Seminar Algorithms for Big Data

Fast Random Integer Generation in an Interval

Based on a paper of the same title by Daniel Lemire

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Supervised by Dr. Manuel Penschuck

29th February 2024 · Algorithm Engineering (Prof. Dr. Ulrich Meyer)

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Motivation

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TBD TBD TBD

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1 Preliminaries

Formal Definition

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- Return $x + a$

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Definition (Power Remainder)

For $W, n \in \mathbb{N}$, we write $\mathcal{R}_n^W$ for $2^W \bmod n$.

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Is the generated number uniform in $[0, n)$?

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Idea: Use rejection sampling to achieve uniformity!

2 Unbiased Algorithms

The OpenBSD Algorithm

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- Number of integer divisions operations equal to number of rounds

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- Return a number in round if $x < 2^W - \mathcal{R}_n^W$
- Happens with probability $\frac{2^W - \mathcal{R}_n^W}{2^W} > \frac{1}{2}$
- Expected number of integer division operations is $\frac{2^W}{2^W - \mathcal{R}_n^W} < 2$

The Bitmask Algorithm

The Bitmask Algorithm

Lemire's Algorithm

Multiply-And-Shift

Multiply-And-Shift

The Algorithm

The Algorithm

4 Conclusion

Summary

Summary

	expected number of integer division operations	maximal number of integer division operations	Unbiased?

Summary

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Conclusion

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End of Talk