Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

New approaches to nowhere-zero flow problems

Diploma Thesis

2026

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Study Programme:	Computer Science
Field of Study:	Computer Science
Department:	Department of Computer Science
Supervisor:	Mgr. Jozef Rajník, PhD.

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ZADANIE ZÁVEREČNEJ PRÁCE

Meno a priezvisko študenta:	Bc. Lukáš Gáborik
Študijný program:	informatika (Jednoodborové štúdium, magisterský II. st., denná forma)
Študijný odbor:	informatika
Typ záverečnej práce:	diplomová
Jazyk záverečnej práce:	anglický
Sekundárny jazyk:	slovenský

Názov: New approaches to nowhere-zero flow problems
Nové prístupy k problémom o nikde-nulových tokoch

Anotácia: Táto práca nadväzuje na výsledky bakalárskej práce rovnakého autora. Novo zavedený pojem viacrozmerných Manhattanských a Čebyševovských tokov ponecháva priestor pre ďalší výskum, ako napríklad hľadanie dolných odhadov pre tokové čísla. Jedným z hlavných výsledkov spomínanej bakalárskej práce je predstavenie hypotézy tvrdiacej, že každý bezmostový graf pripúšťa (1,2)-cirkulačnú dekompozíciu, t. j. 2-cirkuláciu a 4-cirkuláciu takú, že zakaždým keď je 2-cirkulácia nulová na nejakej hrane, tak 4-cirkulácia nemôže nadobúdať 0, +1 alebo -1. To ponúka bohatý priestor na výskum vrátane rôznych zovšeobecnení, v ktorých kladieme ďalšie požiadavky na tokové hodnoty.

Cieľ: 1. Dokázať netriviálne dolné odhady pre dvojrozmerné Čebyševovské tokové číslo grafu.
2. Preskúmať možné spôsoby, ako dokázať hypotézu, že každý bezmostový graf pripúšťa (1, 2)-cirkulačnú dekompozíciu, a iné súvisiace hypotézy. Dokázať túto hypotézu pre niektoré nekonečné triedy snarkov, prípadne pre niektoré snarky, ktoré majú ďaleko od toho, aby boli zafarbiteľné (napr. s nepárnosťou 2, indexom perfektného párenia 4, ...).
3. Preskúmať ďalšie zovšeobecnenia tokov potenciálne užitočné v kontexte vyššie uvedenej hypotézy.

Vedúci:	Mgr. Jozef Rajník, PhD.
Katedra:	FMFI.KI - Katedra informatiky
Vedúci katedry:	prof. RNDr. Martin Škoviera, PhD.
Dátum zadania:	21.11.2024

Dátum schválenia: 05.12.2024

prof. RNDr. Rastislav Kráľovič, PhD.
garant študijného programu

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študent vedúci práce

Comenius University Bratislava
Faculty of Mathematics, Physics and Informatics

THESIS ASSIGNMENT

Name and Surname: Bc. Lukáš Gáborik
Study programme: Computer Science (Single degree study, master II. deg., full time form)
Field of Study: Computer Science
Type of Thesis: Diploma Thesis
Language of Thesis: English
Secondary language: Slovak

Title: New approaches to nowhere-zero flow problems

Annotation: This work builds on the results of the bachelor thesis of the same author. The newly introduced notion of multidimensional Manhattan and Chebyshev flows still leaves some possibilities for further research like finding lower bounds on flow numbers. One of the main results of the mentioned bachelor thesis is the introduction of the conjecture asserting that each bridgeless graph admits a (1,2)-circulation decomposition, that is a 2-circulation and a 4-circulation such that whenever the 2-circulation is zero on any edge, the 4-circulations can not attain 0, +1 or -1. This offers wide possibilities of exploration including various generalisations where further requirements are posed on the flow values.

Aim: 1. Prove nontrivial lower bounds on 2-dimensional Chebyshev flow number of a graph.
2. Explore possible ways of proving the conjecture that each bridgeless graph admits a (1, 2)-circulation decomposition, and other related conjectures. Prove this conjecture for some infinite families of snarks, eventually for some snarks that are far from being colourable (e.g. with oddness 2, perfect matching index 4, …)
3. Research other generalisation of flows potentially useful in the context of the abovementioned conjecture.

Supervisor: Mgr. Jozef Rajník, PhD.
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Assigned: 21.11.2024

Approved: 05.12.2024 prof. RNDr. Rastislav Kráľovič, PhD.
Guarantor of Study Programme

Student Supervisor

Acknowledgments:

Abstrakt

Kľúčové slová:

Abstract

Keywords:

Contents

Introduction

Chapter 1

Flows

In this chapter, we aim to introduce all concepts and current knowledge about flows and related graph theory terms. Mainly, we tend to explain Chebyshev nowhere-zero flows, which have arisen in the bachelor thesis of the author [**TODO**].

Chapter 2

Bounds on Chebyshev flow number

2.1 Upper bound on generalised Blanuša snarks

2.2 General lower bound

Proposition 2.1. *Let Γ denote a snark of order n. Then $\Phi_2^\infty(\Gamma) \geq 2 + 1/\xi$, where $\xi = \left\lfloor \frac{n-2}{4} \right\rfloor$.*

Proof. Assume by contradiction that there exists a 2-dimensional flow φ of Γ such that $\varphi = (\varphi_1(e), \varphi_2(e))$ for each edge $e \in E(\Gamma)$, with $\|\varphi(e)\|_\infty \geq 1$ and $\varphi_i(e) \in (-1 - 1/\xi, 1 + 1/\xi)$ for $i = 1, 2$.

We say an edge $e \in E(\Gamma)$ to be *good* with respect to φ_i if $|\varphi_i(e)| \in [1, 1 + 1/\xi)$, *bad* otherwise. Observe that an edge e can be good with respect to both φ_1 and φ_2, but it cannot be bad with respect to both φ_1 and φ_2, for otherwise $\|\varphi(e)\|_\infty < 1$.

Denote by B_i the subgraph of Γ induced by the bad edges with respect to φ_i and by G_i the one induced by the good edges with respect to φ_i, $i = 1, 2$. By previous observation at least one between B_1 and B_2 has at most $\lfloor |E(\Gamma)|/2 \rfloor = \left\lfloor \frac{3n}{4} \right\rfloor$ edges, say B_1.

B_i is a spanning subgraph of Γ and $\Delta(B_i) \leq 2$, for $i = 1, 2$.

Proof. Observe that $\Delta(G_i) \leq 2$, because the sum of three real numbers all with absolute value in the interval $[1, 1 + 1/\xi)$ cannot give 0 as a result, making the Kirkoff's law impossible to be satisfied by φ around a vertex of Γ. Hence B_i is spanning, for otherwise $\Delta(G_i) = 3$ and $\Delta(B_i) \leq 2$, for otherwise $\Delta(G_{3-i}) = 3$. □

If $C \subseteq E(G_i)$ is an odd edge-cut of Γ for $i = 1, 2$, then $|C| \geq 2\xi + 3$.

Proof. Consider an odd edge-cut C of Γ containing $2k + 1$ good edges, separating components Γ_1, Γ_2. WLOG assume $\varphi_i(e) \geq 0$ for every edge e of C and more edges are directed towards Γ_2. Then, there are at least $k + 1$ edges towards Γ_2, resulting in total inflow at least $(k+1)\cdot 1$. Analogously, the total outflow is strictly less than $k \cdot (1+1/\xi)$. Together with the Kirkoff's law, this leads to $k+1 < k\cdot(1+1/\xi)$, which is equivalent to $k > \xi$. □

A path of length $2k$ cannot be a connected component of B_i, for $k = 1, 2, \ldots, \xi - 1$ and $i = 1, 2$.

Proof. For the sake of contradiction assume there is a path of a length $2k$, $k < \xi$ in B_i. Then the edges adjacent to this path are in G_i and they form an odd edge-cut of Γ, containing at most $2k+3 < 2\xi+3$ edges, which is in contradiction with the Claim 2. □

A cycle of length $2k+1$ cannot be a connected component of B_i, for $k = 1, 2, \ldots, \xi$ and $i = 1, 2$.

Proof. For the sake of contradiction assume there is a cycle of a length $2k+1$, $k < \xi+1$ in B_i. Then the edges adjacent to this cycle are in G_i and they form an odd edge-cut of Γ, containing at most $2k+1 < 2\xi+3$ edges, which is in contradiction with the Claim 2. □

$E(B_i)$ cannot contain a perfect matching of Γ, for $i = 1, 2$.

Proof. For the sake of contradiction assume that $E(B_i)$ contains a perfect matching M of Γ. Then also $E(G_{3-i})$ contains a perfect matching M of Γ. Next, $F = E(\Gamma) \setminus M$ is a 2-factor of Γ. Note that F must contain an odd cycle of length $2k+1 \leq \frac{n}{2}$. This is equivalent with $k \leq \xi$. Then the edges adjacent to this cycle are in G_{3-i} and they form an odd edge-cut of Γ, containing at most $2k+1 \leq 2\xi+1$ edges, which is in contradiction with the Claim 2. □

Note that by the Claim 3, each even path in B_1 contains at least 2ξ edges. Similarly by the Claim 4, each odd cycle in B_1 contains at least $2\xi+3$ edges. Let *odd components* denote odd cycles and even paths. Since $4\xi = 4\lfloor\frac{n-2}{4}\rfloor > \lfloor\frac{3n}{4}\rfloor \geq E(B_1)$ obviously holds for any $n \geq 10$, B_1 may contain at most one odd component. On the other hand, B_1 contains even number of odd componentss. As a result, B_1 contains only odd paths and even cycles, but then also a perfect matching of Γ, which is in contradiction with the Claim 5. □

Conclusion