Seminar Algorithms for Big Data

Fast Random Integer Generation in an Interval

Based on a paper of the same title by Daniel Lemire

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Supervised by Dr. Manuel Penschuck

29th February 2024 · Algorithm Engineering (Prof. Dr. Ulrich Meyer)

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1	2	3	4	5

$\Downarrow$

4	2	1	5	3

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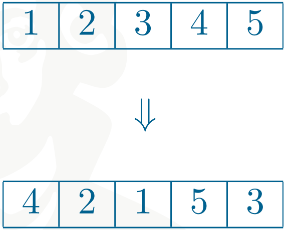
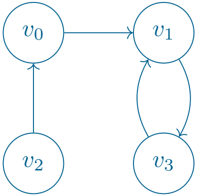
- Graph Generators





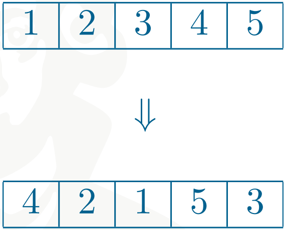
$\mathcal{G}(4,4)$

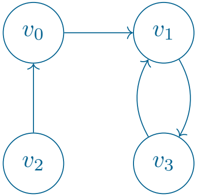
What is our goal?

We want to efficiently draw a uniform random integer in an interval.

Where do we need this?

- Shuffling
- Graph Generators
- Sampling





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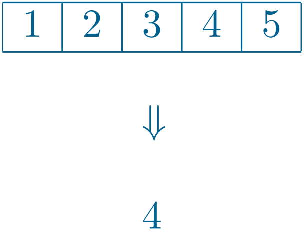

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1

Preliminaries

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- Set $n = b - a$ and draw a uniform random integer $x \in [0, n)$
- Return $x + a \in [a, b)$

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Definition (Power Remainder)

For $W, n \in \mathbb{N}$, we write $\mathcal{R}_n^W$ for $2^W \bmod n$.

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Is the generated number uniform in $[0, n)$?

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Idea: Use rejection sampling to achieve uniformity!

2

Unbiased Algorithms

The OpenBSD Algorithm

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  (2) Return r if $x - r \leq 2^W - n$ else goto **(1)**

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- At least one integer division operation
- Number of integer divisions operations equal to number of rounds

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(1) Draw $x \in [0, 2^W)$ and compute $r = x \bmod n$

(2) Return r if $x - r \leq 2^W - n$ else goto **(1)**

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- At least one integer division operation
- Number of integer divisions operations equal to number of rounds
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- Expected number of integer division operations is $\frac{2^W}{2^W - \mathcal{R}_n^W} < 2$

The Bitmask Algorithm - Representation

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$$n \xrightarrow{\text{binary}} \underbrace{\overbrace{0}^{2^{W-1}}, \ldots, 0,}_{\text{only 0's}} \overbrace{1}^{2^{\lfloor \log_2 n \rfloor}}, \underbrace{0/1, \ldots, \overbrace{0/1}^{2^1}, \overbrace{0/1}^{2^0}}_{\text{series of 0's and 1's}}$$

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- Every number $x \leq n$ only needs the last $\lfloor \log_2 n \rfloor + 1$ bits
- Get these bits with a bitwise-AND with

$$2^{\lfloor \log_2 n \rfloor + 1} - 1 \xrightarrow{\text{binary}} \underbrace{\overbrace{0}^{2^{W-1}}, \ldots, 0,}_{\text{only 0's}} \underbrace{\overbrace{1}^{2^{\lfloor \log_2 n \rfloor}}, 1, \ldots, \overbrace{1}^{2^1}, \overbrace{1}^{2^0}}_{\text{only 1's}}$$

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- Roughly as expensive as a `div` instruction

Lemire's Algorithm

Multiply-And-Shift

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  which is divisible by n
- Every restricted i^{th} interval has $\frac{2^W - \mathcal{R}_n^W}{n} = \lfloor \frac{2^W}{n} \rfloor$ multiples of n
- We can make Multiply-And-Shift uniform by only accepting multiples of n in restricted intervals

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$$\overbrace{\underbrace{0, 1, \ldots, \mathcal{R}_n^W - 1}_{\text{rejected part}}, \underbrace{\mathcal{R}_n^W, \ldots, n, \ldots, 2^W - 1}_{\text{restricted } i^{\text{th}} \text{ interval}}}^{2^W \text{ values}}$$

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- We reject x if $x \bmod 2^W < \mathcal{R}_n^W$

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$\mathcal{R}_n^W \leftarrow 2^W \bmod n$ `/* Compute rejection threshold */`

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R_n^W ← 2^W mod n            /* Compute rejection threshold */
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R_n^W ← 2^W mod n                      /* Compute rejection threshold */
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    x ← rand()
    m ← x · n                          /* Use 2W bits for representation */
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$m \leftarrow x \cdot n$ /* Use $2W$ bits for representation */

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R_n^W ← 2^W mod n                    /* Compute rejection threshold */
while true do
    x ← rand()
    m ← x · n                        /* Use 2W bits for representation */
    l ← m & (2^W − 1)                /* m mod 2^W */
```

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$\mathcal{R}_n^W \leftarrow 2^W \bmod n$ `/* Compute rejection threshold */`

while $true$ **do**

  $x \leftarrow$ `rand()`

  $m \leftarrow x \cdot n$ `/* Use` $2W$ `bits for representation */`

  $l \leftarrow m \;\&\; (2^W - 1)$ `/*` $m \bmod 2^W$ `*/`

  if $l \geq \mathcal{R}_n^W$ **then** `/* Apply rejection rule */`

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  $l \leftarrow m \ \& \ (2^W - 1)$ /* $m \bmod 2^W$ */

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    return $m \gg W$

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Consider the first iteration of the loop:

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- We reject x if $l < \mathcal{R}_n^W$ $\longrightarrow$ we need to compute $\mathcal{R}_n^W$ beforehand

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m ← x · n                    /* Use 2W bits for representation */
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m ← x · n                          /* Use 2W bits for representation */
l ← m & (2^W − 1)                  /* m mod 2^W */
if l < n then                      /* Possibly skip division */
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End of Talk