

Seminar Algorithms for Big Data

Fast Random Integer Generation in an Interval

Based on a paper of the same title by Daniel Lemire

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Supervised by Dr. Manuel Penschuck

29th February 2024 · Algorithm Engineering (Prof. Dr. Ulrich Meyer)

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- Sampling

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1

Preliminaries





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- Set $n = b - a$ and draw a uniform random integer $x \in [0, n)$
- Return $x + a$



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Definition (Power Remainder)

For $W, n \in \mathbb{N}$, we write \mathcal{R}_n^W for $2^W \bmod n$.

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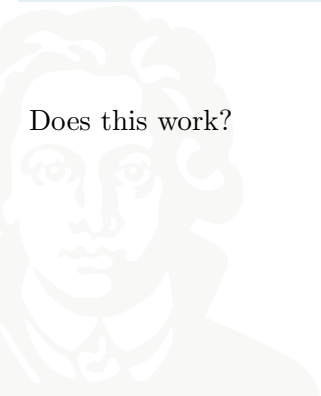
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Is the generated number uniform in $[0, n)$?

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Idea: Use **rejection sampling** to achieve uniformity!

2

Unbiased Algorithms



The OpenBSD Algorithm



The OpenBSD Algorithm

- We shift the **rejection interval** to the left:



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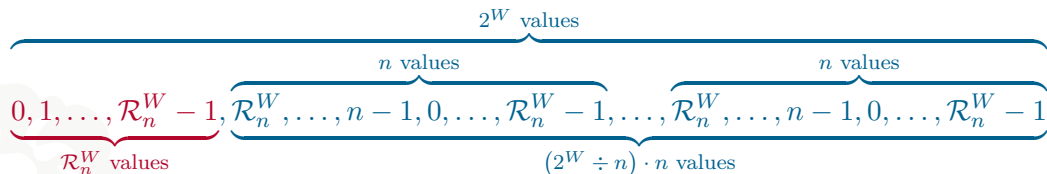
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2^W values

 $(2^W \div n) \cdot n$ values

The OpenBSD Algorithm

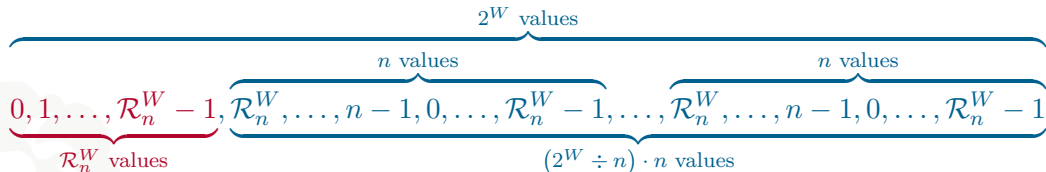
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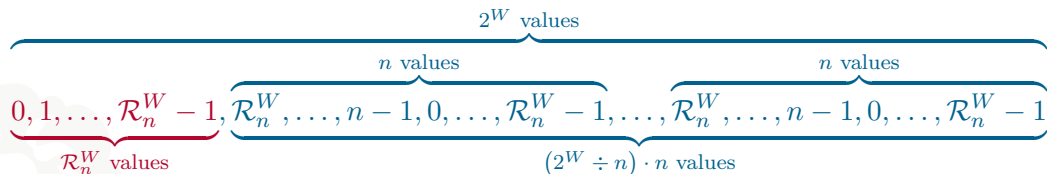
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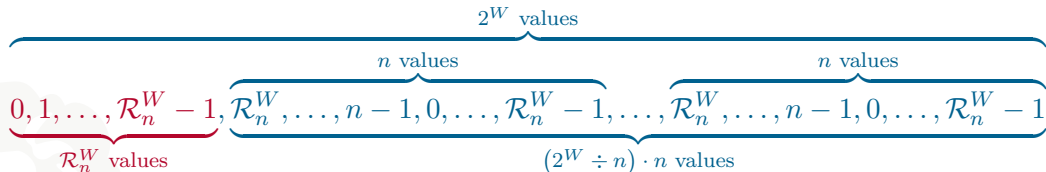


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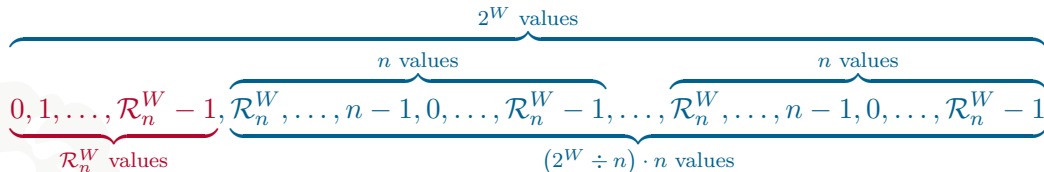
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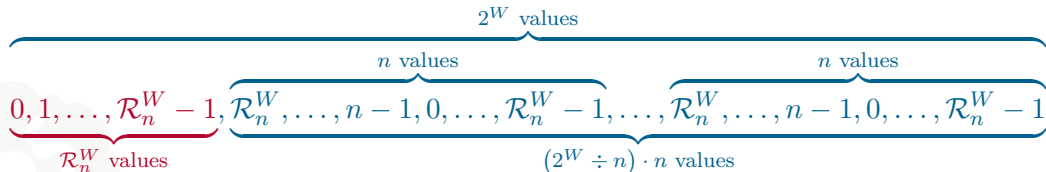
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We require 2 integer division operations: one for computing \mathcal{R}_n^W and one for computing $x \bmod n$.

The Java Algorithm



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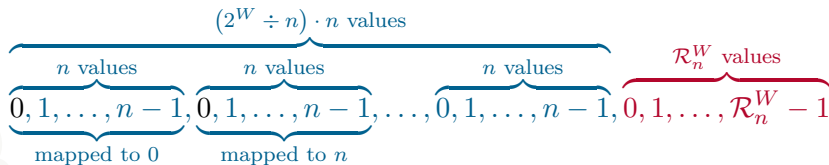
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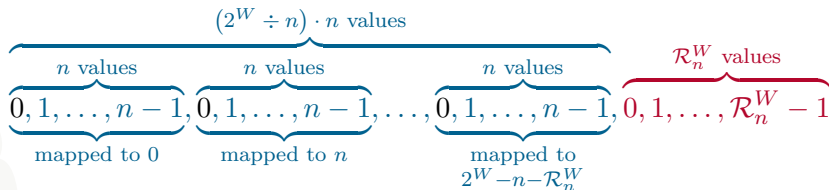
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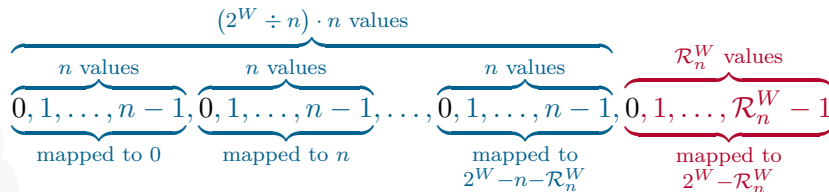
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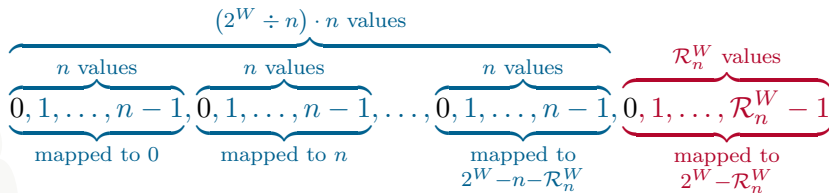
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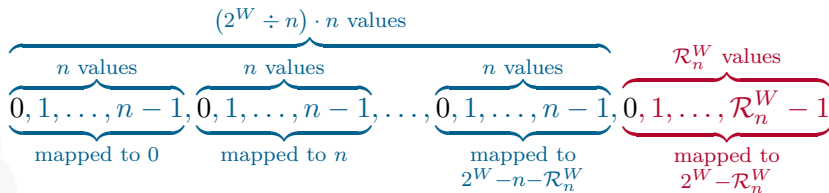
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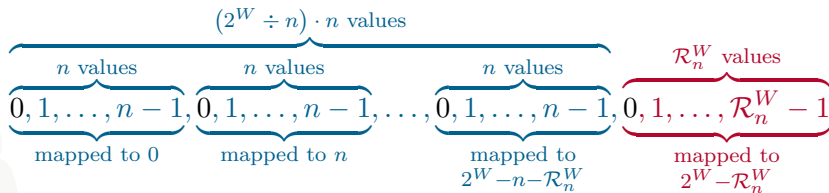
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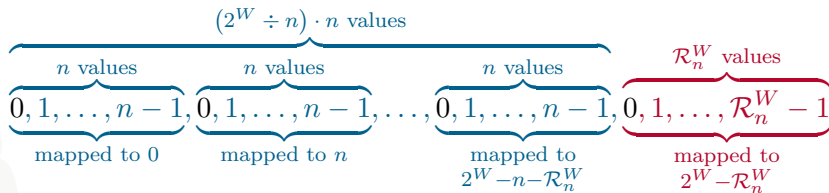
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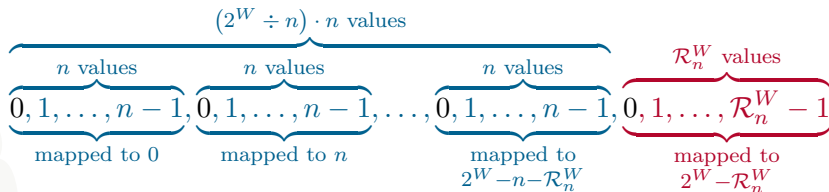
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 - Return r if $x - r > 2^W - n$ else goto (1)

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- Number of integer divisions operations equal to number of rounds

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- Return a number in round if $x < 2^W - \mathcal{R}_n^W$
- Happens with probability $\frac{2^W - \mathcal{R}_n^W}{2^W} > \frac{1}{2}$
- Expected number of integer division operations is $\frac{2^W}{2^W - \mathcal{R}_n^W} < 2$

The Bitmask Algorithm



The Bitmask Algorithm



Lemire's Algorithm



Multiply-And-Shift



Multiply-And-Shift



Lemire's Algorithm

The Algorithm



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4 Conclusion



Conclusion

Summary



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Lemire	$\frac{n}{2^W}$	1	✓

End of Talk

