Seminar Algorithms for Big Data

Fast Random Integer Generation in an Interval

Based on a paper of the same title by Daniel Lemire

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29th February 2024 · Algorithm Engineering (Prof. Dr. Ulrich Meyer)

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- Shuffling
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- Sampling

TBD TBD TBD

Table of Contents

1 Preliminaries
- Formal Definition
- Operations
- The Naive Approach

2 Unbiased Algorithms
- The OpenBSD Algorithm
- The Java Algorithm
- The Bitmask Algorithm

3 Lemire's Algorithm
- Multiply-And-Shift
- The Algorithm

4 Conclusion

1 Preliminaries

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- Set $n = b - a$ and draw a uniform random integer $x \in [0, n)$
- Return $x + a \in [a, b)$

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- Bitwise-And: $x \;\&\; y$

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Definition (Power Remainder)

For $W, n \in \mathbb{N}$, we write $\mathcal{R}_n^W$ for $2^W \bmod n$.

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Is the generated number uniform in $[0, n)$?

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Idea: Use rejection sampling to achieve uniformity!

2 Unbiased Algorithms

The OpenBSD Algorithm

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- Only numbers in leftover interval mapped to $2^W - \mathcal{R}_n^W > 2^W - n$

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- Algorithm:
  (1) Draw $x \in [0, 2^W)$ and compute $r = x \bmod n$
  (2) Return r if $x - r > 2^W - n$ else goto **(1)**

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- Happens with probability $\frac{2^W - \mathcal{R}_n^W}{2^W} > \frac{1}{2}$

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- Number of integer divisions operations equal to number of rounds
- Return number in round if $x < 2^W - \mathcal{R}_n^W$
- Happens with probability $\frac{2^W - \mathcal{R}_n^W}{2^W} > \frac{1}{2}$
- Expected number of integer division operations is $\frac{2^W}{2^W - \mathcal{R}_n^W} < 2$

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$$n \xrightarrow{\text{binary}} \underbrace{\overbrace{0}^{2^{W-1}}, \ldots, 0}_{\text{only 0's}}, \overbrace{1}^{2^{\lfloor \log_2 n \rfloor}}, \underbrace{{}^0\!/_1, \ldots, \overbrace{{}^0\!/_1}^{2^1}, \overbrace{{}^0\!/_1}^{2^0}}_{\text{series of 0's and 1's}}$$

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$$n \xrightarrow{\text{binary}} \underbrace{\overbrace{0}^{2^{W-1}}, \dots, 0,}_{\text{only 0's}} \overbrace{1}^{2^{\lfloor \log_2 n \rfloor}}, \underbrace{{}^0/_1, \dots, \overbrace{{}^0/_1}^{2^1}, \overbrace{{}^0/_1}^{2^0}}_{\text{series of 0's and 1's}}$$

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- Get these bits with a bitwise-AND with

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- Roughly as expensive as a `div` instruction

Lemire's Algorithm

Multiply-And-Shift

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- Mapping is deterministic!
- Mapping can not be uniform for all n!

The Algorithm

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Conclusion

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Lemire	$\frac{n}{2^W}$	1	✓

End of Talk