

Seminar Algorithms for Big Data

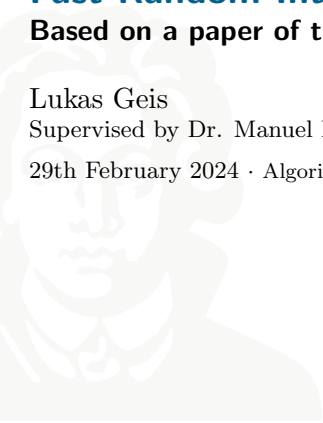
Fast Random Integer Generation in an Interval

Based on a paper of the same title by Daniel Lemire

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Supervised by Dr. Manuel Penschuck

29th February 2024 · Algorithm Engineering (Prof. Dr. Ulrich Meyer)



What is our goal?



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We want to *efficiently* draw a *uniform* random integer in an interval.



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Where do we need this?

- Shuffling
- Complex Graph Generators
- Sampling

TBD

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1 Preliminaries





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- Set $n = b - a$ and draw a uniform random integer $x \in [0, n)$
- Return $x + a$



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Definition (Power Remainder)

For $W, n \in \mathbb{N}$, we write \mathcal{R}_n^W for $2^W \bmod n$.

The Naive Approach



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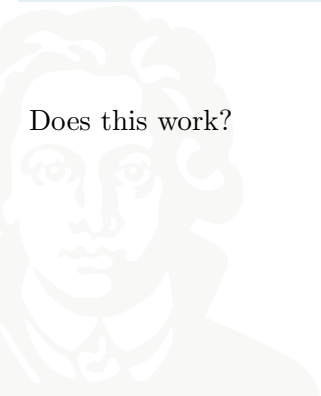
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Is the generated number uniform in $[0, n)$?

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2^W values

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Idea: Use **rejection sampling** to achieve uniformity!

2

Unbiased Algorithms



The OpenBSD Algorithm



The OpenBSD Algorithm

- We shift the **rejection interval** to the left:



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2^W values
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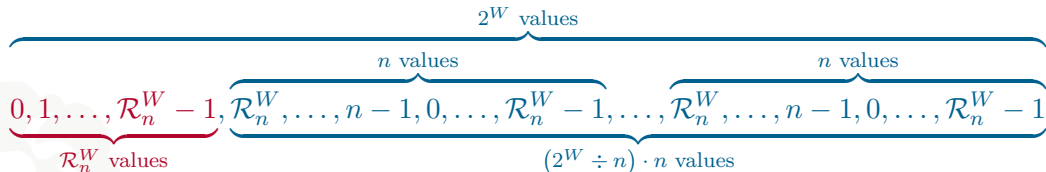
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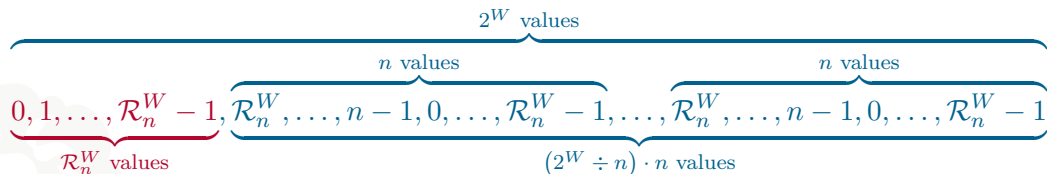


- Algorithm:

- Generate a uniform random number $x \in [0, 2^W)$ until $x \geq \mathcal{R}_n^W$

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Can we do better?

The Java Algorithm



The Java Algorithm

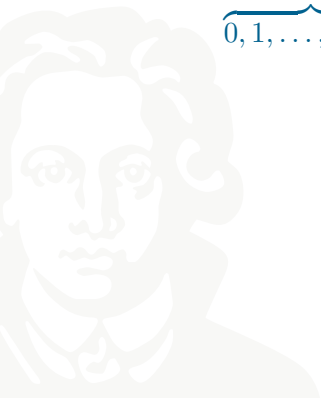
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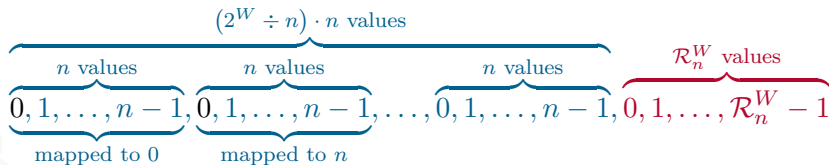
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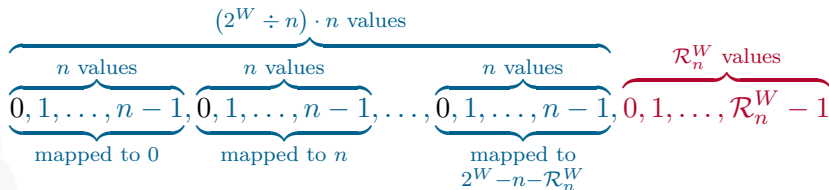
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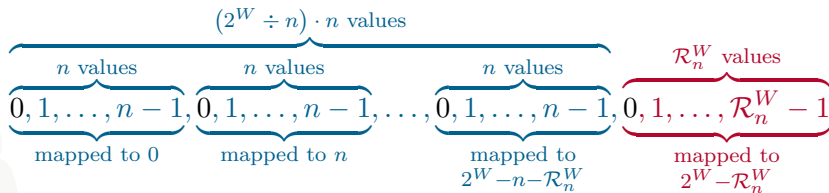
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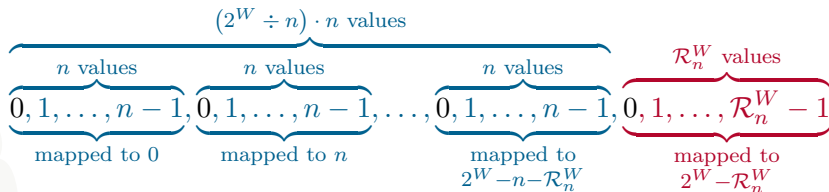
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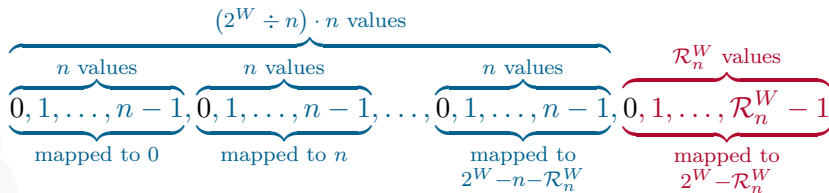
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- Map every number to the next-smallest multiple of n

The Java Algorithm

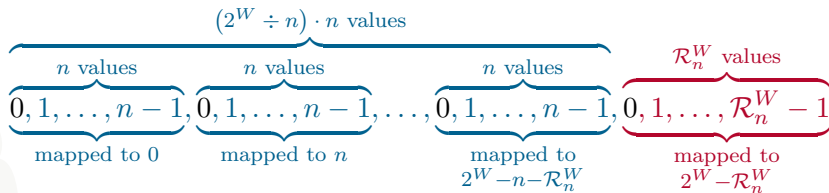
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- Map every number to the next-smallest multiple of n
- Only numbers in **leftover** interval mapped to $2^W - \mathcal{R}_n^W > 2^W - n$

The Java Algorithm

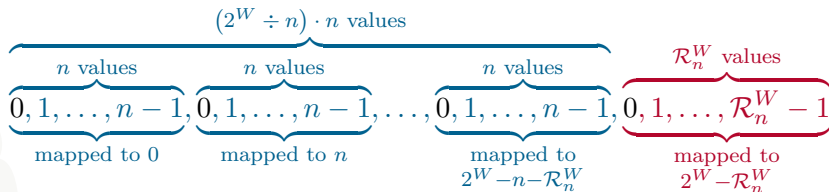
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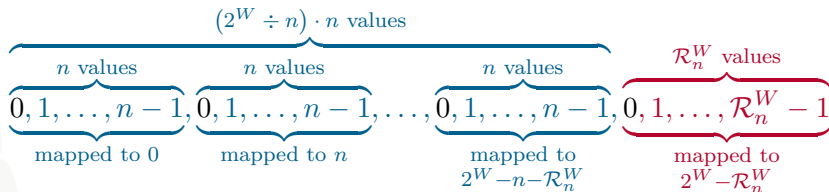
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Efficiency

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Efficiency

- At least one integer division operation

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Efficiency

- At least one integer division operation
- Number of integer divisions operations equal to number of rounds

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- At least one integer division operation
- Number of integer divisions operations equal to number of rounds
- Return number in round if $x < 2^W - \mathcal{R}_n^W$
- Happens with probability $\frac{2^W - \mathcal{R}_n^W}{2^W} > \frac{1}{2}$

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- At least one integer division operation
- Number of integer divisions operations equal to number of rounds
- Return number in round if $x < 2^W - \mathcal{R}_n^W$
- Happens with probability $\frac{2^W - \mathcal{R}_n^W}{2^W} > \frac{1}{2}$
- Expected number of integer division operations is $\frac{2^W}{2^W - \mathcal{R}_n^W} < 2$

The Bitmask Algorithm



The Bitmask Algorithm

- Consider the **binary** representation of n :



The Bitmask Algorithm

- Consider the binary representation of n :

$$\begin{array}{c}
 n \xrightarrow{\text{binary}} \underbrace{\overbrace{0, \dots, 0}^{2^W - 1}}_{\text{only 0's}}, \underbrace{1}_{2^{\lfloor \log_2 n \rfloor}}, \underbrace{0/1, \dots, 0/1}_{\text{series of 0's and 1's}}^{\overbrace{2^1} \quad \overbrace{2^0}}
 \end{array}$$



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- Every number $x \leq n$ only needs the last $\lfloor \log_2 n \rfloor + 1$ bits
- Get these bits with a bitwise-AND with

$$2^{\lfloor \log_2 n \rfloor + 1} - 1 \xrightarrow{\text{binary}} \underbrace{\overbrace{0, \dots, 0}^{2^{W-1}}}_{\text{only 0's}}, \underbrace{\overbrace{1}^{2^{\lfloor \log_2 n \rfloor}}, 1, \dots, \overbrace{1}^{2^1}, \overbrace{1}^{2^0}}_{\text{only 1's}}$$

The Bitmask Algorithm



The Bitmask Algorithm

- How can we compute $2^{\lfloor \log_2 n \rfloor + 1}$?



The Bitmask Algorithm

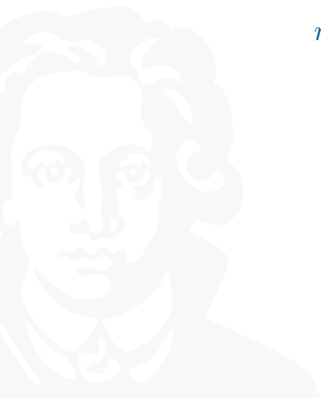
- How can we compute $2^{\lfloor \log_2 n \rfloor + 1}$?
- Count the number ℓ of leading 0's!



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 n \xrightarrow{\text{binary}} \underbrace{\overbrace{0, \dots, 0}^{2^{W-1}}}_{\ell \text{ 0's}}, \overbrace{1}^{2^{\lfloor \log_2 n \rfloor}}, \underbrace{0/1, \dots, 0/1}_{\text{series of 0's and 1's}}^{\overbrace{2^1} \quad \overbrace{2^0}}
 \end{array}$$



- $\lfloor \log_2 n \rfloor = W - \ell - 1$

- $\lceil \log_2 n \rceil = W - \ell - 1$

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The Bitmask Algorithm

- How can we compute $2^{\lfloor \log_2 n \rfloor + 1}$?
- Count the number ℓ of leading 0's!

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- No integer division at all
- Computation of leading 0's requires `clz` instruction/algorithm
- Roughly as expensive as a `div` instruction

Lemire's Algorithm



Multiply-And-Shift



Multiply-And-Shift

- Map `rand()` to $[0, n)$ divisionless with $(\text{rand}() \cdot n) \gg W$:



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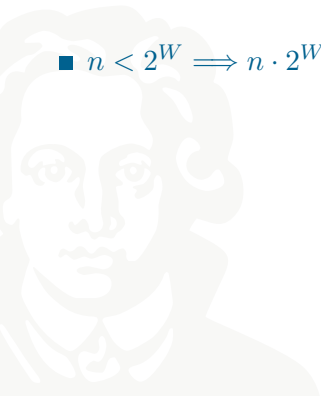


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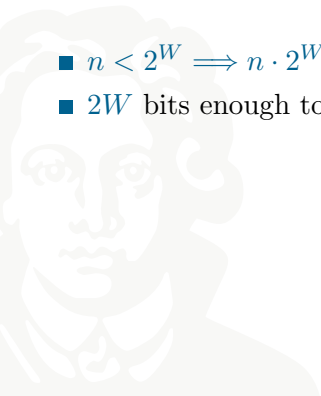


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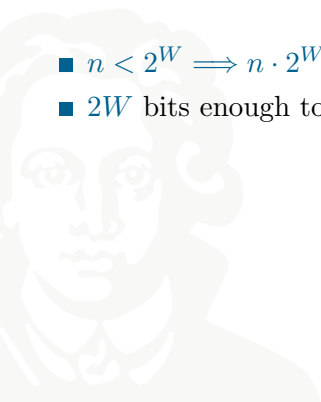


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Is this uniform?

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Is this uniform?

- Mapping is deterministic!
- Mapping can **not** be uniform for all n !

Lemire's Algorithm

The Algorithm



Lemire's Algorithm

The Algorithm



4 Conclusion



Conclusion

Summary



expected number of integer division operations	maximum number of integer division operations	Unbiased?
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Modulo Reduction	1	1	X



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Bitmask	0	0	✓
Lemire	$\frac{n}{2^W}$	1	✓

End of Talk

