Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

New approaches to nowhere-zero flow problems

Diploma Thesis

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Study Programme:	Computer Science
Field of Study:	Computer Science
Department:	Department of Computer Science
Supervisor:	Mgr. Jozef Rajník, PhD.

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ZADANIE ZÁVEREČNEJ PRÁCE

Meno a priezvisko študenta:	Bc. Lukáš Gáborik
Študijný program:	informatika (Jednoodborové štúdium, magisterský II. st., denná forma)
Študijný odbor:	informatika
Typ záverečnej práce:	diplomová
Jazyk záverečnej práce:	anglický
Sekundárny jazyk:	slovenský

Názov: New approaches to nowhere-zero flow problems
Nové prístupy k problémom o nikde-nulových tokoch

Anotácia: Táto práca nadväzuje na výsledky bakalárskej práce rovnakého autora. Novo zavedený pojem viacrozmerných Manhattanských a Čebyševovských tokov ponecháva priestor pre ďalší výskum, ako napríklad hľadanie dolných odhadov pre tokové čísla. Jedným z hlavných výsledkov spomínanej bakalárskej práce je predstavenie hypotézy tvrdiacej, že každý bezmostový graf pripúšťa (1,2)-cirkulačnú dekompozíciu, t. j. 2-cirkuláciu a 4-cirkuláciu takú, že zakaždým keď je 2-cirkulácia nulová na nejakej hrane, tak 4-cirkulácia nemôže nadobúdať 0, +1 alebo -1. To ponúka bohatý priestor na výskum vrátane rôznych zovšeobecnení, v ktorých kladieme ďalšie požiadavky na tokové hodnoty.

Cieľ: 1. Dokázať netriviálne dolné odhady pre dvojrozmerné Čebyševovské tokové číslo grafu.
2. Preskúmať možné spôsoby, ako dokázať hypotézu, že každý bezmostový graf pripúšťa (1, 2)-cirkulačnú dekompozíciu, a iné súvisiace hypotézy. Dokázať túto hypotézu pre niektoré nekonečné triedy snarkov, prípadne pre niektoré snarky, ktoré majú ďaleko od toho, aby boli zafarbiteľné (napr. s nepárnosťou 2, indexom perfektného párenia 4, ...).
3. Preskúmať ďalšie zovšeobecnenia tokov potenciálne užitočné v kontexte vyššie uvedenej hypotézy.

Vedúci:	Mgr. Jozef Rajník, PhD.
Katedra:	FMFI.KI - Katedra informatiky
Vedúci katedry:	prof. RNDr. Martin Škoviera, PhD.
Dátum zadania:	21.11.2024

Dátum schválenia: 05.12.2024

prof. RNDr. Rastislav Kráľovič, PhD.
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THESIS ASSIGNMENT

Name and Surname:	Bc. Lukáš Gáborik
Study programme:	Computer Science (Single degree study, master II. deg., full time form)
Field of Study:	Computer Science
Type of Thesis:	Diploma Thesis
Language of Thesis:	English
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Title: New approaches to nowhere-zero flow problems

Annotation: This work builds on the results of the bachelor thesis of the same author. The newly introduced notion of multidimensional Manhattan and Chebyshev flows still leaves some possibilities for further research like finding lower bounds on flow numbers. One of the main results of the mentioned bachelor thesis is the introduction of the conjecture asserting that each bridgeless graph admits a (1,2)-circulation decomposition, that is a 2-circulation and a 4-circulation such that whenever the 2-circulation is zero on any edge, the 4-circulations can not attain 0, +1 or -1. This offers wide possibilities of exploration including various generalisations where further requirements are posed on the flow values.

Aim: 1. Prove nontrivial lower bounds on 2-dimensional Chebyshev flow number of a graph.
2. Explore possible ways of proving the conjecture that each bridgeless graph admits a (1, 2)-circulation decomposition, and other related conjectures. Prove this conjecture for some infinite families of snarks, eventually for some snarks that are far from being colourable (e.g. with oddness 2, perfect matching index 4, …)
3. Research other generalisation of flows potentially useful in the context of the abovementioned conjecture.

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Acknowledgments:

Abstrakt

Kľúčové slová:

Abstract

Keywords:

Contents

Introduction

Chapter 1

Flows

In this chapter, we aim to introduce all concepts and current knowledge about flows and related graph theory terms. Mainly, we explain Chebyshev nowhere-zero flows, which have arisen in the Student Science Conference paper of the author [3]. Moreover, we present a concept of two smaller flows as a sufficient condition for a graph to have a larger nowhere-zero flow and some Chebyshev flow, also stated in that paper. Then we connect this idea to the theory of group connectivity of a graph established by Jaeger et al. [4] and k-closures and k-bases introduced by Seymour [6] in the paper, where he proved the 6-flow theorem.

1.1 Integral nowhere-zero flows

Definition. Let $G = (V, E)$ be a graph and $\mathcal{A}$ be an abelian group. An $\mathcal{A}$-*circulation* on G is a pair of functions – orientation $o\colon E \to V \times V$ and flow $\varphi\colon E \to \mathcal{A}$ – satisfying the equality

$$\sum_{u \in N_o^+(v)} \varphi(uv) = \sum_{u \in N_o^-(v)} \varphi(uv) \tag{1.1}$$

for each vertex v of G. This equality (1.1) is often referred to as the *flow conservation constraint.*

Definition. Given a graph G and a positive integer $k \geq 2$, a k-*circulation* on G is a $\mathbb{Z}$-circulation (o, φ) with an additional constraint that the inequality $1-k \leq \varphi(e) \leq k-1$ holds for each edge e of the graph. Analogously, a k-*NZF* (o, φ) is a k-circulation satisfying $\varphi(e) \neq 0$ for all edges.

If $e \in E$ is a bridge in Γ, then $\varphi(e) = 0$. Moreover, Γ allows a k-NZF for no natural k.

Definition. Let Γ be a bridgeless graph. A *flow number of* Γ is

$$\Phi(\Gamma) := \min\{k \mid \exists k\text{-NZF on } \Gamma\}.$$

Lemma 1.1. [6, p. 132] *For any bridgeless graph, there exist a 2-flow and a 3-flow such that for each edge e, at least one value from $\varphi_2(e), \varphi_3(e)$ is non-zero.*

Theorem 1.2 (6-flow theorem)**.** [6, p. 133] *There exists a* 6*-NZF on any bridgeless graph.*

Conjecture 1.3. [8, p. 83] *There exists a* 5*-NZF on any bridgeless graph.*

Proposition 1.4. [2, pp. 160, 161] *A cubic graph has a* 4*-NZF if and only if it is* 3*-edge-colourable. Moreover, it has a* 3*-NZF if and only if it is bipartite.*

Definition. A *snark* is a bridgeless cubic graph, which has no cycles of lengths 3 and 4, is cyclically 4-edge connected (at least 4 edges must be removed from the graph to get two components containing a cycle) and is not 3-edge-colourable.

1.2 Chebyshev nowhere-zero flows and sufficient flow-pairs

Definition. Let G be a bridgeless graph and d be a positive integer. A *d-dimensional Manhattan flow number* of G and a *d-dimensional Chebyshev flow number* of G are

$$\Phi_d^{\mathrm{M}}(G) := \inf\{r \mid \exists (r,d)\text{-MNZF on } G\} \text{ and } \Phi_d^{\mathrm{Ch}}(G) := \inf\{r \mid \exists (r,d)\text{-ChNZF on } G\},$$

respectively.

Proposition 1.5. *Each bridgeless graph has a $(3,2)$-ChNZF.*

Theorem 1.6. *A cubic graph has a $(2,2)$-MNZF if and only if it is* 3*-edge-colourable.*

Definition. Let $p \leq q$ be the positive integers. A *(p,q)-circulation decomposition* of a graph G is a 2-circulation (o_2, φ_2) and a $(p+q+1)$-circulation $(o_{p+q+1}, \varphi_{p+q+1})$ such that whenever $\varphi_2(e)$ is zero, the value $|\varphi_{p+q+1}(e)|$ is either q or $p+q$.

Lemma 1.7. *A bridgeless graph G with a (dp,dq)-circulation decomposition also has a (p,q)-circulation decomposition.*

Proposition 1.8. *Consider a bridgeless graph G with a (p,q)-circulation decomposition. Then, there are $\left(2+\frac{p}{q},2\right)$-ChNZF and $\left(4+\frac{2p}{q},1\right)$-NZF on G.*

Conjecture 1.9. *For each bridgeless graph there exists a $(1,2)$-circulation decomposition.*

Lemma 1.10. *Let $p < q$ be positive integers. Consider a snark G with a (p,q)-circulation decomposition. Then the set of edges e with a property $\varphi_2(e) \neq 0$ is a* 2*-factor of G.*

1.3 Closures and bases

Definition. [6, p. 132] Let $\Gamma = (V, E)$ be a graph and $S \subseteq E$ its edge subset. A *k-closure* $\langle S \rangle_k$ of S is a mimimal T such that $S \subseteq T \subseteq E$ and for any circuit $\mathcal{C} \nsubseteq T$ of G, the size of $\mathcal{C} \cap T^c$ is strictly greater than k.

Definition. [1, p. 7] Let $\Gamma = (V, E)$ be a graph. Then $S \subseteq E$ is a *k-base* of Γ if $\langle S \rangle_k = E$.

Any spanning tree of a graph is its inclusion-minimal 1-base.

Lemma 1.11. *[6, p. 133] For any 3-connected graph Γ there exists its even-factor which is also its 2-base.*

Lemma 1.12. *[6, p. 134] For any 3-connected cubic graph $\Gamma = (V, E)$ there exists a vertex partition $E = E_1 \cup E_2$ such that E_1, E_2 are its 1- and 2-bases, respectively.*

1.4 Group connectivity

Chapter 2

Bounds on Chebyshev flow number

2.1 Upper bound on generalised Blanuša snarks

2.2 General lower bound

In one-dimensional flows, there has been proved a lower bound on the flow number of a snark

$$\Phi_1(\Gamma) \geq 4 + \Big/ \left\lceil \frac{n-4}{8} \right\rceil$$

depending on the number n of its vertices [5, p. 14]. Moreover, there is also a meaningful lower bound on the two-dimensional Chebyshev flow number of the Petersen graph P [7, p. 99]. It states that $\Phi_2^{\infty}(P) \geq 5/2$. We provide generalisation of this bound for any snark. Moreover, the bounding number is roughly a half of the bounding number in one dimension.

Proposition 2.1. *Let Γ denote a snark of order n. Then $\Phi_2^{\infty}(\Gamma) \geq 2 + 1/\xi$, where $\xi = \left\lfloor \frac{n-2}{4} \right\rfloor$.*

Proof. Assume by contradiction that there exists a 2-dimensional flow φ of Γ such that $\varphi = (\varphi_1(e), \varphi_2(e))$ for each edge $e \in E(\Gamma)$, with $\|\varphi(e)\|_\infty \geq 1$ and $\varphi_i(e) \in (-1-1/\xi, 1+1/\xi)$ for $i = 1, 2$.

We say an edge $e \in E(\Gamma)$ to be *good* with respect to φ_i if $|\varphi_i(e)| \in [1, 1+1/\xi)$, *bad* otherwise. Observe that an edge e can be good with respect to both φ_1 and φ_2, but it cannot be bad with respect to both φ_1 and φ_2, for otherwise $\|\varphi(e)\|_\infty < 1$.

Denote by B_i the subgraph of Γ induced by the bad edges with respect to φ_i and by G_i the one induced by the good edges with respect to φ_i, $i = 1, 2$. By previous observation at least one between B_1 and B_2 has at most $\lfloor |E(\Gamma)|/2 \rfloor = \left\lfloor \frac{3n}{4} \right\rfloor$ edges, say B_1.

B_i is a spanning subgraph of Γ and $\Delta(B_i) \leq 2$, for $i = 1, 2$.

Proof. Observe that $\Delta(G_i) \leq 2$, because the sum of three real numbers all with absolute value in the interval $[1, 1 + 1/\xi)$ cannot give 0 as a result, making the Kirkoff's law impossible to be satisfied by φ around a vertex of Γ. Hence B_i is spanning, for otherwise $\Delta(G_i) = 3$ and $\Delta(B_i) \leq 2$, for otherwise $\Delta(G_{3-i}) = 3$. □

If $C \subseteq E(G_i)$ is an odd edge-cut of Γ for $i = 1, 2$, then $|C| \geq 2\xi + 3$.

Proof. Consider an odd edge-cut C of Γ containing $2k + 1$ good edges, separating components Γ_1, Γ_2. WLOG assume $\varphi_i(e) \geq 0$ for every edge e of C and more edges are directed towards Γ_2. Then, there are at least $k + 1$ edges towards Γ_2, resulting in total inflow at least $(k + 1) \cdot 1$. Analogously, the total outflow is strictly less than $k \cdot (1 + 1/\xi)$. Together with the Kirkoff's law, this leads to $k + 1 < k \cdot (1 + 1/\xi)$, which is equivalent to $k > \xi$. □

A path of length $2k$ cannot be a connected component of B_i, for $k = 1, 2, \ldots, \xi - 1$ and $i = 1, 2$.

Proof. For the sake of contradiction assume there is a path of a length $2k$, $k < \xi$ in B_i. Then the edges adjacent to this path are in G_i and they form an odd edge-cut of Γ, containing at most $2k + 3 < 2\xi + 3$ edges, which is in contradiction with the Claim 2. □

A cycle of length $2k + 1$ cannot be a connected component of B_i, for $k = 1, 2, \ldots, \xi$ and $i = 1, 2$.

Proof. For the sake of contradiction assume there is a cycle of a length $2k+1$, $k < \xi+1$ in B_i. Then the edges adjacent to this cycle are in G_i and they form an odd edge-cut of Γ, containing at most $2k+1 < 2\xi+3$ edges, which is in contradiction with the Claim 2. □

$E(B_i)$ cannot contain a perfect matching of Γ, for $i = 1, 2$.

Proof. For the sake of contradiction assume that $E(B_i)$ contains a perfect matching M of Γ. Then also $E(G_{3-i})$ contains a perfect matching M of Γ. Next, $F = E(\Gamma) \setminus M$ is a 2-factor of Γ. Note that F must contain an odd cycle of length $2k + 1 \leq \frac{n}{2}$. This is equivalent with $k \leq \xi$. Then the edges adjacent to this cycle are in G_{3-i} and they form an odd edge-cut of Γ, containing at most $2k + 1 \leq 2\xi + 1$ edges, which is in contradiction with the Claim 2. □

Note that by the Claim 3, each even path in B_1 contains at least 2ξ edges. Similarly by the Claim 4, each odd cycle in B_1 contains at least $2\xi+3$ edges. Let *odd components* denote odd cycles and even paths. Since $4\xi = 4\lfloor\frac{n-2}{4}\rfloor > \lfloor\frac{3n}{4}\rfloor \geq E(B_1)$ obviously holds for any $n \geq 10$, B_1 may contain at most one odd component. On the other hand, B_1

contains even number of odd componentss. As a result, B_1 contains only odd paths and even cycles, but then also a perfect matching of Γ, which is in contradiction with the Claim 5. $\square$

Conclusion

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