Seminar Algorithms for Big Data

Fast Random Integer Generation in an Interval

Based on a paper of the same title by Daniel Lemire

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Supervised by Dr. Manuel Penschuck

29th February 2024 · Algorithm Engineering (Prof. Dr. Ulrich Meyer)

Motivation

What is the problem?

0

Motivation

What is the problem?

$$1$$

Motivation

What is the problem?

0

↑

uniform random bit

Motivation

What is the problem?

0

↑

uniform random bit

↑ ↑

$0 \sim 50\%$ $1 \sim 50\%$

Motivation

What is the problem?

$0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0$

What is the problem?

$$1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1$$

$W = 8$ independent uniform bits

Motivation

What is the problem?

1 1 0 1 0 0 0 1

$W = 8$ independent uniform bits

Goal:

Generate a uniform integer between 100 and 200

Motivation

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1 1 0 1 0 0 0 1

interpret as unsigned 8-bit integer

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What is the problem?

2^0

↓

1 1 0 1 0 0 0 1

interpret as unsigned 8-bit integer

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What is the problem?

$$\begin{array}{cccccccc} & & & & & & 2^1 & 2^0 \\ & & & & & & \downarrow & \downarrow \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}$$

interpret as unsigned 8-bit integer

Goal:

Generate a uniform integer between 100 and 200

What is the problem?

$$\begin{array}{cccccccc} & & & & & 2^2 & 2^1 & 2^0 \\ & & & & & \downarrow & \downarrow & \downarrow \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}$$

interpret as unsigned 8-bit integer

Goal:

Generate a uniform integer between 100 and 200

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What is the problem?

$$\begin{array}{cccccccc} 2^7 & 2^6 & 2^5 & 2^4 & 2^3 & 2^2 & 2^1 & 2^0 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}$$

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What is the problem?

209 in binary

$$\begin{array}{cccccccc} 2^7 & 2^6 & 2^5 & 2^4 & 2^3 & 2^2 & 2^1 & 2^0 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}$$

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interpret as uniform 8-bit integer in $[0, 2^8)$

Goal:

Generate a uniform integer between 100 and 200

Table of Contents

1

Preliminaries

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Formal Definition

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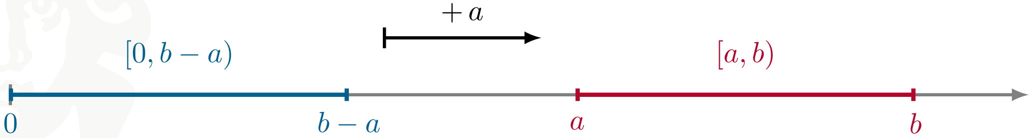

Preliminaries

Operations

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Definition (Common Operations)

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- Integer-Division: $x \div y \quad := \left\lfloor {}^{x}/_{y} \right\rfloor$

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- Bitwise-And: $x \ \& \ y$

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- Bitwise-And: $x \ \& \ y \rightarrow x \bmod 2^W := x \ \& \ (2^W - 1)$

Definition (Power Remainder)

For $W, n \in \mathbb{N}$, we write $\mathcal{R}$ for $2^W \bmod n$.

Preliminaries

The Naive Approach

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$$\texttt{rand()} \bmod n$$

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Is the generated number uniform in $[0, n)$?

Preliminaries

The Naive Approach - Bias

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Every deterministic mapping $f\colon [0, 2^W) \to [0, n)$

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Every deterministic mapping $f\colon [0, 2^W) \to [0, n)$ does not generate uniform random integers in one step

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Deterministic Mappings

Every deterministic mapping $f\colon [0, 2^W) \to [0, n)$ does not generate uniform random integers in one step whenever n does not divide 2^W.

2

Unbiased Algorithms

Unbiased Algorithms

The OpenBSD Algorithm

The OpenBSD Algorithm

- Shift the rejection interval to the left:

The OpenBSD Algorithm

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  - Generate a uniform random number $x \in [0, 2^W)$ until $x \geq \mathcal{R}$

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The OpenBSD Algorithm - Efficiency

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We require 2 integer division operations:

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Efficiency

We require 2 integer division operations:

- one for computing $\mathcal{R}$
- and one for computing $x \bmod n$.

The Java Algorithm

Unbiased Algorithms

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- Map every number to the next-smallest multiple of n

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- Map every number to the next-smallest multiple of n
- Only numbers in leftover interval mapped to $2^W - \mathcal{R} > 2^W - n$

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  (1) Draw $x \in [0, 2^W)$ and compute $r = x \bmod n$

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  (1) Draw $x \in [0, 2^W)$ and compute $r = x \bmod n$
  (2) Return r if $x - r \leq 2^W - n$ else goto **(1)**

The Java Algorithm - Efficiency

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- At least one integer division operation
- Number of integer divisions operations equal to number of rounds

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- At least one integer division operation
- Number of integer divisions operations equal to number of rounds
- Return number in round if $x < 2^W - \mathcal{R}$

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- At least one integer division operation
- Number of integer divisions operations equal to number of rounds
- Return number in round if $x < 2^W - \mathcal{R}$
- Happens with probability $\frac{2^W - \mathcal{R}}{2^W} > \frac{1}{2}$

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Efficiency

- At least one integer division operation
- Number of integer divisions operations equal to number of rounds
- Return number in round if $x < 2^W - \mathcal{R}$
- Happens with probability $\frac{2^W - \mathcal{R}}{2^W} > \frac{1}{2}$
- Expected number of integer division operations is $\frac{2^W}{2^W - \mathcal{R}} < 2$

The Bitmask Algorithm - Representation

Unbiased Algorithms

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- Consider the binary representation of n:

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$$n \xrightarrow{\text{binary}} \underbrace{\overset{\overset{2^{W-1}}{\downarrow}}{0},\dots,0}_{\text{only 0's}},\overset{\overset{2^{\lfloor \log_2 n \rfloor}}{\downarrow}}{1},\underbrace{1,\dots,\overset{\overset{2^1}{\downarrow}}{0},\overset{\overset{2^0}{\downarrow}}{1}}_{\substack{\text{series of 0's}\\\text{and 1's}}}$$

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- Every number $x \leq n$ only needs the last $\lfloor \log_2 n \rfloor + 1$ bits

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- Every number $x \leq n$ only needs the last $\lfloor \log_2 n \rfloor + 1$ bits
- Get these bits with a bitwise-AND with

$$2^{\lfloor \log_2 n \rfloor + 1} - 1 \xrightarrow{\text{binary}} \underbrace{\overset{\substack{2^{W-1}\\ \downarrow}}{0}, \dots, 0}_{\text{only 0's}}, \underbrace{\overset{\substack{2^{\lfloor \log_2 n \rfloor}\\ \downarrow}}{1}, 1, \dots, \overset{\substack{2^1\\ \downarrow}}{1}, \overset{\substack{2^0\\ \downarrow}}{1}}_{\text{only 1's}}$$

The Bitmask Algorithm - Mask

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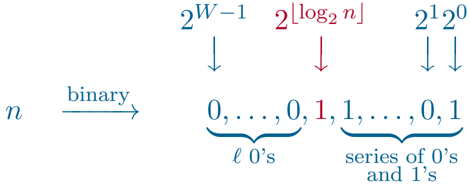

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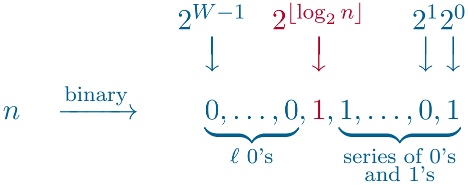

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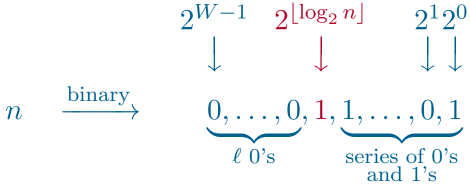

2^{W-1} $2^{\lfloor \log_2 n \rfloor}$ $2^1 2^0$

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The Bitmask Algorithm - Efficiency

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- Roughly as expensive as a `div` instruction

3

Lemire's Algorithm

Lemire's Algorithm

Multiply-And-Shift

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- Map `rand()` to $[0, n)$ divisionless with $(\texttt{rand()} \cdot n) \gg W$:

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- Mapping can not be uniform for all n!

The Algorithm - Intervals

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  which is divisible by n
- Every restricted i^{th} interval has $\frac{2^W - \mathcal{R}}{n} = \lfloor \frac{2^W}{n} \rfloor$ multiples of n
- We can make Multiply-And-Shift uniform by only accepting multiples of n in restricted intervals

Lemire’s Algorithm

The Algorithm - Rejection

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- We reject x if $x \bmod 2^W < \mathcal{R}$

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$\mathcal{R} \leftarrow 2^W \bmod n$                /* Compute rejection threshold */
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$\mathcal{R} \leftarrow 2^W \bmod n$            /* Compute rejection threshold */
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```
$\mathcal{R} \leftarrow 2^W \bmod n$                    /* Compute rejection threshold */
while $true$ do
    $x \leftarrow$ rand()
    $m \leftarrow x \cdot n$                             /* Use $2W$ bits for representation */
```

The Algorithm - Sketch

```
$\mathcal{R} \leftarrow 2^W \mod n$                /* Compute rejection threshold */
while $true$ do
    $x \leftarrow$ rand()
    $m \leftarrow x \cdot n$                       /* Use $2W$ bits for representation */
    $l \leftarrow m \ \& \ (2^W - 1)$              /* $m \mod 2^W$ */
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    if $l \geq \mathcal{R}$ then                /* Apply rejection rule */
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        return $m \gg W$
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The Algorithm - Avoiding Division

Lemire’s Algorithm

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Consider the first iteration of the loop:

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Consider the first iteration of the loop:

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- But we know $\mathcal{R} < n$ $\longrightarrow$ if $l \geq n$ we do not need to know $\mathcal{R}$

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- If $l \geq n$, we accept x without computing $\mathcal{R}$

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With what probability do we need to compute $\mathcal{R}$:

The Algorithm - Avoiding Division

Consider the first iteration of the loop:

- We reject x if $l < \mathcal{R}$ ⟶ we need to compute $\mathcal{R}$ beforehand
- But we know $\mathcal{R} < n$ ⟶ if $l \geq n$ we do not need to know $\mathcal{R}$

We can alter the first iteration of the loop:

- We do not compute $\mathcal{R}$ beforehand
- If $l \geq n$, we accept x without computing $\mathcal{R}$
- If not, we compute $\mathcal{R}$ and proceed as before

With what probability do we need to compute $\mathcal{R}$:

- We assume x to be uniform in $[0, 2^W)$

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- We compute $\mathcal{R}$ if $l < n$ ⟶ happens with probability $\frac{n}{2^W}$

The Algorithm - Pseudocode

Lemire's Algorithm

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$x \leftarrow$ rand()
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4

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Lemire	$\frac{n}{2^W}$	1	✓

End of Talk