Seminar Algorithms for Big Data

Fast Random Integer Generation in an Interval

Based on a paper of the same title by Daniel Lemire

Lukas Geis
Supervised by Dr. Manuel Penschuck

29th February 2024 · Algorithm Engineering (Prof. Dr. Ulrich Meyer)

Motivation

What is the problem?

$$0$$

Motivation

What is the problem?

$$1$$

Motivation

What is the problem?

$$\begin{array}{c} 1 \\ \uparrow \\ \text{uniform random bit} \end{array}$$

What is the problem?

1

↑

uniform random bit

↑ ↑

$0 \sim 50\%$ $1 \sim 50\%$

Motivation

What is the problem?

$1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1$

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$$1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1$$

$W = 8$ independent uniform bits

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1 1 0 1 0 0 0 1

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Goal:

Generate a uniform integer between 100 and 200

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1 1 0 1 0 0 0 1

interpret as unsigned 8-bit integer

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$$\overset{2^0}{\downarrow}$$

$$1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1$$

interpret as unsigned 8-bit integer

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$$\begin{array}{cccccccc} & & & & & & 2^1 & 2^0 \\ & & & & & & \downarrow & \downarrow \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}$$

interpret as unsigned 8-bit integer

Goal:

Generate a uniform integer between 100 and 200

Motivation

What is the problem?

$$\begin{array}{cccccccc} & & & & & 2^2 & 2^1 & 2^0 \\ & & & & & \downarrow & \downarrow & \downarrow \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}$$

interpret as unsigned 8-bit integer

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What is the problem?

$$\begin{array}{cccccccc} 2^7 & 2^6 & 2^5 & 2^4 & 2^3 & 2^2 & 2^1 & 2^0 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}$$

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What is the problem?

209 in binary

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
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interpret as unsigned 8-bit integer

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interpret as uniform 8-bit integer in $[0, 2^8)$

Goal:

Generate a uniform integer between 100 and 200

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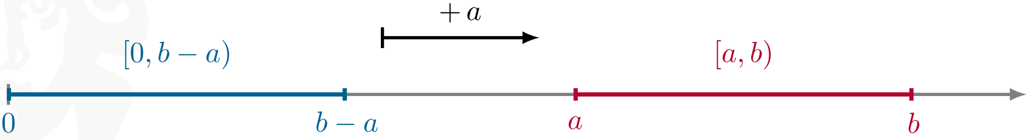

Preliminaries

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Definition (Common Operations)

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- Integer-Division: $x \div y \quad := \left\lfloor {}^{x}/_{y} \right\rfloor$

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Definition (Power Remainder)

For $W, n \in \mathbb{N}$, we write $\mathcal{R}$ for $2^W \bmod n$.

Preliminaries

The Naive Approach

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$$\texttt{rand()} \bmod n$$

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Is the generated number uniform in $[0, n)$?

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The Naive Approach - Bias

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We have a leftover interval that introduces bias.

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Deterministic Mappings

Every deterministic mapping $f\colon [0, 2^W) \to [0, n)$ does not generate uniform random integers in one step whenever n does not divide 2^W.

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Unbiased Algorithms

Unbiased Algorithms

The OpenBSD Algorithm

Unbiased Algorithms

The OpenBSD Algorithm

- Shift the rejection interval to the left:

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Efficiency

We require 2 integer division operations:

- one for computing $\mathcal{R}$
- and one for computing $x \bmod n$.

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- Map every number to the next-smallest multiple of n

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- Map every number to the next-smallest multiple of n
- Only numbers in leftover interval mapped to $2^W - \mathcal{R} > 2^W - n$

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- Map every number to the next-smallest multiple of n
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- Algorithm:
  - **(1)** Draw $x \in [0, 2^W)$ and compute $r = x \bmod n$
  - **(2)** Return r if $x - r \leq 2^W - n$ else goto **(1)**

The Java Algorithm - Efficiency

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- Number of integer divisions operations equal to number of rounds

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- Number of integer divisions operations equal to number of rounds
- Return number in round if $x < 2^W - \mathcal{R}$

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- At least one integer division operation
- Number of integer divisions operations equal to number of rounds
- Return number in round if $x < 2^W - \mathcal{R}$
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- At least one integer division operation
- Number of integer divisions operations equal to number of rounds
- Return number in round if $x < 2^W - \mathcal{R}$
- Happens with probability $\frac{2^W - \mathcal{R}}{2^W} > \frac{1}{2}$
- Expected number of integer division operations is $\frac{2^W}{2^W - \mathcal{R}} < 2$

The Bitmask Algorithm - Representation

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$$n \xrightarrow{\text{binary}} \underbrace{\overset{\overset{2^{W-1}}{\downarrow}}{0}, \dots, 0}_{\text{only 0's}}, \overset{\overset{2^{\lfloor \log_2 n \rfloor}}{\downarrow}}{1}, \underbrace{1, \dots, \overset{\overset{2^1}{\downarrow}}{0}, \overset{\overset{2^0}{\downarrow}}{1}}_{\text{series of 0's and 1's}}$$

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- Every number $x \leq n$ only needs the last $\lfloor \log_2 n \rfloor + 1$ bits
- Get these bits with a bitwise-AND with

$$2^{\lfloor \log_2 n \rfloor + 1} - 1 \xrightarrow{\text{binary}} \underbrace{\overset{\overset{2^{W-1}}{\downarrow}}{0}, \ldots, 0}_{\text{only 0's}}, \underbrace{\overset{\overset{2^{\lfloor \log_2 n \rfloor}}{\downarrow}}{1}, 1, \ldots, \overset{\overset{2^1}{\downarrow}}{1}, \overset{\overset{2^0}{\downarrow}}{1}}_{\text{only 1's}}$$

Unbiased Algorithms

The Bitmask Algorithm - Mask

The Bitmask Algorithm - Mask

- How can we compute $2^{\lfloor \log_2 n \rfloor + 1}$?

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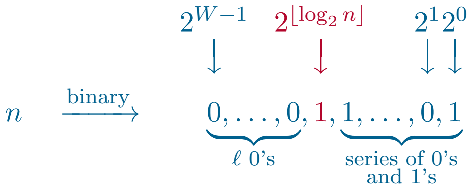

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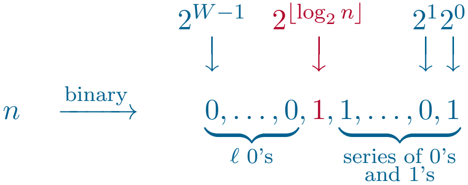

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  (1) Compute ℓ and $\mathcal{M} = 2^{\lfloor \log_2 n \rfloor + 1} - 1$

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  - **(1)** Compute ℓ and $\mathcal{M} = 2^{\lfloor \log_2 n \rfloor + 1} - 1$
  - **(2)** Draw $x \in [0, 2^W)$ and compute $b = x \ \& \ \mathcal{M}$

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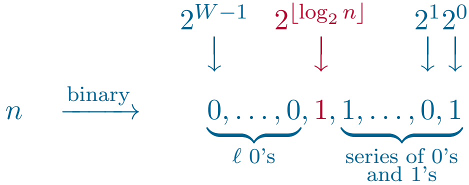

2^{W-1} $2^{\lfloor \log_2 n \rfloor}$ $2^1 2^0$

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Efficiency

- b at most $\mathcal{M} = 2^{\lfloor \log_2 n \rfloor + 1} - 1 < 2n$ ⟶ success probability at least $\approx \frac{1}{2}$

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- At most ≈ 2 rounds in expectation
- No integer division at all
- Computation of leading 0's requires `clz` instruction/algorithm
- Roughly as expensive as a `div` instruction

3

Lemire’s Algorithm

Lemire's Algorithm

Multiply-And-Shift

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Lemire's Algorithm

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Lemire's Algorithm

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- $2W$ bits enough to represent $\texttt{rand()} \cdot n$

Is this uniform?

- Mapping is deterministic!
- Mapping can not be uniform for all n!

Lemire's Algorithm

The Algorithm - Intervals

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- Split $[0, n \cdot 2^W)$ into intervals $[i \cdot 2^W, (i+1) \cdot 2^W)$ for $i < n$

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- Every restricted i^{th} interval has $\frac{2^W - \mathcal{R}}{n} = \lfloor \frac{2^W}{n} \rfloor$ multiples of n

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  which is divisible by n
- Every restricted i^{th} interval has $\frac{2^W - \mathcal{R}}{n} = \lfloor \frac{2^W}{n} \rfloor$ multiples of n
- We can make Multiply-And-Shift uniform by only accepting multiples of n in restricted intervals

The Algorithm - Rejection

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When do we reject $x := \texttt{rand()} \cdot n$?

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$$\overbrace{\underbrace{0, 1, \dots, \mathcal{R} - 1}_{\text{rejected part}}, \underbrace{\mathcal{R}, \dots, n, \dots, 2^W - 1}_{\text{restricted } i^{\text{th}} \text{ interval}}}^{2^W \text{ values}}$$

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- We reject x if $x \bmod 2^W < \mathcal{R}$

Lemire's Algorithm

The Algorithm - Sketch

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$\mathcal{R} \leftarrow 2^W \bmod n$                /* Compute rejection threshold */
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$\mathcal{R} \leftarrow 2^W \bmod n$        /* Compute rejection threshold */
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    $x \leftarrow$ rand()
    $m \leftarrow x \cdot n$        /* Use $2W$ bits for representation */
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$\mathcal{R} \leftarrow 2^W \bmod n$                /* Compute rejection threshold */
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    $x \leftarrow$ rand()
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    $l \leftarrow m \ \& \ (2^W - 1)$               /* $m \bmod 2^W$ */
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    if $l \geq \mathcal{R}$ then                       /* Apply rejection rule */
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$\mathcal{R} \leftarrow 2^W \mod n$                    /* Compute rejection threshold */
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    if $l \geq \mathcal{R}$ then                        /* Apply rejection rule */
        return $m \gg W$
```

The Algorithm - Avoiding Division

Lemire's Algorithm

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Consider the first iteration of the loop:

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- We reject x if $l < \mathcal{R}$

The Algorithm - Avoiding Division

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- We reject x if $l < \mathcal{R}$ ⟶ we need to compute $\mathcal{R}$ beforehand

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The Algorithm - Avoiding Division

Consider the first iteration of the loop:

- We reject x if $l < \mathcal{R}$ ⟶ we need to compute $\mathcal{R}$ beforehand
- But we know $\mathcal{R} < n$ ⟶ if $l \geq n$ we do not need to know $\mathcal{R}$

The Algorithm - Avoiding Division

Consider the first iteration of the loop:

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We can alter the first iteration of the loop:

The Algorithm - Avoiding Division

Consider the first iteration of the loop:

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We can alter the first iteration of the loop:

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We can alter the first iteration of the loop:

- We do not compute $\mathcal{R}$ beforehand
- If $l \geq n$, we accept x without computing $\mathcal{R}$
- If not, we compute $\mathcal{R}$ and proceed as before

Lemire's Algorithm

The Algorithm - Avoiding Division

Consider the first iteration of the loop:

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With what probability do we need to compute $\mathcal{R}$:

The Algorithm - Avoiding Division

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Lemire’s Algorithm

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4

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End of Talk