Seminar Algorithms for Big Data

Fast Random Integer Generation in an Interval

Based on a paper of the same title by Daniel Lemire

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Supervised by Dr. Manuel Penschuck

29th February 2024 · Algorithm Engineering (Prof. Dr. Ulrich Meyer)

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- Shuffling

$$\boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{5}$$

$$\Downarrow$$

$$\boxed{4}\boxed{2}\boxed{1}\boxed{5}\boxed{3}$$

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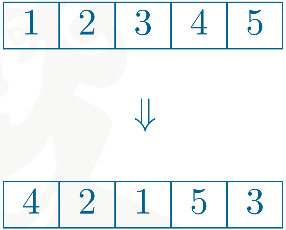
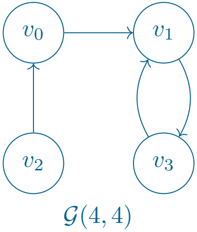
- Graph Generators





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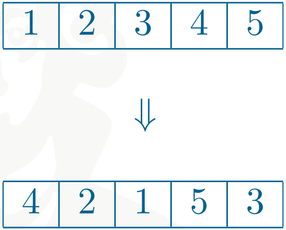
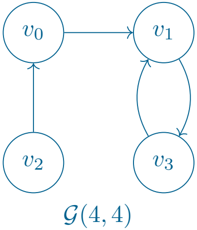
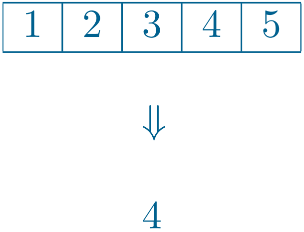
We want to efficiently draw a uniform random integer in an interval.

Where do we need this?

- Shuffling
- Graph Generators
- Sampling







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- Set $n = b - a$ and draw a uniform random integer $x \in [0, n)$
- Return $x + a \in [a, b)$

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Definition (Power Remainder)

For $W, n \in \mathbb{N}$, we write $\mathcal{R}_n^W$ for $2^W \bmod n$.

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Is the generated number uniform in $[0, n)$?

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Idea: Use rejection sampling to achieve uniformity!

2

Unbiased Algorithms

The OpenBSD Algorithm

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- Algorithm:
  (1) Draw $x \in [0, 2^W)$ and compute $r = x \bmod n$
  (2) Return r if $x - r \leq 2^W - n$ else goto **(1)**

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- At least one integer division operation
- Number of integer divisions operations equal to number of rounds

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- At least one integer division operation
- Number of integer divisions operations equal to number of rounds
- Return number in round if $x < 2^W - \mathcal{R}_n^W$

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Algorithm:

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Efficiency

- At least one integer division operation
- Number of integer divisions operations equal to number of rounds
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- Happens with probability $\frac{2^W - \mathcal{R}_n^W}{2^W} > \frac{1}{2}$

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The Bitmask Algorithm - Representation

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- Every number $x \leq n$ only needs the last $\lfloor \log_2 n \rfloor + 1$ bits
- Get these bits with a bitwise-AND with

$$2^{\lfloor \log_2 n \rfloor + 1} - 1 \xrightarrow{\text{binary}} \underbrace{\overbrace{0}^{2^{W-1}}, \dots, 0}_{\text{only 0's}}, \underbrace{\overbrace{1}^{2^{\lfloor \log_2 n \rfloor}}, 1, \dots, \overbrace{1}^{2^1}, \overbrace{1}^{2^0}}_{\text{only 1's}}$$

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- No integer division at all
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- Roughly as expensive as a `div` instruction

Lemire's Algorithm

Multiply-And-Shift

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- Map `rand()` to $[0, n)$ divisionless with $(\texttt{rand()} \cdot n) \gg W$:

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- Map `rand()` to $[0, n)$ divisionless with $(\texttt{rand()} \cdot n) \gg W$:

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- $2W$ bits enough to represent $\texttt{rand()} \cdot n$

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- Mapping can not be uniform for all n!

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  which is divisible by n
- Every restricted i^{th} interval has $\frac{2^W - \mathcal{R}_n^W}{n} = \lfloor \frac{2^W}{n} \rfloor$ multiples of n

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$$(i+1) \cdot 2^W - \left(i \cdot 2^W + \mathcal{R}_n^W\right) = 2^W - \mathcal{R}_n^W$$

  which is divisible by n
- Every restricted i^{th} interval has $\frac{2^W - \mathcal{R}_n^W}{n} = \lfloor \frac{2^W}{n} \rfloor$ multiples of n
- We can make Multiply-And-Shift uniform by only accepting multiples of n in restricted intervals

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$$\overbrace{\underbrace{0, 1, \ldots, \mathcal{R}_n^W - 1}_{\text{rejected part}}, \underbrace{\mathcal{R}_n^W, \ldots, n, \ldots, 2^W - 1}_{\text{restricted } i^{\text{th}} \text{ interval}}}^{2^W \text{ values}}$$

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- We reject x if $x \bmod 2^W < \mathcal{R}_n^W$

The Algorithm - Sketch

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$\mathcal{R}_n^W \leftarrow 2^W \bmod n$ `/* Compute rejection threshold */`

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$\mathcal{R}_n^W \leftarrow 2^W \mod n$ /* Compute rejection threshold */

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```
R_n^W ← 2^W mod n                    /* Compute rejection threshold */
while true do
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The Algorithm - Sketch

```
R_n^W ← 2^W mod n          /* Compute rejection threshold */
while true do
    x ← rand()
    m ← x · n              /* Use 2W bits for representation */
```

The Algorithm - Sketch

$\mathcal{R}_n^W \leftarrow 2^W \bmod n$ /* Compute rejection threshold */
while *true* **do**
    $x \leftarrow \texttt{rand()}$
    $m \leftarrow x \cdot n$ /* Use $2W$ bits for representation */
    $l \leftarrow m \ \& \ (2^W - 1)$ /* $m \bmod 2^W$ */

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$\mathcal{R}_n^W \leftarrow 2^W \bmod n$ `/* Compute rejection threshold */`

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    $x \leftarrow$ `rand()`

    $m \leftarrow x \cdot n$ `/* Use` $2W$ `bits for representation */`

    $l \leftarrow m \ \& \ (2^W - 1)$ `/*` $m \bmod 2^W$ `*/`

    if $l \geq \mathcal{R}_n^W$ **then** `/* Apply rejection rule */`

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  $x \leftarrow$ `rand()`
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  $l \leftarrow m \ \& \ (2^W - 1)$ /* $m \bmod 2^W$ */
  if $l \geq \mathcal{R}_n^W$ **then** /* Apply rejection rule */
    return $m \gg W$

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Consider the first iteration of the loop:

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Consider the first iteration of the loop:

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- But we know $\mathcal{R}_n^W < n$ $\longrightarrow$ if $l \geq n$ we do not need to know $\mathcal{R}_n^W$

The Algorithm - Avoiding Division

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x ← rand()
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l ← m & (2^W − 1)                  /* m mod 2^W */
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End of Talk