

Seminar Algorithms for Big Data

Fast Random Integer Generation in an Interval

Based on a paper of the same title by Daniel Lemire

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Supervised by Dr. Manuel Penschuck

29th February 2024 · Algorithm Engineering (Prof. Dr. Ulrich Meyer)

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We want to *efficiently* draw a *uniform* random integer in an interval.



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- Complex Graph Generators
- Sampling

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1

Preliminaries



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- Set $n = b - a$ and draw a uniform random integer $x \in [0, n)$
- Return $x + a$



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Definition (Power Remainder)

For $W, n \in \mathbb{N}$, we write \mathcal{R}_n^W for $2^W \bmod n$.

The Naive Approach



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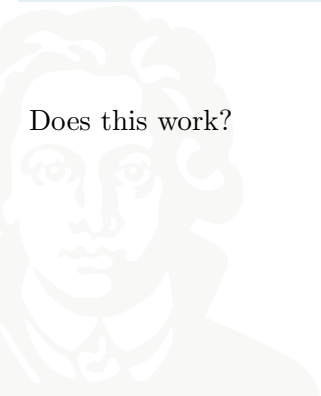
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Is the generated number uniform in $[0, n)$?

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Idea: Use **rejection sampling** to achieve uniformity!

2

Unbiased Algorithms



The OpenBSD Algorithm



The OpenBSD Algorithm

- We shift the **rejection interval** to the left:



The OpenBSD Algorithm

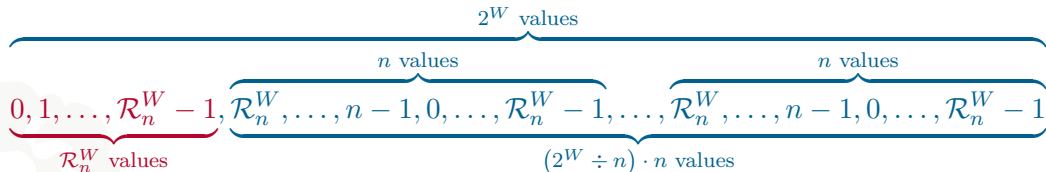
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2^W values
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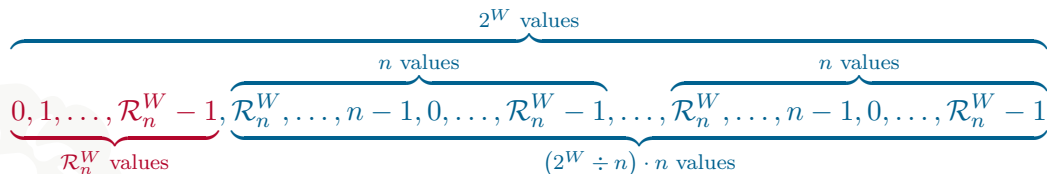
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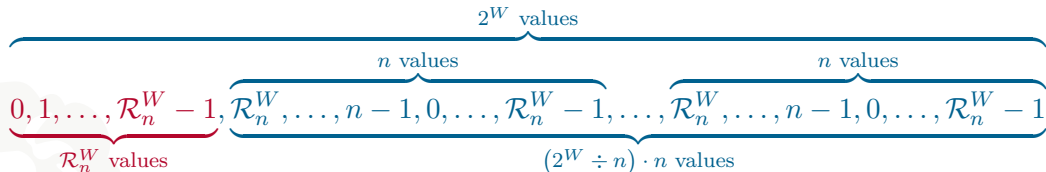


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- Generate a uniform random number $x \in [0, 2^W)$ until $x \geq \mathcal{R}_n^W$

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Can we do better?

The Java Algorithm



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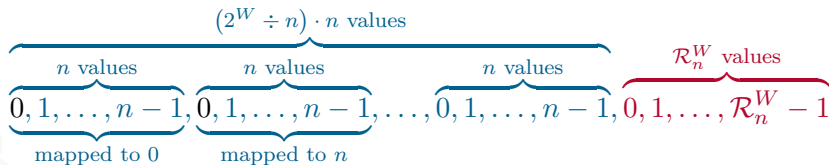
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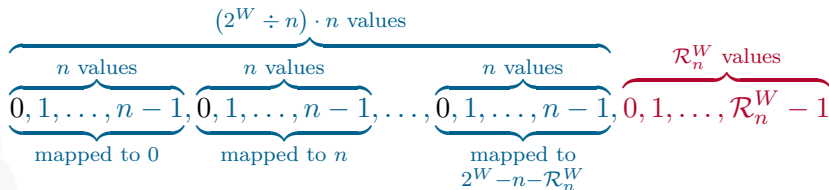
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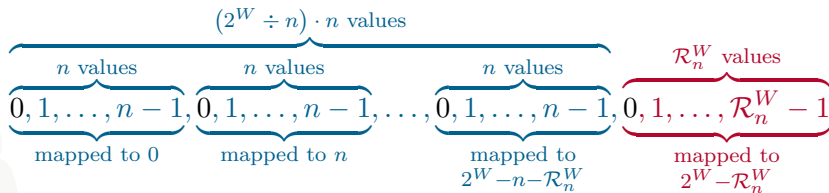
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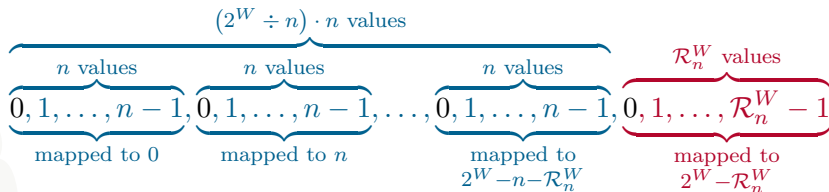
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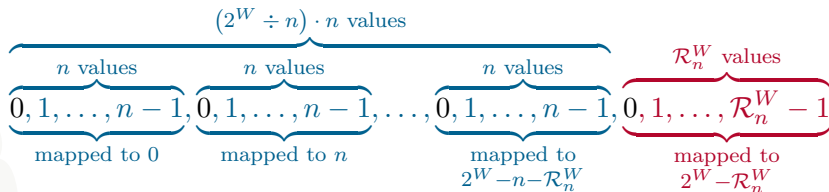
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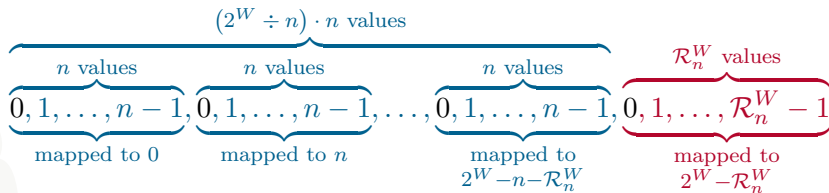
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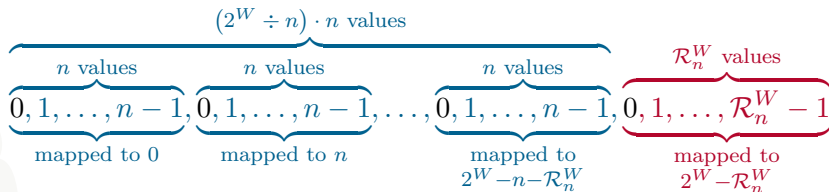
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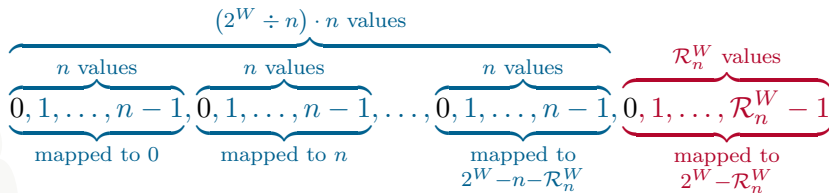
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 - Return r if $x - r > 2^W - n$ else goto (1)

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- Number of integer divisions operations equal to number of rounds

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- Happens with probability $\frac{2^W - \mathcal{R}_n^W}{2^W} > \frac{1}{2}$

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- Return a number in round if $x < 2^W - \mathcal{R}_n^W$
- Happens with probability $\frac{2^W - \mathcal{R}_n^W}{2^W} > \frac{1}{2}$
- Expected number of integer division operations is $\frac{2^W}{2^W - \mathcal{R}_n^W} < 2$

The Bitmask Algorithm



The Bitmask Algorithm

- Consider the **binary** representation of n :



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$$\underbrace{\overbrace{0}^{2^{W-1}}, \dots, 0}_{\text{only 0's}}, \overbrace{1}^{2^{\lfloor \log_2 n \rfloor}}, \underbrace{0/1, \dots, \overbrace{0/1}^{2^1}, \overbrace{0/1}^{2^0}}_{\text{series of 0's and 1's}}$$

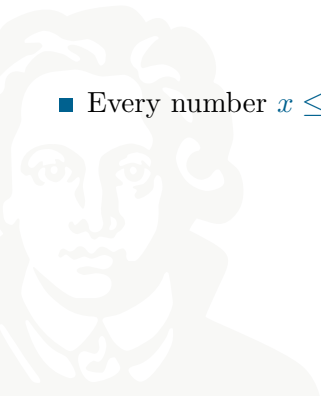


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- Every resulting number $x \ \& \ (2^{\lfloor \log_2 n \rfloor + 1} - 1)$ is at most

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$$\underbrace{\overbrace{0, \dots, 0}^{2^{W-1}}}_{\text{only 0's}}, \overbrace{1}^{2^{\lfloor \log_2 n \rfloor}}, \underbrace{0/1, \dots, 0/1}_{\text{series of 0's and 1's}}^{\overbrace{2^1} \quad \overbrace{2^0}}$$

- Every number $x \leq n$ only needs the last $\lfloor \log_2 n \rfloor + 1$ bits
- Get these bits with a bitwise-AND with $2^{\lfloor \log_2 n \rfloor + 1} - 1$:

$$\underbrace{\overbrace{0, \dots, 0}^{2^{W-1}}}_{\text{only 0's}}, \underbrace{\overbrace{1}^{2^{\lfloor \log_2 n \rfloor}}, 1, \dots, 1}_{\text{only 1's}}^{\overbrace{2^1} \quad \overbrace{2^0}}$$

- Every resulting number $x \ \& \ (2^{\lfloor \log_2 n \rfloor + 1} - 1)$ is at most $2^{\lfloor \log_2 n \rfloor + 1} - 1 < 2n$

The Bitmask Algorithm



The Bitmask Algorithm



Lemire's Algorithm

Multiply-And-Shift



Multiply-And-Shift

$$(\text{rand}() \cdot n) \gg W$$



Multiply-And-Shift

$$(\text{rand}() \cdot n) \gg W = (\text{rand}() \cdot n) \div 2^W$$



Lemire's Algorithm

The Algorithm



Lemire's Algorithm

The Algorithm



4 Conclusion



Conclusion

Summary



expected number of integer division operations	maximum number of integer division operations	Unbiased?
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	expected number of integer division operations	maximum number of integer division operations	Unbiased?
Modulo Reduction	1	1	X



	expected number of integer division operations	maximum number of integer division operations	Unbiased?
Modulo Reduction	1	1	X
Multiply-and-Shift	0	0	X

	expected number of integer division operations	maximum number of integer division operations	Unbiased?
Modulo Reduction	1	1	\times
Multiply-and-Shift	0	0	\times
OpenBSD	2	2	✓

Conclusion

Summary

	expected number of integer division operations	maximum number of integer division operations	Unbiased?
Modulo Reduction	1	1	✗
Multiply-and-Shift	0	0	✗
OpenBSD	2	2	✓
Java	$\frac{2^W}{2^W - (2^W \bmod n)}$	∞	✓

	expected number of integer division operations	maximum number of integer division operations	Unbiased?
Modulo Reduction	1	1	✗
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Bitmask	0	0	✓

	expected number of integer division operations	maximum number of integer division operations	Unbiased?
Modulo Reduction	1	1	✗
Multiply-and-Shift	0	0	✗
OpenBSD	2	2	✓
Java	$\frac{2^W}{2^W - (2^W \bmod n)}$	∞	✓
Bitmask	0	0	✓
Lemire	$\frac{n}{2^W}$	1	✓

End of Talk

