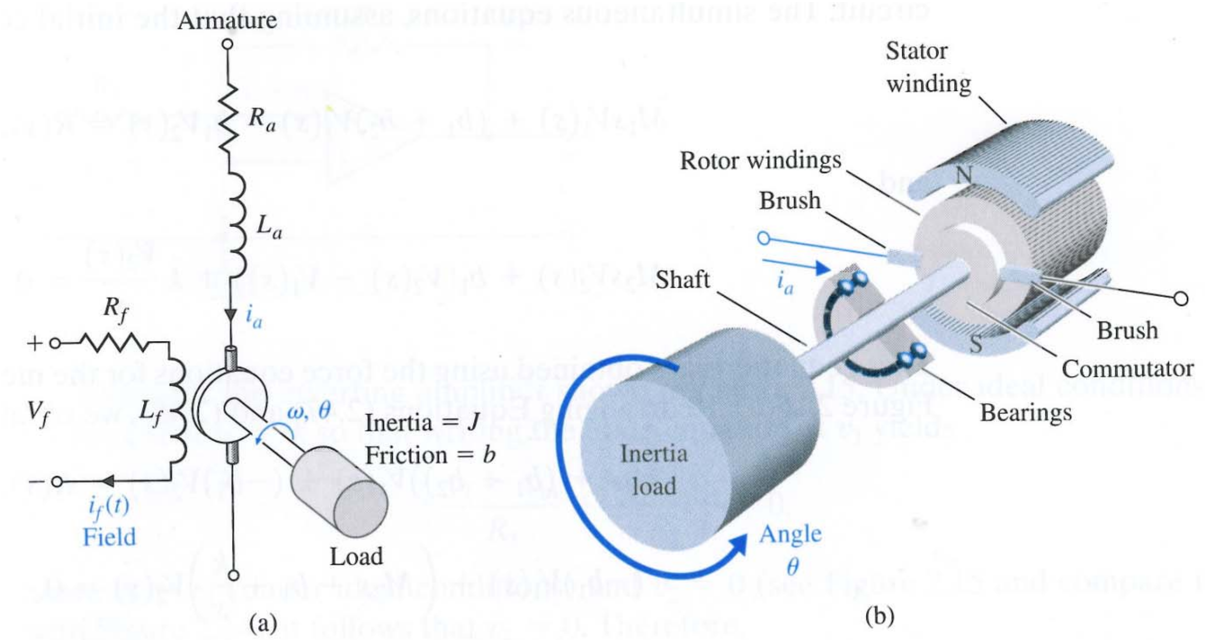
DC Motor

The DC motor is a power actuator device that delivers energy to a load. The DC motor converts direct current (DC) electrical energy into rotational mechanical energy. A major fraction of the torque generated in the rotor (armature) of the motor is available to drive an external load. Because of features such as high torque, speed controllability over a wide range, portability, well-behaved speed-torque characteristics, and adaptability to various types of control methods, DC motors are widely used in numerous control applications, including robotic manipulators, tape transport mechanisms, disk drives, machine tools, and servo-valve actuators.

The transfer function of the DC motor will be developed for a linear approximation to an actual motor, and second-order effects, such as hysteresis and the voltage drop across the brushes, will be neglected. The input voltage may be applied to the field or armature terminals.



The air-gap flux of the motor is proportional to the field current, provided the field is unsaturated, so that

$$\phi = K_f i_f,$$

where i_f is the field current, K_f is the proportional constant, and ϕ is the field flux. The torque derived by the motor is assumed to be related linearly to ϕ and the armature current as follows:

$$T_m(t) = K_1 \phi i_a(t) = K_1 K_f i_f(t) i_a(t),$$

where i_a is the armature current, and K_1 is a constant. It is clear from this equation that, to have a linear system, one current must be maintained constant while the other current becomes the input current. The armature-controlled DC motor uses the armature current as the control variable. The stator field can be established by a field coil and current or a permanent magnet. When a constant field current is established in a field coil, the motor torque in *Laplace transform* notation is

$$T_m(s) = (K_1 K_f I_f) I_a(s) = K_m I_a(s),$$

where K_m is a function of the permeability of the magnetic material.
The armature current is related to the input voltage applied to the armature by

$$V_a(s) = (R_a + L_a s)I_a(s) + V_b(s),$$

where $V_b(s)$ is the back electromotive-force voltage proportional to the motor speed. Therefore, we have

$$V_b(s) = K_b \omega(s),$$

where $\omega(s) = s\theta(s)$ is the transformation of the angular speed.

The motor torque is equal to the torque delivered to the load. This relation may be expressed as

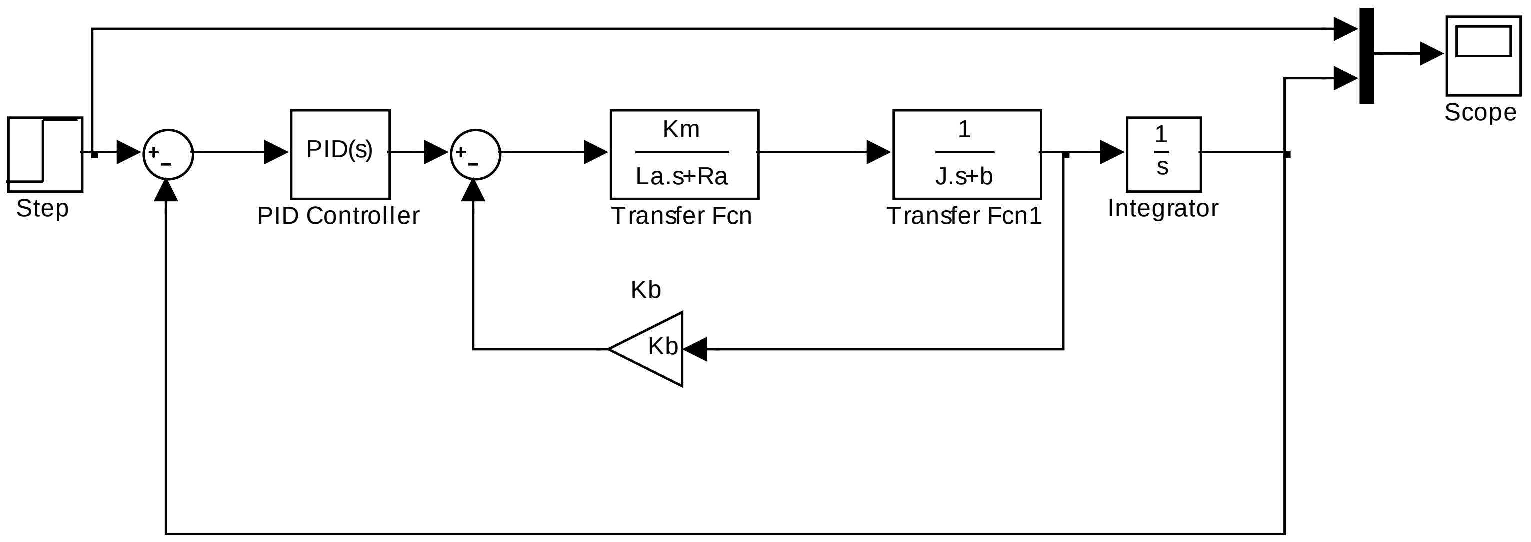
$$T_m(s) = T_L(s) + T_d(s),$$

where $T_L(s)$ is the load torque and $T_d(s)$ is the disturbance torque. The load torque for rotating inertia is written as:

$$T_L(s) = Js^2\theta(s) + bs\theta(s).$$

- Calculate the transfer function $G(s) = \frac{\theta(s)}{V_a(s)}$ using the equations presented above, and letting $T_d(s) = 0$;
- Build a Simulink/Matlab model of the DC motor including a controller (P, PD or PID). ($J = 2, K_m = 10, R_a = 1, L_a = 1, b = 0.5, K_b = 0.1$)

$$\frac{\theta}{V_a} = \frac{K_m}{(Js^2+bs)(R_a+L_as)+K_mK_b s}$$



Robotics 34753

Exercise solutions for week 2

September 2022

Problem 1

What is the rotation matrix for a rotation of 30° about the world z-axis, followed by a rotation of 60° about the world x-axis, followed by a rotation of 90° about the world y-axis?

Solution

The three rotation matrices are

$$R_{z,30^\circ} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 0 \\ \sin 30^\circ & \cos 30^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{x,60^\circ} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 60^\circ & -\sin 60^\circ \\ 0 & \sin 60^\circ & \cos 60^\circ \end{bmatrix}$$

$$R_{y,90^\circ} = \begin{bmatrix} \cos 90^\circ & 0 & \sin 90^\circ \\ 0 & 1 & 0 \\ -\sin 90^\circ & 0 & \cos 90^\circ \end{bmatrix}$$

Rotation wrt. a global (i.e. world) axis means pre-multiplication with the corresponding matrix, therefore the final rotation matrix is

$$R = R_{y,90^\circ} R_{x,60^\circ} R_{z,30^\circ} = \begin{bmatrix} 0.433 & 0.75 & 0.5 \\ 0.25 & 0.433 & -0.866 \\ -0.866 & 0.5 & 0. \end{bmatrix}$$

Problem 2

What is the rotation matrix for a rotation ϕ about the world x-axis, followed by a rotation ψ about the current z-axis, followed by a rotation θ about the world y-axis?

Solution

The three rotation matrices are

$$R_{x,\phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

$$R_{z,\psi} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

The initial rotation matrix $R_{x,\phi}$ is post-multiplied with $R_{z,\psi}$ (rotation about current axis), and the result is then pre-multiplied with $R_{y,\theta}$ (rotation about world axis). The resulting total rotation matrix is

$$\begin{aligned} R &= R_{y,\theta} R_{x,\phi} R_{z,\psi} \\ &= \begin{bmatrix} \cos\theta\cos\psi + \sin\theta\sin\phi\sin\psi & \sin\theta\sin\phi\cos\psi - \cos\theta\sin\psi & \sin\theta\cos\phi \\ \cos\phi\sin\psi & \cos\phi\cos\psi & -\sin\phi \\ -\sin\theta\cos\psi + \cos\theta\sin\phi\sin\psi & \cos\theta\sin\phi\cos\psi + \sin\theta\sin\psi & \cos\theta\cos\phi \end{bmatrix} \end{aligned}$$

Problem 3

Find another sequence of rotations that is different from Prob. 2, but which results in the same rotation matrix.

Solution

For example, rotation with an angle θ about the y-axis (world or current axis, doesn't matter because they coincide), then rotation with an angle ϕ about the **current** x-axis, and then rotation with an angle ψ about the **current** z-axis.

Problem 4

Determine a homogeneous transformation matrix H that represents a rotation with an angle α about the world x-axis, followed by a translation with a length b along the world z-axis, followed by a rotation ϕ about the current y-axis.

Solution

The following three transformation matrices need to be composed appropriately.

$$R_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{z,b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

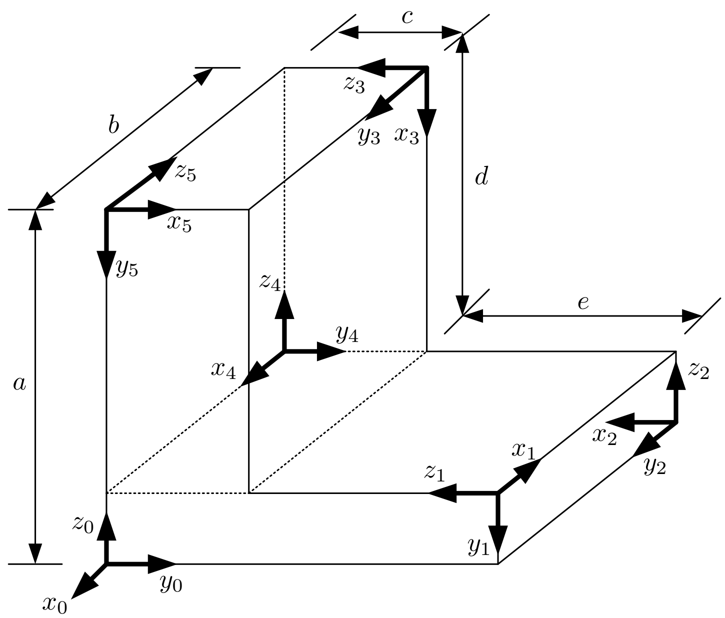
$$R_{y,\phi} = \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The initial rotation matrix $R_{x,\alpha}$ is pre-multiplied with $T_{z,b}$ (world axis translation), and the result is post-multiplied with $R_{y,\phi}$ (current axis rotation). The final transformation matrix is

$$\begin{aligned} T &= T_{z,b} R_{x,\alpha} R_{y,\phi} \\ &= \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ \sin\alpha\sin\phi & \cos\alpha & -\sin\alpha\cos\phi & 0 \\ -\cos\alpha\sin\phi & \sin\alpha & \cos\alpha\cos\phi & b \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Problem 5

For the figure shown below, find the 4×4 homogeneous transformation matrices A_1^0, A_2^1, A_3^2, A_4^3, and A_5^4, as well as A_5^0.



Hint: you can find the answers directly by observation, based on the geometric interpretation of each column in the homogeneous transformation matrix.

Solution

The position vectors of each frame $i = 1, 2, 3, 4, 5$ with respect to the previous frame $i-1$ are easy to find by observation.

$$o_1^0 = \begin{bmatrix} 0 \\ c+e \\ a-d \end{bmatrix}, \quad o_2^1 = \begin{bmatrix} b \\ a-d \\ 0 \end{bmatrix}, \quad o_3^2 = \begin{bmatrix} e \\ 0 \\ a \end{bmatrix}, \quad o_4^3 = \begin{bmatrix} d \\ 0 \\ c \end{bmatrix}, \quad o_5^4 = \begin{bmatrix} b \\ 0 \\ d \end{bmatrix}$$

The position vector of frame 5 wrt. frame 0 is also easy to find.

$$o_5^0 = \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix},$$

These position vectors define the last column of the respective matrices A_1^0, A_2^1, A_3^2, A_4^3, A_5^4, and A_5^0.

The rotation part of these matrices can also be easily determined by observation. Observing that $x_1 = -x_0$, $y_1 = -z_0$ and $z_1 = -y_0$ leads, together with o_1^0, to

$$A_1^0 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & c+e \\ 0 & -1 & 0 & a-d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Observing that $x_2 = z_1$, $y_2 = -x_1$ and $z_2 = -y_1$ leads, together with o_2^1, to

$$A_2^1 = \begin{bmatrix} 0 & -1 & 0 & b \\ 0 & 0 & -1 & a-d \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Observing that $x_3 = -z_2$, $y_3 = y_2$ and $z_3 = x_2$ leads, together with o_3^2, to

$$A_3^2 = \begin{bmatrix} 0 & 0 & 1 & e \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & a \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Observing that $x_4 = y_3$, $y_4 = -z_3$ and $z_4 = -x_3$ leads, together with o_4^3, to

$$A_4^3 = \begin{bmatrix} 0 & 0 & -1 & d \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Observing that $x_5 = y_4$, $y_5 = -z_4$ and $z_5 = -x_4$ leads, together with o_5^4, to

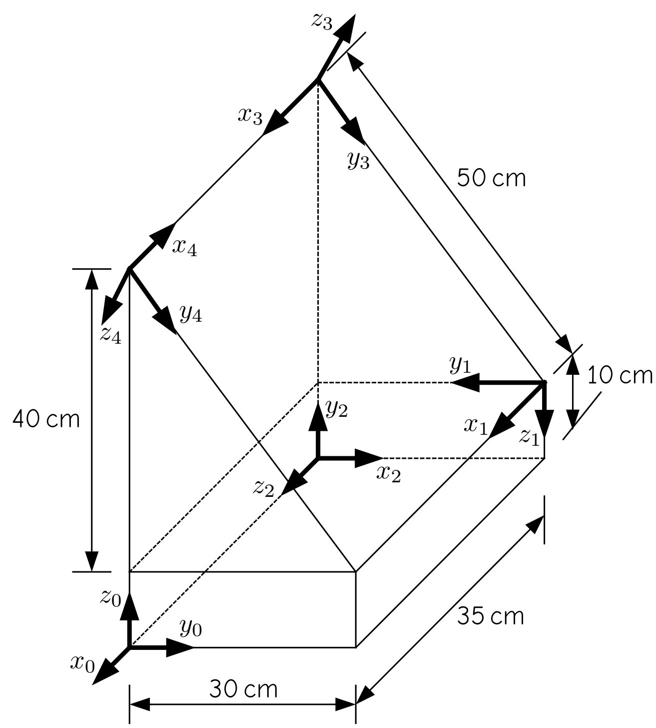
$$A_5^4 = \begin{bmatrix} 0 & 0 & -1 & b \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Finally, observing that $x_5 = y_0$, $y_5 = -z_0$ and $z_5 = -x_0$ leads, together with o_5^0, to

$$A_5^0 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & a \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 6

For the figure shown below, find the 4×4 homogeneous transformation matrices A_1^0, A_2^1, A_3^2, and A_4^3, as well as A_4^0.



Hint: you can find the answers directly by observation, based on the geometric interpretation of each column in the homogeneous transformation matrix.

Solution

Denoting the angle at the top vertex of the triangle with β, trigonometry can be applied to find

$$\sin\beta = 30/50 = 0.6 \quad \text{and} \quad \cos\beta = 40/50 = 0.8$$

Then, all position vectors of each frame $i = 1, 2, 3, 4$ with respect to the previous frame $i-1$ are easy to find by observation.

$$o_1^0 = \begin{bmatrix} -0.35\,\text{m} \\ 0.3\,\text{m} \\ 0.1\,\text{m} \end{bmatrix}, \quad o_2^1 = \begin{bmatrix} 0 \\ 0.3\,\text{m} \\ 0.1\,\text{m} \end{bmatrix}, \quad o_3^2 = \begin{bmatrix} 0 \\ 0.5\,\text{m} \\ 0 \end{bmatrix}, \quad o_4^3 = \begin{bmatrix} 0.35\,\text{m} \\ 0 \\ 0 \end{bmatrix},$$

The position vector of frame 4 wrt. frame 0 is also easy to find.

$$o_4^0 = \begin{bmatrix} 0 \\ 0 \\ 0.5\,\mathrm{m} \end{bmatrix},$$

The rotation part of these matrices can also be easily determined by observation. Observing that $x_1 = x_0$, $y_1 = -y_0$ and $z_1 = -z_0$ leads, together with o_1^0, to

$$A_1^0 = \begin{bmatrix} 1 & 0 & 0 & -0.35 \\ 0 & -1 & 0 & 0.3 \\ 0 & 0 & -1 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Observing that $x_2 = -y_1$, $y_2 = -z_1$ and $z_2 = x_1$ leads, together with o_2^1, to

$$A_2^1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0.3 \\ 0 & -1 & 0 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Observing that $x_3 = z_2$, $y_3 = 0.6x_2 - 0.8y_2$ and $z_3 = 0.8x_2 + 0.6y_2$ leads, together with o_3^2, to

$$A_3^2 = \begin{bmatrix} 0 & 0.6 & 0.8 & 0 \\ 0 & -0.8 & 0.6 & 0.5 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Observing that $x_4 = -x_3$, $y_4 = y_3$ and $z_4 = -z_3$ leads, together with o_4^3, to

$$A_4^3 = \begin{bmatrix} -1 & 0 & 0 & 0.35 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Finally, observing that $x_4 = -x_0$, $y_4 = 0.6y_0 - 0.8z_0$ and $z_4 = -0.8y_0 - 0.6z_0$ leads, together with o_4^0, to

$$A_4^0 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0.6 & -0.8 & 0 \\ 0 & -0.8 & -0.6 & 0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

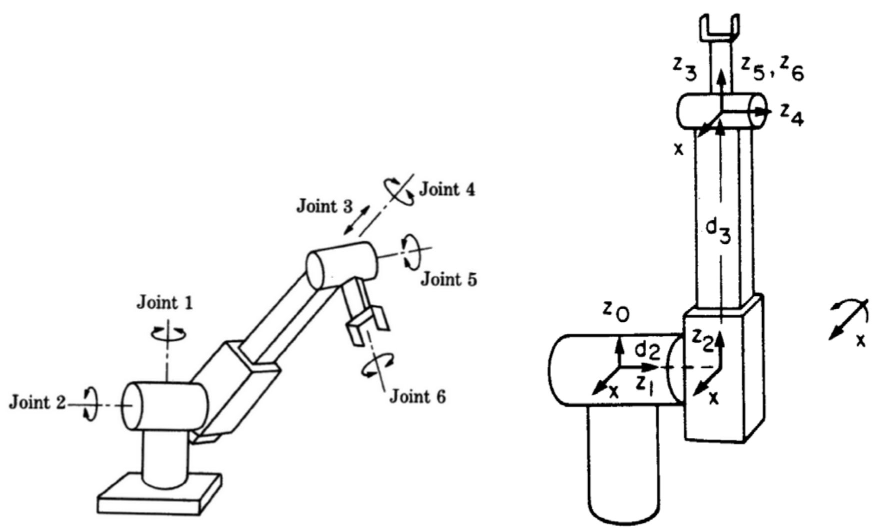
Robotics 34753

Exercise solutions for week 3

September 2023

Problem 1

Given the Stanford arm in the figure below, with $d_2 = 0.1\,\mathrm{m}$, answer the following questions.



Question 1

Find the link parameters for the robotic arm (d_3 is a prismatic joint variable, other joints are rotational joints, the link coordinate frames have been established as shown in the figure). Hint: not all 6 joints are visible in the schematic, you have to deduct the existence of some joints from the corresponding z-axes in the model.

Solution

Joint i	θ_i	d_i	a_i	α_i
1	θ_1^* (0°)	0	0	−90°
2	θ_2^* (0°)	0.1 m	0	90°
3	0°	d_3^*	0	0°
4	θ_4^* (0°)	0	0	−90°
5	θ_5^* (0°)	0	0	90°
6	θ_6^* (0°)	0	0	0°

Question 2

Find the forward kinematic model for the arm and represent it in homogeneous matrix form.

Solution

$$T_1^0 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_2^1 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_3^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_4^3 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_5^4 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_6^5 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_6^0 = T_1^0 T_2^1 T_3^2 T_4^3 T_5^4 T_6^5 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & c_1 d_3 s_2 - 0.1 s_1 \\ r_{21} & r_{22} & r_{23} & d_3 s_1 s_2 + 0.1 c_1 \\ r_{31} & r_{32} & r_{33} & c_2 d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$r_{11} = c_1 \left(c_2 \left(c_4 c_5 c_6 - s_4 s_6\right) - c_6 s_2 s_5\right) - s_1 \left(c_4 s_6 + c_5 c_6 s_4\right)$$
$$r_{12} = c_1 \left(c_2 \left(-c_4 c_5 s_6 - c_6 s_4\right) + s_2 s_5 s_6\right) - s_1 \left(c_4 c_6 - c_5 s_4 s_6\right)$$
$$r_{13} = c_1 \left(c_2 c_4 s_5 + c_5 s_2\right) - s_1 s_4 s_5$$
$$r_{21} = s_1 \left(c_2 \left(c_4 c_5 c_6 - s_4 s_6\right) - c_6 s_2 s_5\right) + c_1 \left(c_4 s_6 + c_5 c_6 s_4\right)$$
$$r_{22} = s_1 \left(c_2 \left(-c_4 c_5 s_6 - c_6 s_4\right) + s_2 s_5 s_6\right) + c_1 \left(c_4 c_6 - c_5 s_4 s_6\right)$$
$$r_{23} = s_1 \left(c_2 c_4 s_5 + c_5 s_2\right) + c_1 s_4 s_5$$
$$r_{31} = -s_2 \left(c_4 c_5 c_6 - s_4 s_6\right) - c_2 c_6 s_5$$
$$r_{32} = c_2 s_5 s_6 - s_2 \left(-c_4 c_5 s_6 - c_6 s_4\right)$$
$$r_{33} = c_2 c_5 - c_4 s_2 s_5$$

Question 3

Represent the orientation of the end-effector with Yaw-Pitch-Roll angles.

Solution

The rotation matrix of the end-effector with respect to the initial frame can be written as a function of the Yaw-Pitch-Roll angles ψ, θ, ϕ

$$R_6^0 = R_{z,\phi} R_{y,\theta} R_{x,\psi} = \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}$$

If $c_\theta \neq 0$, then

$$\theta = \mathrm{Atan2}\left(\pm\sqrt{r_{11}^2 + r_{21}^2}, -r_{31}\right)$$

$$\psi = \mathrm{Atan2}\left(\frac{r_{33}}{c_\theta}, \frac{r_{32}}{c_\theta}\right)$$

$$\phi = \mathrm{Atan2}\left(\frac{r_{11}}{c_\theta}, \frac{r_{21}}{c_\theta}\right)$$

If $c_\theta = 0$ and $s_\theta = 1$, i.e. $\theta = 90°$, then

$$R_6^0 = \begin{bmatrix} 0 & \sin(\psi - \phi) & \cos(\psi - \phi) \\ 0 & \cos(\psi - \phi) & -\sin(\psi - \phi) \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ -1 & 0 & 0 \end{bmatrix}$$

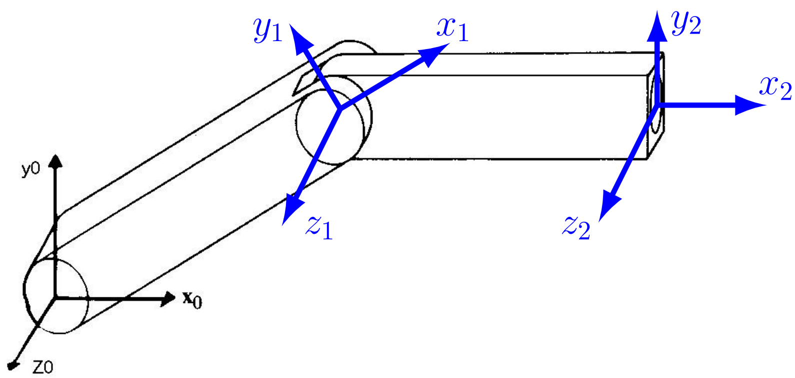
hence $\psi - \phi = \mathrm{Atan2}(r_{22}, r_{12})$
If $c_\theta = 0$ and $s_\theta = -1$, i.e. $\theta = 270°$, then

$$R_6^0 = \begin{bmatrix} 0 & -\sin(\psi + \phi) & -\cos(\psi + \phi) \\ 0 & \cos(\psi + \phi) & -\sin(\psi + \phi) \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 1 & 0 & 0 \end{bmatrix}$$

hence $\psi + \phi = \mathrm{Atan2}(r_{22}, -r_{12})$
In the last two cases, one of the two angles ϕ, ψ can be chosen arbitrarily.

Problem 2

A two degree-of-freedom manipulator is shown in the figure below. Given that the length of each link is 1 m, establish its link coordinate frames and find T_1^0, T_2^1 and the kinematics matrix. Define the z_2-axis of coordinate frame 2 as if there was a revolute joint at the tip of the robotic arm with its axis parallel to z_0.



Question 1

Find the forward kinematics solution for this manipulator, i.e. the homogeneous transformation matrix for the end-effector as a function of the joint angles.

Solution

The coordinate frames 1 and 2 are respectively defined according to the Denavit-Hartenberg convention as shown in the figure above.
For frame 1, the Denavit-Hartenberg parameters are θ_1^*, $d_1 = 0$, $a_1 = 1$ m and $\alpha_1 = 0°$. These lead to the following transformation matrix

$$T_1^0 = \begin{bmatrix} c_1 & -s_1 & 0 & c_1 \\ s_1 & c_1 & 0 & s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For frame 2, the Denavit-Hartenberg parameters are θ_2^*, $d_2 = 0$, $a_2 = 1$ m and $\alpha_2 = 0°$. These lead to the following transformation matrix

$$T_2^1 = \begin{bmatrix} c_2 & -s_2 & 0 & c_2 \\ s_2 & c_2 & 0 & s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In total, the transformation matrix for the end-effector is

$$T_2^0 = T_1^0 T_2^1 = \begin{bmatrix} \cos(\theta_1^* + \theta_2^*) & -\sin(\theta_1^* + \theta_2^*) & 0 & \cos(\theta_1^*) + \cos(\theta_1^* + \theta_2^*) \\ \sin(\theta_1^* + \theta_2^*) & \cos(\theta_1^* + \theta_2^*) & 0 & \sin(\theta_1^*) + \sin(\theta_1^* + \theta_2^*) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix defines the forward kinematics of the robotic arm.
If sufficient elements of the total transformation matrix

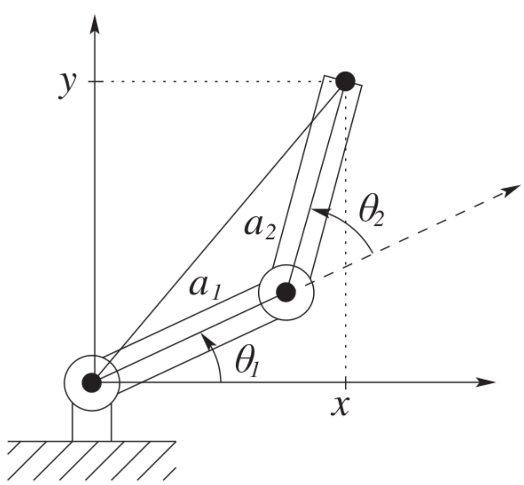
$$T_2^0 = T_1^0 T_2^1 = \begin{bmatrix} r_{11} & r_{12} & 0 & r_{14} \\ r_{21} & r_{22} & 0 & r_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

are known, the inverse kinematics problem can be solved for the angles θ_1^* and θ_2^*.

Question 2

Find the inverse kinematics solution for this manipulator assuming the position of the robot tip is known, i.e. elements r_{14} and r_{24} in the homogeneous transformation matrix. (Hint: use trigonometry and the law of cosines)

Solution



The distance from the base origin to the tip is $\sqrt{x^2 + y^2} = \sqrt{r_{14}^2 + r_{24}^2}$, while both links have length $a_1 = a_2 = 1$.

Using the law of cosines and assuming the elbow down position, i.e. $\theta_2^* > 0$:

$$\cos(\pi - \theta_2^*) = \frac{a_1^2 + a_2^2 - r_{14}^2 - r_{24}^2}{2a_1 a_2} \quad \Rightarrow \quad -\cos(\theta_2^*) = \frac{2 - r_{14}^2 - r_{24}^2}{2} \quad \Rightarrow$$

$$\theta_2^* = \mathrm{Atan2}\,(c_2, s_2) = \mathrm{Atan2}\left(\frac{r_{14}^2 + r_{24}^2}{2} - 1, \sqrt{1 - \left(\frac{r_{14}^2 + r_{24}^2}{2} - 1\right)^2}\right)$$

or using the law of cosines and assuming the elbow up position i.e. $\theta_2^* < 0$:

$$\cos(\pi + \theta_2^*) = \frac{a_1^2 + a_2^2 - r_{14}^2 - r_{24}^2}{2a_1 a_2} \quad \Rightarrow \quad -\cos(\theta_2^*) = \frac{2 - r_{14}^2 - r_{24}^2}{2} \quad \Rightarrow$$

$$\theta_2^* = \mathrm{Atan2}\,(c_2, s_2) = \mathrm{Atan2}\left(\frac{r_{14}^2 + r_{24}^2}{2} - 1, -\sqrt{1 - \left(\frac{r_{14}^2 + r_{24}^2}{2} - 1\right)^2}\right)$$

Then θ_1^* can be found from the equation

$$\theta_1^* + \mathrm{Atan2}\,(a_1 + a_2 c_2, a_2 s_2) = \mathrm{Atan2}\,(r_{14}, r_{24}) \quad \Rightarrow$$

$$\theta_1^* = \mathrm{Atan2}\,(r_{14}, r_{24}) - \mathrm{Atan2}\,(1 + c_2, s_2)$$

where $c_2 = \cos(\theta_2^*)$ and $s_2 = \sin(\theta_2^*)$ are now known.

Robotics 34753

Exercise solutions for week 4

September 2022

Problem 1

Two frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$ are related by the time-independent homogeneous transformation

$$H_1^0 = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A particle has velocity $v^1 = \{3, 1, 0\}^T$ relative to frame $o_1x_1y_1z_1$. What is the velocity of the particle in frame $o_0x_0y_0z_0$?

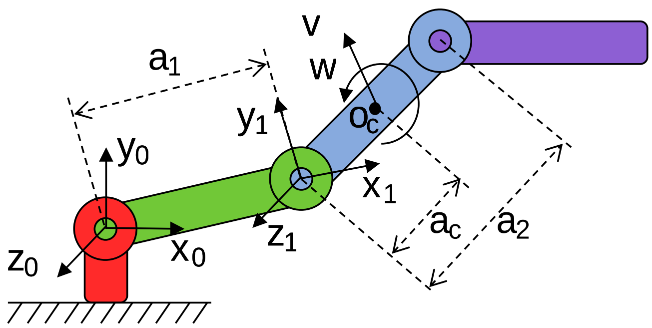
Solution

Since matrix H is independent of time, the two frames 0 and 1 are fixed with respect to each other. The velocity vector, as a free vector, can be expressed in a different frame simply by multiplication with the rotation matrix between the two frames

$$v^0 = R_1^0 v^1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad v^0 = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$$

Problem 2

Consider the three-link planar manipulator shown below.



Compute the linear velocity v and angular velocity ω at the center o_c of link 2, where $a_c = a_2/2$, as functions of the joint variables $\{\theta_1, \theta_2\}$ and joint velocities $\{\dot{\theta}_1, \dot{\theta}_2\}$.

Solution

For this planar robotic manipulator, forward kinematic analysis is rather simple. The positions of origins o_0, o_1 and o_c will be necessary for calculating the Jacobian later. These positions with respect to frame 0 are

$$o_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$o_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix}$$

$$o_c = o_1 + \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_c c_2 \\ a_c s_2 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 c_1 + a_c c_{12} \\ a_1 s_1 + a_c s_{12} \\ 0 \end{bmatrix}$$

where $\{c_1, s_1\} = \{\cos(\theta_1), \sin(\theta_1)\}$ and $\{c_{12}, s_{12}\} = \{\cos(\theta_1 + \theta_2), \sin(\theta_1 + \theta_2)\}$. In these formulas, θ_1 and θ_2 are joint angles respectively for joints 1 and 2, according to the Denavit-Hartenberg convention. This means that when $\theta_1 = \theta_2 = 0$, all three axes x_0, x_1 and x_2 (not shown in the figure) coincide.
The z axes, also needed for calculating the manipulator Jacobian, are trivial.

$$z_0 = z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

All joints of the manipulator are revolute, hence the first two columns of the manipulator Jacobian are

$$J_1 = \begin{bmatrix} z_0 \times (o_c - o_0) \\ z_0 \end{bmatrix} = \begin{bmatrix} -a_1 s_1 - a_c s_{12} \\ a_1 c_1 + a_c c_{12} \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and

$$J_2 = \begin{bmatrix} z_1 \times (o_c - o_1) \\ z_1 \end{bmatrix} = \begin{bmatrix} -a_c s_{12} \\ a_c c_{12} \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The third joint is ignored since it will not affect point o_c and the respective rotation. The manipulator Jacobian for point o_c is

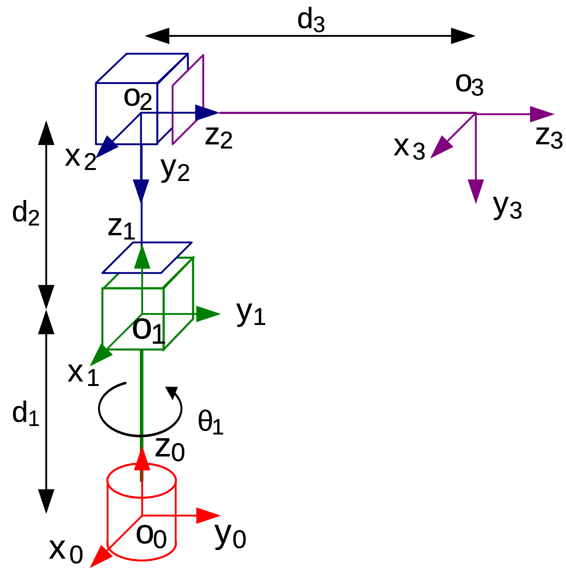
$$J = \begin{bmatrix} -a_1 s_1 - a_c s_{12} & -a_c s_{12} \\ a_1 c_1 + a_c c_{12} & a_c c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

The linear velocity vector v and angular velocity vector ω at o_c are found as

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} -a_1 s_1 - a_c s_{12} & -a_c s_{12} \\ a_1 c_1 + a_c c_{12} & a_c c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

Problem 3

Find the 6×3 Jacobian matrix corresponding to the end-effector frame 3 of the cylindrical manipulator shown below.



Solution

The vectors necessary for the definition of the Jacobian matrix are simple enough to find directly without applying the general Denavit-Hartenberg transformation for each joint

$$o_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$o_3 = \begin{bmatrix} -d_3 s_1 \\ d_3 c_1 \\ d_1 + d_2 \end{bmatrix}$$

$$z_0 = z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$z_2 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix}$$

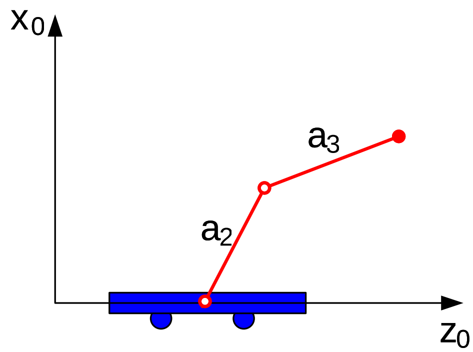
Then the three columns of the Jacobian matrix can be written according to the corresponding definitions for revolute, prismatic and prismatic joints, respectively, as

$$J = \begin{bmatrix} z_0 \times (o_c - o_0) & z_1 & z_2 \\ z_0 & 0_{3\times 1} & 0_{3\times 1} \end{bmatrix} = \begin{bmatrix} -d_3 c_1 & 0 & -s_1 \\ -d_3 s_1 & 0 & c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

In this expression, the common notation $\{c_1, s_1\} = \{\cos(\theta_1), \sin(\theta_1)\}$ has been used.

Problem 4

The planar manipulator with a mobile platform moving in Z direction shown below can be interpreted as a PRR manipulator. The robot operates only in the XZ plane in the working space.

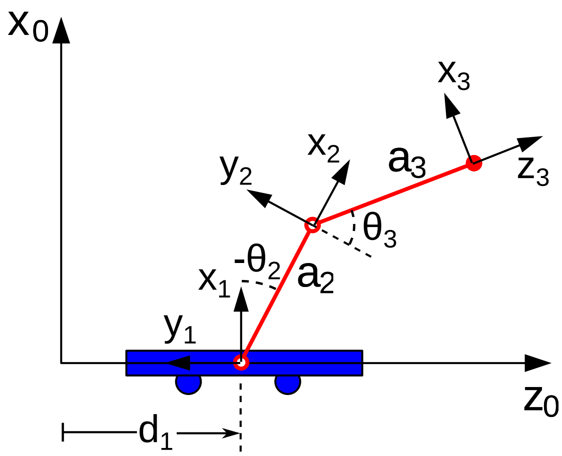


Define the coordinate frames 1 and 2 according to the Denavit-Hartenberg convention. Define frame 3 according to the usual convention for the end-effector, assuming that the grip opens and closes in the out of plane direction. Determine the transformation matrix T_3^0, and find the Jacobian matrix for this mobile robot.
Note: the end-effector frame does not follow the Denavit-Hartenberg convention, hence there is some ambiguity in how you define angle θ_3.

Solution

Following the Denavit-Hartenberg convention for the frame axes leads to the frames 1 and 2 shown below. For the end-effector, the axis z_3 is chosen as the approach direction and axis y_3 as the gripping actuation direction.
The joint parameters d_1 and θ_2 shown in the figure, are according to the Denavit-Hartenberg convention. This means that θ_2 is the angle from x_1 to x_2 around z_1. Although frame 3 is not a Denavit-Hartenberg frame, the choice is made here to also define θ_3 as the angle from x_2 to x_3 around z_2. The Denavit-Hartenberg parameters and variables for the first two frames are summarized in the following table.

i	θ_i	d_i	a_i	α_i
1	0	d_1^*	0	-90
2	θ_2^*	0	a_2	0



Based on the general Denavit-Hartenberg homogeneous transformation matrix

$$A_i = \begin{bmatrix} c_\theta & -s_\theta c_\alpha & s_\theta s_\alpha & a c_\theta \\ s_\theta & c_\theta c_\alpha & -c_\theta s_\alpha & a s_\theta \\ 0 & s_\alpha & c_\alpha & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

the homogeneous transformation matrix from frame 0 to frame 1 is found as

$$T_1^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the homogeneous transformation matrix from frame 1 to frame 2 as

$$T_2^1 = \begin{bmatrix} c_2 & -s2 & 0 & a_2 c_2 \\ s_2 & c2 & 0 & a_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Finally, the last frame is not a Denavit-Hartenberg frame, but its homogeneous transformation matrix is easy to find by considering its axes orientation and its position expressed with respect to frame 2. This leads to

$$T_3^2 = \begin{bmatrix} c_3 & 0 & s_3 & a_3 s_3 \\ s_3 & 0 & -c_3 & -a_3 c_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Multiplying all 3 transformations leads to

$$T_3^0 = T_1^0 T_2^1 T_3^2 = \begin{bmatrix} c_{23} & 0 & s_{23} & a_2 c_2 + a_3 s_{23} \\ 0 & 1 & 0 & 0 \\ -s_{23} & 0 & c_{23} & d_1 - a_2 s_2 + a_3 c_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\{c_2, s_2\} = \{\cos(\theta_2), \sin(\theta_2)\}$ and $\{c_{23}, s_{23}\} = \{\cos(\theta_2 + \theta_3), \sin(\theta_2 + \theta_3)\}$. For the calculation of the Jacobian matrix it is also useful to have the homogeneous transformation matrix

$$T_2^0 = T_1^0 T_2^1 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2 c_2 \\ 0 & 0 & 1 & 0 \\ -s_2 & -c_2 & 0 & d_1 - a_2 s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now for one prismatic and two revolute joints, the Jacobian matrix will be in the form

$$J = \begin{bmatrix} z_0 & z_1 \times (o_3 - o_1) & z_2 \times (o_3 - o_2) \\ 0_{3\times 1} & z_1 & z_2 \end{bmatrix}$$

The three required axes are known

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad z_1 = z_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

(z_1 and z_2 can also be found in T_1^0 and T_2^0, respectively.)
The origins o_1, o_2 and o_3 are respectively extracted from matrices T_1^0, T_2^0 and T_3^0 as

$$o_1 = \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix} \qquad o_2 = \begin{bmatrix} a_2 c_2 \\ 0 \\ d_1 - a_2 s_2 \end{bmatrix} \qquad o_3 = \begin{bmatrix} a_2 c_2 + a_3 s_{23} \\ 0 \\ d_1 - a_2 s_2 + a_3 c_{23} \end{bmatrix}$$

Based on all these known quantities, the result for the Jacobian matrix is

$$J = \begin{bmatrix} 0 & -a_2 s_2 + a_3 c_{23} & a_3 c_{23} \\ 0 & 0 & 0 \\ 1 & -a_2 c_2 - a_3 s_{23} & -a_3 s_{23} \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Note: if you have chosen a different definition for θ_3, you will get some different expressions in the final Jacobian.

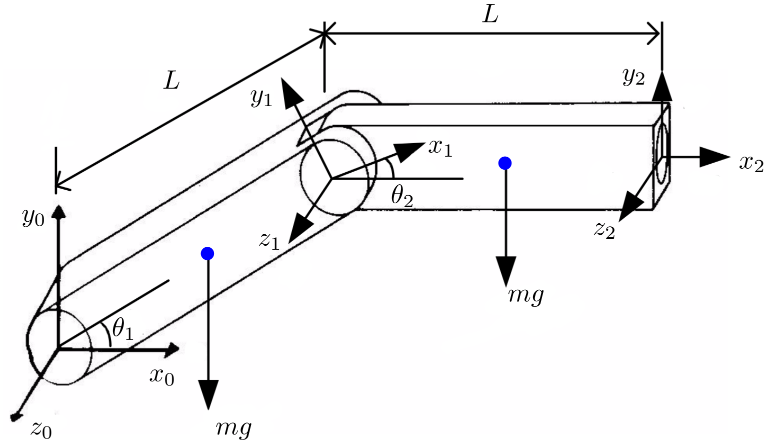
Robotics 34753

Exercise solutions for week 6

December 2023

Problem 1

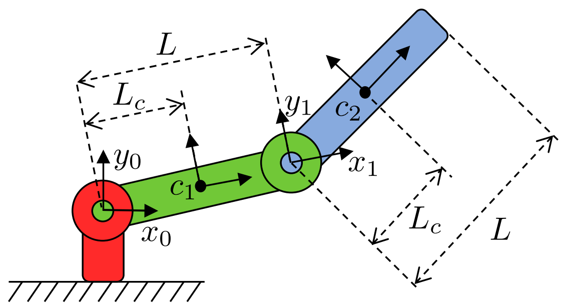
Consider a manipulator of two revolute joints, as shown in the figure below.



Given that the length of each link is L and the mass of each link is m, derive the dynamic model of this two-link robot arm using the Euler-Lagrange method. Approximate the geometry of the two arms as parallelepipeds of dimensions $L \times 0.1L \times 0.1L$ and assume that their mass is uniformly distributed.

Solution

The Euler-Lagrange formulation of the dynamics of the robot relies on knowing the potential and kinetic energy of the robotic arm. The kinetic energy depends on the angular and linear velocities of each link, which means that it is necessary to compute the Jacobian of the center of mass for each link.



The transformation from base frame to the frame at the center of link 1 is

$$T_{c1} = \mathrm{Rot}_z(\theta_1)\,\mathrm{Trans}_x(L_c) = \begin{bmatrix} c_1 & -s_1 & 0 & L_c\,c_1 \\ s_1 & c_1 & 0 & L_c\,s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

the transformation to frame 1 is

$$T_1 = \mathrm{Rot}_z(\theta_1)\,\mathrm{Trans}_x(L) = \begin{bmatrix} c_1 & -s_1 & 0 & L\,c_1 \\ s_1 & c_1 & 0 & L\,s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and the transformation to the frame at the center of link 2 is

$$\begin{aligned} T_{c2} &= T_1\,\mathrm{Rot}_z(\theta_2)\,\mathrm{Trans}_x(L_c) \\ &= \begin{bmatrix} c_1c_2 - s_1s_2 & -c_1s_2 - s_1c_2 & 0 & L_c(c_1c_2 - s_1s_2) + Lc_1 \\ c_1c_2 + s_1c_2 & c_1c_2 - s_1s_2 & 0 & L_c(c_1s_2 + s_1c_2) + Ls_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

In these equations $L_c = L/2$ and the usual abbreviations are used for the sine and cosine of the angles θ_1 and θ_2 with s_1, c_1 and s_2, c_2, respectively.
From the 3rd and 4th columns of these transformation matrices, the following z axes and origins o can be extracted

$$z_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

$$o_{c1} = \begin{bmatrix} L_c\,c_1 & L_c\,s_1 & 0 \end{bmatrix}^T$$

$$o_1 = \begin{bmatrix} Lc_1 & Ls_1 & 0 \end{bmatrix}^T$$
$$o_{c2} = \begin{bmatrix} L_c c_{12} + Lc_1 \\ L_c s_{12} + Ls_1 \\ 0 \end{bmatrix}$$

while it is known that

$$z_0 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \quad \text{and} \quad o_0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$$

Based on these quantities the following Jacobians can be calculated

$$J_{c1} = \begin{bmatrix} z_0 \times (o_{c1} - o_0) & 0 \\ z_0 & 0 \end{bmatrix} = \begin{bmatrix} -L_c s_1 & 0 \\ L_c c_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$J_{c2} = \begin{bmatrix} z_0 \times (o_{c2} - o_0) & z_1 \times (o_{c2} - o_1) \\ z_0 & z_1 \end{bmatrix} = \begin{bmatrix} -Ls_1 - L_c s_{12} & -L_c s_{12} \\ Lc_1 + L_c c_{12} & L_c c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

The kinetic energy for each link is given by a translational and a rotational part

$$\begin{aligned} K_i &= \frac{1}{2} m_i \, v_{ci}^T v_{ci} + \omega_i^T \, I_i \, \omega_i \\ &= \frac{1}{2} \dot{q}^T \big(m_i \, J_{v_{ci}}^T J_{v_{ci}} + J_{\omega_i}^T \, I_i \, J_{\omega_i} \big) \dot{q} \end{aligned}$$

The sum of all links may be described by the matrix multiplication

$$K = \frac{1}{2} \dot{q}^T D \, \dot{q}$$

with

$$D = \big(m_1 \, J_{v_{c1}}^T J_{v_{c1}} + J_{\omega_1}^T \, J_1 \, J_{\omega_1} \big) + \big(m_2 \, J_{v_{c2}}^T J_{v_{c2}} + J_{\omega_2}^T \, J_2 \, J_{\omega_2} \big)$$

The masses of the two links are given

$$m_1 = m_2 = m$$

and their density will therefore be $\rho = m/(L/10\, L/10\, L) = 100m/L^3$.
J_1 and J_2 are the inertia matrices of each link and must be represented in the base frame. By defining them around their local center of mass the matrices are constant, independent of the robot configuration and equal to each other.

$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

Recalling that the x-axis is parallel to the longest dimension L of each link, the three moments of inertia for the rectangular parallelepipeds are

$$I_{xx} = \frac{1}{12}\left((0.1L)^2 + (0.1L)^2\right) m = \frac{mL^2}{600}$$

$$I_{yy} = I_{zz} = \frac{1}{12}\left(L^2 + (0.1L)^2\right) m = \frac{101mL^2}{1200}$$

Hereafter by applying their respective rotation according to the forward kinematics they can be expressed in the base frame

$$J_1 = R_{c1} I R_{c1}^T, \quad J_2 = R_{c2} I R_{c2}^T$$

Since the links only ever rotate in the xy-plane in this case, their moment of inertia around the z-axis will stay the same, equal to I_{zz}.
Carrying out all intermediate calculations, D evaluates to

$$D = \begin{bmatrix} m(L^2 + 2L_c^2 + 2LL_c\cos\theta_2) + 2I_{zz} & m(L_c^2 + LL_c\cos\theta_2) + I_{zz} \\ m(L_c^2 + LL_c\cos\theta_2) + I_{zz} & mL_c^2 + I_{zz} \end{bmatrix} \quad \Rightarrow$$

$$D = \begin{bmatrix} 2002 + 1200\cos\theta_2 & 401 + 600\cos\theta_2 \\ 401 + 600\cos\theta_2 & 401 \end{bmatrix} \frac{mL^2}{1200}$$

i.e.

$$\begin{aligned} d_{11} = &\quad (2002 + 1200\cos\theta_2)\frac{mL^2}{1200} \\ d_{12} = &\quad (401 + 600\cos\theta_2)\frac{mL^2}{1200} \\ d_{22} = &\quad 401\frac{mL^2}{1200} \end{aligned}$$

The Christoffel symbols evaluate to

$$\begin{aligned} c_{111} = &\quad \frac{d_{11,1} + d_{11,1} - d_{11,1}}{2} = \frac{d_{11,1}}{2} = 0 \\ c_{121} = c_{211} = &\quad \frac{d_{12,1} + d_{11,2} - d_{12,1}}{2} = \frac{d_{11,2}}{2} = -\frac{mL^2}{2}\sin\theta_2 \\ c_{221} = &\quad \frac{d_{12,2} + d_{12,2} - d_{22,1}}{2} = d_{12,2} = -\frac{mL^2}{2}\sin\theta_2 \\ c_{112} = &\quad \frac{d_{12,1} + d_{12,1} - d_{11,2}}{2} = -\frac{d_{11,2}}{2} = \frac{mL^2}{2}\sin\theta_2 \\ c_{122} = c_{212} = &\quad \frac{d_{22,1} + d_{12,2} - d_{12,2}}{2} = 0 \\ c_{222} = &\quad \frac{d_{22,2} + d_{22,2} - d_{22,2}}{2} = \frac{d_{22,2}}{2} = 0 \end{aligned}$$

The potential energy of the planar elbow manipulator can be described as

$$P = \sum_i^n P_i = -\sum_i^n m_i \, \mathbf{g}^T o_{ci}$$

where $\mathbf{g} = \{0, -g, 0\}^T$.
This leads to the following potential energy function

$$\begin{aligned}
P_1 &= mg \, L_c \sin(q_1) \\
P_2 &= mg \, \left(L \sin(q_1) + L_c \sin(q_1 + q_2)\right) \\
P &= P_1 + P_2 = mg \, L \, \left(1.5 \sin(q_1) + 0.5 \sin(q_1 + q_2)\right)
\end{aligned}$$

The partial derivatives of the potential energy are computed as

$$\begin{aligned}
g_k &= \frac{\delta P}{\delta q_k} \quad \Rightarrow \\
g_1 &= mg \, L \, \left(1.5 \cos(q_1) + 0.5 \cos(q_1 + q_2)\right) \\
g_2 &= mg \, L \cdot 0.5 \cos(q_1 + q_2)
\end{aligned}$$

Finally, the Euler-Lagrange equations for $q_1(t)$ and $q_2(t)$ can be written as

$$\begin{aligned}
d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + c_{121}\dot{q}_1\dot{q}_2 + c_{211}\dot{q}_1\dot{q}_2 + c_{221}\dot{q}_2{}^2 + g_1 &= \tau_1 \\
d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + c_{112}\dot{q}_1{}^2 + g_2 &= \tau_2
\end{aligned}$$

where τ_1 and τ_2 are moments applied to joints 1 and 2, respectively.