Electrodynamics HT22
Assignment 1

1 Jackson 5.1

Starting with the differential expression

$$\mathrm{d}\mathbf{B} = \underbrace{\frac{\mu_0 I}{4\pi}}_{=:k} \mathrm{d}\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \tag{1}$$

for the magnetic induction at the point P with coordinate $\mathbf{x}$ produced by an increment of current $I\mathrm{d}\mathbf{l}'$ at $\mathbf{x}'$, show explicitly that for a closed loop carrying a current I the magnetic induction at P is

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\nabla\Omega \tag{2}$$

where Ω is the solid angle subtended by the loop at the point P. This corresponds to a magnetic scalar potential, $\Phi_M = -\mu_0 I\Omega/4\pi$. The sign convention for the solid angle is that Ω is positive if the point P views the "inner" side of the surface spanning the loop, that is, if a unit normal $\mathbf{n}$ to the surface is defined by the direction of current flow via the right-hand rule, Ω is positive if $\mathbf{n}$ points away from the point P, and negative otherwise. This is the same convention as in Section 1.6 (Jackson) for the electric dipole layer.

Solution

Let us write the differential equation in terms of vector components

$$dB_i = \frac{\mu_0 I}{4\pi}\hat{x}_i \cdot \left(d\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\right) \tag{3}$$

and then integrate both sides

$$\frac{4\pi}{\mu_0 I}B_i = \oint \hat{x}_i \cdot \left(d\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\right). \tag{4}$$

To evaluate the integral we need to use

$$\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = \nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right). \tag{5}$$

The integral becomes

$$\begin{aligned}\frac{4\pi}{\mu_0 I}B_i &= \oint \hat{x}_i \cdot \left(d\mathbf{l}' \times \nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\right)\\ &= \oint \left(d\mathbf{l}' \cdot \left(\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\times\hat{x}_i\right)\right)\\ &= \int_S \left(\nabla' \times \left(\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\times\hat{x}_i\right)\right)\cdot d\mathbf{a}'\end{aligned} \tag{6}$$

where we used the identity $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$ in row two and Stokes theorem $\oint_C \mathbf{A}\cdot d\mathbf{l} = \int_S(\nabla\times\mathbf{A})\cdot d\mathbf{a}$ in row three.
Next we use $\nabla'\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla'\cdot\mathbf{b}) - \mathbf{b}(\nabla'\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla')\mathbf{a} - (\mathbf{a}\cdot\nabla')\mathbf{b}$.

$$\begin{aligned}\nabla'\times\left(\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\times\hat{x}_i\right) &= \nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)(\nabla'\cdot\hat{x}_i) - \hat{x}_i\left(\nabla'\cdot\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\right)\\ &\quad + (\hat{x}_i\cdot\nabla')\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) - \left(\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\cdot\nabla'\right)\hat{x}_i\\ &= -\hat{x}_i\left(\nabla'\cdot\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\right) + (\hat{x}_i\cdot\nabla')\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\end{aligned} \tag{7}$$

Two of the terms from row one are zero because $\nabla' \cdot \hat{x}_i = 0$ and $\nabla' \hat{x}_i = 0$. The first remaining term is

$$\begin{aligned} -\hat{x}_i \left(\nabla' \cdot \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right) &= -\hat{x}_i \left(\nabla'^2 \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right) \\ &= -\hat{x}_i \left(-4\pi\delta\left(\mathbf{x} - \mathbf{x}'\right) \right) \\ &= 4\pi \hat{x}_i \delta\left(\mathbf{x} - \mathbf{x}'\right) \end{aligned} \tag{8}$$

where we used the Green's function of the Laplace operator

$$\nabla'^2 \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = -4\pi\delta\left(\mathbf{x} - \mathbf{x}'\right) \tag{9}$$

Let us assume that $\mathbf{x} \neq \mathbf{x}'$, i.e. that the point P is not on where the current runs, then the δ-term is zero.
We are left with

$$\begin{aligned} \frac{4\pi}{\mu_0 I} B_i &= \int_S \left(\hat{x}_i \cdot \nabla'\right) \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \cdot d\mathbf{a}' \\ &= \int_S \left(\frac{\partial}{\partial x_i'} \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right) \cdot d\mathbf{a}' \\ &= \int_S \left(-\frac{\partial}{\partial x_i} \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right) \cdot d\mathbf{a}' \\ &= -\frac{\partial}{\partial x_i} \int_S \left(\nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right) \cdot d\mathbf{a}'. \end{aligned} \tag{10}$$

On page 33 in Jackson we have, here $d\mathbf{a}' = \hat{n} da'$,

$$d\mathbf{a}' \cdot \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = \hat{n} \cdot \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) da' = -\frac{\cos(\theta) da'}{|\mathbf{x} - \mathbf{x}'|^2} = -d\Omega'. \tag{11}$$

The sign of $d\Omega$ depends on the direction of $\hat{n}$ as stated in the beginning.
Derivation of the above equation:

$$\begin{aligned} &\nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \\ &\hat{n} \cdot \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} = \hat{n} \cdot \left(\frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} \right) \frac{1}{|\mathbf{x} - \mathbf{x}'|^2} = |\hat{n}| \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} \mid \cos(\theta) \frac{1}{|\mathbf{x} - \mathbf{x}'|^2} = \frac{\cos(\theta)}{|\mathbf{x} - \mathbf{x}'|^2} \\ &d\Omega = \frac{dS}{r^2} \hat{r} \cdot \hat{n} \rightarrow \frac{\cos(\theta) da'}{|\mathbf{x} - \mathbf{x}'|^2} = d\Omega' \end{aligned} \tag{12}$$

This gives

$$\begin{aligned} \frac{4\pi}{\mu_0 I} B_i &= -\frac{\partial}{\partial x_i} \int_S \left(\nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right) \cdot d\mathbf{a}' \\ &= -\frac{\partial}{\partial x_i} \int_S -d\Omega' = \frac{\partial}{\partial x_i} \Omega(\mathbf{x}) \end{aligned} \tag{13}$$

In vector form this is

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \nabla \Omega(\mathbf{x}). \tag{14}$$

2 Linking Number

Consider two curves in space, $C : x_i(s)$ and $C' : x'_i(s')$. They are circles topologically. Define the linking number m as follows: Deform one of the curves to a circle, and count the number of times the second curve passes throught the disk spanned by that circle, counting +1 if it passes in the direction of the normal of the disk and −1 if it passes in the other direction. Use your knowledge of magnetostatics to prove that

$$m = \frac{1}{4\pi}\epsilon_{ijk}\int_C\int_{C'}\frac{\left(x_i - x'_i\right) dl_j dl_k}{|x - x'|^3}. \tag{15}$$

3 Solution

Gaussian units with $c = 1$ are used here.
Say we have a current through the loop C' such that

$$j_i\left(x'\right) d^3x' = I dl'_i. \tag{16}$$

Ampére's law is

$$\int_{\partial S} d\mathbf{l}\cdot\mathbf{B} = 4\pi\int_S d\mathbf{S}\cdot\mathbf{j} \tag{17}$$

It can be written in component form and in terms of the linking number m by looking at the definition for m,

$$\int_C dl_i B_i = 4\pi\int_S dS_i j_i = 4\pi I m \tag{18}$$

We also know

$$\nabla^2 B_i = -4\pi\epsilon_{ijk}\partial_j j_k. \tag{19}$$

The derivation of the previous equation is given below.
We have (component form of Ampère's law): $\epsilon_{ijk}\partial_j B_k = 4\pi j_i$.
Differentiate w.r.t j : $\partial_j\epsilon_{ijk}\partial_j B_k = \partial_j 4\pi j_i \rightarrow \epsilon_{ijk}\partial_j^2 B_k = 4\pi\partial_j j_i$.
Use $\epsilon^{imn}\epsilon_{ijk} = \delta_j^m\delta_k^n - \delta_j^n\delta_k^m$
$\rightarrow \epsilon^{imn}\epsilon_{ijk}\partial_j^2 B_k = \epsilon^{imn}4\pi\partial_j j_i \rightarrow \left(\delta_j^m\delta_k^n - \delta_j^n\delta_k^m\right)\partial_j^2 B_k = \epsilon^{imn}4\pi\partial_j j_i$.
For $j = m, n = k (j \neq n, m \neq k) \rightarrow \partial_j^2 B_k = \epsilon^{ijk}4\pi\partial_j j_i$.
Use $\epsilon_{ijk} = -\epsilon_{kji} \rightarrow \partial_j^2 B_k = -\epsilon_{kji}4\pi\partial_j j_i$.
Rename $i \leftrightarrow k \rightarrow \partial_j^2 B_i = -\epsilon_{ijk}4\pi\partial_j j_k$.
$\Rightarrow \nabla^2 B_i = -4\pi\epsilon_{ijk}\partial_j j_k$ Using the Green's function of the Laplace operator,

$$\nabla'^2 G\left(\mathbf{x},\mathbf{x}'\right) = -4\pi\delta\left(\mathbf{x}-\mathbf{x}'\right) \quad G\left(\mathbf{x},\mathbf{x}'\right) = \frac{1}{|\mathbf{x}-\mathbf{x}'|}, \tag{20}$$

we can rewrite the equation $\nabla^2 B_i = -4\pi\epsilon_{ijk}\partial_j j_k$ like this

$$\begin{aligned}
\nabla^2 B_i(\mathbf{x}) &= -4\pi\epsilon_{ijk}\partial_j j_k(\mathbf{x}) \\
&= -4\pi\epsilon_{ijk}\partial_j\int d^3x' j_k\left(\mathbf{x}'\right)\delta\left(\mathbf{x}-\mathbf{x}'\right) \\
&= \epsilon_{ijk}\partial_j\int d^3x' j_k\left(\mathbf{x}'\right)\left(-4\pi\delta\left(\mathbf{x}-\mathbf{x}'\right)\right) \\
&= \epsilon_{ijk}\partial_j\int d^3x' j_k\left(\mathbf{x}'\right)\nabla'^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \\
&= \epsilon_{ijk}\partial_j\int d^3x' j_k\left(\mathbf{x}'\right)\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \\
&= \nabla^2\left(\epsilon_{ijk}\partial_j\int d^3x' j_k\left(\mathbf{x}'\right)\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \\
\Rightarrow B_i(\mathbf{x}) &= \epsilon_{ijk}\partial_j\int d^3x'\frac{j_k\left(\mathbf{x}'\right)}{|\mathbf{x}-\mathbf{x}'|}
\end{aligned} \tag{21}$$

Let us rewrite it a bit

$$\begin{aligned} B_i(\mathbf{x}) &= \epsilon_{ijk}\partial_j \int d^3x' \frac{j_k(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = \epsilon_{ijk}\int d^3x' \frac{j_k(\mathbf{x}')\left(-\left(x_j - x'_j\right)\right)}{|\mathbf{x}-\mathbf{x}'|^3} \\ &= -\epsilon_{ijk}\int d^3x' \frac{j_k(\mathbf{x}')\left(x_j - x'_j\right)}{|\mathbf{x}-\mathbf{x}'|^3} = -\epsilon_{ijk}\int d^3x' \frac{j_k(\mathbf{x}')\left(x_j - x'_j\right)}{|\mathbf{x}-\mathbf{x}'|^3} \\ &= \epsilon_{jik}\int d^3x' \frac{j_k(\mathbf{x}')\left(x_j - x'_j\right)}{|\mathbf{x}-\mathbf{x}'|^3} \end{aligned} \tag{22}$$

where we used

$$\partial_j \frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{-\left(x_j - x'_j\right)}{|\mathbf{x}-\mathbf{x}'|^3} \tag{23}$$

and $\epsilon_{ijk} = -\epsilon_{jik}$. We change the names $i \leftrightarrow j$ and get

$$B_j(\mathbf{x}) = \epsilon_{ijk}\int d^3x' \frac{j_k(\mathbf{x}')\left(x_i - x'_i\right)}{|\mathbf{x}-\mathbf{x}'|^3}. \tag{24}$$

Next we use

$$j_i(x')\, d^3x' = I dl'_i \tag{25}$$

to get

$$B_j(\mathbf{x}) = \epsilon_{ijk}\int_{C'} d^3x' \frac{j_k(\mathbf{x}')\left(x_i - x'_i\right)}{|\mathbf{x}-\mathbf{x}'|^3} = \epsilon_{ijk}\int_{C'} dl'_k I \frac{\left(x_i - x'_i\right)}{|\mathbf{x}-\mathbf{x}'|^3} \tag{26}$$

Finally we apply Ampére's law and obtain the answer

$$\int_C dl_i B_i = 4\pi I m \quad \Rightarrow m = \frac{1}{4\pi}\int_C dl_j \int_{C'} dl'_k \frac{\epsilon_{ijk}\left(x_i - x'_i\right)}{|\mathbf{x}-\mathbf{x}'|^3} \tag{27}$$

4 Aharonov-Bohm

Define a vector potential on a region of space strictly outside the z-axis, such that $\mathbf{A}(\mathbf{x})$ is independent of z, gives a vanishing magnetic field outside the z-axis, and cannot be gauge transformed to zero. Discuss the last point in some detail, and give a physical interpretation. [1]

Solution

We want a vector field defined on $V = \{(x,y,z) \mid (x,y,z) \neq (0,0,z)\}$ that fulfills

$$\begin{cases} \mathbf{A}(\mathbf{x}) = \mathbf{A}(x,y) \\ \mathbf{B}(\mathbf{x}) = \nabla\times\mathbf{A}(\mathbf{x}) = 0 \\ \mathbf{A}'(\mathbf{x}) = \mathbf{A}(\mathbf{x}) + \nabla\psi(\mathbf{x}) \to \mathbf{A}' \neq \mathbf{0} \end{cases} \tag{28}$$

From the first two conditions we obtain

$$\begin{aligned} &\mathbf{B}(\mathbf{x}) = \nabla\times\mathbf{A}(\mathbf{x}) = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z} \\ &\mathbf{A}(\mathbf{x}) = \mathbf{A}(x,y) = \left(A_x(x,y), A_y(x,y), A_z(x,y)\right) \to \frac{\partial A_z(x,y)}{\partial y} = 0, \frac{\partial A_z(x,y)}{\partial x} = 0 \\ &\mathbf{B}(\mathbf{x}) = 0 \to \frac{\partial A_y(x,y)}{\partial x} = \frac{\partial A_x(x,y)}{\partial y}. \end{aligned} \tag{29}$$

[1]Previous tutor Nadia recommends: When you are done, consult TT Wu and CN Yang, Concept of nonintegrable phase factors and global formulation of gauge fields, Phys. Rev. D12 (1975) 3845 .

Therefore we know A_z must be a constant, $A_z(x, y) = C_z$.
Poincaré's lemma says that a covariant vector field $A_\alpha(x)$ can be written as the gradient of a scalar if and only if its field strentgh $F_{\alpha\beta}$ is zero. This lemma holds for simply connected spaces, i.e spaces where a closed curve can be deformed to a point without leaving the space. For example, $\mathbb{R}^3$ is simply connected but our space V, which excludes the z-axis is not. In V a closed loop around the z-axis cannot be deformed to a point without leaving the space.
For a vector potential that points around the z-axis the field strength can be zero while the vector potential itself cannot be set to zero via a gauge transformation. Poincaré's lemma does not hold in this case.
Let us try a vector potential that points around the z-axis. In cylindrical polar coordinates we can take

$$\mathbf{A} = \frac{1}{\rho}\hat{\phi} \tag{30}$$

which in regular cartesian coordinates is

$$\mathbf{A} = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0\right). \tag{31}$$

This has $\nabla \times \mathbf{A} = \mathbf{0}$. From the cylindrical coordinate form we can see that it looks to have the form of a gradient,

$$\begin{aligned}
&\mathbf{A} = \frac{1}{\rho}\hat{\phi} \\
&\nabla f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z} \\
&\to \mathbf{A} = \frac{1}{\rho}\hat{\phi} = \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\phi} \\
&\to \frac{\partial f}{\partial \phi} = 1 \to f(\phi) = \phi + c.
\end{aligned} \tag{32}$$

$f(x, y)$ is not well defined on the entire space, since the z-axis is excluded. The problem is that $f(\phi) = f(x, y)$ is not single valued since at, for example, $(x > 0, y = 0)$ we have $f(\phi) = 0$ but if we go one closed curve around to $(x > 0, y = 0)$ then $f(\phi) = 2\pi$. Therefore f is not single valued and does not have a well defined gradient.
This is for the global case. If we consider a connected region, where all closed curves can be deformed to points without leaving the space, in the space then locally the potential can be gauge transformed to zero. But not globally.

5 Complementary Notes on Aharonov-Bohm

The Aharonov-Bohm effect is a quantum effect on a charged particle. Say we have a cylinder with a non-zero magnetic field inside but a zero magnetic field outside. We can still have a non-zero vector potential outside, just as we calculated above. The Aharonov-Bohm effect shows that, when considering quantum systems, not only the EM-field strength is enough to describe electromagnetism, also the phase factor

$$\exp\left(\frac{ie}{\hbar c}\oint A_\mu dx^\mu\right) \tag{33}$$

is needed. The phase factor depends on the vector potential, not the magnetic field.
Since this is a quantum effect we need a quantum system to see it. When a charged particle (wavefunction) travels in a region with zero electromagnetic field but non-zero vector potential the particle is affected by the vector potential and it picks up the phase factor. However, the phase factor does not affect how we observe the wavefunction, since for this only $|\psi|$ matters. To see the effect we need a quantum phenomenon, such as interference of wavefunctions.

The Aharonov-Bohm experiment is done by sending a beam of electrons (charged particles) towards a double slit experiment but with a cylinder near the double-slit screen. The cylinder has a non-zero magnetic field inside it, however, outside where the particles travel the field is zero and the potential non-zero. Say we send two of the particles and they take "different paths" and pick up different phase factors. When they interact and form an interference pattern efter passing the double-slits we can measure the phase shift (difference between the phase factors) that they picked up.
The interference fringes depend on the phase factor

$$\exp\left(\frac{ie}{\hbar c}\oint A_\mu dx^\mu\right) = \exp\left(-\frac{ie}{\hbar c}\Omega\right) \tag{34}$$

where Ω is the flux in the cylinder. Two different fluxes can give the same interference pattern if

$$\Omega_a - \Omega_b = \frac{hc}{e}n \tag{35}$$

where n is an integer.
Say we want to find a gauge transformation from a to b, i.e. $\psi_b = e^{i\alpha}\psi_a$. In terms of the vector potential that is

$$\left(A_\mu\right)_b = \left(A_\mu\right)_a + \frac{\hbar c}{e}\frac{\partial\alpha}{\partial x^\mu}. \tag{36}$$

We can write

$$\Delta\alpha = \frac{e}{\hbar c}\oint\left[\left(A_\mu\right)_b - \left(A_\mu\right)_a\right]dx^\mu = \frac{e}{\hbar c}\left(\Omega_b - \Omega_a\right). \tag{37}$$

If $\Omega_a - \Omega_b = \frac{hc}{e}n$, holds then $\Delta\alpha = 2\pi n$, where n is an integer, and the gauge transformation $S = e^{i\alpha}$ is single valued, which means that a and b can be gauge-transformed into each other. And the effect of the gauge transform cannot be physically measured since the interference pattern is the same. This is as it should be since the gauge choice should not affect the physics, neither classical nor quantum.

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