Definition 1. *Convex polytope* $\mathbf{S}$ is the set of all convex combinations of a finite point set $\mathcal{S}$, i.e.

$$\mathbf{S} := \left\{ v \in \mathbb{R}^n : v := \sum_{i=1}^{|\mathcal{S}|} \alpha_i v_i; \forall i = 1, \dots, |\mathcal{S}| : \alpha_i \geq 0, v_i \in \mathcal{S}, \sum_{i=1}^{|\mathcal{S}|} \alpha_i = 1 \right\}.$$

Let us consider two convex polytopes $\mathbf{P}, \mathbf{Q}$ in $\mathbb{R}^d$ defined by the boundary point sets

$$\begin{array}{rcl} \mathcal{P} & := & \{p_1, \dots, p_{n_p}\} \subset \mathbb{R}^d \ , \\ \mathcal{Q} & := & \{q_1, \dots, q_{n_q}\} \subset \mathbb{R}^d \ . \end{array}$$

The problem is to find the shortest distance between these two objects

$$\min_{p \in \mathcal{P}, q \in \mathcal{Q}} \|p - q\| \ . \tag{1}$$

Every interior point of the convex polytope can be expressed as a convex linear combination of given points in sets $\mathcal{P}$ and $\mathcal{Q}$

$$\begin{array}{l} \forall p \in \mathbf{P} \ \exists \alpha_1, \dots, \alpha_{n_p} \in \mathbb{R} : p = \sum\limits_{i=1}^{n_p} \alpha_i p_i \ , \\ \qquad\qquad \text{where } \sum\limits_{i=1}^{n_p} \alpha_i = 1 \text{ and } 0 \leq \alpha_i \leq 1 \ \forall i = 1, \dots, n_p \ , \\ \forall q \in \mathbf{Q} \ \exists \beta_1, \dots, \beta_{n_q} \in \mathbb{R} : q = \sum\limits_{i=1}^{n_q} \beta_i q_i \ , \\ \qquad\qquad \text{where } \sum\limits_{i=1}^{n_q} \beta_i = 1 \text{ and } 0 \leq \beta_i \leq 1 \ \forall i = 1, \dots, n_q \ . \end{array} \tag{2}$$

We denote

$$\begin{array}{rcl} y & := & [\alpha_1, \dots, \alpha_{n_p}, \beta_1, \dots, \beta_{n_q}]^T \in \mathbb{R}^{n_p + n_q} \ , \\ C & := & [p_1, \dots, p_{n_p}, -q_1, \dots, -q_{n_q}] \in \mathbb{R}^{d, n_p + n_q} \ , \\ B & := & \begin{bmatrix} 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 1 \end{bmatrix} \in \mathbb{R}^{2, n_p + n_q} \ , \\ c & := & [1, 1]^T \in \mathbb{R}^2. \end{array}$$

Afterwards, the cost function can be reformulated

$$\|p - q\| = \left\| \sum_{i=1}^{n_p} \alpha_i p_i - \sum_{i=1}^{n_q} \beta_i q_i \right\| = \|Cy\|$$

and the feasible set conditions have the form

$$By = c \quad \wedge \quad y \geq 0 \ .$$

After these notations, the problem (1) can be reformulated as

$$\begin{array}{rcl} \bar{y} & := & \arg \min\limits_{y \in \Omega_E \cap \Omega_I} y^T C^T C y \ , \\ \Omega_E & := & \{y \in \mathbb{R}^{n_p + n_q} : By = c\} \ , \\ \Omega_I & := & \{y \in \mathbb{R}^{n_p + n_q} : y \geq 0\} \ . \end{array} \tag{3}$$

The next step consists of homogenization and orthogonalization. We introduce a substitution

$$x := y - y_{\text{in}} \quad \Rightarrow \quad y = x + y_{\text{in}} \ , \tag{4}$$

where y_{in} is arbitrary point from Ω_E. We can choose

$$y_{\text{in}} := \left[\frac{1}{n_p}, \dots, \frac{1}{n_p}, \frac{1}{n_q}, \dots, \frac{1}{n_q}\right] \in \mathbb{R}^{n_p+n_q} \ .$$

Afterwards, the cost function and conditions have the form

$$\begin{array}{l} f(x) := \frac{1}{2}\|C(x+y_{\text{in}})\|^2 = \frac{1}{2}x^T \overbrace{C^T C}^{=:A} x + x^T \overbrace{C^T C y_{\text{in}}}^{=:-b} + c, \quad c =: \frac{1}{2} y_{\text{in}}^T C^T C y_{\text{in}} = \text{ const.} \ , \\ B(x+y_{\text{in}}) = Bx + By_{\text{in}} = Bx + c \quad \Rightarrow \quad (By = c \ \Leftrightarrow \ Bx = 0) \ , \\ y \geq 0 \quad \Leftrightarrow \quad x \geq -y_{\text{in}} \ . \end{array}$$

Moreover, the matrix B can be orthonormalized using simple process

$$\hat{B} := \left[\begin{array}{cc} \frac{1}{\sqrt{n_p}} & 0 \\ 0 & \frac{1}{\sqrt{n_q}} \end{array} \right] B \ .$$

We obtained QP with homogeneous orthogonal linear equality constraints and bound inequality constraints

$$\begin{array}{rcl} \bar{x} & := & \arg \min\limits_{x \in \Omega_E \cap \Omega_I} \frac{1}{2} x^T A x - b^T x \ , \\ \Omega_E & := & \left\{ x \in \mathbb{R}^{n_p+n_q} : \hat{B}x = 0 \right\} \ , \\ \Omega_I & := & \left\{ x \in \mathbb{R}^{n_p+n_q} : x \geq -y_{\text{in}} \right\} \ . \end{array} \tag{5}$$

After solving this problem, the original solution can be obtained using back substitution (4) to obtain y, i.e. the coeficients of linear combinations (2) of the nearest points from each polytope.

Numerical example We consider two circles discretized by parameter $m \geq 3$, whose boundary points are defined by $P, Q \in \mathbb{R}^{2,m}$ with columns

$$P_{*,i} = \left[\begin{array}{c} \cos(2i\pi/m) - 2 \\ \sin(2i\pi/m) \end{array} \right], \quad Q_{*,i} = \left[\begin{array}{c} \cos(\pi - 2i\pi/m) + 2 \\ \sin(\pi - 2i\pi/m) \end{array} \right], \quad i = 0, \dots, m-1 \ .$$

Examples for $m = 5$ and $m = 7$ can be found in Fig. 1.

The solution of the problem for any m is given by

$$\bar{y} = [\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m]^T = [\underbrace{1, 0, \dots, 0}_{\in \mathbb{R}^m}, \underbrace{1, 0, \dots, 0}_{\in \mathbb{R}^m}]^T \ .$$

In this problem, we can directly compute the regular condition number of Hessian matrix

$$\kappa(A) = \kappa(C^T C) = \kappa(C C^T) = \kappa \left(\left[\begin{array}{cc} \sum\limits_{i=1}^{n_p} P_{i,1}^2 + \sum\limits_{i=1}^{n_q} Q_{i,1}^2 & \sum\limits_{i=1}^{n_p} P_{i,1} P_{i,2} + \sum\limits_{i=1}^{n_q} Q_{i,1} Q_{i,2} \\ \sum\limits_{i=1}^{n_p} P_{i,2} P_{i,1} + \sum\limits_{i=1}^{n_q} Q_{i,2} Q_{i,1} & \sum\limits_{i=1}^{n_p} P_{i,2}^2 + \sum\limits_{i=1}^{n_q} Q_{i,2}^2 \end{array} \right] \right) \ .$$

Moreover, it holds

$$\begin{array}{rcl} \sum\limits_{i=1}^{n_p} P_{i,2} P_{i,1} + \sum\limits_{i=1}^{n_q} Q_{i,2} Q_{i,1} & = & \sum\limits_{i=0}^{m-1} \left(\cos\left(\frac{2i\pi}{m}\right) - 2 \right) \sin\left(\frac{2i\pi}{m}\right) \\ & & \quad + \sum\limits_{i=0}^{m-1} \left(\cos\left(\pi - \frac{2i\pi}{m}\right) + 2 \right) \sin\left(\pi - \frac{2i\pi}{m}\right) = 0 \ , \end{array}$$

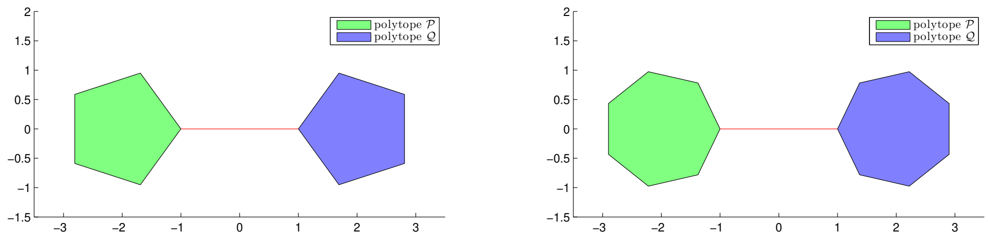

Figure 1: Testing benchmark for polytopes distance with discretization parameter $m = 5$ *(left)* and $m = 7$ *(right)*.

so

$$\kappa(A) = \kappa\left(\begin{bmatrix} \sum\limits_{i=1}^{n_p} P_{i,1}^2 + \sum\limits_{i=1}^{n_q} Q_{i,1}^2 & 0 \\ 0 & \sum\limits_{i=1}^{n_p} P_{i,2}^2 + \sum\limits_{i=1}^{n_q} Q_{i,2}^2 \end{bmatrix}\right) .$$

Afterwards, the condition number can be expressed

$$\kappa(A) = \frac{\sum\limits_{i=1}^{n_p} P_{i,1}^2 + \sum\limits_{i=1}^{n_q} Q_{i,1}^2}{\sum\limits_{i=1}^{n_p} P_{i,2}^2 + \sum\limits_{i=1}^{n_q} Q_{i,2}^2} = \frac{\sum\limits_{i=0}^{m-1} \left(\cos\left(\frac{2i\pi}{m}\right) - 2\right)^2}{\sum\limits_{i=0}^{m-1} \sin^2\left(\frac{2i\pi}{m}\right)} \qquad \forall m \geq 3 .$$

Let us consider $\alpha \in \mathbb{R}$ and let us present a complex number $z \in \mathbb{C}$ by prescription

$$z := \cos\alpha + \mathbf{i}\sin\alpha ,$$

where $\mathbf{i}$ is imaginary unit. Then by De Moivre's formula we can write

$$\sum_{i=0}^{m-1} (\cos(i\alpha) + \mathbf{i}\sin(i\alpha)) = \sum_{i=0}^{m-1} z^i = \frac{z^m - 1}{z - 1} = \frac{\cos(m\alpha) + \mathbf{i}\sin(m\alpha) - 1}{z - 1} .$$

If we choose specific α in previous equality, we obtain next

$$\begin{aligned} \alpha := \tfrac{2\pi}{m} \quad &\Rightarrow \quad \sum_{i=0}^{m-1} \left(\cos\left(\tfrac{2i\pi}{m}\right) + \mathbf{i}\sin\left(\tfrac{2i\pi}{m}\right)\right) = 0 \quad \Rightarrow \quad \sum_{i=0}^{m-1} \cos\left(\tfrac{2i\pi}{m}\right) = 0, \\ \alpha := \tfrac{4\pi}{m} \quad &\Rightarrow \quad \sum_{i=0}^{m-1} \left(\cos\left(\tfrac{4i\pi}{m}\right) + \mathbf{i}\sin\left(\tfrac{4i\pi}{m}\right)\right) = 0 \quad \Rightarrow \quad \sum_{i=0}^{m-1} \cos\left(\tfrac{4i\pi}{m}\right) = 0. \end{aligned} \tag{6}$$

Now, we return back to regular condition number and $\forall m \geq 3$ we can write (using (6))

$$\begin{aligned}\kappa(A) = & \frac{\sum\limits_{i=0}^{m-1}\left(\cos\left(\frac{2i\pi}{m}\right)-2\right)^2}{\sum\limits_{i=0}^{m-1}\sin^2\left(\frac{2i\pi}{m}\right)} = \frac{\sum\limits_{i=0}^{m-1}\cos^2\left(\frac{2i\pi}{m}\right)-4\cos\left(\frac{2i\pi}{m}\right)+4}{\sum\limits_{i=0}^{m-1}\sin^2\left(\frac{2i\pi}{m}\right)} \\ = & \frac{\sum\limits_{i=0}^{m-1}\left(\frac{1+\cos\left(\frac{4i\pi}{m}\right)}{2}-4\cos\left(\frac{2i\pi}{m}\right)+4\right)}{\sum\limits_{i=0}^{m-1}\frac{1-\cos\left(\frac{4i\pi}{m}\right)}{2}} = \frac{\frac{1}{2}\sum\limits_{i=0}^{m-1}\cos\left(\frac{4i\pi}{m}\right)-4\sum\limits_{i=0}^{m-1}\cos\left(\frac{2i\pi}{m}\right)+\sum\limits_{i=0}^{m-1}\frac{9}{2}}{-\frac{1}{2}\sum\limits_{i=0}^{m-1}\cos\left(\frac{4i\pi}{m}\right)+\sum\limits_{i=0}^{m-1}\frac{1}{2}} = 9\ .\end{aligned}$$