Exercise 1.13

2025-08-28

Prove that Fib(n) is the closest integer to $\phi^n/\sqrt{5}$, where $\phi = (1+\sqrt{5})/2$. Hint: Let $\psi = (1-\sqrt{5})/2$. Use induction and the definition of the Fibonacci numbers to prove that $\text{Fib}(n) = (\phi^n - \psi^n)/\sqrt{5}$.

$$\text{Fib}(n) = \begin{cases} 0, & \text{if } n = 0, \\ 1, & \text{if } n = 1, \\ \text{Fib}(n-1) + \text{Fib}(n-2) & \text{otherwise.} \end{cases}$$

Proposition. $\text{Fib}(n) = (\phi^n - \psi^n)/\sqrt{5}$.

Proof. We will prove this with mathematical induction.

(1) If $n = 0$, $\text{Fib}(0) = 0$. $\phi^0 = (\frac{(1+\sqrt{5})}{2})^0 = 1$ and $\psi^1 = (\frac{(1-\sqrt{5})}{2})^0 = 1$.

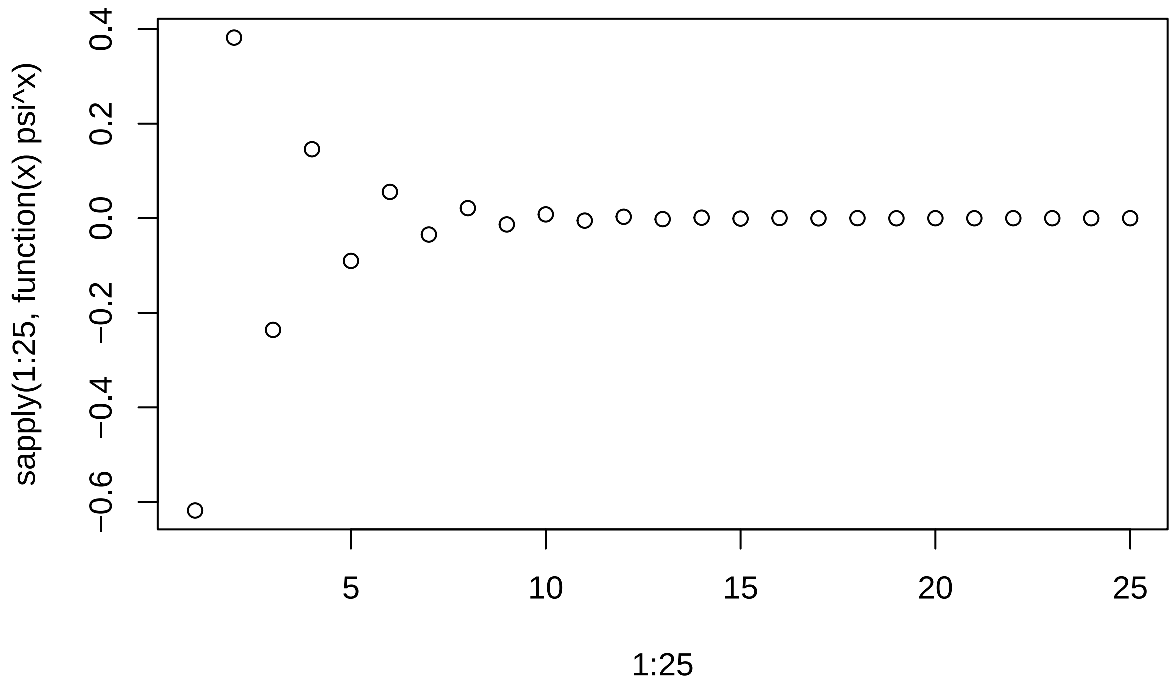
$\frac{\phi^1 - \psi^1}{5} = \frac{1-1}{5} = \frac{0}{5} = 0$

(2) Say $k \in \mathbb{N}$. Using direct proof, we will show that $\text{Fib}(k) = (\phi^k - \psi^k)/\sqrt{5}$ implies that $\text{Fib}(k+1) = \frac{\phi^{k+1} - \psi^{k+1}}{\sqrt{5}}$
Using the definition of the Fibonacci numbers, $\text{Fib}(k+1) = \text{Fib}((k+1)-1) + \text{Fib}((k+1)-2) = \text{Fib}(k) + \text{Fib}(k-1)$.
$\text{Fib}(k+1) = \frac{\phi^k - \psi^k}{\sqrt{5}} + \frac{\phi^{k-1} - \psi^{k-1}}{\sqrt{5}} = \frac{\phi^k - \psi^k + \phi^{k-1} - \psi^{k-1}}{\sqrt{5}} = \frac{\phi^{k-1}(1+\phi) - \psi^{k-1}(1+\psi)}{\sqrt{5}}$
Remember, $\phi^2 = \phi + 1$
We can also show that $\psi^2 = \psi + 1$
$\psi^2 = (\frac{1-\sqrt{5}}{2})^2 = \frac{(1-\sqrt{5})(1-\sqrt{5})}{4} = \frac{1-2\sqrt{5}+5}{4} = \frac{6-2\sqrt{5}}{4} = \frac{3-\sqrt{5}}{2} = \frac{1-\sqrt{5}+2}{2} = \frac{1-\sqrt{5}}{2} + \frac{2}{2} = \frac{1-\sqrt{5}}{2} + 1 = \psi + 1$
$\text{Fib}(k+1) = \frac{\phi^{k-1}(\phi^2) - \psi^{k-1}(\psi^2)}{\sqrt{5}} = \frac{\phi^{k+1} - \psi^{k+1}}{\sqrt{5}}$
Therefore, $\text{Fib}(k+1) = \frac{\phi^{k+1} - \psi^{k+1}}{\sqrt{5}}$. It follows by induction that $\text{Fib}(n) = (\phi^n - \psi^n)/\sqrt{5}$ for every $n \in \mathbb{N}$.

Now, we will show that Fib(n) is the closest integer to $\phi^n/\sqrt{5}$.

As $\lim_{n\to\infty} \psi^n = 0$, so $\lim_{n\to\infty} \frac{\phi^n}{\sqrt{5}} = \lim_{n\to\infty} \frac{\phi^n - \psi^n}{\sqrt{5}}$. This is because $\psi \approx -0.618$ and as it gets exponentiated to higher power, it approaches zero.

```r
psi <- (1 - sqrt(5)) / 2
plot(x = 1:25, y = sapply(1:25, function(x) psi^x))
```

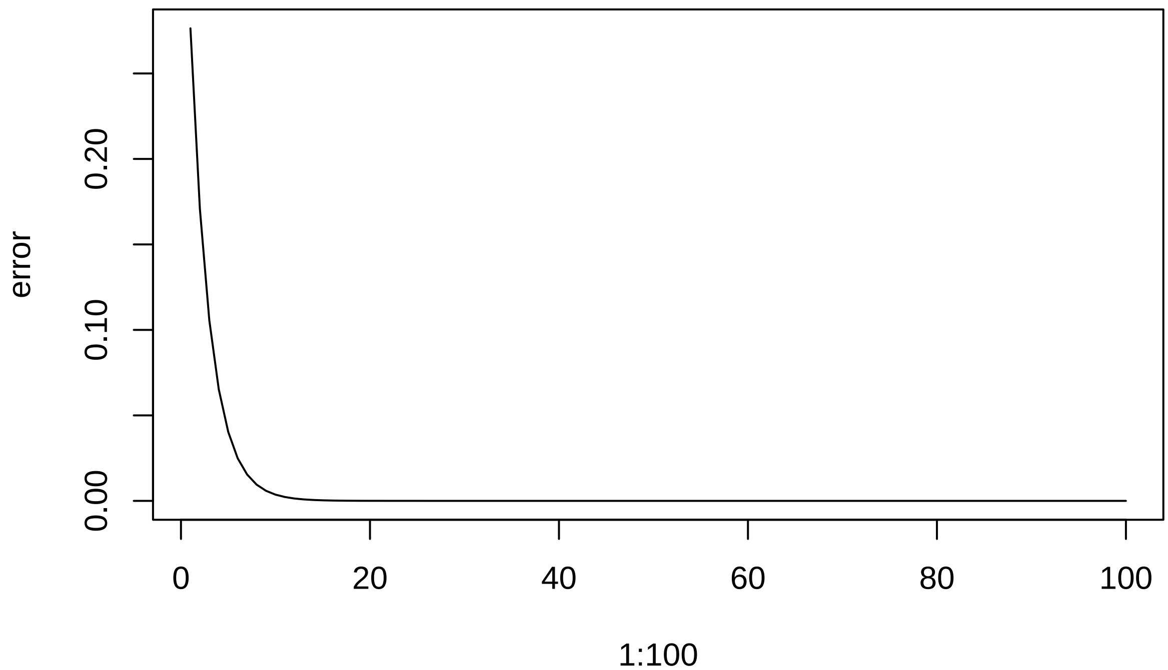


The proof can also be visualized here, where the difference between approximation and Fibonacci number is displayed on the y-axis.

```
phi <- (1 + sqrt(5)) / 2
fib <- function(x) {
  (phi^x - psi^x) / sqrt(5)
}
approx <- function(x) {
  phi^x / sqrt(5)
}

error <- abs(sapply(1:100, fib) - sapply(1:100, approx))

plot(x = 1:100, y = error, type = "l")
```



As seen on the graph, the largest error at $n = 0$ is less than 0.5, and for all $n \in \mathbb{N}$, $\phi^n/\sqrt{5}$ is the closest integer to $\text{Fib}(n)$.

```r
fib(0)
```

```
## [1] 0
```

```r
approx(0)
```

```
## [1] 0.4472136
```