New approaches to nowhere-zero flow problems

Master thesis

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Nowhere-zero k-flows

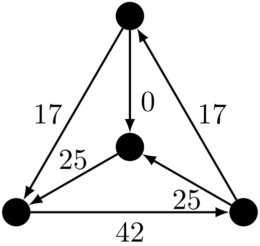


Figure: 43-flow

- assignment of values $0, 1, 2, \ldots, k-1$ to edges
- Kirchoff's law in vertices

Nowhere-zero k-flows

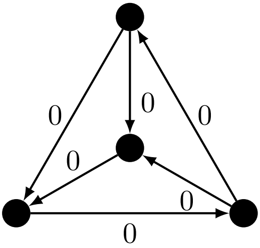


Figure: 1-flow

- assignment of values $0, 1, 2, \ldots, k-1$ to edges
- Kirchoff's law in vertices
- this allowing trivial cases
- restrict zero flow values

Nowhere-zero k-flows

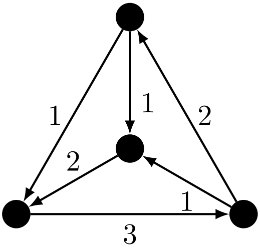


Figure: NZ 4-flow, $\Phi(K_4) = 4$

- assignment of values $0, 1, 2, \ldots, k-1$ to edges
- Kirchoff's law in vertices
- this allowing trivial cases
- restrict zero flow values
- graph with NZ k-flow has also a NZ $(k+1)$-flow
- rate of graph complexity
- flow number $\Phi(\Gamma)$ – minimum

Chebyshev NZ r-flows

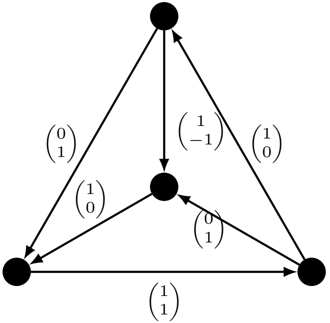


Figure: 2D ChNZ 2-flow, $\Phi_2^\infty(\Gamma) = 2$

- generalisation of one-dimensional flows
- $\left\|(x_1, x_2)^T\right\|_\infty = \max\{|x_1|, |x_2|\} \in [1, r-1]$
- flow number $\Phi_2^\infty(\Gamma)$

Known bounds

- generally proved $\Phi(\Gamma) \leq 6$
- generally conjectured $\Phi(\Gamma) \leq 5$
- for cubic graphs, 3-colourable iff $\Phi(\Gamma) \leq 4$
- generally proved $\Phi_2^\infty(\Gamma) \leq 3$
- generally conjectured $\Phi_2^\infty(\Gamma) \leq 5/2$
- for cubic graphs, 3-colourable iff $\Phi_2^\infty(\Gamma) = 2$

Sufficient flow-pairs

- NZ 6-flow constructed by Seymour from 2-flow and 3-flow

value of 2-flow	admitted values of 3-flow
0	1, 2
1	0, 1, 2

- using same flow pair we can construct 2D ChNZ 3-flow
- conjectured 2-flow and 4-flow, implying NZ 5-flow and 2D ChNZ $5/2$-flow

value of 2-flow	admitted values of 4-flow
0	2, 3
1	0, 1, 2, 3

Seymour's k-closures and k-bases

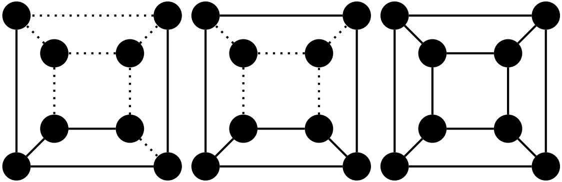


Figure: 2-base of Y_4, its 1-closure and 2-closure

- k-closure – add a circuit, which has at most k missing edges
- k-base – its k-closure is the original graph
- spanning tree is a 1-base

Flows and k-bases

- for a k-base $X \subseteq E$, there exists a $(k+1)$-flow that is non-zero on $E \setminus X$
- "each" graph has a 2-base, that is a "cycle"

Group connectivity

- analogue of list colourings
- we can deny any flow value for each of the edges

Forbidding more values

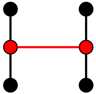
- k-base $X \subseteq E$, n-flow $\rightarrow$ we can forbid **strictly less than** n/k values for edges from $E \setminus X$
- inequality not strict $\Rightarrow$ 2-flow and 4-flow with $0, 1$ forbidden on 0 edges of the 2-flow $\Rightarrow$ conjectures on NZ 5-flow and ChNZ 2.5-flow proved
- prove for at least some classes of graphs

Lower bound on Φ_2^∞ for snarks

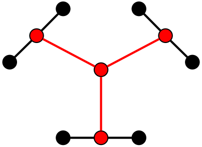
- we proved
$$\Phi_2^\infty \geq 2 + 1 \Big/ \left\lfloor \frac{n-2}{4} \right\rfloor$$
- tight for Petersen snarks, Blanuša snarks, but no other equality cases known
- can we do better?

Generation of snarks with given girth g

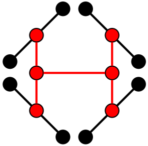
- naive algorithm – adding K_2



- last level of generation lasts the most – try adding tripod instead of last two levels



- even better with 6-vertex subgraph $\mathcal{H}$



Properties of generation

- each graph with $g \geq 5$ has a reducible tripod, after reduction, $g' \geq g - 1$
- each graph with $g \geq 5$ has a reducible $\mathcal{H}$, after reduction, $g' \geq g - 2$

Thank you for your attention!

Any questions?