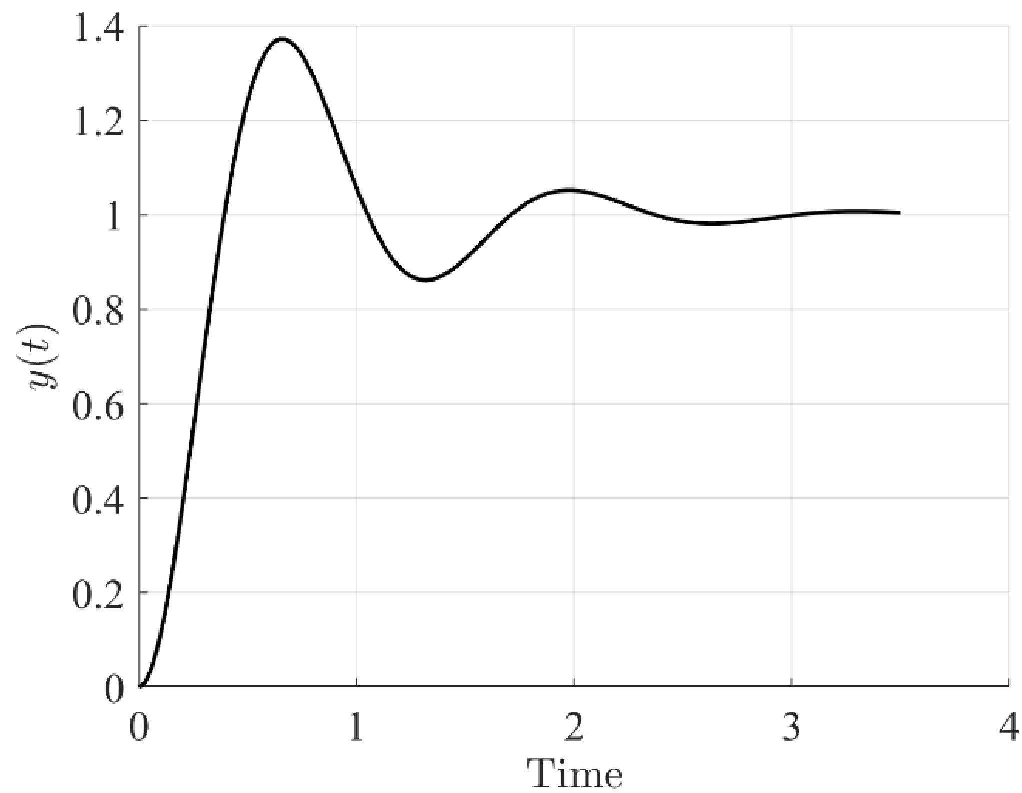
ans =

$$\frac{{\beta_2}^2\,\delta^2\,{\sigma_u}^2}{2\,\lambda_2}+\frac{{\beta_1}^2\,{\beta_2}^2\,{\sigma_u}^2}{2\,{\lambda_1}^2\,\lambda_2+2\,\lambda_1\,{\lambda_2}^2}+\frac{\beta_1\,{\beta_2}^2\,\delta\,{\sigma_u}^2}{{\lambda_2}^2+\lambda_1\,\lambda_2}$$

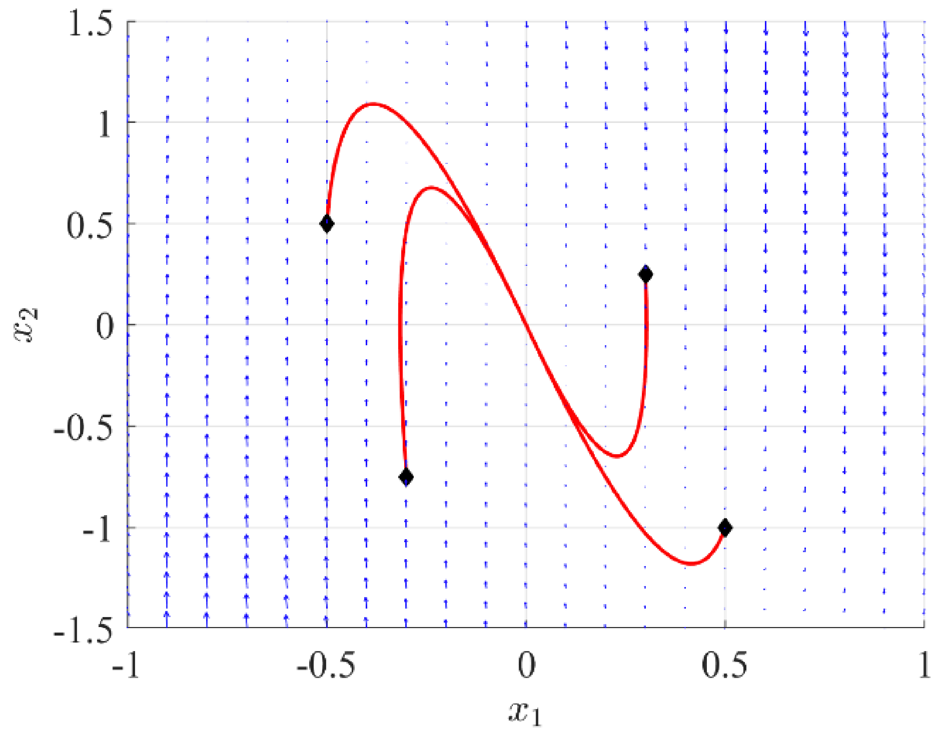
Consider the 2nd order LTI system

$$\Sigma : \begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \omega_0^2 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} \end{cases}$$

where $\omega_0 > 0$ and $0 < \zeta < 1$. The open loop step response of the system to a unit step is shown in the following figure



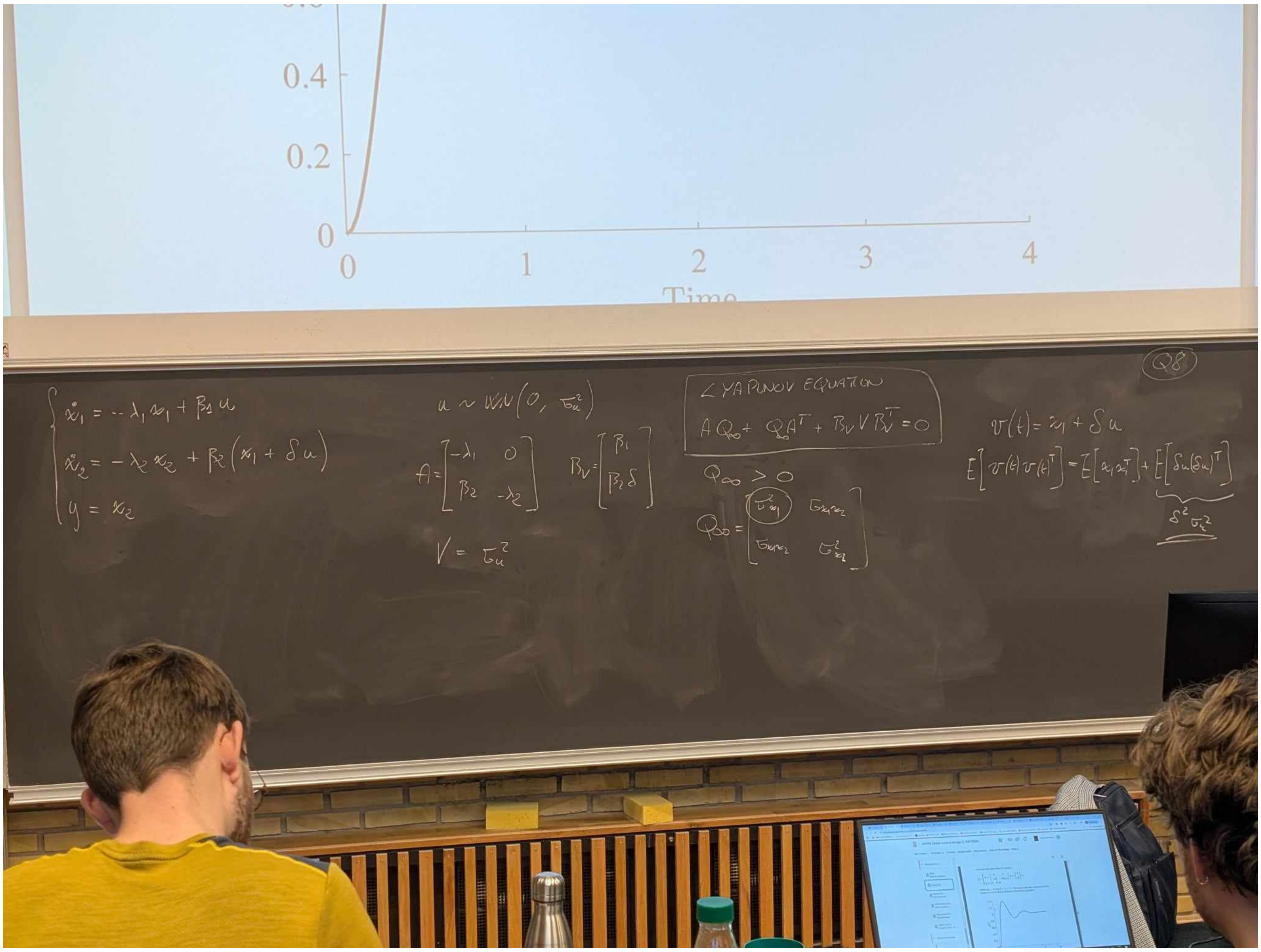
A full state feedback controller $u = -\mathbf{K}\mathbf{x}$ is designed such that the dynamical behaviour shown in the phase portrait is achieved.



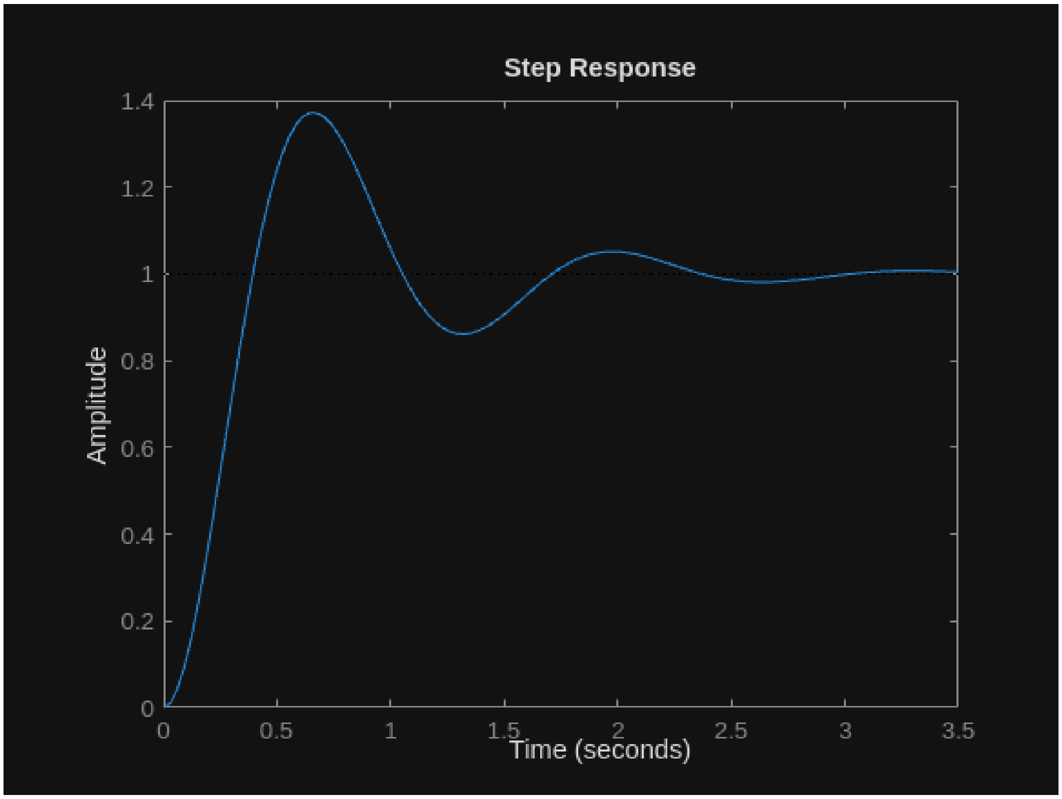
Which of the following full state feedback controller matrix $\mathbf{K} = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ achieves the closed-loop behaviour shown in the phase portrait?

✓ The closed-loop behaviour can be achieved for $k_1 > -1$ and $k_2 > k_2^\star$, where

$k_2^\star = \frac{2}{\omega_0}(\sqrt{1 + k_1} - \zeta)$




```
w0 = 5;
gamma = 0.3;
A = [0 1;-w0^2 -2*gamma*w0];
B = [0; w0^2];
C = [1 0];
D = 0;
sys = ss(A,B,C,D);
step(sys)
```




```
syms k1 k2 omega0 zeta lambda x1 x2
A = [0 1; -omega0^2 -2*zeta*omega0];
B = [0; omega0^2];
C = [1 0];
K = [k1 k2];
x = [x1; x2];
u = -K*x;

eqn = A*x + B*u
```

eqn =

$$\begin{pmatrix} x_2 \\ -{\omega_0}^2\,(k_1\,x_1 + k_2\,x_2) - {\omega_0}^2\,x_1 - 2\,\omega_0\,x_2\,\zeta \end{pmatrix}$$

```
arr = collect(eqn, [x1 x2])
```

arr =

$$\begin{pmatrix} x_2 \\ \left(-k_1\,{\omega_0}^2 - {\omega_0}^2\right)\,x_1 + \left(-k_2\,{\omega_0}^2 - 2\,\zeta\,\omega_0\right)\,x_2 \end{pmatrix}$$

```
mtx = [0 1; -k1*omega0^2-omega0^2 -k2*omega0^2-2*zeta*omega0]
```

mtx =

$$\begin{pmatrix} 0 & 1 \\ -k_1\,\omega_0{}^2 - \omega_0{}^2 & -k_2\,\omega_0{}^2 - 2\,\zeta\,\omega_0 \end{pmatrix}$$

```
e = simplify(eig(mtx))
```

e =

$$\begin{pmatrix} -\dfrac{\omega_0\left(2\,\zeta + k_2\,\omega_0 + \sqrt{k_2{}^2\,\omega_0{}^2 + 4\,k_2\,\omega_0\,\zeta + 4\,\zeta^2 - 4\,k_1 - 4}\right)}{2} \\ -\dfrac{\omega_0\left(2\,\zeta + k_2\,\omega_0 - \sqrt{k_2{}^2\,\omega_0{}^2 + 4\,k_2\,\omega_0\,\zeta + 4\,\zeta^2 - 4\,k_1 - 4}\right)}{2} \end{pmatrix}$$

```
k2s = (2/omega0)*(sqrt(1 + k1) - zeta);

% 1)
simplify(subs(e, [k1 k2], [k1 k2s]))
```

ans =

$$\begin{pmatrix} -\omega_0\,\sqrt{k_1+1} \\ -\omega_0\,\sqrt{k_1+1} \end{pmatrix}$$

```
solve(-sqrt(k1+1) == 0) %false
```

ans = -1

```
% 2)
simplify(subs(e, [k1 k2], [1 -1]))
```

ans =

$$\begin{pmatrix} -\dfrac{\omega_0\left(2\,\zeta - \omega_0 + \sqrt{\omega_0{}^2 - 4\,\omega_0\,\zeta + 4\,\zeta^2 - 8}\right)}{2} \\ \dfrac{\omega_0\left(\omega_0 - 2\,\zeta + \sqrt{\omega_0{}^2 - 4\,\omega_0\,\zeta + 4\,\zeta^2 - 8}\right)}{2} \end{pmatrix}$$

```
% 3)
simplify(subs(e, [k1 k2], [k1 -k2s]))
```

ans =

$$\begin{pmatrix} -\omega_0\left(2\,\zeta - \sqrt{k_1+1} + 2\,\sqrt{\zeta\,\left(\zeta - \sqrt{k_1+1}\right)}\right) \\ \omega_0\left(\sqrt{k_1+1} - 2\,\zeta + 2\,\sqrt{\zeta\,\left(\zeta - \sqrt{k_1+1}\right)}\right) \end{pmatrix}$$

```
% 5)
simplify(subs(e, [k1 k2], [-4, k2s+1]))
```

```
ans =
```

$$\begin{pmatrix} -\dfrac{\omega_0\left(\omega_0+2\,\sqrt{k_1+1}+\sqrt{4\,k_1+{\omega_0}^2+4\,\omega_0\,\sqrt{k_1+1}+16}\right)}{2} \\ -\dfrac{\omega_0\left(\omega_0+2\,\sqrt{k_1+1}-\sqrt{4\,k_1+{\omega_0}^2+4\,\omega_0\,\sqrt{k_1+1}+16}\right)}{2} \end{pmatrix}$$

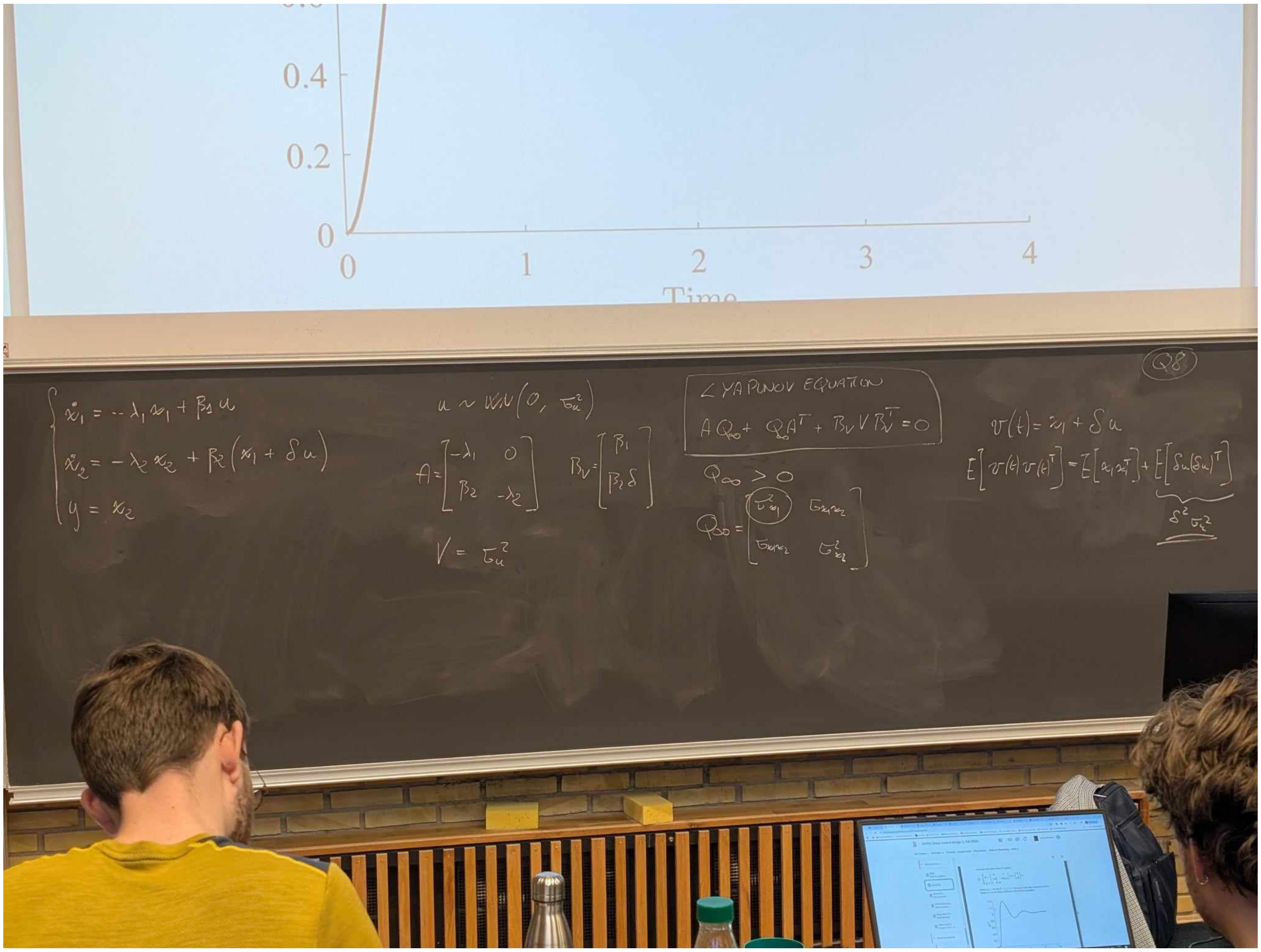
Consider the 2nd order LTI system

$$\Sigma : \begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix} \mathbf{x} + \begin{bmatrix} \delta \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} \end{cases}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ and $\delta \in \mathbb{R} \setminus \{0\}$. The full state feedback controller $u = \mathbf{Kx}$ with $\mathbf{K} = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, $k_1, k_2 \in \mathbb{R}_+$ is designed to stabilize the system. Which of the following statements is correct?

Choose one answer

- [x] If $\gamma < 0$ then the control law $u = \mathbf{Kx}$ can stabilize the system Σ by arbitrary eigenvalue assignment only to the subsystem $\dot{x}_1 = \alpha x_1 + \beta x_2 + \delta u$.
- [ ] If $\gamma \geq 0$ then the control law $u = \mathbf{Kx}$ can stabilize the system Σ by arbitrary eigenvalue assignment only to the subsystem $\dot{x}_1 = \alpha x_1 + \beta x_2 + \delta u$.
- [ ] The control law $u = \mathbf{Kx}$ can stabilize the system Σ by arbitrary eigenvalue assignment because the open loop system is observable.
- [ ] If $\alpha < 0$ then the control law $u = \mathbf{Kx}$ can stabilize the system Σ by arbitrary eigenvalue assignment only to the subsystem $\dot{x}_2 = \gamma x_2$.
- [ ] The control law $u = \mathbf{Kx}$ can stabilize the system Σ by arbitrary eigenvalue assignment because the open loop system is controllable.




```
syms alpha beta gamma sigma
A = [alpha beta; 0 gamma];
B = [sigma;0];
C = [1,0];
D = 0;

% Controllability has nothing to do with controller design last option is
% out
% B only has 1 controllable row meaning its not fully controllable
e = eig(A)
```

e =

$$\begin{pmatrix} \alpha \\ \gamma \end{pmatrix}$$

```
% sigma we cant control: x2_dot = gamma x2 is the non controllable part
% stability only if gamma < 0
% Therefore the 1. option
```

Consider the 2nd order LTI system

$$\Sigma : \begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\alpha & -\beta \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \gamma \end{bmatrix} u + \begin{bmatrix} 0 \\ \delta \end{bmatrix} d \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} \end{cases}$$

where $\alpha, \beta, \in \mathbb{R}_+$ and $\gamma, \delta \in \mathbb{R} \setminus \{0\}$. The system is subject to an unknown disturbance $d(t)$.

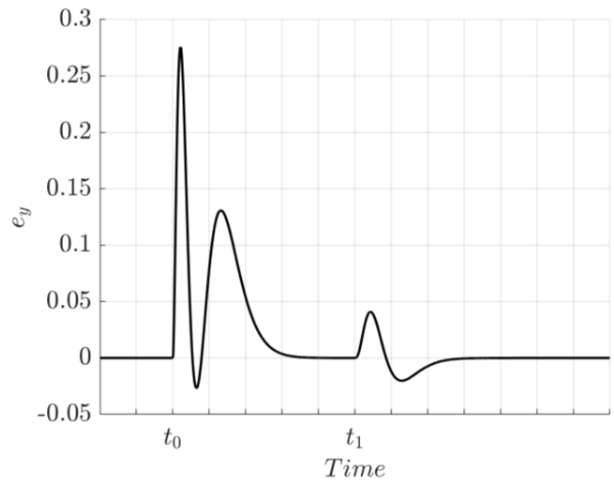
The following observer is designed under the assumption that the disturbance acting on the system is constant, i.e.

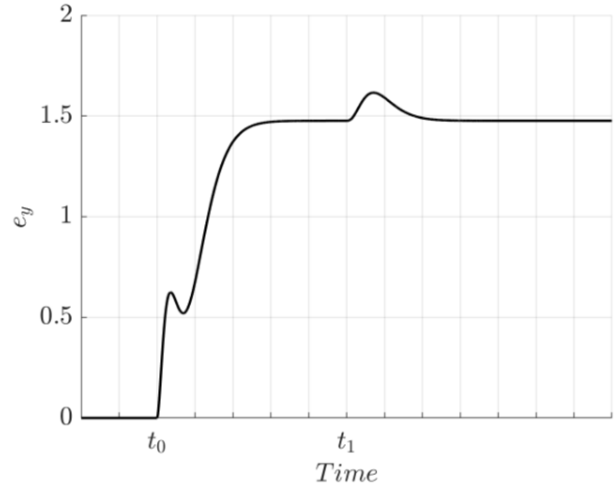
$$\Sigma_o : \begin{cases} \dot{\hat{\mathbf{x}}}_o = \begin{bmatrix} 0 & 1 & 0 \\ -\alpha & -\beta & \delta \\ 0 & 0 & 0 \end{bmatrix} \hat{\mathbf{x}}_o + \begin{bmatrix} 0 \\ \gamma \\ 0 \end{bmatrix} u + \mathbf{L}(y - \hat{y}_o) \\ \hat{y}_o = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \hat{\mathbf{x}}_o \end{cases}$$

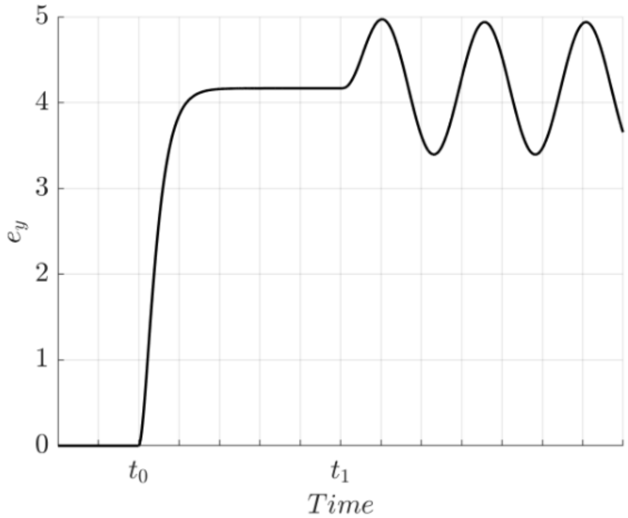
where the state of the observer is $\hat{\mathbf{x}}_o = [\hat{x}_1, \hat{x}_2, \hat{d}\,]^T$. The following disturbance profile acts on the system

$$d(t) = \begin{cases} 0 & 0 \leq t < t_0 \\ d_0 & t_0 \leq t < t_1 \\ d_0 + A_d \sin(\omega_d t + \varphi_d) & t \geq t_1 \end{cases}$$

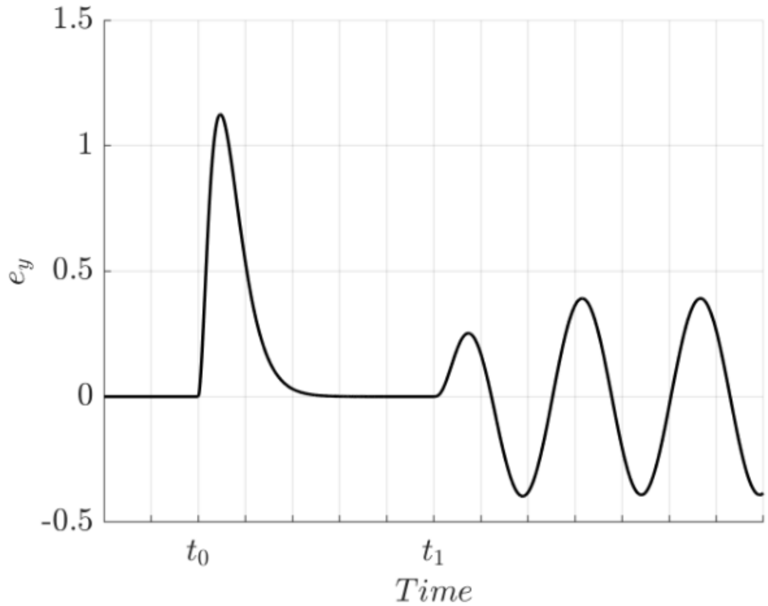
where $d_0, A_d, \omega_d, \varphi_d \in \mathbb{R}_+$. Under the assumption that the observer initial condition matches the system initial condition, i.e. $\hat{\mathbf{x}}_o(0) = [\hat{x}_1(0), \hat{x}_2(0), \hat{d}\,(0)]^T = [x_1(0), x_2(0), 0]^T$, and that $e_y = y - \hat{y}_o$ is the output estimation error, which of the following plots shows the correct behaviour of $e_y(t)$ for all $t \geq 0$?







assumes ramp error, our controller expects constant error



```
syms alpha beta gamma sigma
A = [0 1;-alpha -beta];
B = [0;gamma]
C = [1,0]
d = [0,sigma]

% at T = 0 no disurbance meaning no effect
% at t0 disturbance = d0, lim -> inf error goes to 0 again
% at t1 sin error, lin -> inf constant
```

Consider the 2nd order LTI system

$$\Sigma : \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} g(t) \\ y = \mathbf{C}\mathbf{x} + n \end{cases} \qquad \mathbf{x} \in \mathbb{R}^2, u \in \mathbb{R}, y \in \mathbb{R}$$

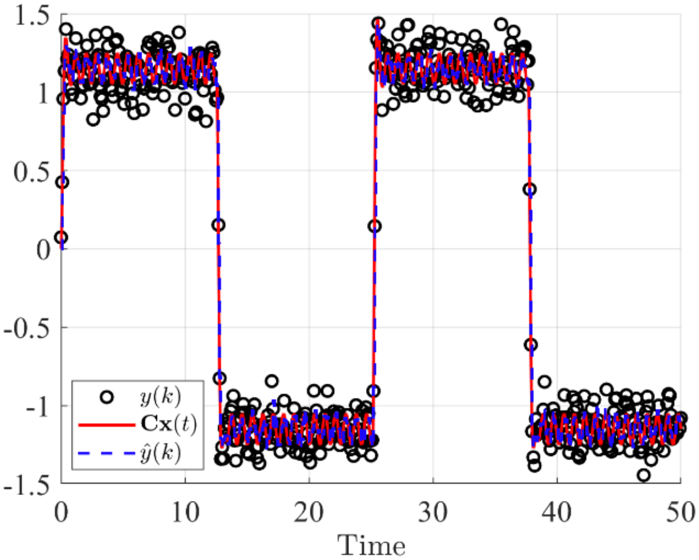
where $g(t)$ is an unknown function of time, and n is zero mean white Gaussian noise with noise intensity σ_n^2.

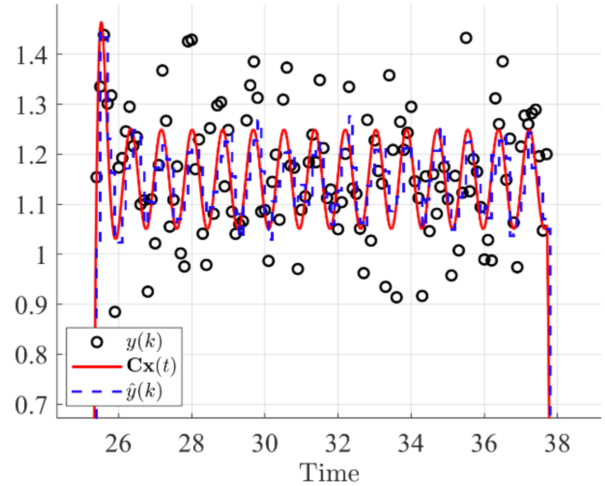
A discrete time Kalman filter is designed to reconstruct the state $\mathbf{x}$ based on noisy measurements y, i.e.

$$\Sigma_{KF} : \begin{cases} \hat{\mathbf{x}}(k+1) = \mathbf{F}\hat{\mathbf{x}}(k) + \mathbf{G}u(k) + \mathbf{L}'(y(k) - \hat{y}(k)) \\ \hat{y}(k) = \mathbf{C}\hat{\mathbf{x}}(k) \end{cases}$$

where $\mathbf{F} = e^{\mathbf{A}T_s}$, $\mathbf{G} = \int_0^{T_s} e^{\mathbf{A}t}\mathbf{B}\,\mathrm{d}t$, $\mathbf{L}' = \mathbf{F}\mathbf{L}$ is the steady state Kalman gain in predictive form, and T_s is the sampling time. The Kalman gain $\mathbf{L}$ is designed based on the variance of the measurement noise n and the variance of the process noise v. The process noise is used to account for the model uncertainty introduced by the function $g(t)$ on the second state equation. The process noise is zero mean white Gaussian noise with noise intensity σ_v^2.

The estimated output $\hat{y}$ is shown in the following figures (second figure is a zoom in), when the system Σ is excited by a square wave and subject to the function $g(t) = A_g \sin \omega_g t$, where both A_g and ω_g are unknown.





Based on the comparison of the estimated output $\hat{y}$ with the true output $\mathbf{Cx}$, which of the following statements is correct?

Choose one answer

- ○ The Kalman filter estimates the unknown dynamics of the system Σ by tuning the process noise intensity to be equal to the measurement noise intensity, i.e. $\sigma_v^2 = \sigma_n^2$.
- ○ The Kalman filter estimates the unknown dynamics of the system Σ by setting the process noise intensity to zero, i.e. $\sigma_v^2 = 0$.
- ○ The Kalman filter estimates the unknown dynamics of the system Σ by tuning the process noise intensity to be much smaller than the measurement noise intensity, i.e. $\sigma_v^2 \ll \sigma_n^2$.
- ✓ The Kalman filter estimates the unknown dynamics of the system Σ by tuning the process noise intensity to be much larger than the measurement noise intensity, i.e. $\sigma_v^2 \gg \sigma_n^2$.
- ○ The Kalman filter estimates the unknown dynamics of the system Σ regardless of the value of the process and measurement noise intensities.

```
% Look at the dashed line vs the full line.. the filter vs the system.
% they follow each other therefore they are equal.
```