The Kolmogorov equation in $L^p(\mathbb{R}^n)$
Semigroup, spectrum, maximal regularity?

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The (fractional) Kolmogorov equation

$$\begin{cases} \partial_t u + v \cdot \nabla_x u + (-\Delta_v)^{\frac{\beta}{2}} u = f, & t > 0 \\ u(0) = g, \end{cases} \tag{1}$$

where $u = u(t, x, v)$ and $\beta \in (0, 2]$ has gained more and more interest in the past years. This is mainly due to the following three phenomena. First, it can be seen as the prototype of a kinetic partial differential equation similar to the famous Boltzmann or the Landau equation. Second, even though the Laplacian only acts in half of the variables, i.e. the equation is degenerate, solutions of this equation admit good regularity properties. Last, but not least, it serves as an excellent example to study the regularity transfer, a special feature of kinetic equations.

The Kolmogorov equation admits a fundamental solution, which has been discovered already in 1934 by Kolmogorov for $\beta = 2$, [2]. Hence, we can define the corresponding semigroup for example in $L^p(\mathbb{R}^{2n})$ and study its properties, see also [3]. For example, we can consider the generator, which turns out to be

$$Au = \Delta_v u - v \cdot \nabla_x u$$

with domain $D(A) = \{u \in L^p(\mathbb{R}^{2n}) \colon \Delta_v u, v \cdot \nabla_x u \in L^p(\mathbb{R}^{2n})\}$ for $\beta = 2$.

The degeneracy of the equation can also be seen at the level of the spectrum of A. It can be shown that $i\mathbb{R} \subset \sigma(A)$ so that A can't be the generator of a holomorphic semigroup in $L^p(\mathbb{R}^{2n})$. Actually, the spectrum can be even characterized to be the complex half-plane $\{z \in \mathbb{C} \colon \mathrm{Re}(z) \leq 0\}$ as shown in [1, 5]. In particular, this proves that maximal L^p-regularity is unavailable. However, there is a way to overcome this by incorporating the kinetic structure of the equation, see [4].

In this project, we are going to study equation (1) starting at the fundamental solution and the corresponding semigroup, we will characterize the spectrum and take a look on how to overcome the lack of analyticity. Furthermore, we may look at related nonlinear problems or study the spectrum of the fractional Kolmogorov equation. We are mainly going to work with [1, 3].

References

[1] S. Fornaro, G. Metafune, D. Pallara, and R. Schnaubelt. L^p-spectrum of degenerate hypoelliptic ornstein-uhlenbeck operators. *Journal of Functional Analysis*, 280(2), 2021.

[2] A. Kolmogoroff. Zufallige Bewegungen (Zur Theorie der Brownschen Bewegung). *Annals of Mathematics*, 35(1):116–117, 1934.

[3] L. Lorenzi. *Analytical methods for Kolmogorov equations.* Monographs and research notes in mathematics. CRC Press/Taylor & Francis Group, Boca Raton, second edition edition, 2017.

[4] L. Niebel and R. Zacher. Kinetic maximal L^p-regularity with temporal weights and application to quasilinear kinetic diffusion equations. *Journal of Differential Equations*, 307, 2022.

[5] G. Metafune. L^p-spectrum of Ornstein-Uhlenbeck operators. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, 30(1), 2001.