Definition 1. The Fibonnaci numbers are defined as the following.

$$\mathrm{Fib}(n) := \begin{cases} 0 \text{ if } n = 0 \\ 1 \text{ if } n = 1 \\ \mathrm{Fib}(n-1) + \mathrm{Fib}(n-2) \text{ otherwise.} \end{cases}$$

Theorem 2. Fib(n) is the closest integer to $\frac{\varphi^n}{\sqrt{5}}$, where $\varphi = \frac{1+\sqrt{5}}{2}$.

Proof. Let $\psi = \frac{1-\sqrt{5}}{2}$. We will prove Theorem 2 by first showing Lemma 3 and then using Lemma 6 to show that $\frac{\psi^n}{\sqrt{5}}$ is small enough such that Fib(n) is the closest integer to $\frac{\varphi^n}{\sqrt{5}}$.

Lemma 3. Fib(n) = $\frac{\varphi^n - \psi^n}{\sqrt{5}}$.

Proof. By strong induction. The induction hypothesis $P(n)$, will be Lemma 3.

Base case: Fib(n) = 0, by Definition 1. $P(0) = \frac{\varphi^0 - \psi^0}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = \frac{0}{\sqrt{5}} = 0$.

Inductive step: Assume $\forall m \in Z^+, m \leq n, P(m)$ and show that $P(n+1)$.
Fib($n+1$) = Fib(n) + Fib($n-1$), by Definition 1.
By the strong induction hypothesis, we know that Fib(n) $= \frac{\varphi^n - \psi^n}{\sqrt{5}}$ and Fib($n-1$) $= \frac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}}$.
Fib($n+1$) $= \frac{\varphi^n - \psi^n}{\sqrt{5}} + \frac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}} = \frac{\varphi^{n-1} + \varphi^n - \psi^{n-1} - \psi^n}{\sqrt{5}} = \frac{\varphi^{n-1}(1+\varphi) - \psi^{n-1}(1+\psi)}{\sqrt{5}}$.

Remark 4. φ is defined by the *golden ratio*, $\varphi^2 = \varphi + 1$. Similarily, we can show that $\psi^2 = \psi + 1$.

Lemma 5. $\varphi^2 = \varphi + 1$ and $\psi^2 = \psi + 1$.

Proof. The former is given by the text. The latter is shown by direct proof.
$\psi^2 = \left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{1-2\sqrt{5}+5}{4} = \frac{6-2\sqrt{5}}{4} = \frac{3-\sqrt{5}}{2} = \frac{1-\sqrt{5}+2}{2} = \frac{1-\sqrt{5}}{2} + 1 = \psi + 1$. □

By Lemma 5, we can simplify Fib($n+1$) to $\frac{\varphi^{n-1}(\varphi^2) - \psi^{n-1}(\psi^2)}{\sqrt{5}} = \frac{\varphi^{n+1} - \psi^{n+1}}{\sqrt{5}} = P(n+1)$. □

By Lemma 3, Fib(n) = $\frac{\varphi^n}{\sqrt{5}} - \frac{\psi^n}{\sqrt{5}}$. We can rewrite this as $\frac{\varphi^n}{\sqrt{5}}$ = Fib(n) + $\frac{\psi^n}{\sqrt{5}}$.
Now, we can show Theorem 2 by proving Lemma 6.

Lemma 6. $|\frac{\psi^n}{\sqrt{5}}| < \frac{1}{2}, \forall n \in Z^+$.

Proof. First, we show the absolute value of the numerator is less than 1.

Lemma 7. $|\psi^n| < 1, \forall n \in Z^+$.

Proof. $|\frac{1-\sqrt{5}}{2}| = |-\frac{\sqrt{5}-1}{2}| = \frac{\sqrt{5}-1}{2}$. If $\sqrt{5} - 1 < 2$, then $\frac{\sqrt{5}-1}{2} < 1$. $\sqrt{5} < 3 \iff \sqrt{5} < \sqrt{9}$, therefore $\frac{\sqrt{5}-1}{2} < 1$ and $|\psi^n| < 1$. □

By Lemma 7, $|\frac{\psi^n}{\sqrt{5}}| < \frac{1}{\sqrt{5}} < \frac{1}{2} \iff \sqrt{5} > 2 \iff \sqrt{5} > \sqrt{4}$. □

Remark 8. Because $\psi^n < 1$, as $n \to \infty$, $\psi^n \to 0$.

By Lemma 6, $\forall n \in Z^+$, $\frac{\varphi^n}{\sqrt{5}}$ is the closest integer to $\frac{\varphi^n - \psi^n}{\sqrt{5}}$. By Lemma 3, that is the closest integer to Fib(n). □