Seminar Algorithms for Big Data

Fast Random Integer Generation in an Interval

Based on a paper of the same title by Daniel Lemire

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Supervised by Dr. Manuel Penschuck

29th February 2024 · Algorithm Engineering (Prof. Dr. Ulrich Meyer)

What is the problem?

$$0$$

What is the problem?

1

What is the problem?

$$\begin{array}{c} 1 \\ \uparrow \\ \text{uniform random bit} \end{array}$$

What is the problem?

$$1$$

$$\uparrow$$

uniform random bit

$$\uparrow \qquad\qquad \uparrow$$

$$0 \sim 50\% \qquad 1 \sim 50\%$$

What is the problem?

1 1 0 1 0 0 0 1

What is the problem?

$$1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1$$

$W = 8$ independent uniform bits

What is the problem?

1 1 0 1 0 0 0 1

$W = 8$ independent uniform bits

Goal:

Generate a uniform integer between 100 and 200

What is the problem?

1 1 0 1 0 0 0 1

interpret as unsigned 8-bit integer

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Generate a uniform integer between 100 and 200

What is the problem?

$$2^0$$
$$\downarrow$$

$$1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1$$

interpret as unsigned 8-bit integer

Goal:

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What is the problem?

$$\begin{array}{cccccccc} & & & & & & 2^1 & 2^0 \\ & & & & & & \downarrow & \downarrow \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}$$

interpret as unsigned 8-bit integer

Goal:

Generate a uniform integer between 100 and 200

What is the problem?

$$\begin{array}{cccccccc} & & & & & 2^2 & 2^1 & 2^0 \\ & & & & & \downarrow & \downarrow & \downarrow \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}$$

interpret as unsigned 8-bit integer

Goal:

Generate a uniform integer between 100 and 200

What is the problem?

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
1	1	0	1	0	0	0	1

interpret as unsigned 8-bit integer

Goal:

Generate a uniform integer between 100 and 200

What is the problem?

209 in binary

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
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interpret as unsigned 8-bit integer

Goal:

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What is the problem?

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interpret as uniform 8-bit integer in $[0, 2^8)$

Goal:

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1 Preliminaries

Formal Definition

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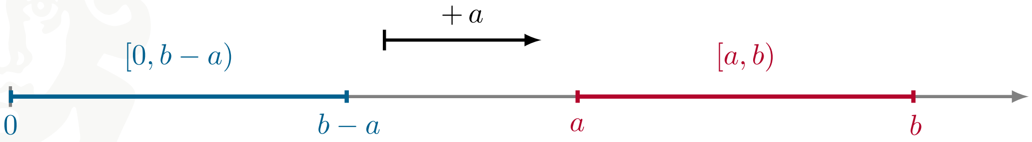

Operations

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Definition (Common Operations)

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- Integer-Division: $x \div y \quad := \left\lfloor {}^{x}\!/_{y} \right\rfloor$

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- Bitwise-And: $x \;\&\; y$

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Definition (Power Remainder)

For $W, n \in \mathbb{N}$, we write $\mathcal{R}$ for $2^W \bmod n$.

The Naive Approach

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$$\texttt{rand()} \bmod n$$

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Is the generated number uniform in $[0, n)$?

The Naive Approach - Bias

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We have a leftover interval that introduces bias.

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Deterministic Mappings

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Every deterministic mapping $f\colon [0, 2^W) \to [0, n)$ does not generate uniform random integers in one step

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Deterministic Mappings

Every deterministic mapping $f\colon [0, 2^W) \to [0, n)$ does not generate uniform random integers in one step whenever n does not divide 2^W.

2 Unbiased Algorithms

The OpenBSD Algorithm

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- Shift the rejection interval to the left:

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$$\overbrace{\underbrace{0,1,\dots,\mathcal{R}-1}_{\mathcal{R}\text{ values}},\underbrace{\overbrace{\mathcal{R},\dots,n-1,0,\dots,\mathcal{R}-1}^{n\text{ values}},\dots,\overbrace{\mathcal{R},\dots,n-1,0,\dots,\mathcal{R}-1}^{n\text{ values}}}_{(2^W \div n)\cdot n\text{ values}}}^{2^W\text{ values}}$$

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- Algorithm:
  - Generate a uniform random number $x \in [0, 2^W)$ until $x \geq \mathcal{R}$

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We require 2 integer division operations:

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We require 2 integer division operations:

- one for computing $\mathcal{R}$

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Efficiency

We require 2 integer division operations:

- one for computing $\mathcal{R}$
- and one for computing $x \bmod n$.

The Java Algorithm

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$$\overbrace{\underbrace{\overbrace{0, 1, \ldots, n-1}^{n \text{ values}}}_{\text{mapped to } 0}, \overbrace{0, 1, \ldots, n-1}^{n \text{ values}}, \ldots, \overbrace{0, 1, \ldots, n-1}^{n \text{ values}}}^{(2^W \div n) \cdot n \text{ values}}, \overbrace{0, 1, \ldots, \mathcal{R}-1}^{\mathcal{R} \text{ values}}$$

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- Map every number to the next-smallest multiple of n

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- Map every number to the next-smallest multiple of n
- Only numbers in leftover interval mapped to $2^W - \mathcal{R} > 2^W - n$

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- Only numbers in leftover interval mapped to $2^W - \mathcal{R} > 2^W - n$
- Algorithm:
  - **(1)** Draw $x \in [0, 2^W)$ and compute $r = x \bmod n$
  - **(2)** Return r if $x - r \leq 2^W - n$ else goto **(1)**

The Java Algorithm - Efficiency

Algorithm:

(1) Draw $x \in [0, 2^W)$ and compute $r = x \bmod n$

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Efficiency

- At least one integer division operation

The Java Algorithm - Efficiency

Algorithm:

(1) Draw $x \in [0, 2^W)$ and compute $r = x \bmod n$

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Efficiency

- At least one integer division operation
- Number of integer divisions operations equal to number of rounds

The Java Algorithm - Efficiency

Algorithm:

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- At least one integer division operation
- Number of integer divisions operations equal to number of rounds
- Return number in round if $x < 2^W - \mathcal{R}$

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- At least one integer division operation
- Number of integer divisions operations equal to number of rounds
- Return number in round if $x < 2^W - \mathcal{R}$
- Happens with probability $\frac{2^W - \mathcal{R}}{2^W} > \frac{1}{2}$

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Algorithm:

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Efficiency

- At least one integer division operation
- Number of integer divisions operations equal to number of rounds
- Return number in round if $x < 2^W - \mathcal{R}$
- Happens with probability $\frac{2^W - \mathcal{R}}{2^W} > \frac{1}{2}$
- Expected number of integer division operations is $\frac{2^W}{2^W - \mathcal{R}} < 2$

3 Lemire's Algorithm

Multiply-And-Shift

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- Map `rand()` to $[0, n)$ divisionless with $(\texttt{rand()} \cdot n) \gg W$:

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$$(\texttt{rand()} \cdot n) \gg W$$

Multiply-And-Shift

- Map `rand()` to $[0, n)$ divisionless with $(\texttt{rand()} \cdot n) \gg W$:

$$(\texttt{rand()} \cdot n) \div 2^W$$

Multiply-And-Shift

- Map `rand()` to $[0, n)$ divisionless with $(\texttt{rand()} \cdot n) \gg W$:

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Multiply-And-Shift

- Map `rand()` to $[0, n)$ divisionless with $(\texttt{rand()} \cdot n) \gg W$:

$$\underbrace{(\texttt{rand()} \cdot n)}_{\in[0, n \cdot 2^W)} \div 2^W$$

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$$\underbrace{(\texttt{rand()} \cdot n)}_{\in[0, n \cdot 2^W)} \div 2^W$$

- $n < 2^W \implies n \cdot 2^W < 2^W \cdot 2^W = 2^{2W}$

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- $n < 2^W \implies n \cdot 2^W < 2^W \cdot 2^W = 2^{2W}$
- $2W$ bits enough to represent $\texttt{rand()} \cdot n$

Multiply-And-Shift

- Map `rand()` to $[0, n)$ divisionless with $(\texttt{rand()} \cdot n) \gg W$:

$$\underbrace{(\texttt{rand()} \cdot n) \div 2^W}_{\in[0,n)}$$

- $n < 2^W \implies n \cdot 2^W < 2^W \cdot 2^W = 2^{2W}$
- $2W$ bits enough to represent $\texttt{rand()} \cdot n$

Multiply-And-Shift

- Map `rand()` to $[0, n)$ divisionless with $(\texttt{rand()} \cdot n) \gg W$:

$$\underbrace{(\texttt{rand()} \cdot n) \div 2^W}_{\in[0,n)}$$

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- Every restricted i^{th} interval has $\frac{2^W - \mathcal{R}}{n} = \lfloor \frac{2^W}{n} \rfloor$ multiples of n
- We can make Multiply-And-Shift uniform by only accepting multiples of n in restricted intervals

The Algorithm - Rejection

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- We reject x if $x \bmod 2^W < \mathcal{R}$

The Algorithm - Sketch

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$\mathcal{R} \leftarrow 2^W \bmod n$ `/* Compute rejection threshold */`

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```
R ← 2^W mod n                          /* Compute rejection threshold */
while true do
    x ← rand()
    m ← x · n                          /* Use 2W bits for representation */
    l ← m & (2^W − 1)                  /* m mod 2^W */
    if l ≥ R then                      /* Apply rejection rule */
```

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$\mathcal{R} \leftarrow 2^W \mod n$ /* Compute rejection threshold */
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    $m \leftarrow x \cdot n$ /* Use $2W$ bits for representation */
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    if $l \geq \mathcal{R}$ **then** /* Apply rejection rule */
        return $m \gg W$

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- We assume x to be uniform in $[0, 2^W)$ $\longrightarrow$ l is also uniform in $[0, 2^W)$
- We compute $\mathcal{R}$ if $l < n$ $\longrightarrow$ happens with probability $\frac{n}{2^W}$

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End of Talk

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The Bitmask Algorithm - Representation

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$$
\begin{array}{c}
\qquad\qquad 2^{W-1} \qquad 2^{\lfloor \log_2 n \rfloor} \qquad 2^1 2^0 \\
\qquad\qquad \downarrow \qquad\quad \downarrow \qquad\qquad \downarrow\ \downarrow \\
n \xrightarrow{\text{binary}} \underbrace{0,\ldots,0}_{\text{only 0's}}, 1, \underbrace{1,\ldots,0,1}_{\substack{\text{series of 0's}\\ \text{and 1's}}}
\end{array}
$$

- Every number $x \leq n$ only needs the last $\lfloor \log_2 n \rfloor + 1$ bits

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- Every number $x \leq n$ only needs the last $\lfloor \log_2 n \rfloor + 1$ bits
- Get these bits with a bitwise-AND with

$$\begin{array}{ccc} & & 2^{W-1} \quad 2^{\lfloor \log_2 n \rfloor} \quad 2^1 2^0 \\ & & \downarrow \qquad \downarrow \qquad \downarrow \downarrow \\ 2^{\lfloor \log_2 n \rfloor + 1} - 1 & \xrightarrow{\text{binary}} & \underbrace{0, \ldots, 0}_{\text{only 0's}}, \underbrace{1, 1, \ldots, 1, 1}_{\text{only 1's}} \end{array}$$

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- $\lfloor \log_2 n \rfloor = W - \ell - 1$

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- $\lfloor \log_2 n \rfloor = W - \ell - 1 \quad \longrightarrow \quad 2^{\lfloor \log_2 n \rfloor + 1} = 1 \ll (W - \ell)$

The Bitmask Algorithm - Mask

- How can we compute $2^{\lfloor \log_2 n \rfloor + 1}$?
- Count the number ℓ of leading 0's in n!

$$
\begin{array}{ccccccc}
 & & 2^{W-1} & & 2^{\lfloor \log_2 n \rfloor} & & 2^1 \; 2^0 \\
 & & \downarrow & & \downarrow & & \downarrow \; \downarrow \\
n & \xrightarrow{\text{binary}} & \multicolumn{5}{c}{\underbrace{0, \ldots, 0}_{\ell \text{ 0's}}, {\color{red}1}, \underbrace{1, \ldots, 0, 1}_{\substack{\text{series of 0's} \\ \text{and 1's}}}}
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(1) Compute ℓ and $\mathcal{M} = 2^{\lfloor \log_2 n \rfloor + 1} - 1$

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- Algorithm:
  (1) Compute ℓ and $\mathcal{M} = 2^{\lfloor \log_2 n \rfloor + 1} - 1$
  (2) Draw $x \in [0, 2^W)$ and compute $b = x \ \& \ \mathcal{M}$

The Bitmask Algorithm - Mask

- How can we compute $2^{\lfloor \log_2 n \rfloor + 1}$?
- Count the number ℓ of leading 0's in n!

$$n \xrightarrow{\text{binary}} \underbrace{\overset{2^{W-1}}{\downarrow}\hspace{-0.8em}0, \ldots, 0}_{\ell \text{ 0's}}, \overset{2^{\lfloor \log_2 n \rfloor}}{\downarrow}\hspace{-0.8em}1, \underbrace{1, \ldots, \overset{2^1}{\downarrow}\hspace{-0.5em}0, \overset{2^0}{\downarrow}\hspace{-0.5em}1}_{\text{series of 0's and 1's}}$$

- $\lfloor \log_2 n \rfloor = W - \ell - 1 \qquad \longrightarrow \qquad 2^{\lfloor \log_2 n \rfloor + 1} = 1 \ll (W - \ell)$
- Algorithm:
  (1) Compute ℓ and $\mathcal{M} = 2^{\lfloor \log_2 n \rfloor + 1} - 1$
  (2) Draw $x \in [0, 2^W)$ and compute $b = x \;\&\; \mathcal{M}$
  (3) Return b if $b < n$ else goto **(2)**

Summary

The Bitmask Algorithm - Efficiency

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- Computation of leading 0's requires `clz` instruction/algorithm
- Roughly as expensive as a `div` instruction