Critical trajectories in kinetic geometry

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Kolmogorov equation

Here: $(t, x, v) \in \Omega_T = (0, T) \times \Omega_x \times \Omega_v \subset \mathbb{R}^{1+2d}$,
Study particle density $f = f(t, x, v)\colon \Omega_T \to \mathbb{R}$ solution to

$$(\partial_t + v \cdot \nabla_x) f = \nabla_v \cdot (\mathfrak{a}(t, x, v) \nabla_v f)$$

with $\mathfrak{a}\colon \Omega_T \to \mathbb{R}^{d \times d}$ measurable such that

(H1) $\quad 0 < \lambda := \inf\limits_{\substack{0 \neq \xi \in \mathbb{R}^d \\ (t,x,v) \in \Omega_T}} \frac{\langle \mathfrak{a}(t,x,v)\xi, \xi \rangle}{|\xi|^2}$

(H2) $\quad \Lambda := \sup\limits_{\substack{0 \neq \xi \in \mathbb{R}^d \\ (t,x,v) \in \Omega_T}} \frac{|\mathfrak{a}(t,x,v)\xi|^2}{\langle \mathfrak{a}(t,x,v)\xi, \xi \rangle} < \infty.$

Weak solutions

Definition:

A function $f \in \mathrm{L}^\infty_t \, \mathrm{L}^2_{x,v}(\Omega_T) \cap \mathrm{L}^2_{t,x} \, \dot{\mathrm{H}}^1_v(\Omega_T)$ is a weak (sub-, super-) solution to (1) if for all $\varphi \in \mathrm{C}^\infty_c(\Omega_T)$ with $\varphi \geq 0$ we have

$$\int_{\Omega_T} \Big[- f(\partial_t + v \cdot \nabla_x)\varphi + \langle \mathfrak{a} \nabla_v f, \nabla_v \varphi \rangle \Big] \mathrm{d}(t,x,v) = \,(\leq, \,\geq)\, 0.$$

Literature:

- Regularity, existence and uniqueness of weak solutions together with P. Auscher and C. Imbert 24
- previous works: Degond 86, Albritton-Armstrong-Mourrat-Novack 24, N.-Zacher 21, Nyström-Litsgård 21

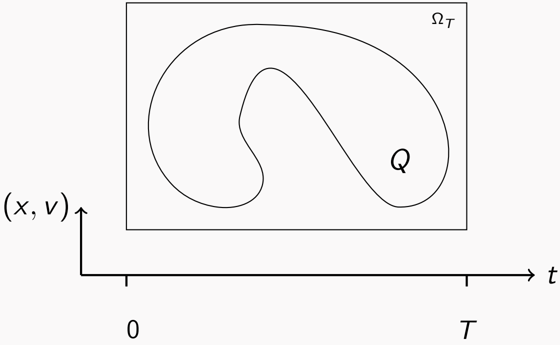
Kinetic De Giorgi-Nash-Moser theory

Hölder continuity

Theorem (Zhang 11, Wang & Zhang 09,11):

Let f be a weak solution to (1) and $Q \subset\subset \Omega_T$. Then, there exist constants $\varepsilon, C > 0$ such that $f \in \dot{\mathsf{C}}^{\varepsilon}_{\mathrm{kin}}(\bar{Q})$ and

$$\|f\|_{\dot{\mathsf{C}}^{\varepsilon}_{\mathrm{kin}}(\bar{Q})} \leq C \|f\|_{\mathsf{L}^\infty(\Omega_T)}.$$



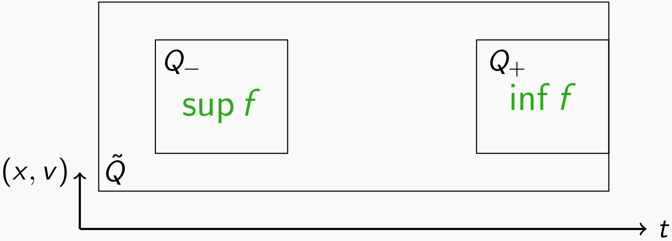
Harnack inequality

(1) $\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (\mathfrak{a} \nabla_v f)$

Theorem (GIMV 19, GI 22, GM 22):

There exists a universal const $C = C(d, \lambda, \Lambda) > 0$ such that for any nonnegative weak solution f of (1) in $\tilde{Q}$ we have

$$\sup_{Q_-} f \leq C \inf_{Q_+} f.$$



Weak L^1-estimate for $\log f$

$(1)\ \partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (\mathfrak{a} \nabla_v f)$

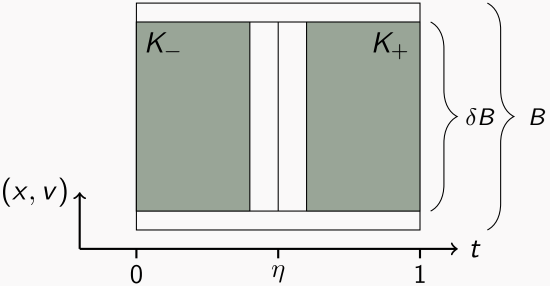
Theorem (DMNZ 25):

Let $\delta, \eta \in (0,1)$. Then for any supersolution $f > 0$ to (1) there exists a constant $C = C(d, \delta, \eta) > 0$ such that

$$s\,|\{(t,x,v) \in K_- \colon \log f(t,x,v) - c(f) > s\}| \le C\left(\tfrac{1}{\lambda} + \Lambda\right)$$

$$s\,|\{(t,x,v) \in K_+ \colon c(f) - \log f(t,x,v) > s\}| \le C\left(\tfrac{1}{\lambda} + \Lambda\right)$$

for all $s > 0$ with $c(f) = \dfrac{1}{c_\varphi} \displaystyle\int_B \log f(\eta, y, w) \varphi^2(y,w) \mathrm{d}(y,w)$.



The logarithm

Suppose that f is a positive weak supersolution

$$(\partial_t + v \cdot \nabla_x) f \geq \nabla_v \cdot (\mathfrak{a} \nabla_v f)$$

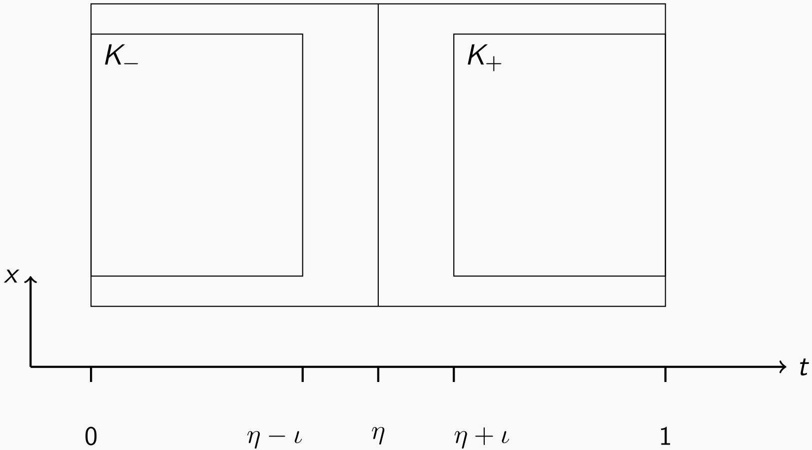
then the $g = \log f$ is a weak supersolution to

$$(\partial_t + v \cdot \nabla_x) g \geq \nabla_v \cdot (\mathfrak{a} \nabla_v g) + \langle \mathfrak{a} \nabla_v g, \nabla_v g \rangle.$$

Proof of the weak L^1-estimate

Unit scale. $\mathfrak{a} = \mathrm{Id}$ for simplicity. $f \geq \varepsilon$ and $\varepsilon \to 0^+$. Goal:

$$s\,|\{(t,x,v) \in K_- : \log f(t,x,v) - c(f) > s\}| \leq C, \quad s > 0$$



Proof of the weak L^1-estimate

Recall

$$c(f) = \frac{1}{c_\varphi} \int_B [\log f](\eta, y, w)\varphi^2(y, w)\mathrm{d}(y, w).$$

where

$$c_\varphi = \int_B \varphi^2(y, w)\mathrm{d}(y, w).$$

Proof of the weak L^1-estimate

Recall

$$c(f) = \frac{1}{c_\varphi} \int_B [\log f](\eta, y, w) \varphi^2(y, w) \mathrm{d}(y, w).$$

Note that

$$\begin{aligned} & s\,|\{(t,x,v) \in K_- \colon \ \log(f) - c(f) > s\}| \\ & \leq \int_0^{\eta-\iota} \int_B ([\log f](t,x,v) - c(f))_+ \mathrm{d}(t,x,v) \end{aligned}$$

Proof of the L^1-estimate

Recall

$$c(f) = \frac{1}{c_\varphi} \int\limits_B [\log f](\eta, y, w)\varphi^2(y, w)\mathrm{d}(y, w).$$

Goal: estimate

$$\int\limits_0^{\eta-\iota} \int\limits_B ([\log f](t, x, v) - c(f))_+ \mathrm{d}(t, x, v) \leq C$$

by a constant.

L^1-Poincaré inequality in spacetime without a gradient.

Proof of the L^1-estimate

Recall

$$c(f) = \frac{1}{c_\varphi} \int_B [\log f](\eta, y, w)\varphi^2(y, w)\mathrm{d}(y, w).$$

Goal: estimate

$$\int_0^{\eta-\iota} \int_B ([\log f](t, x, v) - c(f))_+ \mathrm{d}(t, x, v) \leq C$$

by a constant.

L^1-Poincaré inequality in spacetime without a gradient.

Recall: if f is supersolution to (1), then $g = \log f$ is a supersolution to

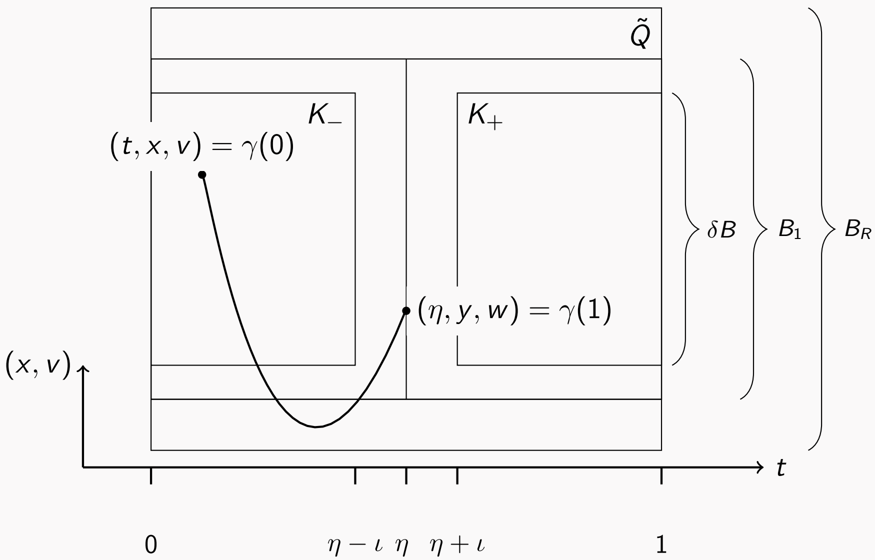
$$(\partial_t + v \cdot \nabla_x)g = \Delta_v g + |\nabla_v g|^2$$

Proof of the L^1-estimate

For $g = \log f$ we have

$$
\begin{aligned}
& g(t,x,v) - c(f) \\
&= \frac{1}{c_\varphi} \int_B \left(g(t,x,v) - g(\eta,y,w) \right) \varphi^2(y,w) \mathrm{d}(y,w) \\
&= -\frac{1}{c_\varphi} \int_B \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r)) \mathrm{d}r \; \varphi^2(y,w) \mathrm{d}(y,w)
\end{aligned}
$$

Critical kinetic trajectories



Proof of the L^1-estimate

For $g = \log f$ we have

$$
\begin{aligned}
&g(t,x,v) - c(f) \\
&= \frac{1}{c_\varphi} \int_B \left(g(t,x,v) - g(\eta,y,w))\right) \varphi^2(y,w) \mathrm{d}(y,w) \\
&= -\frac{1}{c_\varphi} \int_B \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r)) \mathrm{d}r \ \varphi^2(y,w)\mathrm{d}(y,w) \\
&= -\frac{1}{c_\varphi} \int_B \int_0^1 \dot{\gamma}_t(r)[(\partial_t + v \cdot \nabla_x)g](\gamma(r)) + \dot{\gamma}_v(r) \cdot [\nabla_v g](\gamma(r)) \mathrm{d}r \ \varphi^2 \mathrm{d}(y,w)
\end{aligned}
$$

Proof of the L^1-estimate

(1) $\partial_t g + v \cdot \nabla_x g \geq \Delta_v g + |\nabla_v g|^2$

For $g = \log f$ we have

$$
\begin{aligned}
&g(t,x,v) - c(f) \\
&= \frac{1}{c_\varphi} \int_B \left(g(t,x,v) - g(\eta,y,w))\right) \varphi^2(y,w) \mathrm{d}(y,w) \\
&= -\frac{1}{c_\varphi} \int_B \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r)) \mathrm{d}r\ \varphi^2(y,w) \mathrm{d}(y,w) \\
&= -\frac{1}{c_\varphi} \int_B \int_0^1 \dot{\gamma}_t(r) [(\partial_t + v \cdot \nabla_x) g](\gamma(r)) + \dot{\gamma}_v(r) \cdot [\nabla_v g](\gamma(r)) \mathrm{d}r\ \varphi^2 \mathrm{d}(y,w) \\
&\leq -\frac{\eta - t}{c_\varphi} \int_B \int_0^1 [\Delta_v g](\gamma(r)) + |\nabla_v g|^2 (\gamma(r)) \mathrm{d}r\ \varphi^2(y,w) \mathrm{d}(y,w) \\
&\quad - \frac{1}{c_\varphi} \int_B \int_0^1 \dot{\gamma}_v(r) \cdot [\nabla_v g](\gamma(r)) \mathrm{d}r\ \varphi^2(y,w) \mathrm{d}(y,w)
\end{aligned}
$$

Proof of the L^1-estimate

(1) $\partial_t g + v \cdot \nabla_x g \geq \Delta_v g + |\nabla_v g|^2$

For $g = \log f$ we have

$$
\begin{aligned}
& g(t,x,v) - c(f) \\
&= \frac{1}{c_\varphi} \int_B \left(g(t,x,v) - g(\eta,y,w) \right) \varphi^2(y,w) \mathrm{d}(y,w) \\
&= -\frac{1}{c_\varphi} \int_B \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r)) \mathrm{d}r \; \varphi^2(y,w) \mathrm{d}(y,w) \\
&= -\frac{1}{c_\varphi} \int_B \int_0^1 \dot{\gamma}_t(r) [\partial_t g + v \cdot \nabla_x g](\gamma(r)) + \dot{\gamma}_v(r) \cdot [\nabla_v g](\gamma(r)) \mathrm{d}r \; \varphi^2 \mathrm{d}(y,w) \\
&\leq -\frac{\eta - t}{c_\varphi} \int_B \int_0^1 [\Delta_v g](\gamma(r)) + |\nabla_v g|^2 (\gamma(r)) \mathrm{d}r \; \varphi^2(y,w) \mathrm{d}(y,w) \\
&\quad - \frac{1}{c_\varphi} \int_B \int_0^1 \dot{\gamma}_v(r) \cdot [\nabla_v g](\gamma(r)) \mathrm{d}r \; \varphi^2(y,w) \mathrm{d}(y,w)
\end{aligned}
$$

Idea: use quadratic gradient term to absorb all gradients

The forcing terms

Recall that $|\dot{\gamma}_v| \lesssim r^{-\frac{1}{2}}$, hence

$$-\frac{1}{c_\varphi}\int_B \int_0^1 \dot{\gamma}_v(r)\cdot[\nabla_v g](\gamma(r))\mathrm{d}r\ \varphi^2(y,w)\mathrm{d}(y,w)$$
$$\lesssim \int_B \int_0^1 r^{-\frac{1}{2}}\left|\nabla_v g\right|(\gamma(r))\mathrm{d}r\ \varphi(y,w)\mathrm{d}(y,w)$$

Partial integration

$$\int_B [\Delta_v g](\gamma(r))\varphi^2(y,w)\mathrm{d}(y,w)$$

Partial integration

Substitute $(\tilde{y}, \tilde{w}) = \Phi(y, w) = \Phi_{r,t,x,v,\eta}(y, w) := (\gamma_x(r), \gamma_v(r))$.

$$\int_B [\Delta_v g](\gamma(r))\varphi^2(y, w)\mathrm{d}(y, w)$$

$$= \int_{\Phi(B)} [\Delta_v g](\gamma_t(r), \tilde{y}, \tilde{w})\varphi^2(\Phi^{-1}(\tilde{y}, \tilde{w}))\, |\det \mathcal{A}|^{-1}\, \mathrm{d}(\tilde{y}, \tilde{w})$$

Partial integration

Substitute $(\tilde{y}, \tilde{w}) = \Phi(y, w) = \Phi_{r,t,x,v,\eta}(y, w) := (\gamma_x(r), \gamma_v(r))$.

$$\int_B [\Delta_v g](\gamma(r))\varphi^2(y, w)\mathrm{d}(y, w)$$

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$$= -\int_{\Phi(B)} \langle [\nabla_v g](\gamma_t(r), \tilde{y}, \tilde{w}), \nabla_v \varphi^2(\Phi^{-1}(\tilde{y}, \tilde{w}))\rangle\,|\det \mathcal{A}|^{-1}\,\mathrm{d}(\tilde{y}, \tilde{w})$$

$$= -2\int_{\Phi(B)} \langle [\nabla_v g](\gamma_t(r), \tilde{y}, \tilde{w}), [\nabla\varphi]^T(\Phi^{-1}(\tilde{y}, \tilde{w}))(\mathcal{A}(r)^{-1})_{\cdot;2}\rangle$$

$$\varphi(\Phi^{-1}(\tilde{y}, \tilde{w})) \cdot |\det \mathcal{A}|^{-1}\,\mathrm{d}(\tilde{y}, \tilde{w})$$

Partial integration

Substitute $(\tilde{y}, \tilde{w}) = \Phi(y, w) = \Phi_{r,t,x,v,\eta}(y, w) := (\gamma_x(r), \gamma_v(r))$.

$$
\begin{aligned}
&\int_B [\Delta_v g](\gamma(r))\varphi^2(y, w)\mathrm{d}(y, w) \\
&= \int_{\Phi(B)} [\Delta_v g](\gamma_t(r), \tilde{y}, \tilde{w})\varphi^2(\Phi^{-1}(\tilde{y}, \tilde{w}))\, |\det \mathcal{A}|^{-1}\, \mathrm{d}(\tilde{y}, \tilde{w}) \\
&= -\int_{\Phi(B)} \langle [\nabla_v g](\gamma_t(r), \tilde{y}, \tilde{w}), \nabla_v \varphi^2(\Phi^{-1}(\tilde{y}, \tilde{w}))\rangle\, |\det \mathcal{A}|^{-1}\, \mathrm{d}(\tilde{y}, \tilde{w}) \\
&= -2\int_{\Phi(B)} \langle [\nabla_v g](\gamma_t(r), \tilde{y}, \tilde{w}), [\nabla\varphi]^T(\Phi^{-1}(\tilde{y}, \tilde{w}))(\mathcal{A}(r)^{-1})_{\cdot;2}\rangle \\
&\qquad\qquad \varphi(\Phi^{-1}(\tilde{y}, \tilde{w})) \cdot |\det \mathcal{A}|^{-1}\, \mathrm{d}(\tilde{y}, \tilde{w}) \\
&= -2\int_B \langle [\nabla_v g](\gamma(r)), [\nabla\varphi]^T(y, w)(\mathcal{A}(r)^{-1})_{\cdot;2}\rangle \varphi(y, w)\mathrm{d}(y, w) \\
&\lesssim r^{-1/2} \int_B |\nabla_v g|\, (\gamma(r))\varphi(y, w)\mathrm{d}(y, w),
\end{aligned}
$$

Distributing the good term

$$(g(t,x,v) - c(f))_+$$
$$\lesssim \int_0^1 \int_B \Big(M r^{-1/2} \left| \nabla_v g \right| (\gamma(r)) \varphi(y,w) - \left| \nabla_v g \right|^2 (\gamma(r)) \varphi^2(y,w) \Big)_+ \mathrm{d}(y,w) \mathrm{d}r$$

for some constant $M > 0$.

Integrating on K_-

$$\int_0^{\eta-\iota}\int_B (g(t,x,v)-c(f))_+ \mathrm{d}(t,x,v)$$
$$\leq \int_0^{\eta}\int_B\int_0^1\int_B \left(Mr^{-1/2}\left|\nabla_v g\right|(\gamma(r))\varphi(y,w)-\left|\nabla_v g\right|^2(\gamma(r))\varphi^2(y,w)\right)_+ \mathrm{d}(y,w)\mathrm{d}r\mathrm{d}(t,x,v)$$

Integrating on K_-

$$
\begin{aligned}
&\int_0^{\eta-\iota}\int_B (g(t,x,v)-c(f))_+ \mathrm{d}(t,x,v)\\
&\leq \int_0^{\eta}\int_B\int_0^1\int_B \Big(Mr^{-1/2}\left|\nabla_v g\right|(\gamma(r))\varphi(y,w)-\left|\nabla_v g\right|^2(\gamma(r))\varphi^2(y,w)\Big)_+ \mathrm{d}(y,w)\mathrm{d}r\mathrm{d}(t,x,v)\\
&=\int_0^{\eta}\int_B\int_0^{\frac{1}{2}}\int_B \Big(Mr^{-1/2}\left|\nabla_v g\right|(\gamma(r))\varphi(y,w)-\left|\nabla_v g\right|^2(\gamma(r))\varphi^2(y,w)\Big)_+ \mathrm{d}(y,w)\mathrm{d}r\mathrm{d}(t,x,v)\\
&+\int_0^{\eta}\int_B\int_{\frac{1}{2}}^1\int_B \Big(Mr^{-1/2}\left|\nabla_v g\right|(\gamma(r))\varphi(y,w)-\left|\nabla_v g\right|^2(\gamma(r))\varphi^2(y,w)\Big)_+ \mathrm{d}(y,w)\mathrm{d}r\mathrm{d}(t,x,v)
\end{aligned}
$$

Integrating on K_-

$$
\begin{aligned}
&\int_0^{\eta-\iota}\int_B (g(t,x,v)-c(f))_+ \mathrm{d}(t,x,v)\\
&\leq \int_0^{\eta}\int_B\int_0^1\int_B \Big(Mr^{-1/2}\left|\nabla_v g\right|(\gamma(r))\varphi(y,w)-\left|\nabla_v g\right|^2(\gamma(r))\varphi^2(y,w)\Big)_+ \mathrm{d}(y,w)\mathrm{d}r\mathrm{d}(t,x,v)\\
&=\int_0^{\eta}\int_B\int_0^{\frac{1}{2}}\int_B \qquad\qquad \dots \qquad\qquad \mathrm{d}(y,w)\mathrm{d}r\mathrm{d}(t,x,v)\\
&+\int_0^{\eta}\int_B\int_{\frac{1}{2}}^1\int_B \Big(Mr^{-1/2}\left|\nabla_v g\right|(\gamma(r))\varphi(y,w)-\left|\nabla_v g\right|^2(\gamma(r))\varphi^2(y,w)\Big)_+ \mathrm{d}(y,w)\mathrm{d}r\mathrm{d}(t,x,v)\\
&\leq\int_0^{\eta}\int_B\int_0^{\frac{1}{2}}\int_B \qquad\qquad \dots \qquad\qquad \mathrm{d}(y,w)\mathrm{d}r\mathrm{d}(t,x,v)\\
&+C =: I_1 + C
\end{aligned}
$$

for some $C > 0$ by Cauchy-Schwarz inequality.

Estimating I_1

Substitute $(\tilde{x}, \tilde{v}) = \Psi_{r,t,\eta,y,w}(x, v) := \gamma_{x,v}(r)$ and $\tilde{t} = t + r(\eta - t)$.

$$I_1 = \int_0^\eta \int_B \int_0^{\frac{1}{2}} \int_B \Big(M r^{-1/2} \left|\nabla_v g\right| (\gamma(r)) \varphi(y, w) - \left|\nabla_v g\right|^2 (\gamma(r)) \varphi^2(y, w) \Big)_+ \mathrm{d}(y, w) \mathrm{d}r \mathrm{d}(x, v) \mathrm{d}t$$

Estimating I_1

Substitute $(\tilde{x}, \tilde{v}) = \Psi_{r,t,\eta,y,w}(x, v) := \gamma_{x,v}(r)$ and $\tilde{t} = t + r(\eta - t)$.

$$I_1 \leq \int_0^{\frac{1}{2}} \int_B \int_0^{\eta} \int_{\Psi(B)} \Big(M r^{-1/2} \left|\nabla_v g\right| (\tilde{t}, \tilde{x}, \tilde{v}) \varphi(y, w) - \left|\nabla_v g\right|^2 (\tilde{t}, \tilde{x}, \tilde{v}) \varphi^2(y, w) \Big)_+ \frac{1}{1-r} \left|\det \mathcal{B}(r)\right|^{-1} \mathrm{d}(\tilde{x}, \tilde{v}) \mathrm{d}\tilde{t} \mathrm{d}(y, w) \mathrm{d}r$$

Estimating I_1

Substitute $(\tilde{x}, \tilde{v}) = \Psi_{r,t,\eta,y,w}(x, v) := \gamma_{x,v}(r)$ and $\tilde{t} = t + r(\eta - t)$.

$$I_1 \leq \int_0^{\frac{1}{2}} \int_B \int_0^{\eta} \int_{\tilde{B}} \Big(M r^{-1/2} \left|\nabla_v g\right| (\tilde{t}, \tilde{x}, \tilde{v}) \varphi(y, w) - \left|\nabla_v g\right|^2 (\tilde{t}, \tilde{x}, \tilde{v}) \varphi^2(y, w) \Big)_+ \mathrm{d}(\tilde{x}, \tilde{v}) \mathrm{d}\tilde{t} \mathrm{d}(y, w) \mathrm{d}r$$

as $\Psi(B) \subset \tilde{B}$ and $\det \mathcal{B}(r) \sim 1$ on $(\frac{1}{2}, 1)$.

Estimating I_1

Substitute $(\tilde{x}, \tilde{v}) = \Psi_{r,t,\eta,y,w}(x, v) := \gamma_{x,v}(r)$ and $\tilde{t} = t + r(\eta - t)$.

$$I_1 \leq \int_0^{\frac{1}{2}} \int_B \int_0^{\eta} \int_{\tilde{B}} \left(M r^{-1/2} \left| \nabla_v g \right| (\tilde{t}, \tilde{x}, \tilde{v}) \varphi(y, w) - \left| \nabla_v g \right|^2 (\tilde{t}, \tilde{x}, \tilde{v}) \varphi^2(y, w) \right)_+ \mathrm{d}(\tilde{x}, \tilde{v}) \mathrm{d}\tilde{t} \mathrm{d}(y, w) \mathrm{d}r$$

as $\Psi(B) \subset \tilde{B}$ and $\det \mathcal{B}(r) \sim 1$ on $(\frac{1}{2}, 1)$.

Calculating the r-integral from 0 to $\min\{1/2, M^2/p^2\}$ yields

$$\int_0^{1/2} \left(r^{-1/2} M p - p^2 \right)_+ \mathrm{d}r \lesssim M^2$$

for all $p > 0$. Here $p = \left| \nabla_v g \right| (\tilde{t}, \tilde{x}, \tilde{v}) \varphi(y, w)$.

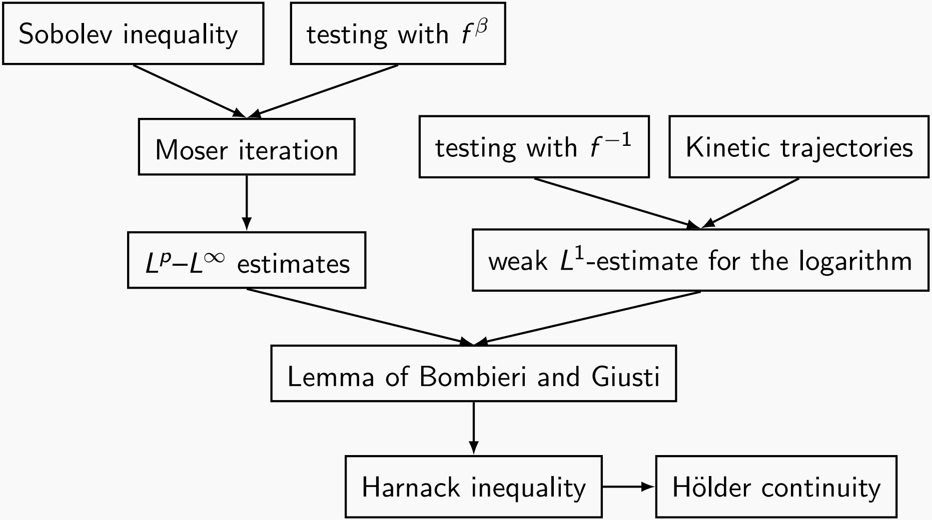
Conclusion of the L^1-estimate

We have proven

$$\int_0^{\eta-\iota}\int_B ([\log f](t,x,v) - c(f))_+ \mathrm{d}(t,x,v) \leq C$$

for universal constant $C > 0$.

Moser's 1971 method in kinetic theory



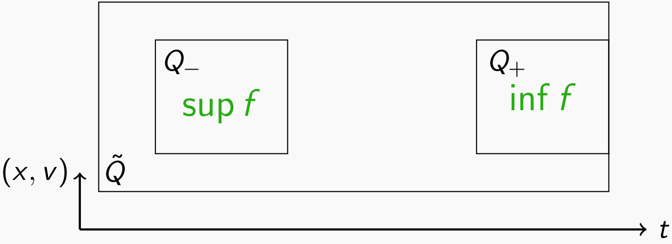
Harnack inequality

Theorem (DMNZ 25):

There exists a universal constant $C = C(d) > 0$ such that for any nonnegative weak solution f of (1) in $\tilde{Q}$ we have

$$\sup_{Q_-} f \leq C^{\mu} \inf_{Q_+} f.$$

Here, $\mu = \frac{1}{\lambda} + \Lambda$. Optimal in μ!



Weak Harnack inequality

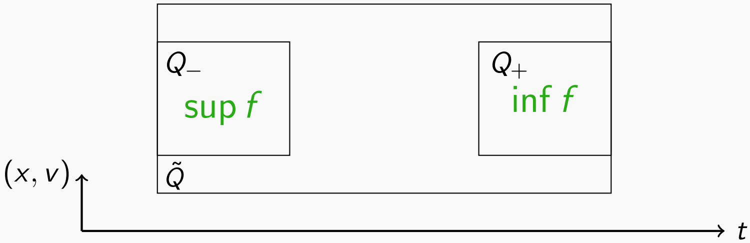
$(1)\ \partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (\mathfrak{a} \nabla_v f)$

Theorem (DMNZ 25):

There exists a universal $C(d,\mu) > 0$ such that for all $p \in (0, 1 + \frac{1}{2d})$ and any nonnegative weak supersolution f to (1) in $\tilde{Q}$ we have

$$\left(\int_{Q_-} f^p \mathrm{d}(t,x,v) \right)^p \leq C \inf_{Q_+} f.$$

Optimal range for p.



Kinetic mollification

Consider $\gamma^{\mathrm{m}} \colon \mathbb{R} \to \mathbb{R}^{1+2d}$ with $\mathrm{m} = (\mathrm{m}_0, \mathrm{m}_1, \mathrm{m}_2) \in \mathbb{R}^{1+2d}$, $\mathrm{m}_0 \neq 0$ as

$$\gamma^{\mathrm{m}}(r;(t,x,v)) = \begin{pmatrix} \gamma^{\mathrm{m}}_t(r) \\ \gamma^{\mathrm{m}}_x(r) \\ \gamma^{\mathrm{m}}_v(r) \end{pmatrix} = \begin{pmatrix} t + m_0 r \\ \mathcal{E}_{m_0}(r) \begin{pmatrix} x \\ v \end{pmatrix} + \mathcal{D}_{m_0} \mathcal{W}(r) \mathcal{D}_{m_0}^{-1} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \end{pmatrix}$$

with

$$\mathcal{W}(r) := \begin{pmatrix} g_1(r) & g_2(r) \\ \dot{g}_1(r) & \dot{g}_2(r) \end{pmatrix}, \; \mathcal{D}_{m_0} := \begin{pmatrix} m_0 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \mathcal{E}_{m_0}(r) = \begin{pmatrix} 1 & m_0 r \\ 0 & 1 \end{pmatrix}.$$

Define the kinetic mollification operator as

$$[S_r(f)](t,x,v) = \frac{1}{c_\varphi} \int_{\mathbb{R}^{1+2d}} f(\gamma^{\mathrm{m}}(r;(t,x,v)))\varphi^2(\mathrm{m}) \mathrm{d}\mathrm{m}.$$

Kinetic Sobolev embedding

Theorem (DMNZ 25):

Let $f \in \mathrm{L}^2(\mathbb{R}^{1+d}; \mathrm{H}^1(\mathbb{R}^d))$ such that $(\partial_t + v \cdot \nabla_x) f = \nabla_v \cdot S$ for some $S \in \mathrm{L}^2(\mathbb{R}^{1+2d}; \mathbb{R}^d)$, then

$$\|f\|_{\mathrm{L}^{2\kappa}(\mathbb{R}^{1+2d})} \leq C \left(\|\nabla_v f\|_{\mathrm{L}^2(\mathbb{R}^{1+2d})} + \|S\|_{\mathrm{L}^2(\mathbb{R}^{1+2d};\mathbb{R}^d)} \right)$$

with $\kappa = 1 + \frac{1}{2d}$ and $C = C(d) > 0$.

Local versions. No fundamental solution needed.

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