Seminar Algorithms for Big Data

Fast Random Integer Generation in an Interval

Based on a paper of the same title by Daniel Lemire

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29th February 2024 · Algorithm Engineering (Prof. Dr. Ulrich Meyer)

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- Sampling

TBD TBD TBD

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Preliminaries

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- Set $n = b - a$ and draw a uniform random integer $x \in [0, n)$
- Return $x + a$

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Definition (Power Remainder)

For $W, n \in \mathbb{N}$, we write $\mathcal{R}_n^W$ for $2^W \bmod n$.

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Is the generated number uniform in $[0, n)$?

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Idea: Use rejection sampling to achieve uniformity!

2 Unbiased Algorithms

The OpenBSD Algorithm

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  (1) Draw $x \in [0, 2^W)$ and compute $r = x \bmod n$
  (2) Return r if $x - r > 2^W - n$ else goto **(1)**

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- At least one integer division operation
- Number of integer divisions operations equal to number of rounds

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- Return a number in round if $x < 2^W - \mathcal{R}_n^W$
- Happens with probability $\frac{2^W - \mathcal{R}_n^W}{2^W} > \frac{1}{2}$
- Expected number of integer division operations is $\frac{2^W}{2^W - \mathcal{R}_n^W} < 2$

The Bitmask Algorithm

The Bitmask Algorithm

Lemire's Algorithm

Multiply-And-Shift

Multiply-And-Shift

The Algorithm

The Algorithm

4 Conclusion

Summary

Summary

	expected number of integer division operations	maximum number of integer division operations	Unbiased?

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Lemire	$\frac{n}{2^W}$	1	✓

End of Talk