Sequential Pattern Retrieval in the Hopfield Model

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The Hopfield model is a remarkably simple model of associative memory that abstracts away much of the confounding biological complexity. However, this model fails in one crucial aspect of associative memory. It is unable to retrieve sequences of patterns when endowed with asymmetric synaptic connections. At some point in the transition between patterns an instability occurs and the network settles on a combination of multiple patterns. The cause of the instability is investigated analytically for the case of three patterns and is then generalized to many patterns. The results of the analysis is verified through simulation, and two modifications which allow for sequential pattern recall are reviewed and tested through simulation.

I. INTRODUCTION

The Hopfield model, originally proposed in [1], is an associative memory model which makes extreme biological simplification in favor of mathematical and physical beauty and simplicity. The model consists of a set of fully connected neurons whose synaptic connections (weights) are determined by the patterns that are to be stored in the network. The state of the network at a given time is denoted by $S^{(t)} = \{s_1, s_2, \ldots, s_N\}$ with neuron $s_i = \pm 1$. Each of the neuron in the network is connected to every other neuron by the weight matrix whose entries are

$$J_{ij} = \frac{1}{N} \sum_\mu \xi_i^\mu \xi_j^\mu, \tag{1}$$

where J_{ij} is the synaptic connection between neurons i and j in pattern ξ^μ, and the diagonal entries of the matrix are set to zero to avoid self connections. Connecting the weights in this manner creates a basin of attraction about each pattern, and the evolution of the network at a given time will be towards the pattern with the smallest Hamming distance to the network. Thus, beginning the network closest to pattern ξ^μ will ensure $S^{(t)} = \xi^\mu$ at some future t. More explicitly, the network evolves according to the equation

$$S_i^{(t+1)} = \text{sgn}\left(\sum_j J_{ij} S_j^{(t)} + \theta\right), \tag{2}$$

where θ is the activation threshold of the neurons, which is taken to be zero unless explicitly stated. The evolution of this equation may be realized asynchronously through a Monte-Carlo simulation, or synchronously where at each time step the every neuron in the network is updated.

In [1] Hopfield suggests that adding an asymmetric coupling between neurons, defined as

$$\tilde{J}_{ij} = \frac{\lambda}{N} \sum_\mu \xi_i^{\mu+1} \xi_j^\mu, \tag{3}$$

which Eq. 2 is modified to include as

$$S_i^{(t+1)} = \text{sgn}\left(\sum_j \left(J_{ij} + \tilde{J}_{ij}\right) S_j^{(t)} + \theta\right), \tag{4}$$

could induce transitions between patterns ξ^μ and $\xi^{\mu+1}$. The asymmetric weight matrix also has it's diagonal elements set to zero, and $\mu + 1$ is calculated modulo the number of patterns p. As it turns out, $\lambda > 1$ is sufficient to induce a transition.

However, as Hopfield briefly remarks, the transitions are increasingly unstable as more patterns are added in the sequence. This paper aims to analytically determine why the asymmetric couplings $\tilde{J}_{ij}$ are unable to produce stable patterns. Only the case of asynchronous updating is considered, as synchronous updating is able to produce transitions between patterns. The results of the analysis are then justified through simulation. Subsequently, two variations of the Hopfield network which succeed in generating stable temporal sequences are overviewed, and their results are verified through simulation.

II. STABILITY OF ASYMMETRIC COUPLINGS IN A THREE PATTERN HOPFIELD MODEL

Before we begin, it will be beneficial to define a useful quantity which forms the basis of the analysis. The overlap between pattern ξ^μ and the network $S^{(t)}$ is defined as

$$C^{\mu,(t)} = \frac{1}{N} \sum_i \xi_i^\mu S_i^{(t)}, \tag{5}$$

and $C^{\mu,(t)}$ ranges from -1, indicating perfect overlap with the inverse pattern $-\xi^\mu$, to +1 indicating perfect overlap with ξ^μ. When $C^{\mu,(t)} = 0$ the system is completely uncorrelated with ξ^μ.

Consider a network of three uncorrelated patterns ξ^0, ξ^1, and ξ^2. Suppose that the weights of the network S have been constructed in the manner discussed above, and that the network $S^{(t)}$ is in the process of transitioning from ξ^0 to ξ^1. Using Eqs. 1, 3, and 5, Eq. 4 may be

expressed as

$$S_i^{(t+1)} = \text{sgn}\left(\sum_\mu \xi_i^{\mu+1}\left(C^{\mu+1,(t)} + \lambda C^{\mu,(t)}\right)\right). \quad (6)$$

The transition of the network from one pattern to another can be broken up into three sections: (i) When the network is leaving the initial pattern, (ii) when the network is approximately midway between patterns, and (iii) when the network is approaching it's destination pattern. Only (i) and (ii) are considered, as the behavior during (ii) is guaranteed to generate an instability. Case (i) is included for completeness.

A. Initial Transition

During (i), when the network is nearby pattern ξ^0, the overlaps of the network are such that $C^{0,(t)} \gg C^{1,(t)} \approx C^{2,(t)}$ since all of the patterns are uncorrelated. Since the neurons are binary variables there are 2^3 ways of arranging the values of the i-th value in each pattern, and because the patterns are uncorrelated each arrangement is expected to occur with equal probability. Moreover, since the value of $S_i^{(t+1)}$ is determined by the sign of a linear combination of ξ_i^μ, and $\text{sgn}(x)$ is an odd function, only 2^2 arrangements need be considered. The discarded arrangements are those which are the inverse of some other arrangement. Explicitly enumerating the arrangements that must be considered yields:

$$\xi_i^0 = \xi_i^1 = \xi_i^2, \quad (7a)$$

$$\xi_i^0 = \xi_i^1 = -\xi_i^2, \quad (7b)$$

$$\xi_i^0 = -\xi_i^1 = \xi_i^2, \quad (7c)$$

$$-\xi_i^0 = \xi_i^1 = \xi_i^2. \quad (7d)$$

When (i) holds, each of Eqs. 7 gives the expected result of $S_i^{(t+1)} = \xi_i^1$ when Eq. 6 is evaluated. Additionally, explicitly evaluating Eq. 6 for each of Eqs. 7, indicates that $S_i^{(t)}$ makes no net change towards or away from ξ_i^2. From this we may conclude that so long as the system overlaps only with the pattern that is transitioning away from to a high degree, the system will evolve towards only the next pattern in the sequence determined by the asymmetric couplings.

B. Approaching the Midpoint

As the network considered in case (i) continues to evolve it's overlap with ξ^1 will increase. That is to say that $C^{1,(t)}$ cannot be considered negligible as it was in case (i). For case (ii) we adopt the relationship

$$C^{1,(t)} = C^{0,(t)} + \varepsilon, \quad (8a)$$

$$C^{2,(t)} = \delta, \quad (8b)$$

which may be interpreted as $C^{0,(t)}$ and $C^{1,(t)}$ being separated by some amount ε, and $C^{2,(t)}$ being some small, random overlap δ. Letting $C^{0,(t)} = C$, we find that for Eqs. 7a, 7b, and 7d the quantity $C * \xi_i^1$ dominates the argument of Eq. 6, and so $S_i^{(t+1)} = \xi_i^1$. For 7a and 7d and $C^{2,(t)}$ is increasing while for 7b $C^{2,(t)}$ is decreasing.

This trajectory of the network when 7c applies is not so clear. Substituting Eq. 7c and Eqs. 8 into Eq. 6, we find that

$$S_i^{(t+1)} = \text{sgn}\left(\varepsilon(\lambda - 1) + \delta(\lambda + 1)\right), \quad (9)$$

taking $\xi_i^0 = 1$ without loss of generality. Thus, the state of $S_i^{(t+1)}$ is dependent on the signs and relative magnitudes of ε and δ. More concretely, the future state of the network is dependent on how close the network is to the midpoint of the transition and which side of the midpoint it is on, as well as the sign and magnitude of the small, random overlap δ.

First, we consider the argument of Eq. 9 as the network approaches the midpoint. In this regime $\varepsilon < 0$ dominates the expression and the network evolves towards ξ_i^1.

Upon closer approach to the midpoint, δ dominates over ε. When $\delta < 0$ the network will evolve towards ξ_i^1 as expected, since the argument of Eq. 9 is negative. In contrast, when $\delta > 0$ the network will evolve towards ξ_i^0 and ξ_i^2 rather than ξ_i^1. Combining this with the results from analyzing Eqs. 7a, 7b, and 7d, we find that the network's overlap with ξ^2 will increase with a probability of 0.75 for a given neuron update in this regime. This occurs over a short time-frame where $|\delta| > |\varepsilon|$ and $\delta > 0$, but is nonetheless enough to destabilize the system by introducing an overlap with ξ_i^2 that effects the evolution of the network. The fact that this is enough to destabilize the network will be verified through simulation.

Finally, when $\varepsilon > 0$ and the system has passes the midpoint, the argument of Eq. 9 is positive so the system will transition towards ξ^0 and ξ^2. And after accounting for the remaining Eqs. 7, the probability of such a transition occurring per network update if 0.75. Thus, the system will inevitably be destabilized during a transition involving three patterns.

III. STABILITY OF ASYMMETRIC COUPLINGS FOR MANY STORED PATTERNS

For a Hopfield model with p random patterns the argument is not generally different from the $p = 3$ case described above. For p patterns there are 2^p ways of configuring $\xi_i^\mu = \pm 1$ over $\mu = 0, \ldots, p-1$. And since half of these configurations are related to another by a change in sign, only 2^{p-1} configurations need be considered.

The initial transition away from where the network starts in no different. The overlap of the network with the starting pattern will dominate and the network will evolve towards the next pattern in the temporal sequence. However, due to there being $p - 2$ small random overlaps, the duration for which the effects of these

overlap may be ignored decreases as more patterns are added.

Once these overlaps become sufficiently large Eqs. 8 and 9 may be modified to include the effects of an arbitrary number of patterns as

$$C^{1,(t)} = C^{0,(t)} + \varepsilon, \tag{10a}$$

$$C^{\mu,(t)} = \delta_\mu, \tag{10b}$$

with $2 \le \mu \le p - 1$, and

$$S_i^{(t+1)} = \mathrm{sgn}\left(\varepsilon(\lambda - 1) + (\lambda + 1) \sum_{\mu=2}^{p-1} \delta_\mu \right). \tag{11}$$

Since all but two of the overlaps are small and random, only the i-th value of three patterns need be considered: ξ^0, ξ^1, ξ^2. So, of all the 2^{p-1} configurations that must be considered, only those where ξ_i^0, ξ_i^1, ξ_i^2 have distinct relationships impact the evolution of the system in distinct ways. And these distance relationships are exactly those described in Eqs. 7. Like the three pattern case, the p pattern case behaves as expected ($S_i^{(t+1)} \to \xi_i^1$) except in the case of $\xi_i^0 = -\xi_i^1 = \xi_i^2$, where Eq. 11 applies.

We are now tasked with determining how the p pattern case behaves under these circumstances. Keeping in mind that δ_μ is a symmetric zero-mean random variable, initially the contribution of the second term in the argument of Eq. 11 is approximately zero. Next, we should examine the statistical fluctuations of δ_μ. Since all of the δ_μ are uncorrelated random variables, the variance of their sum will be the sum of their variances. Thus, we can expect that the fluctuations of the sum of δ_μ will increase as more patterns are stored in the network, which will result in the instability occurring at an earlier time. And when $\varepsilon > 0$ dominates an instability is bound to occur.

The result of this analysis is that the manner in which the instability observed in the many pattern case comes about is not qualitatively different from that of the three pattern case. Since the configuration of values in 7c stipulates that $\xi_i^1 = -\xi_i^2$, the network will certainly move closer to ξ_2 whenever the sum over δ_μ dominates and is greater than zero or $\varepsilon > 0$ dominates in Eq. 11. And furthermore, since only four distinct equally probable configurations of ξ_i^μ determine the fate of the network, the probability of an unstable transition occurring in the regime where the above analysis applies is 0.75. As the number of patterns increases, the impact of δ_μ must be accounted for at earlier times in the transition between patterns.

IV. EXPERIMENTAL VALIDATION

A. Three Patterns

How can we verify that an instability does occur in the manner described previously? Synthesizing all of the
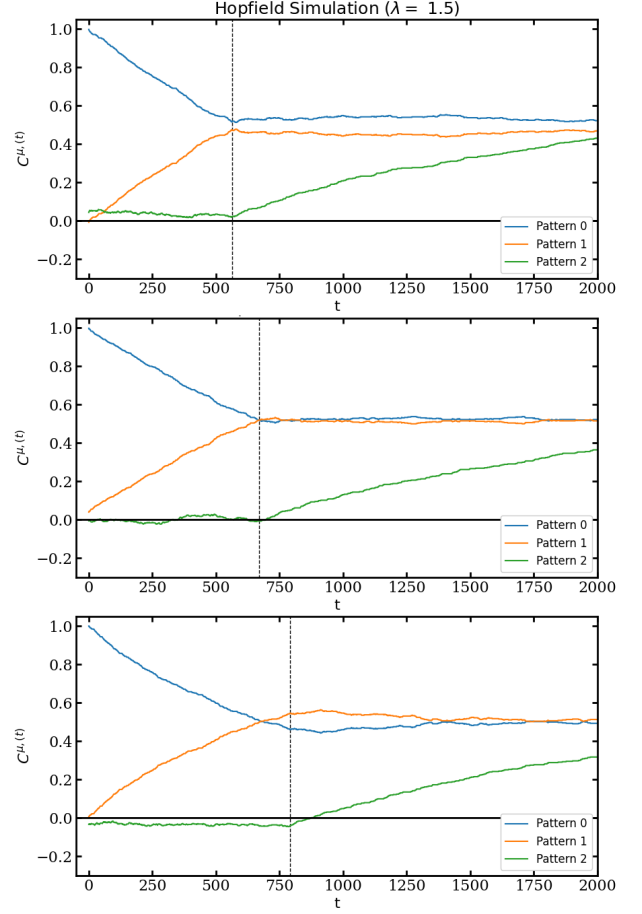

FIG. 1. **Three pattern simulation results** Simulations of a system of three random, uncorrelated patterns is depicted. The three cases shown are when $C^{2,(t)} > 0$ (top), when $C^{2,(t)} \approx 0$ (middle), and when $C^{2,(t)} < 0$ (bottom). Each simulation was performed with an asynchronous updating scheme on a network composed of 900 neurons, and asymmetric couplings of $\lambda = 1.5$ The dashed vertical line marks the approximate time of the instability occurring.

results so far, we may expect that the destabilization will occur before or after the midpoint of the transition depending on the sign of $C^{2,(t)}$, and how far from the midpoint will be determined by the magnitude of $C^{2,(t)}$. More specifically, if $C^{2,(t)} > 0$ the destabilization should occur before the midpoint while if $C^{2,(t)} < 0$ the destabilization should occur after, and the larger $|C^{2,(t)}|$ the further the transition will occur from the midpoint.

This prediction is tested through simulation. A network composed of 900 neurons and three stored patterns is simulated asynchronously. Three cases are tested: $C^{2,(t)} > 0$, $C^{2,(t)} \approx 0$, and $C^{2,(t)} < 0$. The results of the simulation are depicted in Fig. 1. From the results, it is clear that the time of destabilization of the system is dependent on $C^{2,(t)}$. When $C^{2,(t)} > 0$, the system becomes unstable before $C^{1,(t)} = C^{2,(t)}$, while for $C^{2,(t)} > 0$

the instability occurs sometime after $C^{1,(t)} = C^{2,(t)}$, and when $C^{2,(t)} \approx 0$ the instability is coincident with $C^{1,(t)} = C^{2,(t)}$, as predicted.

B. Many Patterns

For the many pattern case the driving force behind when the instability occurs is the number of patterns in the system. And to determine the validity of the many pattern analysis we examine the behavior of $\sum \delta_\mu$ described in Eq. 11 as the network evolves with three different numbers of patterns stored.
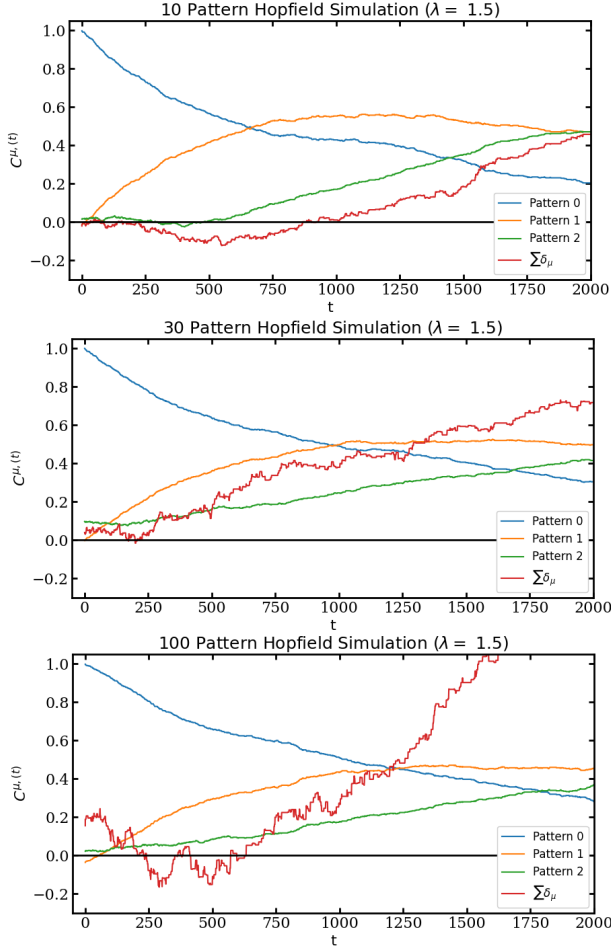

FIG. 2. **Many pattern simulation results** Simulations of a system of 10, 30, and 100 random, uncorrelated patterns is depicted. Each simulation was performed on a network of 900 neurons with asynchronous updating and $= 1.5$. For the $\sum \delta_\mu$ trace the overlap of the network with pattern 2 is included.

Fig. 2 shows the simulation results for a network storing 10, 30, and 100 patterns. As predicted, the variance of the sum of all of the small overlaps increases as the number of patterns stored in the system increases. And

the inferred impact of this increase in variance is also observed; as the number of patterns in the system increases the time period for which the system is stable decreases. For the 10 pattern simulation the transition is stable for longer than both the 30 and 100 pattern cases, and for the 100 pattern simulation the network becomes unstable almost immediately.

Interestingly however, the dynamics of patterns 0, 1, and 2 are more or less the same over the range of patterns tested. The $C^{0,(t)}$ decreasing is coincident with $C^{1,(t)}$ increasing. Then, after a period, $C^{1,(t)}$ increases while $C^{0,(t)}$ and $C^{1,(t)}$ decreases. The reason for this is beyond the scope of this paper, which only analyzes the manner in which the initial instability occurs.

V. MODIFICATIONS TO PRESERVE TEMPORAL STABILITY

The following sections will consist of a brief overview of two ways that the Hopfield network may be modified to include temporal instability. Additionally, the ability of each modification to follow stable temporal sequences is verified through simulation. Only one of the proposed solutions are able to generate temporal sequences for asynchronous updates, but both are able to follow temporal sequences for synchronous updating. This is still an improvement over the traditional Hopfield network which is stable for synchronous updating but is unable to remain at a pattern for a period of time.

A. Time Delayed Synaptic Connections

In [2], Sompolinsky and Kanter propose a small change to the original Hopfield model to produce sequential pattern recall. The authors propose that each neuron in the network should be endowed with both a slow and fast response connection. The fast connections are exactly the same as Eq. 1 and have the effect of stabilizing the network at a stored pattern, while the slow connections takes the form of Eq. 3 and induces for transitions between patterns. The main difference from the traditional Hopfield network is that the inputs into a neuron are calculated differently for fast and slow connections. The inputs to a neuron i are

$$h_i^{(\text{slow})} = \sum_j \tilde{J}_{ij} \bar{S}_j^{(t)}, \tag{12a}$$

$$h_i^{(\text{fast})} = \sum_j J_{ij} S_j^{(t)}. \tag{12b}$$

The $\bar{S}_j^{(t)}$ term is a measure of the history of spin j in the network, which is found by

$$\bar{S}_j^{(t)} = \int_{-\infty}^{t} S_j^{(t')} w(t - t') dt', \tag{13}$$

where $w(t - t')$ is a normalized distribution which will decrease contributions to $\bar{S}_j^{(t)}$ from neuron states further into the past. Some choices for $w(t - t')$ are a step function, delta function, or exponential decay with characteristic time τ. In Fig. 3 sequential pattern retrieval is exhibited using both asynchronous and synchronous update dynamics and a step function.
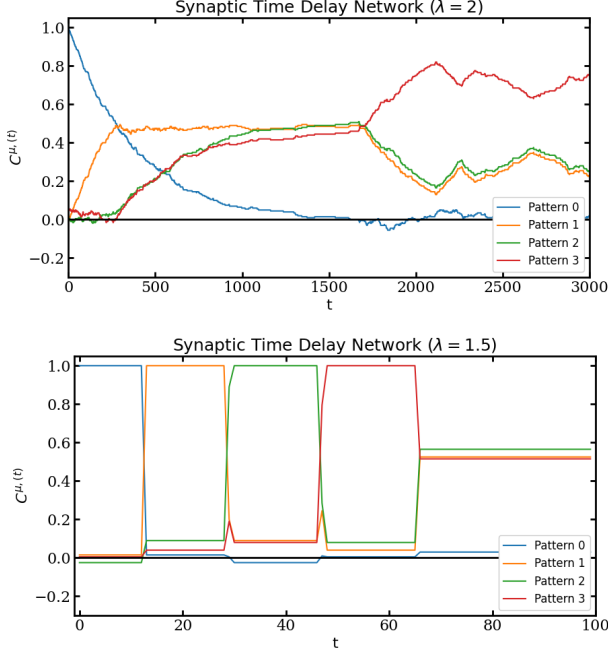

FIG. 3. **Synaptic time delay network simulation results** Simulations of a 400 neuron storing four patterns was performed with asynchronous updating and $\lambda = 2$, and synchronous updating with $\lambda = 1.5$. The asynchronous τ is equal to $16/400$ and the synchronous was $\tau = 16$. The differing values are because in the asynchronous case τ has to be measured per neuron. included.

Clearly the modified network is able to recall patterns when updated synchronously, and the amount of time spent at each pattern is approximately τ. However, when updated asynchronously the intermediate transitions are cut short, and the only full transition that is made is to the last pattern.

B. Dynamical Activation Threshold

In [3] Horn and Usher introduce temporal dynamics to the activation threshold in order to generate temporal sequences of patterns. This introduces a time dependence into θ of Eq. 4, and the authors define θ for neuron i recursively as

$$\theta_i(t + 1) = \frac{\theta_i(t)}{c} + S_i^{(t+1)}, \qquad (14)$$

where c is a free parameter which determines the saturation value of the activation threshold. The intuitive

interpretation for this threshold is that neuron i is continually active the activation threshold will increase until it saturates at $c/c - 1$. The input to a neuron i is then

$$h_i = \sum_j \left(J_{ij} + \lambda \tilde{J}_{ij} \right) S_j^{(t)} + b\, \theta_i(t), \qquad (15)$$

where b scales the activation threshold. From here we may characterize the activation using the quantity $bc/(c - 1)$ which represents the maximum value of the activation and shall be denoted as g.

Depicted in Fig. 4 is a pattern retrieval sequence generated using synchronous and asynchronous updating. Asynchronous updating is able to generate pattern recall here since the dynamical threshold is allowed to be negative, allowing for transitions even when $\lambda < 1$. This allows the network to pass through the midpoint of the pattern transition (analyzed in sections II and III) without an instability occurring. Synchronous updating also allows for pattern retrieval, but the network does not linger at a pattern as in the time delay network.
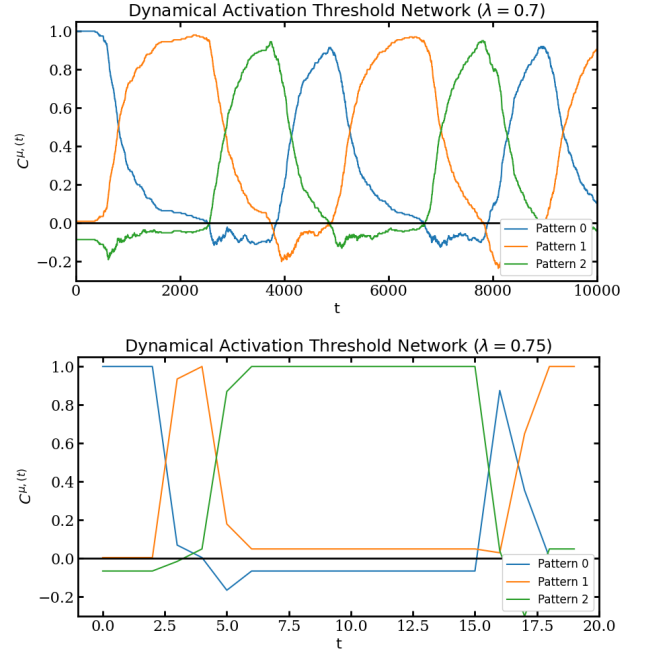

FIG. 4. **Dynamical threshold simulation results** Simulation of a network with 400 neurons storing three patterns. With $b = 0.5$, and $g = 1.5$ temporal pattern recall is achieved in both the asynchronous (top) and synchronous (bottom) case.

C. Conclusions

We analyzed why the Hopfield model is unable to recall patterns sequentially when endowed with asymmetric couplings for a system of three patterns and updated

asynchronously. The results were then shown to generalize to systems of many patterns, and both of the analyses are supported through simulations. Subsequently, two modifications to the Hopfield network are reviewed, and their capacity to follow temporal sequences for synchronous and asynchronous updating schemes are tested through simulation.

We find that a modification consisting of endowing each connection with fast and slow components is able to recall patterns sequentially for synchronous updating but not asynchronous. In contrast, we find that a modification which uses asymmetric couplings in addition to a dynamical threshold calculated from the activity history a neuron is sufficient to recall sequences of patterns in both synchronous and asynchronous update processes.

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