Kinetic trajectories

and a kinetic Poincaré inequality

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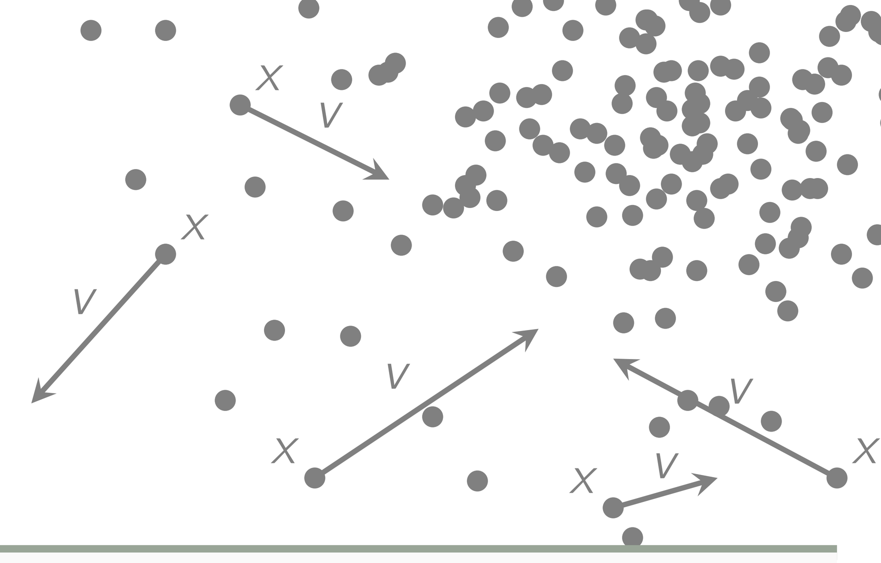


A kinetic Poincaré inequality

Theorem 1 ([1, 2]):

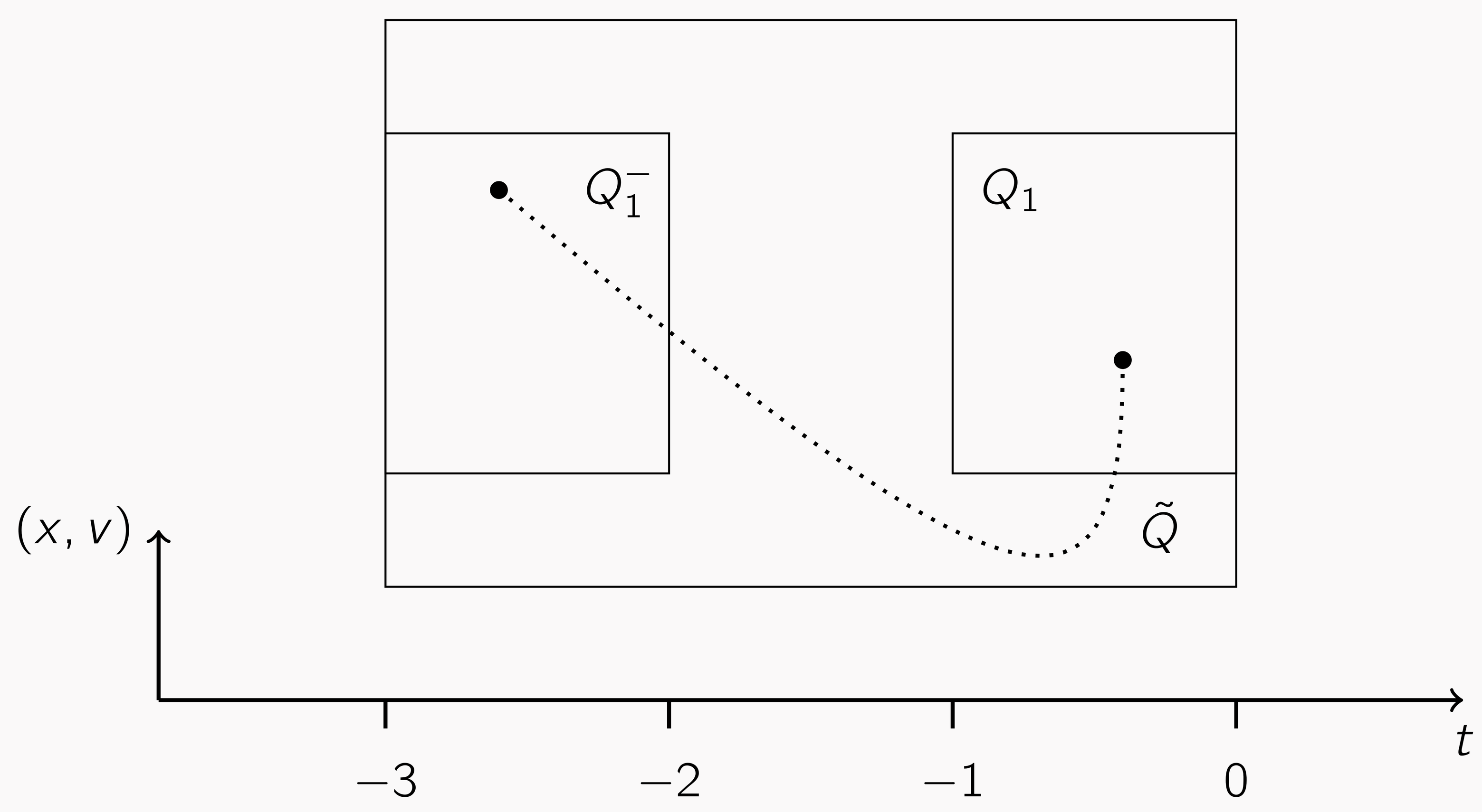
There exists $R > 1$ such that the following holds for any $0 \leq \varphi \in C_c^\infty(B_1 \times B_1)$ nonzero. Let $f \in L^1_{t,x}((-3,0] \times B_R; W^{1,1}_v(B_R))$ be such that

$$(\partial_t + v \cdot \nabla_x) f \leq \nabla_v \cdot S$$

in the distributional sense for some $S \in L^1_{t,x,v}((-3,0] \times B_R \times B_R; \mathbb{R}^d)$. Then,

$$\left\| \left(f - \frac{1}{c_\varphi} \int_{Q_1^-} f\varphi \, \mathrm{d}(s,y,w) \right)_+ \right\|_{L^1(Q_1)} \leq C \left(\|\nabla_v f\|_{L^1(\tilde{Q})} + \|S\|_{L^1(\tilde{Q})} \right),$$

where $C = C(d,\varphi) > 0$, $c_\varphi = \int_{Q_1^-} \varphi \mathrm{d}(s,y,w)$, $Q_1^- = (-3,-2] \times B_1 \times B_1$, $Q_1 = (-1,0] \times B_1 \times B_1$, and $\tilde{Q} = (-3,0] \times B_R \times B_R$.



Kinetic trajectories

Definition:

Let $(t_0, x_0, v_0), (t_1, x_1, v_1) \in \mathbb{R}^{1+2d}$ with $t_0 \neq t_1$. A kinetic trajectory is a map $\gamma \in C([0,1]; \mathbb{R}^{1+2d})$ differentiable in $(0,1)$ with $\gamma(0) = (t_0, x_0, v_0)$ and $\gamma(1) = (t_1, x_1, v_1)$ satisfying $\dot\gamma_x = \dot\gamma_t \gamma_v$ in $(0,1)$.

Important property:

For $g \colon \mathbb{R}^{1+2d} \to \mathbb{R}$ smooth we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r} g(\gamma(r)) &= \dot\gamma_t(r)[\partial_t g] + \dot\gamma_x(r) \cdot [\nabla_x g](\gamma(r)) + \dot\gamma_v(r) \cdot [\nabla_v g](\gamma(r)) \\ &= \dot\gamma_t(r)[\partial_t g + v \cdot \nabla_x g](\gamma(r)) + \dot\gamma_v(r) \cdot [\nabla_v g](\gamma(r)). \end{aligned}$$

Construction ([2, 3]):

Ansatz: set $\gamma_t = t_0 + (t_1 - t_0) r$ (kinetic scaling $r \mapsto (rt, r^{\frac{3}{2}} x, r^{\frac{1}{2}} v)$) and

$$\dot\gamma_v(r) = \ddot{g}_0(r) \mathbf{m}_0 + \ddot{g}_1(r) \mathbf{m}_1$$

for two forcings $\ddot{g}_0, \ddot{g}_1 \colon [0,1] \to \mathbb{R}$ and vectorial parameters $\mathbf{m}_0, \mathbf{m}_1 \in \mathbb{R}^d$.

Integration and $\dot\gamma_x = \dot\gamma_t \gamma_v$ yields

$$\begin{cases} \gamma_t(r) = t_0 + (t_1 - t_0) r \\ \gamma_x(r) = (t_1 - t_0) g_0(r) \mathbf{m}_0 + (t_1 - t_0) g_1(r) \mathbf{m}_1 + (t_1 - t_0) r v_0 + x_0 \\ \gamma_v(r) = \dot{g}_0(r) \mathbf{m}_0 + \dot{g}_1(r) \mathbf{m}_1 + v_0. \end{cases}$$

Ansatz: $g_0(r) \sim r^{p_0}$ and $g_1(r) \sim r^{p_1}$ for $p_0 \neq p_1 \in (1,2)$. This is needed to have enough linear independence at $r = 1$ and for good properties of γ.

If this is the case, solving $\gamma(1) = (t_1, x_1, v_1)$ yields $\mathbf{m}_0, \mathbf{m}_1$.

Proof

Take kinetic trajectory connecting $(t,x,v) \in Q_1$ with $(s,y,w) \in Q_1^-$. Then

$$\begin{aligned} f - \frac{1}{c_\varphi} \int_{Q_1^-} f\varphi \, \mathrm{d}(s,y,w) &= \frac{1}{c_\varphi} \int_{Q_1^-} (f(t,x,v) - f(s,y,w)) \varphi(y,w) \mathrm{d}(s,y,w) \\ &= -\frac{1}{c_\varphi} \int_{Q_1^-} \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} f(\gamma(r)) \mathrm{d}r \varphi(y,w) \mathrm{d}(s,y,w) \\ &= -\frac{1}{c_\varphi} \int_0^1 \int_{Q_1^-} \dot\gamma_t(r) [(\partial_t + v \cdot \nabla_x) f](\gamma(r)) \varphi(y,w) \mathrm{d}(s,y,w) \mathrm{d}r \\ &\quad - \frac{1}{c_\varphi} \int_0^1 \int_{Q_1^-} \dot\gamma_v(r) \cdot [\nabla_v f](\gamma(r)) \varphi(y,w) \mathrm{d}(s,y,w) \mathrm{d}r =: I_1 + I_2. \end{aligned}$$

As $\dot\gamma_t(r) = (s - t) \leq 0$, we can use the subsolution property. Substitute

$$\begin{pmatrix} \tilde{y} \\ \tilde{w} \end{pmatrix} = \Phi_{r,t,x,v,s}(y,w) = \begin{pmatrix} \gamma_x(r) \\ \gamma_v(r) \end{pmatrix} = \mathcal{A}_{s-t}(r) \begin{pmatrix} y \\ w \end{pmatrix} + \mathcal{B}_{s-t}(r) \begin{pmatrix} x \\ v \end{pmatrix}$$

to perform a partial integration as follows

$$\begin{aligned} I_1 &\lesssim \int_0^1 \int_{Q_1^-} [\nabla_v \cdot S](\gamma(r)) \varphi(y,w) \mathrm{d}(s,y,w) \mathrm{d}r \\ &= \int_0^1 \int_{-3}^{-2} \int_{\Phi(B)} [\nabla_v \cdot S](\gamma_t(r), \tilde{y}, \tilde{w}) \varphi(\Phi^{-1}(\tilde{y}, \tilde{w})) \, |\det \mathcal{A}(r)|^{-1} \, \mathrm{d}(\tilde{y}, \tilde{w}) \mathrm{d}s \mathrm{d}r \\ &= \int_0^1 \int_{-3}^{-2} \int_{\Phi(B)} S(\gamma_t(r), \tilde{y}, \tilde{w}) \cdot ([\nabla\varphi]^T (\Phi^{-1}(\tilde{y}, \tilde{w})) [\mathrm{D}_{\tilde{w}} \Phi^{-1}](\tilde{y}, \tilde{w})) \\ &\qquad \cdot |\det \mathcal{A}(r)|^{-1} \, \mathrm{d}(\tilde{y}, \tilde{w}) \mathrm{d}s \mathrm{d}r \\ &= \int_0^1 \int_{Q_1^-} S(\gamma(r)) \cdot ([\nabla\varphi]^T (y,w) (\mathcal{A}^{-1})_{:,2}) \mathrm{d}(s,y,w) \mathrm{d}r \\ &\lesssim \int_0^1 \int_{Q_1^-} r^{1 - \max\{p_0, p_1\}} \, |S| \, (\gamma(r)) \mathrm{d}(s,y,w) \mathrm{d}r. \end{aligned}$$

Moreover,

$$\begin{aligned} I_2 &\lesssim \int_0^1 \int_{Q_1^-} |\dot\gamma_v(r)| \, |\nabla_v u| \, (\gamma(r)) \varphi(y,w) \mathrm{d}(s,y,w) \mathrm{d}r \\ &\lesssim \int_0^1 \int_{Q_1^-} r^{\min\{p_0, p_1\} - 2} \, |\nabla_v u| \, (\gamma(r)) \mathrm{d}(s,y,w) \mathrm{d}r. \end{aligned}$$

The singularities are integrable due to the careful construction. Next, integrate on Q_1 split the r integral in $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ and change coordinates with the opposite end of the trajectory.

Literature

- Theorem 1 was proven first in [1] using piecewise trajectories along the vector fields $\partial_t + v \cdot \nabla_x, \nabla_v$, and ∇_x where control on the transfer of regularity is needed.
- A similar Poincaré inequality for weak solutions is proven in Albritton-Armstrong-Mourrat-Novack '24, see also Guerand-Imbert-Mouhot '24.
- Follow-up work [3] studying a Kolmogorov equation with more commutators.
- Theorem 1 is an important tool in the context of the De Giorgi-Nash-Moser theory for the Kolmogorov equation with rough coefficients, see [1].
- Trajectories and Moser's approach to the regularity of weak solutions to elliptic/parabolic equations with rough coefficients is studied in [4].

References

[1] Jessica Guerand and Clément Mouhot. Quantitative De Giorgi methods in kinetic theory. *Journal de l'École polytechnique. Mathématiques*, 2022.

[2] Lukas Niebel and Rico Zacher. On a kinetic Poincaré inequality and beyond. *Journal of Functional Analysis*, 2025.

[3] Francesca Anceschi, Helge Dietert, Jessica Guerand, Amélie Loher, Clément Mouhot, and Annalaura Rebucci. Poincaré inequality and quantitative De Giorgi method for hypoelliptic operators. arXiv:2401.12194, 2024.

[4] Lukas Niebel and Rico Zacher. A trajectorial interpretation of Moser's proof of the Harnack inequality. *To appear in Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie V*, 2025.