GOETHE UNIVERSITÄT FRANKFURT AM MAIN

Seminar Algorithms for Big Data

Fast Random Integer Generation in an Interval

Based on a paper of the same title by Daniel Lemire

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Supervised by Dr. Manuel Penschuck

29th February 2024 · Algorithm Engineering (Prof. Dr. Ulrich Meyer)

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TBD

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TBD TBD

What is our goal?

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- Shuffling
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- Sampling

TBD TBD TBD

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Formal Definition

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- Set $n = b - a$ and draw a uniform random integer $x \in [0, n)$
- Return $x + a$

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- Bitwise-And: $x \;\&\; y$

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- Bitwise-And: $x \ \& \ y \quad \rightarrow x \bmod 2^W := x \ \& \ (2^W - 1)$

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Is the generated number uniform in $[0, n)$?

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We have a leftover interval that introduces bias.

Every approach that maps every integer in $[0, 2^W)$ to a single number in $[0, n)$ does not generate uniform random integers in one step.

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Unbiased Algorithms

The OpenBSD Algorithm

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The Java Algorithm

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The Fast-Dice-Roller Algorithm

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The Bitmask Algorithm

The Bitmask Algorithm

Lemire's Algorithm

Multiply-And-Shift

Multiply-And-Shift

The Algorithm

The Algorithm

4 Conclusion

Summary

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End of Talk