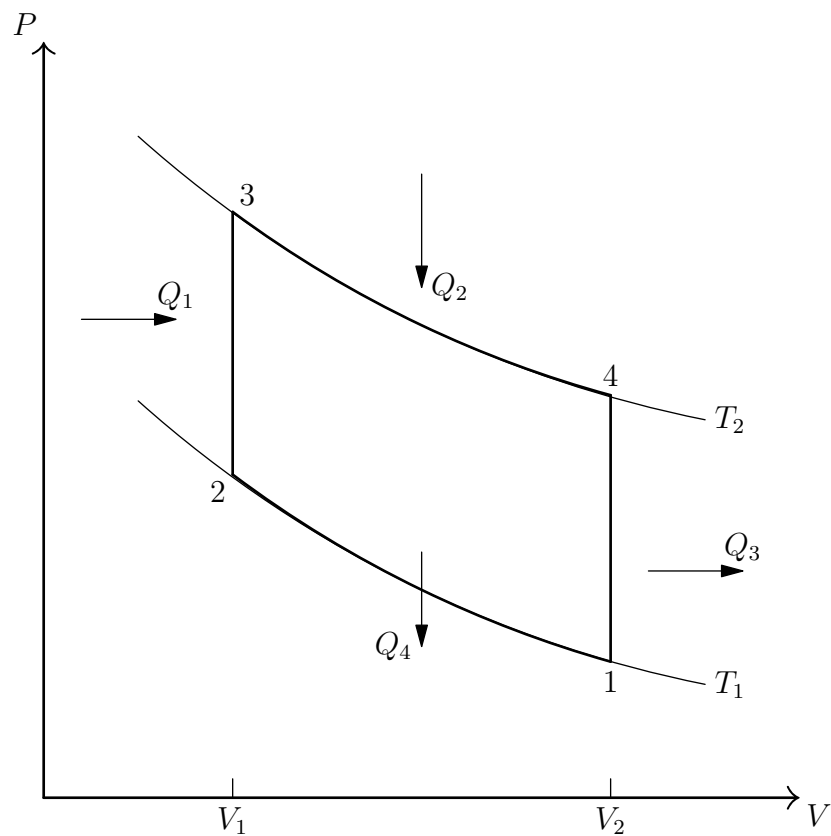


# Solutions to Jaan Kalda's Problems in Thermodynamics

With detailed diagrams and walkthroughs

Edition 1.2.1

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## Preface

Jaan Kalda's [handouts](#) are beloved by physics students both in for a quick challenge, to students preparing for international Olympiads. As of writing, the current [thermodynamics](#) handout has 84 unique problems and 20 main 'ideas'.

This solutions manual came as a pilot project from the online community at [artofproblemsolving.com](#). Although there were detailed hints provided, full solutions have never been written. The majority of the solutions seen here were written on a private forum given to those who wanted to participate in making solutions. In an amazing show of an online collaboration, students from around the world came together to discuss ideas and methods and created what we see today.

This project would not have been possible without the countless contributions from members of the community. Online usernames were used for those who did not wish to be named:

*Jonathan Qin, Anant Lunia, Ameya Deshmukh and Hermab Podar.*

## Structure of The Solutions Manual

Each chapter in this solutions manual will be directed towards a section given in Kalda's mechanics handout. There are three major chapters: statics, dynamics, and revision problems. If you are stuck on a problem, cannot make progress even with the hint, and come here for reference, look at only the start of the solution, then try again. Looking at the entire solution wastes the problem for you and ruins an opportunity for yourself to improve.

## Contact Us

Despite editing, there is almost zero probability that there are *no* mistakes inside this book. If there are any mistakes, you want to add a remark, have a unique solution, or know the source of a specific problem, then please contact us at [hello@physoly.tech](mailto:hello@physoly.tech). The most current and updated version can be found on our website [physoly.tech](#)

Please feel free to contact us at the same email if you are confused on a solution. Chances are that many others will have the same question as you.

# 1 Solutions to Heat and Motion Problems

This section will consist of the solutions to problems from problem 1-8 of the handout. Heat and temperature is typically the analysis of objects interacting with each other via thermal energy. These problems usually involve conservation of energy, heat flux, Stefan-Boltzmann's Law, analysis of graphs, and more.

**pr 1.** Applying idea 1, we have:

$$P = C \frac{dT}{dt}$$

where

$$\frac{dT}{dt} = \frac{aT_0}{4}(1 + a(t - t_0))^{-3/4}$$

Since we have that,

$$T(t) = T_0[1 + a(t - t_0)]^{1/4}$$

dividing by  $T_0$  gives us

$$\frac{T}{T_0} = [1 + a(t - t_0)]^{1/4}.$$

We then get

$$\frac{dT}{dt} = T_0^4(a/4)(1/T^3)$$

Plugging everything back in, we get:

$$C = \frac{4PT^3}{aT_0^4}$$

**pr 2.** The ice melts when the temperature of the kettle begins to drop. All the heat that was supplied to the kettle was used in melting the ice and bringing it up to the temperature of the rest of the water. Meanwhile the water already present lost some heat to the surroundings, and thus the graph dips at that point.

The total time for the temperature to recover to  $T = 75^\circ \text{C}$  is approximately  $t = 37 \text{ s}$ . The heating rate of water is the slope of the graph or  $dT/dt$ . This means that the power at  $T = 70^\circ \text{C}$  is  $P' = P \tan 75^\circ \approx 500 \text{ W}$ . Therefore, from energy conservation, we have that

$$mL + mc\Delta T = Pt \implies m = \frac{Pt}{L + c\Delta T} \approx \boxed{28 \text{ g}}.$$

**pr 3.** Over one complete oscillation of the voltage, the heat lost by the filament must equal the heat gained by it. Let the resistance of the filament be  $R$ . The heat gained by the filament is  $\frac{U_1^2}{R} \frac{T}{2}$  (because the voltage is applied only for  $\frac{T}{2}$ ). Let the rate at which heat is lost to the surrounding be  $r$ . The heat lost to the surroundings is  $rT$  therefore

$$rT = \frac{U_1^2}{R} \frac{T}{2} \implies r = \frac{U_1^2}{2R}.$$

From  $t = 0.5T$  to  $T$ , the heat lost takes the temperature from the maximum temperature to the minimum temperature, a change of  $2\Delta T$  (beware, the  $\Delta T$  is the amplitude of the temperature while  $T$  is time

period of voltage oscillations). This implies that

$$r \frac{T}{2} = 2mc\Delta T \implies \Delta T = \frac{U_1^2 T}{8Rmc}.$$

However,  $R = \frac{\rho_{\text{el}} \ell}{A}$  and  $m = \rho \ell A$ , where  $A$  is cross-section area of the wire. Substituting these values gives

$$\Delta T = \frac{U_1^2 T}{8c\rho_{\text{el}}\rho\ell^2} = \frac{(17)^2(0.01)}{8(235)(9.95 \times 10^{-7})(18200)(0.05)^2} = \boxed{33.8 \text{ K}}$$

**pr 4.** Applying fact 6, we find that the initial heat flux is proportional to  $\Delta T_1 = T_1 - T_0$  as the temperature change is minimal (in fact smaller than one Kelvin). Likewise, the final heat flux is proportional to  $\Delta T_2 = T_2 - T_0$ . Therefore, we have that the ratio of powers is

$$\frac{P + P_{\text{man}}}{P} = \frac{\Delta T_2}{\Delta T_1}$$

Solving for  $P_{\text{man}}$  and plugging in our numbers, we find that

$$P_{\text{man}} = P \left( \frac{T_2 - T_1}{T_1} \right) = \boxed{52.6 \text{ W}}.$$

**pr 5.** a) The heat flux (or energy flux) density is  $\Phi = \frac{P}{S}$  and the thermal resistivity is:

$$\rho = \frac{1}{\Phi} \frac{dT}{dx} = \frac{S}{P} \frac{dT}{dx}$$

Separating variables, we have:

$$\Delta T = \int_0^d \frac{\rho P}{S} dx = 11.7 \text{ K}$$

b) Again, we separate variables. This time however, our expression becomes:

$$\frac{\Delta T S}{P} = \int_0^\ell \rho(x) dx.$$

The integral can be approximated as the area under the curve. In this case, we can see that the average value is approximately  $0.14 \text{ K} \cdot \text{m/W}$  and then use this value to approximate the integral as a rectangle. Solving for  $P$  from here gives us

$$P = 1.8 \times 10^{-2} \text{ W}.$$

**pr 6.** According to Stefan-Boltzmann's Law, the power per unit area emitted from the surface of a blackbody at temperature  $T$  is  $\sigma T^4$ .

The total power emitted from the sun, considered a blackbody for the sake of the problem, is therefore,

$$P_\odot = \sigma T_\odot^4 (4\pi R_\odot^2)$$

Due to the inverse square law, the solar flux stays constant through any closed surface. The portion of energy that reaches the satellite is given by the ratio between the cross-sectional area of the satellite and

the area of an imaginary sphere centered around the Sun with a radius of  $L$  ( $R_{\oplus}$  is the radius of the satellite)

$$\gamma = \frac{\pi R_{\oplus}^2}{4\pi L^2} = \left(\frac{R_{\oplus}}{2L}\right)^2$$

According to Prevost's theory of exchange, in order to maintain thermal equilibrium, any object must emit the same energy as it receives. If this was not true, then it would continuously lose or gain energy until it is at equilibrium. Now let the emissivity and absorptivity factors of the satellite be  $\epsilon$  and  $a$ . Since we know that these two parameters essentially have the same value at a particular wavelength, we have

$$P_{\text{in}} = P_{\text{out}}$$

By Stefan-Boltzmann Law, we have

$$P_{\text{output}} = 4\pi R_{\oplus}^2 \sigma T^4$$

Equating  $P_{\text{in}}$  to  $P_{\text{out}}$  gives us

$$\begin{aligned} P_{\text{output}} &= P_{\odot} \gamma_{\text{eff}} \\ \epsilon_{\times} 4\pi R_{\oplus}^2 \sigma T^4 &= a \times \sigma T_{\odot}^4 (4\pi R_{\odot}^2) \left(\frac{R_{\oplus}}{2L}\right)^2 \\ \cancel{\epsilon}_{\times} 4\pi R_{\oplus}^2 \sigma T^4 &= \cancel{a} \times \sigma T_{\odot}^4 (4\pi R_{\odot}^2) \left(\frac{R_{\oplus}}{2L}\right)^2 \\ T^4 &= T_{\odot}^4 (R_{\odot}^2) \left(\frac{1}{2L}\right)^2 \\ T^4 &= \frac{T_{\odot}^4 R_{\odot}^2}{4L^2} \\ T &= T_{\odot} \sqrt{\left(\frac{R_{\odot}}{2L}\right)} \end{aligned}$$

Plugging in our given constants gives us  $\approx \boxed{290 \text{ K}}$ .

**pr 7.** We solve the system for the heat shield made up of  $N$  thin black plates thermally isolated from each other in stationary conditions. Of course, since the plates are black and we deal with the system when stationary conditions have been obtained. Let the  $i^{\text{th}}$  plate from the left plane have a temperature  $T_i$ , and  $W_0$  be the initial heat radiation flux density. Hence, since the flux density remains constant between the planes, we have

$$W = \sigma(T_h^4 - T_1^4)$$

$$W = \sigma(T_1^4 - T_2^4)$$

$$W = \sigma(T_2^4 - T_3^4)$$

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$$W = \sigma(T_N^4 - T_l^4)$$

Adding all the equations, we have

$$(N + 1)W = \sigma(T_h^4 - T_l^4) = W_0 \Rightarrow W = \frac{W_0}{N + 1}$$

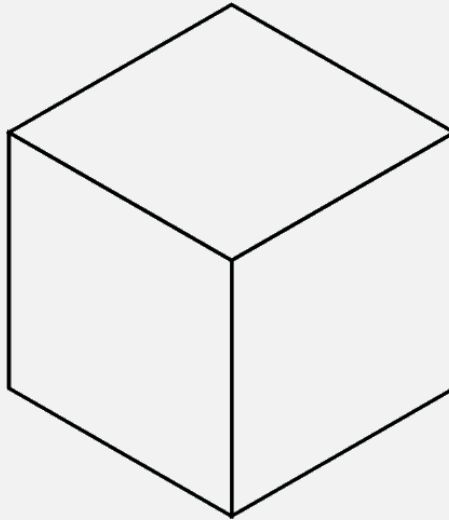
which means that the heat flow has reduced by a factor of  $\xi = \frac{1}{N + 1}$ . For 2 plates, this is simply  $\xi = \frac{1}{3}$

**pr 8.** According to Prevost's theory of exchange, in order to maintain thermal equilibrium, any object must emit the same energy as it receives. The power recieved is  $IA_{\text{eff}}$  and the power emmited is  $\sigma 6s^2T^4$ , where  $s$  is the sidelength of the cube. This then tells us that

$$IA_{\text{eff}} = \sigma 6s^2T^4 \Rightarrow T = \sqrt[4]{\frac{IA_{\text{eff}}}{\sigma 6s^2}}.$$

We now have to analyze the extremas of  $A_{\text{eff}}$ .

1.  $A_{\text{eff}}$  is minimum when the cube recieves the beam directly on the center of its face. This means that  $A_{\text{min}} = s^2 \Rightarrow T = \sqrt{\frac{I}{6\sigma}}$ .
2. The maximum is achieved when two space opposite vertices are aligned with the direction of the beam. This would then mean that the projection is a regular hexagon as shown below.



However, the side length of this hexagon is not  $s$  because the edges are not parallel to the projection plane. However, if you take the three vertices of the hexagon closest to you and connect these points to make an equilateral triangle, the sides of this triangle will be parallel to the projection plane. The sides of the equilateral triangle are face diagonals and have length  $s\sqrt{2}$ . The area of the hexagon is twice the area of the equilateral triangle and therefore has area  $\frac{\sqrt{3}}{2}(s\sqrt{2})^2 = s^2\sqrt{3}$ . Therefore

$$T_{\text{max}} = \sqrt[4]{\frac{I\sqrt{3}}{\sigma 6}}$$

**Solution 2.** Once again, according to Prevost's theory of exchange, in order to maintain thermal equilibrium, any object must emit the same energy as it receives. Let the power recieved be  $\alpha IA$  and

the power emitted is  $\sigma 6s^2 T^4$ . We then have that

$$\alpha I A = \sigma 6s^2 T^4 \implies T = \sqrt[4]{\frac{\alpha I}{6\sigma}}.$$

The projection of area  $A = A\vec{n} \cdot \vec{i}$  where  $\vec{i}$  is unit vector in direction of incoming flux, and  $\vec{n}$  is unit vector in direction perpendicular to plane containing the mentioned area. The projection of area is  $A = A\vec{n} \cdot \vec{i}$  where  $\vec{i}$  is unit vector in direction of incoming flux, and  $\vec{n}$  is unit vector in direction perpendicular to plane containing the mentioned area. Therefore, we have that  $\alpha = \sum \vec{n} \cdot \vec{i}$ . There can be at most 3 faces with positive projections of area. Let these 3 faces be in  $x, y, z$  directions. Then  $\alpha = (\vec{n}_x + \vec{n}_y + \vec{n}_z) \cdot \vec{i}$  where  $n_x = n_y = n_z = 1$  as these are unit vectors. Let  $i = \vec{i}_x + \vec{i}_y + \vec{i}_z$ . Then  $\alpha = i_x + i_y + i_z$ . Since  $i$  is unit vector  $i_x^2 + i_y^2 + i_z^2 = 1$ . Also,  $0 \leq i_x, i_y, i_z \leq 1$  as  $\beta^2 \geq 0$  for  $\beta \in R$  (Note that  $i_x, i_y, i_z$  cannot be all equal to zero). Therefore,  $i_x^2 \leq i_x, i_y^2 \leq i_y, i_z^2 \leq i_z \implies i_x^2 + i_y^2 + i_z^2 \leq i_x + i_y + i_z$ . We then have by power mean inequality,

$$\frac{i_x + i_y + i_z}{3} \leq \left( \frac{i_x^2 + i_y^2 + i_z^2}{3} \right)^{1/2}$$

This then means that

$$i_x^2 + i_y^2 + i_z^2 \leq i_x + i_y + i_z \leq \sqrt{3(i_x^2 + i_y^2 + i_z^2)}$$

which then tells us

$$1 \leq \alpha \leq \sqrt{3}.$$

Therefore,

$$T_{\max} = \sqrt[4]{\frac{\sqrt{3}I}{6\sigma}} \text{ and } T_{\min} = \sqrt[4]{\frac{I}{6\sigma}}$$

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## 2 Solutions to Gases

This chapter will focus on problems 9-14 of the handout.

**pr 9.** Let us assume that  $-dm'$  mass sublimates at some instant. ( $m'$  is the total mass of the cup.) Initially it is moving with  $v$ , the velocity of the vessel. Finally its velocity *with respect to the vessel*, in the *direction of motion* of the vessel becomes  $-\frac{\sqrt{\frac{3RT}{\mu}}}{\sqrt{3}} = -\sqrt{\frac{RT}{\mu}}$ .<sup>a</sup> Thus, it provides an impulse,  $-(-dm'\Delta v) = -dm'\sqrt{\frac{RT}{\mu}}$  to the vessel

$$\therefore v = \int dv = \int_{M+m}^M \frac{-dm'\sqrt{\frac{RT}{\mu}}}{m'} = \sqrt{\frac{RT}{\mu}} \ln \left( 1 + \frac{m}{M} \right) \approx \sqrt{\frac{RT}{\mu} \frac{m}{M}}$$

<sup>a</sup>Since the rms speed is randomly directed in  $x, y, z$  directions and we only want one speed, we divide by the magnitude of the unit vectors.

**pr 10.** During each process for a short time interval  $\Delta t$ , the number of molecules passing through the hole will be proportional to:

$$N \sim An^*u_x \propto \frac{n^*}{\sqrt{M}}$$

where  $n^*$  is the number density  $M$  is the molar mass of the molecule. For the two compounds,  $\text{U}^{235}\text{F}_6$  and  $\text{U}^{238}\text{F}_6$ , the ratio of their number densities,  $\frac{n_1^*}{n_2^*}$ , will change by a factor of  $\left(\frac{M_2}{M_1}\right)^{\frac{1}{2}}$  after each process.

$$\therefore \left( \frac{[\text{U}^{235}\text{F}_6]}{[\text{U}^{238}\text{F}_6]} \right)_f = \left( \frac{[\text{U}^{235}\text{F}_6]}{[\text{U}^{238}\text{F}_6]} \right)_i \left( \frac{M_2}{M_1} \right)^{\frac{N}{2}}$$

Substituting and solving gives us

$$1.4 \leq 0.7 \left( \frac{238 + 19 \cdot 6}{235 + 19 \cdot 6} \right)^{\frac{N}{2}} \implies N \geq 2 \cdot \log_{\frac{352}{349}} 2 \approx 161.96 \implies N = \boxed{162}.$$

**pr 11.** Let us consider the time period during which all the molecules complete one round trip. Consider a cross-section between the plates: all the molecules would have crossed it during this time interval. ( $\lambda \gg L$ ) The net energy transfer through this section is equal to  $((\text{K.E.})_{T_1} - (\text{K.E.})_{T_2})N$  towards the colder plate.

$$\therefore \Phi A \left( \frac{L}{v_{x,T_1}} + \frac{L}{v_{x,T_2}} \right) = ((\text{K.E.})_{T_1} - (\text{K.E.})_{T_2})N$$

$$\Phi A = \frac{m \left( \frac{3R}{\sqrt{2\pi M}} (T_2 - T_1) \right) N}{L \sqrt{\frac{M}{R}} \left( \frac{1}{\sqrt{T_2}} - \frac{1}{\sqrt{T_1}} \right)}.$$

Using the fact that  $mN = mN_A n A L = M n A L$ , we can get

$$\therefore \Phi = \frac{3Rn}{\sqrt{2\pi}} \left( \sqrt{T_2} + \sqrt{T_1} \right) (\sqrt{T_2 T_1}) \sqrt{\frac{R}{M}}$$



Given that,  $T_1 \gg T_2 \implies \sqrt{T_2} + \sqrt{T_1} \approx \sqrt{T_1}$

$$\Phi \approx \frac{3RnT_1}{\sqrt{2\pi}} \sqrt{\frac{RT_2}{M}}.$$

**pr 12.** We can take  $s = E$ , because the only restriction is the distance between two steel balls, the angle between them can be changed. As there are  $E$  edges, number of restrictions is  $E$ . By Euler's formula,

$$N = E - F + 2 \Rightarrow 3N = 3F - 3E + 6$$

$$3N - E = 2E - 3F + 6 \implies j = 2E - 3F + 6$$

Every edge is part of exactly 2 faces. Every face contains atleast 3 edges.

$$\therefore 2E \geq 3F \text{ with equality in case of only triangles}$$

$$\therefore j \geq 6 \text{ with equality in case of only triangles}$$

In case of rigid body 6 coordinates are needed to describe it. If the number of degrees of freedom were larger, it would be needed to use more parameters than in the case of a rigid body to describe its state, i.e. it cannot be rigid. Therefore, a convex polyhedron is rigid then and only then if all the faces of the polyhedron are triangles.

**pr 13.** The total mass of the balloon is  $Mg + m_{H_2}g$  which is equal to the mass of the air that is holding the balloon up  $\rho_{\text{air}}Vg$ . From the ideal gas law, we can write

$$\rho_{\text{air}} = \frac{p\mu_{\text{air}}}{RT_{\text{air}}}$$

and similarly, we can write the mass of the hydrogen gas  $m_{H_2}$  as

$$m_{H_2} = \frac{p\mu_{H_2}V}{RT_{\text{air}}}.$$

We can then write that

$$\rho_{\text{air}}Vg = Mg + m_{H_2}g \implies \frac{p\mu_{\text{air}}}{RT_{\text{air}}}Vg = Mg + \frac{p\mu_{H_2}V}{RT_{\text{air}}}g.$$

This means that

$$M_0 = \frac{pV_0}{RT_{\text{air}}}(\mu_{\text{air}} - \mu_{H_2}).$$

The mass of the balloon when it reaches a volume  $V_1$  is then given by

$$M_1 = \frac{pV_1}{RT_{\text{air}}}(\mu_{\text{air}} - \mu_{H_2})$$

and by Charle's law, note that

$$\frac{V_1}{T_{\text{air}}} = \frac{V_0}{T_1}$$

which means that

$$M_1 = \frac{pV_0}{RT_1}(\mu_{\text{air}} - \mu_{H_2}).$$

The ballast needed to be thrown out is then

$$\Delta m = M_0 - M_1 = (\mu_{\text{air}} - \mu_{H_2}) \frac{pV_0}{R} \left( \frac{1}{T_{\text{air}}} - \frac{1}{T_1} \right).$$

**pr 14.** Initially, the gas doesn't take up any volume inside the balloon. In the final state of our system, all of the gas occupies  $V_b$  and therefore we have our change in volume  $\Delta V = V_b$ . This means that the work done is  $W = -p\Delta V_b = -pV_b$ . By the ideal gas law, we have that  $pV_b = nRT$ . We then have by the first law of thermodynamics that

$$\Delta U = Q - W$$

but since the heat flux through the balloon and the heat capacity of the balloon is neglected, then the heat flown into the system is 0 which means that

$$\Delta U = -W \implies \nu C_V(T' - T) = nRT.$$

Substituting  $C_V = \frac{1}{\gamma-1}R$ , we have that

$$\frac{1}{\gamma-1}(T' - T) = T \implies T' = \gamma T = \boxed{1.4T}.$$

where  $\gamma = 1.4$  for air.

### 3 Solutions to Adiabatic processes

This chapter will focus on problems 15-24 of the handout.

**pr 15.** Since the wavelength of the universe grows proportionally to the linear size of the Universe, then the wavelength  $\lambda$  will be two times larger if the size of the universe becomes two times larger. The wavelength is inversely proportional to temperature by Wien's Law. In other words,

$$\lambda \propto T^{-1} \implies T_f = \frac{T_0}{2}.$$

Since we have that

$$PV^\gamma = TV^{\gamma-1} = \text{const.}$$

then, we can substitute values to get (note that  $V$  becomes 8 times larger since all three dimensions get two times larger)

$$T_0 V^{\gamma-1} = T_f (8V)^{\gamma-1}.$$

Simplifying gives us

$$8^{\gamma-1} = 2 \implies \gamma - 1 = \frac{1}{3} \implies \gamma = \boxed{\frac{4}{3}}.$$

**pr 16.** Using Idea 11, we have:

$$T^\gamma \propto P^{\gamma-1}$$

Therefore, a slight change in temperature would imply:

$$\gamma \left( \frac{\delta T}{T} \right) = (\gamma - 1) \left( \frac{\delta P}{P} \right)$$

The crux of this problem is that the pressure change is so small thus by idea 13, it can be approximated with  $\Delta p = -\rho gh$ . Therefore, we can isolate for  $\delta T$  to be:

$$\delta T = \frac{\gamma - 1}{\gamma} \left( \frac{-\rho_0 g H_0}{P_0} \right) T_0$$

**pr 17.** From the prelude, we have the work as:

$$W = \frac{m}{2}(v_2^2 - v_1^2) + \Delta U$$

The internal energy changes both in gravitational energy and in internal energy so:

$$\begin{aligned} \Delta U &= mg\Delta h + nC_v\Delta T \\ &= mg\Delta h + (nR\Delta T) \frac{C_v}{R} \\ &= mg\Delta h + \Delta(PV) \frac{C_v}{R} \end{aligned}$$

The compression work  $W$  is:

$$W = -\Delta(PV)$$

so putting everything together gives:

$$0 = \frac{m}{2}(v_2^2 - v_1^2) + mg\Delta h + \Delta(PV) \left( \frac{C_v + R}{R} \right)$$

Having  $C_P = C_V + R$ ,  $\Delta(PV) = nR\Delta T$ , and letting the molar mass  $\mu \equiv m/n$ , we can simplify this to:

$$0 = \frac{1}{2}(v_2^2 - v_1^2) + g(h_2 - h_1) + \frac{C_P(T_2 - T_1)}{\mu} \implies \frac{1}{2}v^2 + gh + c_p T = 0$$

**pr 18. a)** Let the height of the streamline flow far away from the wing to be  $h_0$  and the height of the streamline flow at point  $P$  to be  $h_P$ . Therefore, the continuity expression gives us:

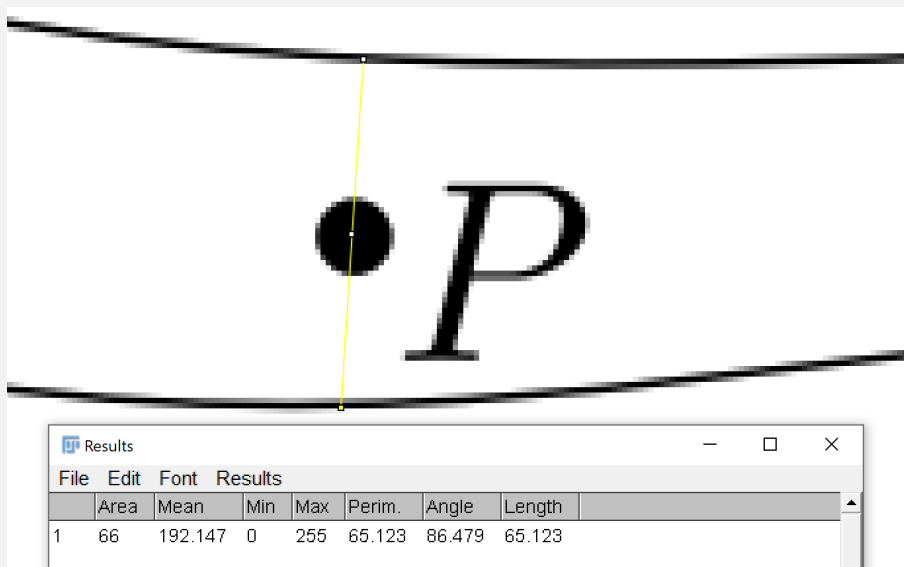
$$u_0 h_0 = u_P h_P \implies u_P = u_0 \frac{h_0}{h_P}$$

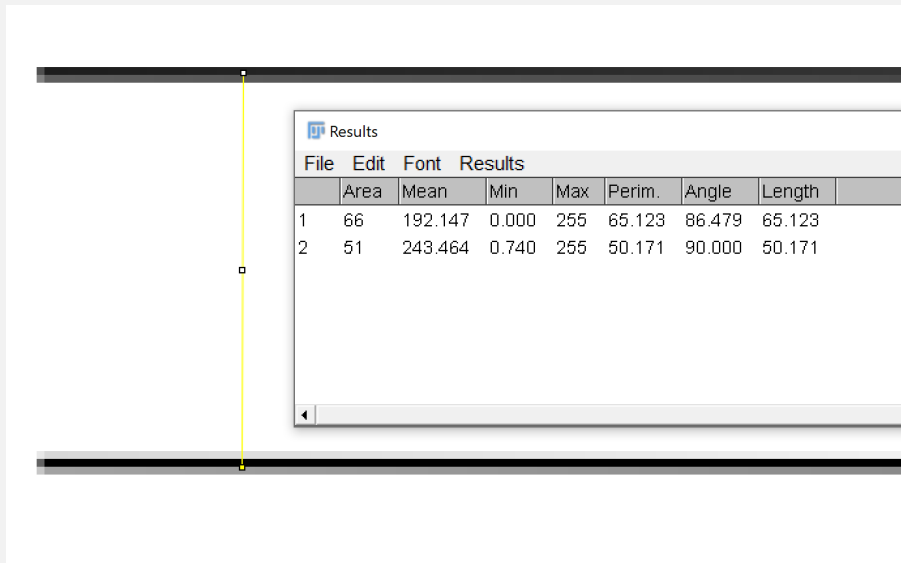
here we are assuming that the airflow is completely two-dimensional, which means the area is proportional to the height. Note that we can only use this continuity expression with assumption (d) as the dynamic pressure is much smaller, which means effects caused by compression can be negligible. We can measure  $h_0/h_P = 0.77$  so we have  $u_0 = 77$  m/s. However, this is the speed of the air as measured by the plane. We can relate these quantities through:

$$\vec{v}_{\text{air relative ground}} = \vec{v}_{\text{air relative plane}} + \vec{v}_{\text{plane relative ground}}$$

Far away from the plane,  $\vec{v}_{\text{air relative ground}} = 0$  (no wind) so then the airplane must be travelling at a speed of  $u_0$ . Therefore, the speed of the air at  $P$  relative to the ground is:

$$u_0 - u_0 \left( \frac{h_0}{h_P} \right) = \boxed{23 \text{ m/s}}$$





b) Using definition 10, condensation into water vapor is favored when the dynamic pressure  $\frac{1}{2}\rho v^2$  is high. This occurs when  $v$  is at a maximum, or when the streamline lines are close together. From the diagram, they are closest together at the top left corner of the wing.

c) From problem 17, we find that

$$\frac{1}{2}v^2 + c_p T = 0.$$

We find from part a that,

$$\frac{1}{2}\Delta v^2 = \frac{1}{2}v_{\text{crit}}^2 \left( \frac{h_0^2}{h_Q^2} - 1 \right) = c_p \Delta T \implies v_{\text{crit}} = \sqrt{\frac{2c_p \Delta T}{h_0^2/h_Q^2 - 1}}.$$

To finally solve this problem, we have to find  $\Delta T$ . By relating the ratios, we find that

$$\frac{p_{\text{sa}} - p_w}{T_a - T} = \frac{p_{\text{sb}} - p_{\text{sa}}}{T_b - T_a} \implies T = \frac{p_{\text{sa}} - p_w}{p_{\text{sb}} - p_{\text{sa}}} (T_b - T_a) - T_a$$

where  $p_w = r p_{\text{sa}}$ . We note that  $\Delta T = T_a - T$ , thus by substituting this into our final expression of  $v_{\text{crit}}$ , we find that

$$v_{\text{crit}} = \sqrt{\frac{2c_p}{h_0^2/h_Q^2 - 1} \left( \frac{p_{\text{sa}}(1-r)}{p_{\text{sb}} - p_{\text{sa}}} (T_b - T_a) \right)}.$$

**pr 19.** Consider a horizontal displacement of a gas. The impulse the gas experiences is:

$$J = F_{\text{net}} dt = \Delta p S dt = (p_1 - p_2) S dt$$

where  $S$  is the cross sectional area. Take  $p_1 > p_2$  such that the gas is moving towards the right (which we arbitrarily set as the positive direction). This gives the change in momentum to be:

$$\Delta p = (\rho S x_2) v_2 - (\rho S x_1) v_1 = (\rho S) v_2^2 dt - (\rho S x_1) v_1^2 dt$$

Setting these two expressions equal gives:

$$p_1 - p_2 = \rho v_2^2 - \rho v_1^2$$

Since this is true for any two intervals, then the quantity

$$p + \rho v^2 = \text{constant}$$

must be preserved.

**pr 20.** Let us consider a sound wave which propagates in the direction of  $x$ -axis; then, the air density  $\rho = \rho(x - c_s t)$ . Following the idea 16, we consider a frame which moves with speed  $c_s$ , with coordinate axis  $x' = x - c_s t$ . In that frame, the density perturbation remains constant in time,  $\rho = \rho(x')$ . This means that we can use idea 14, so that we obtain two equations:

$$\begin{aligned} (\rho + \Delta\rho)(v + c_s) &= \rho c_s \\ \frac{1}{2}(v + c_s)^2 + c_p T &= \frac{1}{2}c_s^2 + c_p T \end{aligned}$$

where  $v \ll c_s$  is the speed of the gas in the laboratory frame. Note that we also have our momentum equation to be

$$P + (\rho_0 + \Delta\rho)(v + c_s)^2 + c_p T = P_0 + \rho_0 c_s^2.$$

From the problem statement, we have that  $\Delta\rho_0 = \rho_0$  which means that from our first equation we have

$$2\rho_0(v + c_s) = \rho_0 c_s \implies v = -\frac{c_s}{2}.$$

Going into our momentum equation, we have that

$$P + 2\rho_0 \left(c_s - \frac{c_s}{2}\right)^2 = P_0 + \rho_0 c_s^2 \implies P = P_0 + \frac{\rho_0 c_s^2}{2}.$$

Also, note that by the ideal gas law, we have:

$$PV = nRT \implies P\mu = \rho RT \implies T = \frac{P\mu}{\rho R}.$$

We can now use these results to go into our second equation (energy) to find that

$$\frac{1}{2} \left(c_s - \frac{c_s}{2}\right)^2 + c_p \left(\frac{P_0 + \rho_0 c_s^2/2}{2R\rho}\right) = \frac{1}{2}c_s^2 + c_p T.$$

We can simplify further to get

$$\frac{1}{8}c_s^2 + c_p \left(\frac{P_0\mu}{2R\rho_0} + \frac{c_s^2\mu}{4R}\right) = \frac{1}{2}c_s^2 + c_p T$$

then expanding tells us

$$\frac{c_s^2}{8} + \frac{c_p T_0}{2} + \frac{c_p c_s^2 \mu}{4R} = \frac{1}{2}c_s^2 + c_p T.$$

From the problem, we see that  $c_p = 5R/2\mu$  and solving gives us the equation

$$c_s = \sqrt{\frac{5RT_0}{2\mu}} = \sqrt{\frac{5}{2} \left(\frac{\gamma RT_0}{\mu}\right)} \implies \text{the speed of the wave becomes } \sqrt{2.5} \text{ times faster.}$$

**pr 21. a)** Let us assume that the temperature stays roughly constant. This means that the sublimation rate is also constant and exerts some pressure  $p$  on the vapour. We know that the saturation vapour pressure  $p_0$  is defined such that the rate of sublimation = rate of deposition. This means that the pressure exerted by the sublimation is also  $p_0$ . Therefore the force is:

$$p_0 A = Ma \implies a = \boxed{\frac{M}{p_0 A}}$$

**b)** Both evaporation and condensation apply the same pressure at saturation ( $p_0/2$ , to be exact), but since the particles escape never to come back (because  $\lambda \gg$  the length of the vessel), there is no condensation and thus only half the pressure is applied. Therefore,

$$\frac{p_0}{2} A = Ma \implies a = \boxed{\frac{p_0 A}{2M}}.$$

**pr 22. a)** First off, we find the pressure at 20° C on the graph. At this point, the pressure is approximately given by 2.3 kPa. We are told that the relative humidity is 90% which means that the relative pressure is given by

$$2.3 \text{ kPa} \cdot 0.9 = 2.07 \text{ kPa}.$$

The temperature on the graph when it is approximately 2.07 kPa is around 18.5° C. This then tells us that the temperature difference is

$$20^\circ \text{ C} - 18.5^\circ \text{ C} = \boxed{1.5^\circ \text{ C}}$$

**b)** We are given the equations

$$\begin{aligned} Q_c &= a(T_0 - T) \\ Q_e &= b[p_s(T) - p_a] \end{aligned}$$

Dividing these two equations through gives us

$$\frac{Q_c}{Q_e} = \frac{a}{b} \frac{T_0 - T}{p_s(T) - p_a}$$

from here, we know immediately that  $a/b = 65 \text{ Pa/K}$  and  $T_0 = 20^\circ \text{ C}$ . Because  $r = 0$ , then  $p_a = 0$ , and because  $r = 0$ , then  $Q_c = Q_e$ . Therefore, our new equation is

$$1 = 65 \frac{20 - T}{p_s(T)} \implies p_s(T) = 65(20 - T).$$

From here, we find the intersection point with this line is (6.5, 0.87), which implies the temperature is  $\boxed{6.5^\circ \text{ C}}$ .

**c)** In steady state, we have that

$$Q_c = Q_e \implies \frac{a}{b}(T_0 - T) = p_s(T) - p_a.$$

Substituting  $a/b = 65 \text{ Pa/K}$  and  $T_0 = 20^\circ \text{ C}$  and  $T \approx 2300r \text{ kPa}$  where  $r$  is the relative humidity gives us

$$65(20 - T) = p_s(T) - 2300r.$$

- When  $r = 1$ , we have the equation the line as

$$p_s = 3600 - 65T.$$

The intersection of this line with the given curve is at  $T = 20^\circ \text{C}$  and  $p_s = 2300 \text{ Pa}$ .

- When  $r = 0.8$ , we have the equation of the line as

$$p_s = 3140 - 65T.$$

The intersection of this line with the given curve is at  $T = 18.75^\circ \text{C}$  and  $p_s = 2000 \text{ Pa}$ .

Since  $p \propto r$  and we have the values of pressure at two different values of  $r$ , we can find a linear relation between pressure and relative humidity to get the equation

$$p_s = 1500r + 800 \implies 65\Delta T = 800(1 - r) \implies \Delta T = \boxed{12.3(1 - r)}.$$

d) For the boundary condition, heat dissipated through evaporation. Therefore,

$$k \frac{dT}{dt} = b(p_s - 2300r) = 800(1 - r) \implies \dot{T} \propto 1 - r.$$

We then see that

$$\frac{\dot{T}_{80}}{\dot{T}_{35}} = \frac{1 - 0.8}{1 - 0.35} = \boxed{4}.$$

**pr 23.** The water in the reservoir will boil if the saturation pressure exceeds the excess pressure:

$$p_s > p_{\text{atm}} + \rho gh$$

This threshold is  $p_s > 0.982 \text{ MPa}$ , which using the graph corresponds to a boiling point of  $183^\circ \text{C}$ . Once the water has reached this temperature, it will start boiling, sending steam up and pushing up the water in the channel such that the pressure suddenly drops down to  $0.1 \text{ MPa}$ . This sudden drop in temperature will cause the superheated water to boil violently until it settles down to  $100^\circ \text{C}$  at which point cool water will slowly fill up within the channel.

The sudden drop in temperature of  $\Delta T = 83^\circ \text{C}$  released an energy of

$$E_1 = Mc\Delta T$$

where  $M$  is the mass of the entire reservoir. This energy went into turning water into steam, which takes up an energy:

$$m\lambda = Mc\Delta T \implies \frac{m}{M} = \frac{c\Delta T}{\lambda}$$

where  $m$  is the mass of the water that got turned into steam. Therefore  $m/M$  is the ratio we seek, which we can determine to be  $m/M = \boxed{15.4\%}$ .



**pr 24.** a) We are given that

$$\ln \frac{p_i}{p_0} = \frac{a_i}{T} + b_i$$

Substituting the values from the table for  $A$  we get,

$$\ln 0.284 = \frac{a_A}{313} + b_A$$

and

$$\ln 1.476 = \frac{a_A}{363} + b_A.$$

Solving these we get  $a_A = -3748.49K$  and  $b_A = 10.72$ . The boiling temperature is the temperature at which the saturated vapour pressure equals the atmospheric pressure, i.e.  $\frac{p_A}{p_0} = 1$  This gives

$$\ln 1 = \frac{a_A}{T_A} + b_A \implies T_A = \frac{a_A}{b_A} \approx 350K$$

Similarly solving for  $B$  gives,

$$a_B = -5121.64$$

$$b_B = 13.735$$

$$T_B \approx 373K$$

b) Firstly, evaporation will start at the interface. We use fact 15 to conclude that  $\frac{p_A}{p_0} + \frac{p_B}{p_0} = 1$ , at the time  $t_1$

$$\implies e^{\frac{a_A}{t_1} + b_A} + e^{\frac{a_B}{t_1} + b_B} - 1 = 0.$$

**Bisection method** (or you could just randomly put some values less than 370 K, to zero in on the root) can be used to find the root of this equation, which gives  $t_1 \approx 340K = 67^\circ$  The saturated vapour pressures for the two liquids at this temperature are

$$p_A \approx 0.734p_0$$

$$p_B \approx 0.267p_0$$

Now let  $m_A$  and  $m_B$  be the mass of liquid  $A$  and  $B$  that escape in a bubble. We have  $\frac{m_A}{\rho_A} = \frac{m_B}{\rho_B} \implies \frac{m_A}{m_B} = \frac{p_A M_A}{p_B M_B} \implies \frac{m_A}{m_B} = 22$ . Here  $\rho_A$  and  $\rho_B$  are the densities of the vapours of  $A$  and  $B$ . Thus the rate at which  $A$  evaporates is 22 times that of  $B$ . Therefore the temperature starts increasing again when  $A$  is completely evaporated. The amount of  $B$  that has evaporated during this time is  $\frac{100}{22} = 4.5g$ . Thus at  $\tau_1$ , there will be no  $A$  left, while 95.56g of  $B$  will be left. Also, the temperature  $t_2$  is the boiling point of  $B$ .

$$t_2 = 373K = 100^\circ C$$

**pr 25.** We consider the changes in energy along the three interfaces. Consider moving the edge inwards by a distance  $a \ll L$  where  $L$  is the length of the boundary. Between the solid and the gas, the initial contact area is  $(1-r)aL$  and the final contact area is  $aL$  so the change in energy is:

$$\Delta E_1 = \sigma_{sg} r AL$$

Between the solid and the liquid, the initial contact area is  $raL$  and the final contact area is 0, so the change in energy is:

$$\Delta E_2 = -\sigma_{sl} r AL$$

Between the liquid and the gas, the initial contact area is  $aL \cos \alpha + (1-r)aL$ , and the final contact area is 0, so the change in energy is:

$$\Delta E_3 = -\sigma_{gl}aL \cos \alpha - \sigma_{gl}(1-r)aL$$

At equilibrium  $\frac{dU}{da} = 0$ , or:

$$\Delta E_1 + \Delta E_2 + \Delta E_3 = 0 \implies r(\sigma_{sg} - \sigma_{sl}) = \sigma_{gl} \cos \alpha + \sigma_{gl}(1-r)$$

Letting  $\sigma_{sg} - \sigma_{sl} = \sigma_{gl} \cos \alpha_0$  gives:

$$r\sigma_{gl} \cos \alpha_0 = \sigma_{gl} \cos \alpha + \sigma_{gl}(1-r) \implies \cos \alpha = r \cos \alpha_0 - (1-r).$$

Letting  $\alpha_0 = 110^\circ$  and  $r = 0.006$  gives:  $\boxed{\alpha = 174.9^\circ \approx 175^\circ}$ .

**pr 26.** We have a meniscus as shown in figure, and separate a fraction of water (depicted in grey) by a fictitious horizontal plane passing the flat bottom of the meniscus, and consider the force balance for the grey volume. At the separation plane inside the liquid, the hydrostatic pressure equals to the atmospheric one. Indeed, at the bottom of the meniscus, through the flat water-air interface, there is no capillary pressure, hence the hydrostatic gauge pressure must be zero; inside the liquid, the hydrostatic pressure is a function of height only, so the pressure remains equal to the atmospheric one through the horizontal plane. So, the volume depicted by grey in the figure is surrounded by atmospheric pressure, i.e. there is no extra net force acting on it due to pressure.



By idea 17, the gauge pressure due to capillary forces across a curved interface is  $\rho gh = \frac{\sigma}{r}$  in cylindrical geometry where  $H = V/(\pi r^2)$  gives the radius of the cylinder. We can determine the volume through a force balance:

$$\rho \Delta V g = \sigma(2\pi r)$$

Plugging in our expression for  $r$  gives:

$$\Delta V = \frac{2\sigma}{\rho g} \sqrt{\frac{V\pi}{H}} \approx 0.59 \text{ mL}$$

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It is believed that the symbolic answer to this problem in the handout is incorrect as it is not dimensionally correct.

**pr 27.** Consider a circular frustum with radii  $r_1$  and  $r_2$ . Let  $\theta_1$  and  $\theta_2$  be the angles the tangents at these two circular sections make with the vertical. Therefore, balancing forces in the horizontal direction, we must have:

$$\sigma(2\pi r_1) \cos \theta_1 = \sigma(2\pi r_2) \cos \theta_2$$

Since this applies for any arbitrary  $r$  and  $\theta$ , we can say that:

$$r \cos \theta = \text{constant}$$

An alternative way of phrasing this is:

$$\frac{d}{dx} r \cos \theta = 0 \implies \frac{dr}{dx} \cos \theta - r \sin \theta \frac{d\theta}{dx} = 0 \implies \frac{dr}{d\theta} = r \tan \theta$$

Separating variables:

$$\begin{aligned} \int \frac{1}{r} dr &= \int \tan \theta d\theta \\ \ln(r) &= \ln(\sec \theta) + C \\ r &= r_0 \sec \theta \\ r &= r_0 \sqrt{1 + \tan^2 \theta} \\ r &= r_0 \sqrt{1 + \left(\frac{dr}{dx}\right)^2} \end{aligned}$$

Solving for  $\frac{dr}{dx}$ , we can separate variables again and integrating:

$$x = r_0 \cosh^{-1} \left( \frac{r}{r_0} \right) \implies r = r_0 \cosh \left( \frac{x}{r_0} \right)$$

If the separation is  $L$ , we get:

$$R = r_0 \cosh(L/2r_0) \implies L = 2r_0 \cosh^{-1}(R/r_0)$$

where  $R$  is the radius of the ends. Finding the maximum value of  $L$  by using a calculator gives us 13.3 cm

**Solution 2.** We want to minimize the energy of the soap film  $E = S\gamma$ . This is achieved when the area  $S$  is at a minimum. The area is:

$$S = \int_{-L/2}^{L/2} 2\pi r \sqrt{1 + r'^2} dr$$

We want to minimize  $r\sqrt{1 + r'^2}$  which we can take to be our Lagrangian. We can apply the Euler Lagrange equations, except instead of  $r$  being dependent on time, it's dependent on  $x$ . This equivalent is:

$$\begin{aligned} \frac{d}{dx} \left( \frac{\partial L}{\partial r'} \right) &= \frac{\partial L}{\partial r} \\ \frac{d}{dx} \left( 2\pi r \frac{r'}{\sqrt{1 + r'^2}} \right) &= 2\pi \sqrt{1 + r'^2} \end{aligned}$$

which can be simplified to:

$$1 + r'^2 = Ar^2$$

where  $A = 1/r_0^2$ . The solution is well known. Motivated by the fact that  $1 + \sinh^2 x = \cosh^2 x$ , we can guess the solution:

$$r(x) = r_0 \cosh(x/r_0)$$

**pr 28.** (a) We can apply idea 18 by breaking the droplet up into a fictitious surface by imagining a circular cut at the tip of the needle. The length is  $\pi d$  where  $d$  is the diameter so the upwards force is:

$\pi\sigma d$ . This force supports the weight of the droplet  $mg$  so we have:

$$m = \frac{\pi\sigma d}{g}$$

(b) If the droplet is approximated to be a sphere of radius  $R$ , then by fact 17 the gauge pressure inside the droplet will be  $\Delta p = \frac{2\sigma}{R}$ . The droplet is surrounded by air except for at the tip of the syringe, an area of  $\pi\left(\frac{d}{2}\right)^2$ , where the droplet is in contact with water with gauge pressure  $\Delta p$ . Thus the net force on the droplet due to pressure from outside is  $\pi\left(\frac{d}{2}\right)^2 \Delta p$  in the downwards direction. Then our adjusted equilibrium equation becomes

$$mg + \pi\left(\frac{d}{2}\right)^2 \Delta p = \pi d\sigma$$

The capillary/gauge pressure can also be thought as the extra pressure needed from the inside of the substance to resist the surface tension forces

In order to calculate  $\Delta p$  we must first find  $R$  using our first order approximation from part (a) and our assumption that the droplet is a sphere:

$$\begin{aligned} \rho \frac{4}{3}\pi R^3 &= m = \frac{\pi d\sigma}{g} \\ R &= \left(\frac{3d\sigma}{4\rho g}\right)^{\frac{1}{3}} \implies \Delta p = 2\sigma \left(\frac{4\rho g}{3d\sigma}\right)^{\frac{1}{3}} \end{aligned}$$

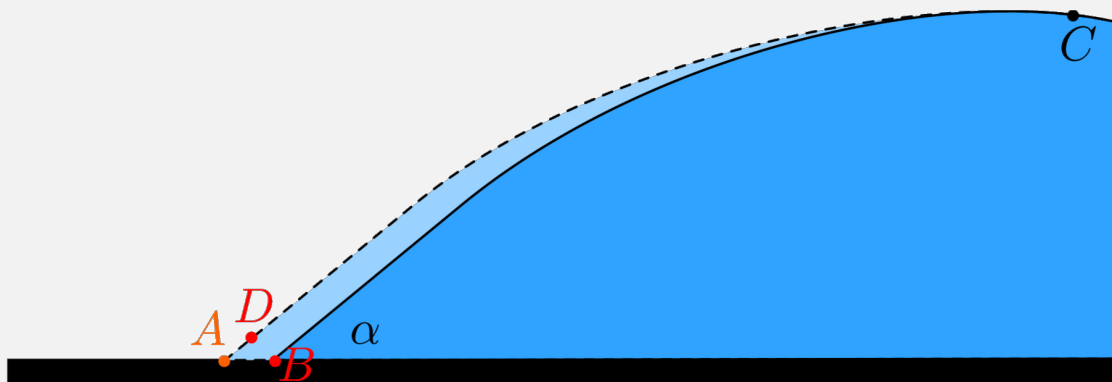
Substituting this into (1) we get

$$\begin{aligned} mg &= \pi d\sigma - \pi\left(\frac{d}{2}\right)^2 2\sigma \left(\frac{4\rho g}{3d\sigma}\right)^{\frac{1}{3}} \\ &= \pi d\sigma - \frac{\pi d^2 \sigma (4\rho g)^{\frac{1}{3}}}{2(3d\sigma)^{\frac{1}{3}}} \end{aligned}$$

Hence,

$$g = \frac{\pi d\sigma}{g} - \frac{\pi d^2 \sigma (4\rho g)^{\frac{1}{3}}}{2g(3d\sigma)^{\frac{1}{3}}}$$

**pr 29.** If the water was displaced a small distance  $s$  from equilibrium (distance  $|AB|$ ), the surface tension forces would have to cancel out by the principal of virtual work.



This means that

$$\sigma s + \sigma s' = 0$$

where  $s'$  is the distance the water has expanded (distance  $|AD|$ ) after moving away a distance  $s$  from equilibrium. We express the total energy as

$$E = \sigma S + \sigma' S + mg \frac{h}{2}$$

where  $S$  is the surface area,  $m$  is the mass of the puddle, and  $h = V/S$  is the thickness of the puddle. We then substitute these relations to get

$$E = \sigma S(1 - \cos \theta) + \frac{\rho g V^2}{2S}.$$

Minimizing this function for  $E$  gives us

$$\frac{dE}{dS} = \sigma(1 - \cos \theta) - \frac{\rho g V^2}{2S^2} = 0 \implies S = \sqrt{\frac{\rho g V^2}{2\sigma(1 - \cos \theta)}}.$$

Note that  $h = V/S$ , therefore,

$$h = \frac{V}{S} = \sqrt{\frac{2\sigma}{\rho g}(1 - \cos \theta)}.$$

**Solution 2.** We consider equilibrium of the surface defined by the two endpoints B and C in the figure of the solution before. There will be two surface tension forces directed at both endpoints of magnitude  $\sigma x$  where  $x$  is the total length of the surface. Taking forces into horizontal equilibrium gives us

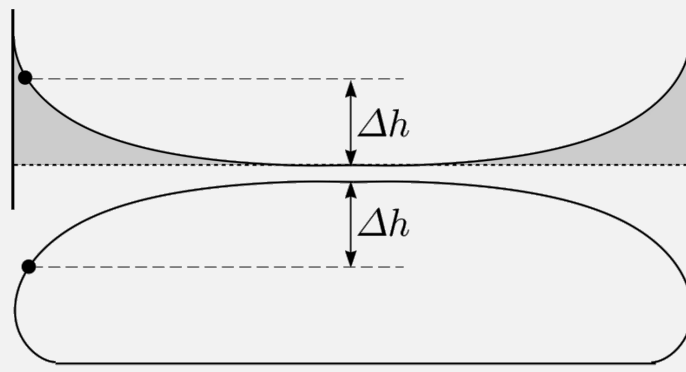
$$\sigma x(1 - \cos \alpha) = \int Px \cdot ds \sin \alpha = \int P x dy$$

where  $P$  is the external pressure. Taking this result into account tells us that

$$\begin{aligned} \sigma x(1 - \cos \alpha) &= \int_0^h \rho g y x dy \\ \sigma(1 - \cos \alpha) &= \frac{1}{2} \rho g y^2 \implies y = \sqrt{\frac{2\sigma}{\rho g}(1 - \cos \alpha)} \end{aligned}$$

It is believed that the symbolic answer to this problem is incorrect as the same answer derived here is given in [Wikipedia](#)

**pr 30.** From idea 19 we conclude that the shape of the cross-section of the meniscus is identical to the cross-section of the pool of liquid laying on the desk (cf. problem 29). This is because on each point of the surface, we have the surface tension and gauge pressure cancelling out each other to result into the same type of the surface except inverted as shown in the picture below (the result is also rather intuitive if you think about it). From fact 17 we conclude that we only have to look at half of the height of the pool of liquid since in cylindrical geometry the pressure becomes halved.



From problem 29, we can conclude that the height is given by

$$h = \sqrt{\frac{2\sigma}{\rho g}(1 - \cos \theta)}.$$

Since the contact angle is  $180^\circ$ , we have that  $\cos 180^\circ = -1 \implies h = 2\sqrt{\frac{\sigma}{\rho g}}$  we can then find that the height of the meniscus is half of this or in other words  $h_m = \boxed{\sqrt{\frac{\sigma}{\rho g}}}$ .

**pr 31.** Since the potential energy in a sphere is lower, let us assume that the potential energy stored in the surface is going to be completely transferred to kinetic energy instantaneously. Let us determine the radius of this sphere. Since the volume is equal, we have:

$$\pi(d/2)^2 h = \frac{4}{3}\pi r^3 \implies r = \frac{1}{2} \left( \frac{3d^2 h}{2} \right)^{1/3}$$

Therefore, the change in energy is:

$$\sigma \Delta S = \sigma \pi d h - \sigma 4\pi \left( \frac{1}{2} \left( \frac{3d^2 h}{2} \right)^{1/3} \right)^2 = \sigma \pi h \sqrt[3]{h} \left( \sqrt[3]{h} - \sqrt[3]{9d/4} \right)$$

Let us assume this change in energy causes half of the liquid to move at a speed of  $v$ . Then:

$$\frac{1}{2}(0.5m)v^2 = \sigma \pi h \sqrt[3]{h} \left( \sqrt[3]{h} - \sqrt[3]{9d/4} \right) \implies v = 4\sqrt{\frac{\sigma}{\rho d}} \sqrt{\frac{\sqrt[3]{h} - \sqrt[3]{9d/4}}{\sqrt[3]{h}}}$$

using the fact that  $m = \rho \pi (d/2)^2 h$ . If we take  $v$  to be the average velocity, then the characteristic time would be given by:

$$t = \frac{h}{v} = \frac{1}{4} \sqrt{\frac{\rho d}{\sigma}} \sqrt{\frac{\sqrt[3]{h^7}}{\sqrt[3]{h} - \sqrt[3]{9d/4}}}$$

We see that the characteristic time depends on the height  $h$  of the original cylinder. Since we want the time for the most unstable perturbations, we want to maximize  $t$  and we can do this by taking the derivative. Doing so gives us:

$$h = \left( \frac{7}{6} \right)^3 \frac{9}{4} d \approx 3.57d$$

Therefore, the characteristic time is:

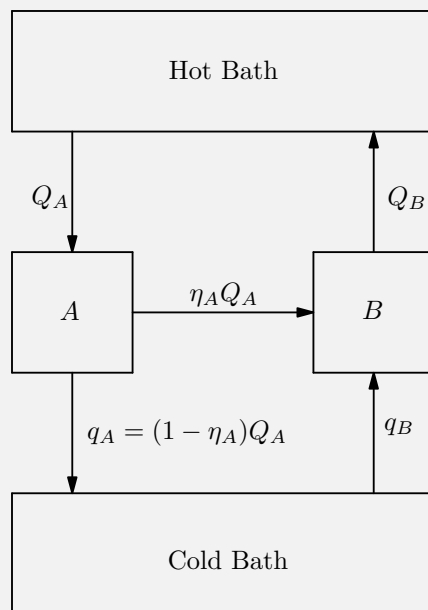
$$t = \frac{1}{4} \sqrt{\frac{\rho d}{\sigma}} \sqrt{\frac{\sqrt[3]{(3.57d)^7}}{\sqrt[3]{3.57d} - \sqrt[3]{9d/4}}} \approx 0.0088 \text{ s}$$

**Solution 2.** Dimensional analysis tells us that the characteristic time is:

$$t = \sqrt{\frac{\rho d^3}{\sigma}} = 0.003727 \text{ s}$$

**pr 32.** For the sake of contradiction, let there be 2 reversible engines  $A$  and  $B$  with different efficiencies  $\eta_A$  and  $\eta_B$  where  $\eta_A > \eta_B$ . Although they will be working with the same hot and cold baths, suppose this different efficiency is achieved through using a different design or using a different gas. We will prove that this cannot be the case and that the efficiencies must be the same no matter how it is designed.

Let engine  $A$  takes a heat  $Q_A$  from the hot bath and delivers a heat  $q_A$  to the cold bath and engine  $B$  takes a heat  $Q_B$  and delivers a heat  $q_B$  to the cold bath. Since the Carnot Cycle is reversible, we will use engine  $A$  to drive engine  $B$  backwards so that  $B$  acts like a heat pump, as depicted below.



The efficiency of engine  $A$  is:

$$\eta_A = 1 - \frac{q_A}{Q_A}$$

so the heat delivered to the cold bath is  $q_A = (1 - \eta_A)Q_A$ . Let us assume the leftover work this produces  $W = \eta_A Q_A$  goes into driving engine  $B$  backwards. Due to conservation of energy:

$$q_B + \eta_A Q_A = Q_B$$

Coupled with the efficiency equation for engine  $B$  we get:

$$Q_B = \frac{\eta_A}{\eta_B} Q_A$$

Therefore, since  $\eta_A > \eta_B$ , the hot bath is gaining heat and the cold bath is getting colder. This clearly violates the second law of thermodynamics which states that for a reversible process  $\Delta S = 0$ . The only

way for this analysis to be true is if our assumption that the efficiencies are different is incorrect and that  $Q_B = Q_A$ . Therefore, no matter how the engines are constructed as long as they are reversible, their efficiencies can only depend on the temperatures of the hot and cold baths.

**pr 33.** Assuming that idea 2 is correct, it may be proved that between the quantity of heat  $Q$ , which in a cyclical process-of the kind described above is transformed into work (or, where the process is in the reverse order, generated by work), and the quantity of heat  $Q_2$  which is transferred at the same time from a hotter to a colder body (or vice versa), there exists a relation independent of the nature of the variable body which acts as the medium of the transformation and transfer; and thus that, if several cyclical processes are performed, with the same reservoirs of heat  $K_1$  and  $K_2$ , but with different variable bodies, the ratio 3. will be the same for all. If we suppose the processes so arranged, according to their magnitude, that the quantity of heat  $Q$ , which is transformed into work, has in all of them a constant value, then we have only to consider the magnitude of the quantity of heat  $Q_2$  which is transferred, and the principle which is to be proved takes the following form:

If where two different variable bodies are used, the quantity of heat  $Q$  transformed into work is the same, then the quantity of heat  $Q_2$  which is transferred, will also be the same.

Let there, if possible, be two bodies A and A' (e.g. the perfect gas and the combined mass of liquid and vapour, described above) for which the values of  $Q$  are equal, but those of the transferred quantities of heat are different, and let these different values be called  $Q_2$ , and  $Q'_2$ , respectively:  $Q'_2$ , being the greater of the two. Now let us in the first place subject the body a to a cyclical process, such that the quantity of heat  $Q$  is transformed into work, and the quantity  $Q$  is transferred from  $K_2$  to  $K_1$ . Next let us subject A' to a cyclical process of the reverse description, so that the quantity of heat  $Q$  is generated out of work, and the quantity  $Q'_2$  is transferred from  $K_2$  to  $K_1$ . Then the above two changes, from heat into work, and work into heat, will cancel each other since we may suppose that when in the first process the heat  $Q$  has been taken from the body  $K_1$  and transformed into work, this same work is expended in the second process in producing the heat  $Q$ , which is then returned to the same body  $K_1$ . In all other respects also the bodies will have returned, at the end of the two operations, to their original condition, with one exception only. The quantity of heat  $Q'_2$  transferred from  $K_1$ , to  $K_2$  has been assumed to be greater than the quantity  $Q_2$  transferred from  $K_1$  to  $K_2$ . Hence, these two do not cancel each other, but there remains at the end a quantity of heat, represented by the difference  $\Delta Q_2$ , which has passed over from  $K_1$  to  $K_2$ . Hence a passage of heat will have taken place from a colder to a warmer body without any other compensating change. But this contradicts the fundamental principle. Hence the assumption that  $Q'_2$  is greater than  $Q_2$ , must be false.

Again, if we make the opposite assumption, that  $Q'_2$  is less than  $Q_2$  we may suppose the body A' to undergo the cyclical process in the first, and a in the reverse direction. We then arrive similarly at the result that a quantity of heat  $Q_2 - Q'_2$ , has passed from the colder body  $K_2$  to the hotter  $K_1$  which is again contrary to the principle.

Since then  $Q'_2$ , can be neither greater nor less than  $Q_2$ , it must be equal to  $Q_2$ ; which was to be proved. We will now give to the result thus obtained the mathematical form most convenient for our subsequent reasoning. Since the quotient  $Q/Q_2$  is independent of the nature of the variable body (fact 2), it can only depend on the temperature of the two bodies  $K_1$  and  $K_2$  which act as heat reservoirs. The same will of course be true of the sum

$$1 + \frac{Q}{Q_2} = \frac{Q + Q_2}{Q_2} = \frac{Q_1}{Q_2}.$$

This last ratio, which is that between the whole heat received and the heat transferred, we shall select for further consideration; and shall express the result obtained in this section as follows:



The ratio  $Q_1/Q_2$  can only depend on the temperatures  $T_1$  and  $T_2$ .

This leads to the equation:

$$\frac{Q_1}{Q_2} = f(T_1, T_2).$$

Since the process is isothermal, we can obtain the equation

$$\frac{Q_1}{Q_2} = \frac{nRT_1 \ln \frac{V_f}{V_i}}{nRT_2 \ln \frac{V_f}{V_i}} = \frac{T_1}{T_2}.$$

Using this definition, the Carnot's Cycle efficiency can then be rewritten as

$$\eta_C = 1 - \frac{Q_2}{Q_1} = 1 - \frac{T_2}{T_1}.$$

□

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Part of this solution is taken from Rudolf Clausius' original work "The Mechanical Theory of Heat".

**pr 34.** According to Prevost's theory of exchange, in order to maintain thermal equilibrium, any object must emit the same energy as it receives. Thus the absorption and emission of light must be the same at all frequencies.

A consequence of this problem is that radiation and absorption properties of a material must be identical throughout the entire spectrum. It can be similarly shown that a partially reflecting material must have equal transmittance from both sides. It may seem that dark window glasses are more transparent when looking from inside of a darkly lit room, but this is a mere illusion: when looking from outside, a small fraction of reflected abundant outside light can easily dominate over the transmitted part of the light coming from inside, but the opposite is not true. The effect can be enhanced by overlaying an absorbing and reflecting layers and turning the reflecting layer outside. Then, while total transmittance is equal from both sides, the reflectance from outside is larger because from outside, reflected light does not pass through the absorbing layer.

**pr 35.**

*Proof.* We analyze 4 processes on a Carnot cycle working in phase:

- We have our substance undergo an isothermal expansion at a constant temperature  $T$ . The substance undergoes isothermal expansion since it becomes in contact with a reservoir at a temperature  $T$ . A bit of the mass of the liquid evaporates during this process and as a result, the total volume that the substance works on changes from  $V_1$  to  $V_2$ . We can approximate this change in volume by a straight line.
- We have our substance undergo adiabatic expansion where the temperature and pressure increase to  $T + dT$  and  $p + dp$ .
- The substance is compressed isothermally at temperature  $T + dT$  when it is put in thermal contact with a reservoir of temperature  $T + dT$ . In this process, the vapour mass condenses and therefore, the pressure of this process is constant at  $p + dp$ .
- The substance is compressed adiabatically and it's pressure and temperature goes to  $(p + dp, T + dT) \rightarrow (p, T)$ .

The graph that this cycle creates is a parallelogram with height  $-dp$  and edges of lengths  $m\Delta V$ . The work done by this Carnot cycle is then  $-m\Delta V dp$ . Furthermore, this substance receives heat during the first process which is of value of  $mL$ . We then can write the efficiency  $\eta$  of this thermodynamic process as:

$$\eta = \frac{W}{Q} = \frac{-m\Delta V dp}{mL} = -\frac{\Delta V dp}{L}.$$

Assuming all reversible processes means that this in turn must equal to the efficiency of a Carnot cycle or

$$\eta_{\text{Carnot}} = \frac{T_h}{T_h - T_\ell} = \frac{T + dT}{dT} = -\frac{dT}{T}.$$

Therefore, equating both of these together gives us

$$-\frac{dT}{T} = -\frac{\Delta V dp}{L} \implies \frac{dp}{dT} = \frac{L}{T\Delta V} = p_s \frac{\lambda\mu}{RT^2}.$$

□

Solving the differential equation, we have:

$$\ln\left(\frac{p}{p_0}\right) = \frac{\lambda\mu}{R} \left(\frac{1}{T_i} - \frac{1}{T_f}\right)$$

We can assume that  $\Delta T = T_f - T_i \ll T$ , allowing us to simplify our relationship to:

$$P = P_0 \exp\left(\frac{\lambda\mu\Delta T}{RT^2}\right) \implies \Delta P = P_0 \left(\exp\left(\frac{\lambda\mu\Delta T}{RT^2}\right) - 1\right)$$

Using the first order expansion, we get:

$$\Delta P = \frac{P_0 \lambda\mu \Delta T}{RT^2} \approx 350 \text{ Pa}.$$

Let's consider one mole of some material at the phase boundary so that it is equally stable as a liquid and gas alike. Therefore, its Gibbs free energy must be the same at the phase boundary. Let us denote  $dG_l$  by the Gibb's free energy of the liquid state and  $dG_g$  by the Gibb's free energy of the gas. We require that  $G_l = G_g$ . Consider raising the temperature by  $dT$  and the pressure by  $dP$  such that both phases remain equally stable therefore we require  $dG_l = dG_g$ . Also, since

$$dG = -SdT + Vdp$$

We have that

$$-S_l dT + V_l dP = -S_g dT + V_g dP.$$

Where  $(V_1, S_1)$  is the volume and entropy of the liquid and  $(V_2, S_2)$  is the volume and entropy of the gas. We then have

$$dp(V_2 - V_1) = dT(S_2 - S_1) \implies \frac{dp}{dT} = \frac{S_g - S_l}{V_g - V_l}.$$

It is easier to write the difference in entropies as  $S_g - S_l = \lambda\mu/T$  and  $V = \frac{RT}{p}$  from the ideal gas law. Therefore,

$$\frac{dp_s}{dT} = p_s \frac{\lambda\mu}{RT^2}.$$

Let us expand on entropy  $S = S(V, T)$  as a function of state with the multivariable chain rule:

$$dS = \left( \frac{\partial S}{\partial V} \right)_T dV + \left( \frac{\partial S}{\partial T} \right)_V dT.$$

In a closed system, temperature and pressure remain constant and therefore,  $\left( \frac{\partial S}{\partial T} \right)_V dT = 0$ . We are then left with

$$dS = \left( \frac{\partial S}{\partial V} \right)_T dV.$$

We then use Maxwell's relations:

$$\left( \frac{\partial S}{\partial V} \right)_T = \left( \frac{\partial p}{\partial T} \right)_V \implies dS = \left( \frac{\partial p}{\partial T} \right)_V dV.$$

Next, note that by using the multivariable chain rule on  $P = P(V, T)$  we have:

$$dP = \left( \frac{\partial p}{\partial T} \right)_V dT + \left( \frac{\partial p}{\partial V} \right)_T dV$$

and  $\left( \frac{\partial p}{\partial V} \right)_T = 0$  as a change in phase transition does not change the pressure with constant parameters. Therefore,

$$dP = \left( \frac{\partial p}{\partial T} \right)_V dT \implies \frac{dp}{dT} = \left( \frac{\partial p}{\partial T} \right)_V.$$

This means that - after going back to the expression of entropy - we have

$$dS = \frac{dp}{dT} dV \implies \frac{dp}{dT} = \frac{\Delta S}{\Delta V} = \frac{\Delta H}{TV}.$$

Note that  $\Delta H = \lambda\mu$  and  $V = \frac{RT}{p}$  from the ideal gas law. Therefore,

$$\frac{dp_s}{dT} = p_s \frac{\lambda\mu}{RT^2}.$$

**pr 36.** The Clausius-Clapeyron equation for water is:

$$\frac{dp_s}{dT} = p_s \frac{L\mu}{RT^2}.$$

Separating variables gives:

$$\frac{1}{p_s} \frac{dp_s}{dT} = \frac{L\mu}{RT^2}.$$

Integrating this gives us

$$\int_{p_{s1}}^{p_{s2}} \frac{dp_s}{p_s} = \int_{T_1}^{T_2} L\mu \frac{dT}{RT^2} \implies \ln \frac{p_{s2}}{p_{s1}} = \frac{Lm_v}{R} \left( \frac{1}{T_1} - \frac{1}{T_2} \right).$$

Simplifying this result finally gives us

$$p_{s2} = p_{s1} \exp \left[ \frac{Lm_v}{R} \left( \frac{1}{T_1} - \frac{1}{T_2} \right) \right].$$

where  $p_{s1}$  and  $T_1$  are the initial points of integration. We are given,  $p_s = p_0 e^{-U/k_B T}$  and therefore,

$$p_{s2} = p_0 e^{-U/k_B T_2}, \quad p_{s1} = p_0 e^{-U/k_B T_1}.$$

This mean that

$$p_{s_2} = p_{s_1} \exp \left[ \frac{U}{k_B} \left( \frac{1}{T_1} - \frac{1}{T_2} \right) \right].$$

Comparing the 2 equations gives,  $\boxed{U = L\mu/N_A}$  as  $k_B = R/N_A$ .

To interpret this equation, note that we can rewrite it as the latent heat of evaporation per mole. We can imagine the gas phase to have an energy of zero and thus the latent heat of evaporation represents the energy "stored" inside the bonds in the water molecules. Thus, it represents the work that needs to be done to evaporate one mole of water

**pr 37.** First, let us convert the temperatures from celsius to kelvin.

$$T_1 = 20 + 273 = 293 \text{ K}$$

$$T_2 = 0 + 273 = 273 \text{ K}$$

Furthermore, converting the time from hours to seconds gives us  $t = 36000 \text{ s}$  Note that the heat flux as given in Kalda's introduction to thermoelectricity is

$$\Phi = \frac{P}{\eta_C}$$

where  $\eta_C = 1 - \frac{T_c}{T_h}$ . For a small increment of time  $dt$  the mass gained is  $dm$  and therefore the heat flux is  $\frac{dm}{dt}\lambda$ . Equating these two expressions together gives us

$$\frac{dm}{dt}\lambda = I^2 R \frac{T_h}{T_h - T_c} \implies \int_0^m dm = \int_0^t I^2 R \frac{T_h}{T_h - T_c} dt.$$

Integrating through and dividing both sides by  $\lambda$  tells us that

$$m = \frac{I^2 R t T_h}{(T_h - T_c)\lambda} \approx 1.5 \text{ g}.$$

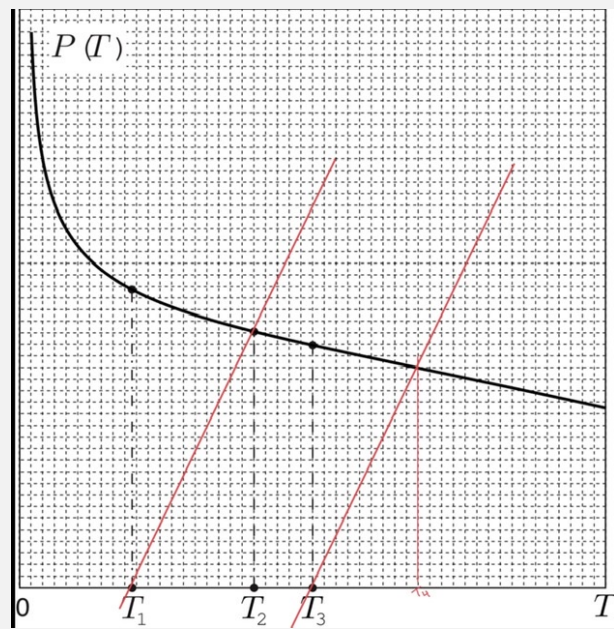
**pr 38.** Heat leaving the room in stable state is equal to heat produced by heater by Fourier's law.

$$P(T_2) = k(T_2 - T_1)$$

In the second case,

$$P(T_4) = k(T_4 - T_3)$$

Therefore, point  $(T_4, P(T_4))$  must be on the line which passes through point  $(T_3, 0)$  with the same slope as the line passing through  $(T_2, P(T_2))$  and  $(T_1, 0)$  This gives  $\boxed{T_4 = 1.4T_3}$



## 4 Solutions to Revision Problems

This section will contain problem 38-86 of the handout. Revision problems take concepts and ideas from earlier problems and places them in a new context. As a result, many of the problems in this section will seem familiar. This however, does not mean that all the problems in this section are easy. Some of the hardest problems originate in this section.

**pr 39.** At nominal voltage, we have two relations of  $P = I_0 V_0$  and  $R = \frac{V_0}{I_0}$ . As resistivity is proportional to temperature, we then have the temperature at which the filament is supposed to emit light to be

$$\frac{R}{R_0} = \frac{T}{T_0} \implies T = \frac{RT_0}{R_0} = \boxed{\frac{V_0}{I_0 R_0} T_0}.$$

The power  $P$  radiated according to Stefan-Boltzmann law is

$$\frac{P}{A} = k\sigma T^4 \implies P = k\sigma T^4 \cdot \pi l d \implies l d = \frac{V_0 I_0}{\pi k \sigma T^4}.$$

The initial resistance is given by

$$R = \frac{\rho_0 l}{A} = \frac{\rho_0 l}{\pi d^2/4} \implies l = \boxed{\frac{\pi R_0 d^2}{4 \rho_0}}.$$

Therefore,

$$\frac{\pi R_0}{4 \rho_0} d^3 = \frac{V_0 I_0}{\pi k \sigma T^4} \implies d = \boxed{\sqrt[3]{\frac{4 V_0 I_0 \rho_0}{k \pi^2 R_0 \sigma T^4}}}.$$

**pr 40.** The rate at which the temperature rises or falls is proportional to total heat output. Until  $t = 1100$  min heater is not switched on as temperature decreases continuously. After  $t = 1100$  min power of heater is added to heat lost i.e.

$$P_{\text{total}} = P_{\text{heater}} + P_{\text{heat lost}}$$

where  $P_{\text{heat lost}}$  is the power lost to surroundings. The total Power is proportional to rate of change in temperature which is the slope of given graph. For the heater, it is is change in slope at  $t = 1100$  min. Now applying calculations gives us:

$$\dot{T}_{\text{heater}} = \dot{T}_{\text{total}} - \dot{T}_{\text{heat lost}} \implies \dot{T}_{\text{heater}} = \frac{2}{69} - \left( -\frac{2}{218.18} \right) = 0.038^\circ\text{C/min}.$$

Therefore, equilibrium temperature is when power of heat losses is equal to power of heater (in magnitude) i.e. the temperature at which the  $\dot{T}_{\text{heat lost}} = 0.038^\circ\text{C/min}$ . This can be found in the graph before  $t = 1100$  min between  $T = 20^\circ\text{C}$  to  $T = 24^\circ\text{C}$ .

**pr 41.** Before tackling this problem, we have to think of what the setup tells us. It may be tempting to use the adiabatic  $pV^\gamma = \text{const}$  relation to help us solve this problem but note that there is heat exchange being done with the surrounding and furthermore, the process is being done at constant pressure so we cannot take this to be an adiabatic process. This process is instead isobaric since it is being done at constant pressure. Also, to make our naming shorter, we denote:

1. Process I: Helium receives a certain amount of heat, because of which the piston moves up by  $d_1 = 5$  cm.
2. Process II: After waiting for some longer period of time, an additional displacement  $d_2$  of the piston was observed.

Since both processes are isobaric, note that the heat added to the surroundings can be accurately predicted with the formula

$$Q_1 = C_p n \Delta T.$$

In process I, the heat added to the surroundings is given by

$$Q_1 = C_{p_{\text{He}}} n_{\text{He}} \Delta T$$

since only Helium receives the amount of heat and not Hydrogen. The change in temperature can be accurately predicted by the ideal gas law:

$$p \Delta V = n R \Delta T \implies \Delta T = \frac{p \Delta V}{n_{\text{He}} R} = \frac{p S d_1}{n_{\text{He}} R}$$

where  $S$  is the cross-sectional area of the container. Therefore,

$$Q_1 = C_{p_{\text{He}}} n_{\text{He}} \frac{p S d_1}{n_{\text{He}} R} = \frac{C_{p_{\text{He}}} p S d_1}{R}.$$

In process II, the piston goes to an equilibrium temperature  $T_{\text{eq}}$ . Both Hydrogen and Helium are changing volume and temperature to achieve this equilibrium state so therefore, we denote the heat added by

$$Q_2 = (C_{p_{\text{He}}} n_{\text{He}} + C_{p_{H_2}} n_{H_2}) \Delta T_{\text{eq}}.$$

By ideal gas law, we can write

$$\begin{aligned} p V_0 &= n_H R T \\ p(3V_0) &= n_{\text{He}} R T \end{aligned}$$

This implies that  $n_{\text{He}} = 3n_{H_2}$ . Furthermore, once again by ideal gas law note that

$$p \Delta V = (n_{\text{He}} + n_H) R \Delta T_{\text{eq}} \implies \Delta T_{\text{eq}} = \frac{p \Delta V}{(n_{\text{He}} + n_H) R} = \frac{p S (d_1 + d_2)}{\frac{4}{3} n_{\text{He}} R}.$$

Next, note that the heat added in process I must be the same as the heat added in process II by the second law of thermodynamics. Therefore,

$$\frac{C_{p_{\text{He}}} p S d_1}{R} = (C_{p_{\text{He}}} n_{\text{He}} + C_{p_{H_2}} n_{H_2}) \Delta T_{\text{eq}}.$$

Substituting the relations we found from the ideal gas law tells us that

$$\frac{C_{p_{\text{He}}} p S d_1}{R} = n_{\text{He}} (C_{p_{\text{He}}} + \frac{1}{3} C_{p_{H_2}}) \frac{p S (d_1 + d_2)}{\frac{4}{3} n_{\text{He}} R}.$$

Simplifying gives us the expression

$$\frac{4}{3} C_{p_{\text{He}}} d_1 = (C_{p_{\text{He}}} + \frac{1}{3} C_{p_{H_2}}) (d_1 + d_2).$$

Substituting  $C_{p_{H_2}} = 7/2 R$  and  $C_{p_{\text{He}}} = 5/2 R$  tells us that

$$d_2 = \frac{\frac{4}{3} C_{p_{\text{He}}} - C_{p_{\text{He}}} - \frac{1}{3} C_{p_{H_2}}}{C_{p_{\text{He}}} + \frac{1}{3} C_{p_{H_2}}} d_1 = \frac{\frac{4}{3} \frac{5}{2} R - \frac{5}{2} R - \frac{1}{3} \frac{7}{2} R}{\frac{5}{2} R + \frac{1}{3} \frac{7}{2} R} d_1 = -\frac{1}{11} d_1.$$

This tells us that the piston moves down by  $\frac{1}{11} d_1 = 0.45$  cm.

**pr 42. (a)**

*Proof.* We note that the total change in energy is given by

$$\begin{aligned}\Delta U &= nC_V\Delta T \\ &= C_Vn_1(T - T_1) + C_Vn_2(T - T_2)\end{aligned}$$

By the ideal gas law,  $pV = nRT$ , we have that

$$\Delta U = \frac{C_V p_0 (V - V_1 - V_2)}{R}$$

we also see that internal energy change must be equal to the work produced by external pressure. Therefore,

$$\frac{C_V p_0 (V - V_1 - V_2)}{R} = p_0 (V - V_1 - V_2).$$

Since  $C_V/R$  is non-zero we find that

$$V - V_1 - V_2 = 0 \implies V = V_1 + V_2.$$

□

**(b)** We note that by the ideal gas law

$$pV = nRT \implies n = \frac{pV}{RT}.$$

This tells us that the moles of gas in our scenario is given as

$$n = \frac{p_0}{R} \left( \frac{V_1}{T_1} + \frac{V_2}{T_2} \right) = \frac{p_0}{R} \left( \frac{V_1 + V_2}{T'} \right)$$

therefore, we find that

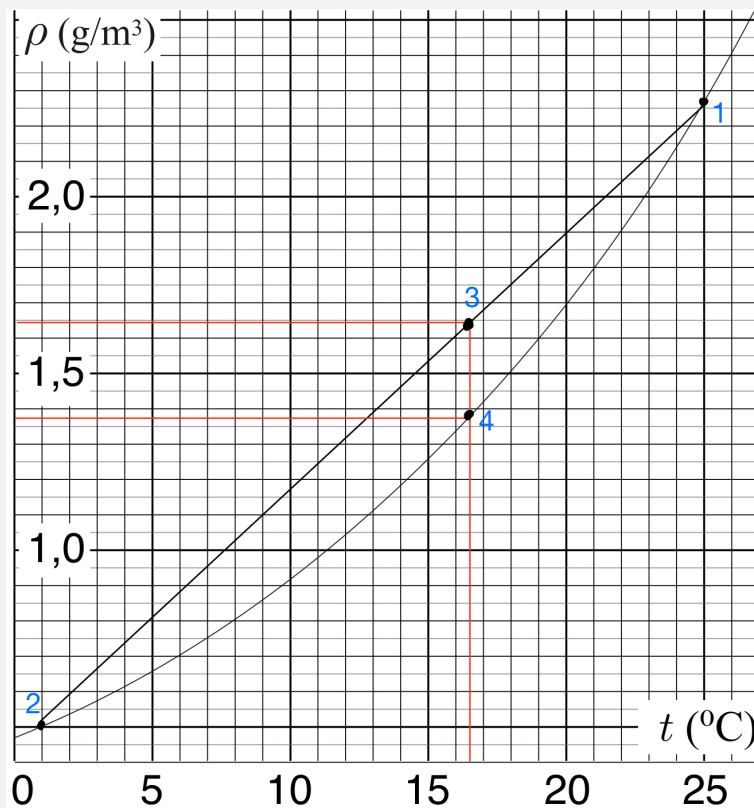
$$T' = \frac{V_1 + V_2}{V_1/T_1 + V_2/T_2} \approx \boxed{16.5^\circ \text{ C}}$$

**(c)** Note that relative humidity is denoted as the mass of water vapor divided by the mass of water vapor in saturation. In this problem, we have to calculate the relative humidity of the mixed air. First, note that y-axis of the graph that is given in part (c) represents density. We have to convert our mass relation to density which is fairly easy as the density is directly proportional to mass which means that  $r = \frac{\rho_v}{\rho_{vs}}$ .

Next, we need to determine these two densities. First, note that there were two individual mixtures of air, one, inside the room at temperature  $t_1 = 25^\circ \text{ C}$  and the other outside of the room at temperature  $t_2 = 1^\circ \text{ C}$ . We mark these two points on our graph as point 1 and 2 respectively. We then connect these two points with a line. The temperature of the mixed air as shown in part (b) is  $16.5^\circ \text{ C}$ . The density of the water vapor will be the intersection of the line  $x = 16.5$  and the line connecting the two points 1 and 2. By finding the intersection of these two points, we find that  $\rho_v$  is at point 3 with a density of  $1.65 \text{ g/cm}^3$ . The mass of the water vapor in saturation is represented by the intersection of the graph and the line  $x = 16.5$  (this is because the graph shows the dependence of saturated vapor density of water as a function of temperature). This gives us the point 4 with a density of  $\rho_{vs} = 1.37 \text{ g/cm}^3$ . We now finally obtain the relative humidity to be

$$r = \frac{\rho_v}{\rho_{vs}} = \frac{1.65 \text{ g/cm}^3}{1.37 \text{ g/cm}^3} = 1.20 = 120\%.$$





(d) The mass will have to undergo a small temperature change to condensate. This means that it will have to transfer heat to its surroundings and with that information, we can attempt to balance the heat going out into the system and the heat going into the system. The heat going out of the system moves to the surrounding and will go to the outside air that has a given density  $\rho_0$ . With a temperature change denoted by  $\Delta t$ , we write  $Q_{\text{out}} = mc\Delta T = \rho_0 V c \Delta t$ . The heat going into the surroundings changes the temperature of the mixed air. If  $t' = 16.5^\circ \text{C}$  is the initial temperature of the mixed air, we write  $Q = q(\rho_v(t') - \rho_{vs}(t' + \Delta t))$ . We conserve heat to get the equation

$$\rho_0 c \Delta t = q(\rho_v(t') - \rho_{vs}(t' + \Delta t)).$$

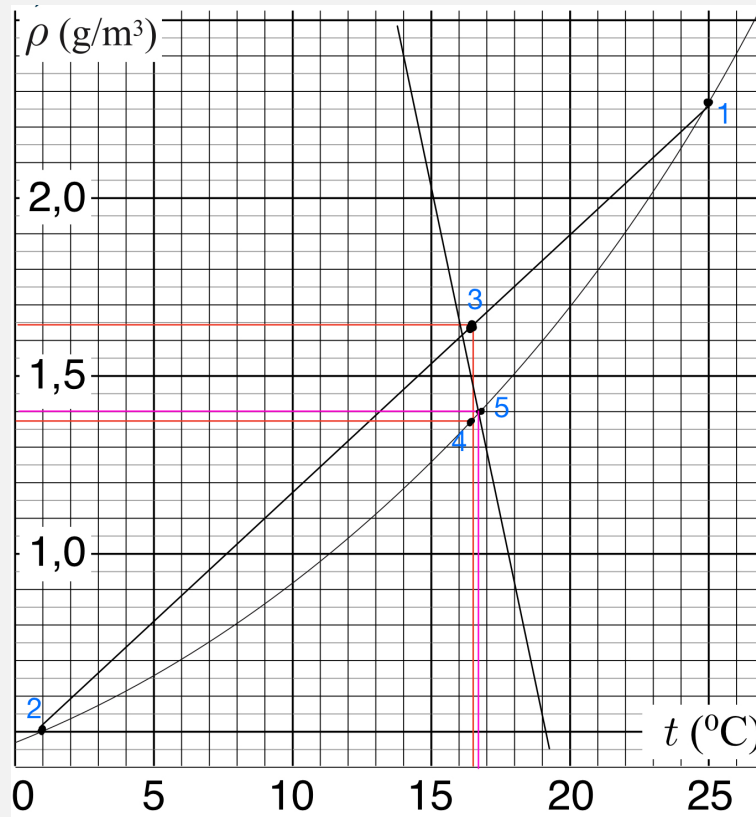
We then find by rearranging variables that

$$\rho_{vs}(t' + \Delta t) = \rho_v(t') - \frac{\rho_0 c_p \Delta t}{q}.$$

Let us denote  $T = t' + \Delta t$ . This means that  $\Delta t = T - t'$ . Our equation then looks like

$$\rho_{vs}(T) = 1.65 - 0.47(T - 16.5).$$

This is an equation of a line, and the intersection of this line with the graph of the saturated vapor density gives us the new condensated mass. As shown in the picture below, we have our new point 5 to be the density of the new mass. Remember that we have to find the new mass of the condensated mass which will be given by  $m = \Delta \rho(V_1 + V_2)$ . We see that  $\Delta \rho \approx 0.25 \text{ g/m}^3$  from the graph so  $m \approx 7.5 \text{ g}$ .



**pr 43.** (a) Note that the total force on a cross-sectional area of the cylinder is given by

$$F = p_0 \pi r^2 + mg \implies p = p_0 + \frac{mg}{\pi r^2}.$$

By ideal gas law we note that

$$p \Delta V = n R \Delta T \implies \Delta T = \frac{p \Delta V}{n R} = \frac{p_0 (\pi r^2 \Delta s)}{n R} = \frac{(p_0 \pi r^2 + mg) \Delta s}{n R}.$$

The total temperature after collision will then be given as

$$T = T_0 + \Delta T = T_0 + \frac{(p_0 \pi r^2 + mg) \Delta s}{n R}.$$

(b) Note that work is given by  $p \Delta V$ . The change in volume is  $\pi r^2 \Delta s$  so all in all, we have  $W = (p_0 \pi r^2 + mg) \Delta s$ .

(c) The process is isobaric since it occurs at a constant pressure which means that the energy is given by the heat converted to the system or  $Q = c_p n \Delta T$ . We know from part (a) that

$$\Delta T = \frac{(p_0 \pi r^2 + mg) \Delta s}{n R}$$

which means that

$$Q = c_p n \frac{(p_0 \pi r^2 + mg) \Delta s}{n R} = \frac{c_V + R}{R} n \frac{(p_0 \pi r^2 + mg) \Delta s}{n} = \Delta (p_0 \pi r^2 + mg) \left( \frac{c_V}{R} + 1 \right).$$

(d) The power is given by  $P = \frac{dQ}{dt}$  which means that for a unit time  $\Delta t$  it is

$$P = \left( \frac{c_V}{R} + 1 \right) \frac{\Delta s}{\Delta t} (p_0 \pi r^2 + mg).$$

The number of molecules for a unit time is given by power over energy (this is justified as it is dimensionally correct). The energy of a single photon will be  $E_p = h\nu = \frac{hc}{\lambda}$  which means

$$\dot{n} = \frac{P}{E_p} = P \frac{\lambda}{hc}.$$

(e) The efficiency of the process is given by  $\eta = \frac{W}{Q}$ . The change in mechanical energy of the gas  $mg\Delta s$  serves as the change in work so therefore,

$$\eta = \frac{mg\Delta s}{\left(1 + \frac{c_V}{R}\right) (p_0 \pi r^2 + mg)\Delta s} = \frac{1}{\left(\frac{p_0 \pi r^2}{mg} + 1\right) \left(1 + \frac{c_V}{R}\right)}.$$

(f) When the cylinder is rotated, the gas undergoes adiabatic expansion to a new pressure  $p_0$  due to the lack of pressure from external surroundings. Noting that  $T^\gamma P^{1-\gamma} = \text{const}$  the new temperature of the gas is

$$T_f = T_0 \left( \frac{p_0}{p} \right)^{\frac{\gamma-1}{\gamma}}.$$

**pr 44.** The tire does not exchange any heat with its surroundings which means that it undergoes an adiabatic expansion/compression. Since we are relating two state variables  $p$  and  $T$ , we use the relation

$$p^{1-\gamma} T^\gamma = \text{const}.$$

Substituting both final and initial variables shows us

$$p_0^{1-\gamma} T_0^\gamma = p_f^{1-\gamma} T_f^\gamma \implies T_f = T_0 \left( \frac{p_f}{p_0} \right)^{\frac{\gamma-1}{\gamma}} = T_0 \left( \frac{p_0 + p_1}{p_0} \right)^{\frac{\gamma-1}{\gamma}} = 3^{\frac{\gamma-1}{\gamma}} T_0 \approx 400 \text{ K}.$$

**pr 45.** (a) Let us analyze each step in the cycle:

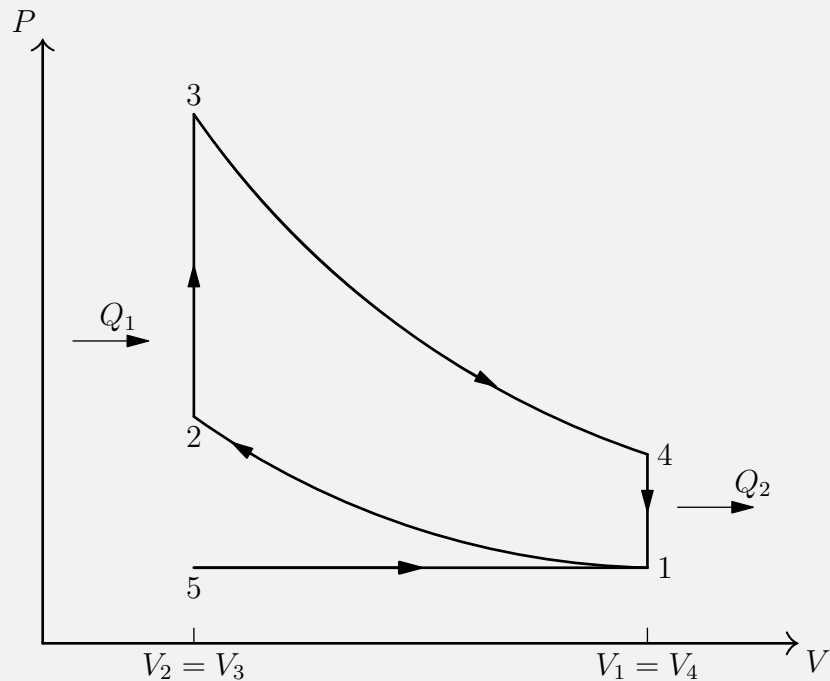
I. This is an isobaric process (or in other words, constant pressure). This is because no thermodynamic parameter is changing in the piston apart from the change in volume from  $5 \rightarrow 1$ . Fresh air comes into the valve and fills the cylinder.

II. For process  $1 \rightarrow 2$ , we have a reversible adiabatic process. Both parameters of pressure and volume are changing and the air is being compressed adiabatically.

III. For process  $2 \rightarrow 3$ , we have an isothermal process. This is because the combustion spark into the valve is only for a very short time so we essentially have a constant volume. We then have heat absorbed from a series of reservoirs of temperatures  $T_2$  and  $T_3$ .

IV. Now the gas is disposed of and the pressure and volume again change adiabatically from  $3 \rightarrow 4$ . This process then repeats once again periodically through an isothermal process of  $4 \rightarrow 1$ .

We can now draw the thermodynamic cycle.



(b) The heat out of the isotherm on  $2 \rightarrow 3$  is given by

$$Q_1 = C_V(T_3 - T_2)$$

while the heat out of the isotherm on  $4 \rightarrow 1$  is given by

$$Q_2 = C_V(T_3 - T_1).$$

Since efficiency is given by

$$\eta = \frac{W}{Q_2} = 1 - \frac{Q_1}{Q_2}$$

we find that

$$\eta = 1 - \frac{T_4 - T_1}{T_3 - T_2}.$$

On an adiabat,  $TV^{\gamma-1}$  is constant or in other words,

$$T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1}, \quad T_4 V_1^{\gamma-1} = T_3 V_2^{\gamma-1}$$

which implies that

$$(T_4 - T_1) V_1^{\gamma-1} = (T_3 - T_2) V_2^{\gamma-1} \implies \frac{T_4 - T_1}{T_3 - T_2} = \left( \frac{V_2}{V_1} \right)^{\gamma-1}.$$

Hence:

$$\eta = 1 - \frac{1}{(V_2/V_1)^{\gamma-1}} = \boxed{1 - k^{1-\gamma}}.$$

**Remark:** This problem is essentially resemblant of the Otto cycle, a four-stroke cycle in internal combustion engines that is found in cars, electrical generators, etc.

**pr 46.** First, let us figure out how to calculate the energy cost. It is given in the problem that the electrical energy cost is  $c = 0.1$  EUR/kWh so if we find the amount of energy dissipated, we can then convert this to monetary value with using the electricity cost constant via dimensional analysis.

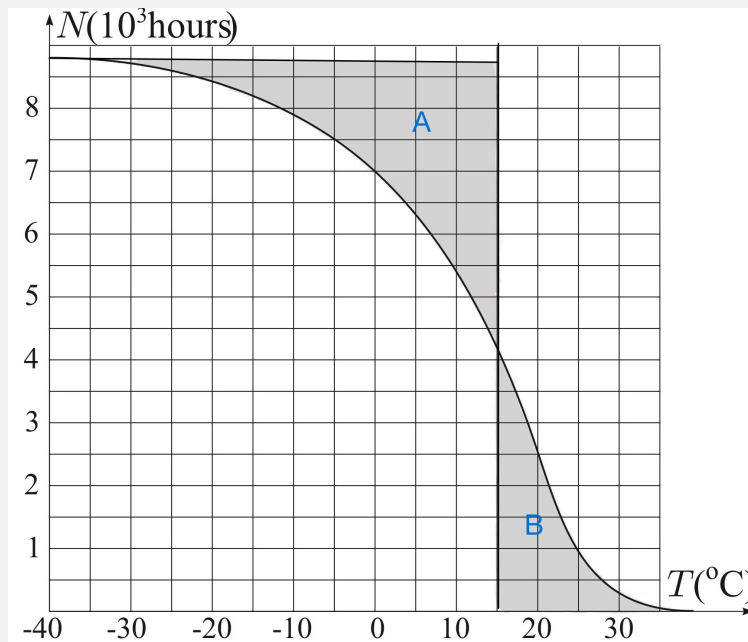
Let us make use of idea 4 in Kalda's handout. If a system is described by a parameter  $x$  (which can be time, coordinate, velocity, etc.) and a quantity  $A$  can be expressed as  $A = \sum_i F_i \Delta x$ , where  $\Delta x$  is a small interval of the parameter  $x$ , the sum is taken over all the small intervals, and  $F_i$  is a function of  $x$ . In this case, our parameter is temperature. We divide the graph into small chunks where the temperature at each endpoint of a singular chunk is  $T_i$  where  $i \in \mathbb{Z}^+$ . For the cooling period, during the number of days  $\Delta N_i$  for which the temperature stayed in the (small) range between  $T_i$  and  $T_i + \Delta T_i$ , the heat loss is  $Q_c = (T_i - T_0)C\Delta N_i \cdot 3600$  s/h. Note that  $(T_i - T_0)\Delta N_i$  is a horizontal narrow rectangular region between the graph and the vertical line  $T = T_0$ . Therefore, the net energy can be represented as

$$Q_{\text{tot}} = Q_A + Q_B = C \left( \sum_{T_i < T_0} (T_0 - T_i)\Delta N_i + \sum_{T_i > T_0} (T_i - T_0)\Delta N_i \right) = C(A_A + A_B)$$

where  $A_A$  and  $A_B$  are the respective areas shown in the graph below. Next, we need to make use of the efficiencies as the total power is not just all we need. Note that when the device is used as an air conditioner (i.e  $T_i < T_0$ ) we use the COP  $\eta_c$  and when the device is used to heat the room (i.e  $T_i > T_0$ ) we use the efficiency  $\eta_h$ . Therefore, our effective heating power is given by

$$Q_{\text{eff}} = \frac{Q_A}{\eta_c} + \frac{Q_B}{\eta_h} = C \left( \frac{A_A}{\eta_c} + \frac{A_B}{\eta_h} \right).$$

By measuring areas, we can approximate the effective energy to be  $Q_{\text{eff}} \approx 2900$  kWh and multiplying by  $c$  gives us the answer of 290 EUR.



**pr 47.** (a) The power dissipated by the wire is

$$P = \frac{U_0^2}{R} = \frac{U_0^2 S}{\rho_{\text{el}} l}.$$

By idea 1, we can write

$$P \equiv \frac{dQ}{dT} = C \frac{dT}{dt} = mc \frac{dT}{dt}$$

which means that

$$mc \frac{dT}{dt} = \frac{U_0^2 S}{\rho_{\text{el}} l} \implies \int mc \rho_{\text{el}} l dT = \int U_0^2 S dt.$$

By integrating and solving this equation for  $t$ , we find

$$\frac{U_0^2 A t}{\rho_{\text{el}} l} = A l \rho_{20} c (T_1 - T_2) \implies t = \frac{\rho_{20} \rho_{\text{el}} c l^2 (T_1 - T_0)}{U_0^2} = 25 \text{ ms.}$$

(b) As  $U_1$  is large,  $\rho_{\text{el}}$  changes significantly with temperature. Applying the same initial equations as in part (a), we find that

$$\int mc \rho_{\text{el}} l dT = \int U_1^2 S dt \implies t = \frac{\rho_{20} c l^2}{U_1^2} \times (\text{Area under } \rho_{\text{el}} - T \text{ graph}).$$

The area under the graph is approximately  $\frac{11.6 \pm 0.4}{2} \times 3400 \cdot 10^{-7}$  and therefore  $t \approx 1.6 \text{ ms}$ .

**pr 48.** (a) The wording of the temperatures  $T_1, T_2$ , and  $T_0$  may be slightly confusing so let us clarify this before solving the problem.  $T_2$  is the temperature of the cold air *going* into the room while  $T_0$  is the warm air from the room and  $T_1$  is the cold air that is already inside the room.

Let us assume that there is a constant difference of temperature across the opposite sides of the plates given by  $\Delta T \equiv T_0 - T_2$ . By fact 6, we note that for small tempera difference  $\Delta T \equiv T_0 - T_2$ , the heat flux is proportional to  $\Delta T$ . In other words,

$$\dot{Q} \propto T_0 - T_2 = \text{const.}$$

The heat flux is also equal to

$$\dot{Q} = mc_p \dot{T} = \rho V c_p \dot{T} = \rho s h c_p \dot{T}$$

where  $s$  is the cross sectional area for an air element of volume  $V$ . We remember that the thermal conductance of the metal is  $\sigma$  (the heat flux through a unit area of the plate per unit time, assuming that the temperature drops by one degree per unit thickness of the plate). This means that we can write

$$\dot{Q} = \frac{\sigma s (T_0 - T_2)}{d}.$$

Since the heat flux is proportional to the difference in temperature which is constant, this means that the temperature gradient is linear with respect to position. If the velocity of the air is  $v$ , we write with dimensional arguments that

$$\dot{T} = \frac{v(T_2 - T_1)}{x}$$

we write  $\Delta T \equiv T_2 - T_1$  here since we are looking at the temperature difference horizontally from the cold air going into the room and the cold air that is already in the room. Substituting  $\dot{T}$  into our initial expression of  $\dot{Q}$  and equating that to our other expression with thermal conductance, we result in the equation

$$\rho s h c_p \frac{v(T_2 - T_1)}{x} = \frac{\sigma s (T_0 - T_2)}{d}.$$

To solve this equation for  $T_2$ , we can cross multiply to get

$$\rho shc_p(T_2 - T_1) = \sigma sx(T_0 - T_2) \implies T_2(\rho hc_p v + \sigma x) = \rho shc_p dT_1 + \sigma sxT_0.$$

Dividing over gives

$$T_2 = \frac{x\sigma T_0 + \rho hc_p dT_1}{x\sigma + \rho hc_p v d}.$$

(b) Since the tempera difference is very large, the temperature gradient is not linear by fact 6. This is because (a) heat conductivity of the materials may depend on the temperature, (b) the heat flux due to heat radiation is a non-linear function of  $T_1$  and  $T_2$  (however, it can be still linearized for small values of  $\Delta T$ ); (c) large temperature differences may cause convection of air and fluids which will enhance heat flux in a nonlinear way. Therefore, we have to rely on the graph to carry out calculations. Note that by idea 1:

$$P \equiv \frac{dQ}{dT} = C \frac{dT}{dt}.$$

By integrating, we find that

$$\int_0^t dt = \int_{T_2}^{T_1} \frac{C}{P} dT \implies t = 12C = 120 \text{ s}.$$

**pr 49.** (a) The adiabatic index is defined as the ratio of  $c_p$  and  $c_v$  so that  $\gamma = \frac{c_p}{c_v}$ . Note that

$$c_p = c_v + R \implies c_v = c_p - R$$

which means that upon substitution

$$\gamma = \frac{c_p}{c_p - R} \implies \gamma(c_p - R) = c_p \implies c_p = \frac{\gamma}{\gamma - 1} R.$$

(b) Remember that from the ideal gas law, we have that  $pV = nRT$  where  $n = \frac{m}{M}$ . Therefore,

$$p_0 V = \frac{m}{M} RT \implies \rho = \frac{m}{V} = \frac{p_0 M}{RT}.$$

(c) We can conserve momentum of a cross-sectional area of air  $S$  for a small interval of time  $dt$  assuming that the air moves at a velocity  $v$ . Momentum is  $p = mv = (\rho \times V)v$ . So we can write for a small interval of time, velocity remains approximately constant and that  $p = (\rho \times Sv \cdot dt)v$ . We can also write that  $p = Ft = S\Delta p \cdot dt$  where  $\Delta p$  is the difference in pressure. Therefore,

$$S\Delta p \cdot dt = (\rho \times Sv \cdot dt)v \implies \Delta p = \rho v^2.$$

The difference in pressure  $\Delta p$  in terms of density inside the pipe of length  $L$  can be written as  $\Delta p = \Delta \rho gL = (\rho_0 - \rho)gL$ . This means that

$$(\rho_0 - \rho)gL = \rho v^2 = \left( \frac{p_0 M}{RT} - \rho \right) gL = \rho v^2.$$

(d) From idea 1, we can write  $P \equiv \frac{dQ}{dt}$ . The process is isobaric so

$$P = \frac{d}{dt} (C_V n \Delta T) = C_V \Delta T \frac{dn}{dt}.$$

From the ideal gas law:

$$n = \frac{\rho V}{M} \implies \frac{dn}{dt} = \frac{\rho S \cdot dx}{M \cdot dt} = \frac{\rho S v}{M}.$$

Therefore, from part (a) we can substitute to write

$$P = C_V \Delta T \frac{\rho S v}{M} = \frac{\gamma}{\gamma - 1} R (T - T_0) \frac{\rho S v}{M}.$$

(e) From part (b), we know that  $\rho = \frac{p_0 M}{RT}$  which means that

$$(\rho_0 - \rho) = \frac{p_0 M}{R} \left( \frac{1}{T_0} - \frac{1}{T} \right) = \frac{p_0 M}{R} \left( \frac{T - T_0}{T_0 T} \right) \implies \frac{\Delta \rho}{\rho} = \frac{\Delta T}{T_0}.$$

From part (c), we write

$$\frac{\Delta \rho}{\rho} = \frac{v^2}{gL} \implies \Delta T = \frac{v^2}{gL} T_0$$

which means that when substituting into our equation in (d), we have that

$$P = \frac{\gamma}{\gamma - 1} R \frac{v^3}{gL} T_0 \frac{\rho S}{M}.$$

Remember that from the ideal gas law:

$$p_0 V = \frac{m}{M} RT \implies \rho = \frac{m}{V} = \frac{p_0 M}{RT}$$

wich means that

$$v^3 = \frac{\gamma}{\gamma - 1} \frac{gL}{S} \frac{P}{p_0}.$$

Remember that

$$\Delta T = \frac{v^2}{gL} T_0 \implies T = T_0 \left( 1 + \frac{v^2}{gL} \right) = T_0 \left[ 1 + \frac{1}{gL} \left( \frac{\gamma}{\gamma - 1} \frac{gL}{S} \frac{P}{p_0} \right)^{2/3} \right].$$

#### pr 50.

1. The temperature dependence of the bubble volume is  $w = w_0 - \alpha(T - T_0)V$ . Therefore, by ideal gas law,

$$\frac{pw}{T} = \frac{p_0 w_0}{T_0} \implies p = \frac{p_0 T}{T_0} \frac{w_0}{w_0 - V \alpha (T - T_0)}.$$

2. The equality no longer holds when the volume is negative or

$$w_0 - V \alpha (T - T_0) \leq 0 \implies \frac{w_0}{V \alpha} + T_0 \leq T.$$

As  $T$  approaches  $T_{\max}$  pressure increases. Due to high pressure compressibility of glycerin and elasticity of walls must be considered.



**pr 51.** The saturated vapor pressure at  $80^\circ = 50$  kPa while the saturated vapor pressure at  $20^\circ$  is 2.4 kPa. When cap is open, the inside pressure is equal to the outside pressure which implies that the partial pressure of air in bottle is equal to  $(101 - 50)$  kPa = 51 kPa. When the cap is closed, the volume of gaseous mixture is constant. Also, note that the number of moles of air trapped in bottle are constant. Let final partial pressure of air in bottle be  $x$ . By ideal gas equation we yield:

$$\frac{P_0 V_0}{T_0} = \frac{P_1 V_1}{T_1} \Rightarrow \frac{51 \cdot V}{273 + 80} = \frac{x \cdot V}{273 + 20} \Rightarrow x = 42.33 \text{ kPa.}$$

Therefore, the total pressure is the sum of both pressures at  $20^\circ$  or  $(42.33 + 2.4)$  kPa = 44.73 kPa.

**pr 52.** (a) Since the process is adiabatic, we have:

$$T_0^\gamma P_0^{1-\gamma} = T_1^\gamma P_1^{1-\gamma} \Rightarrow T_1 = \left( \frac{P_0}{P_1} \right)^{\frac{1-\gamma}{\gamma}} T_0$$

Solving, we get  $T_1 = 279.2$  K.

(b) We know from the ideal gas law that:

$$\Delta \rho = \frac{M}{R} \left( \frac{P_1}{T_1} - \frac{P_0}{T_0} \right)$$

Since the change is linear, we have  $\rho_1 = \rho_0 - \alpha h \Rightarrow \Delta \rho = -\alpha h$ . The change in pressure is given by:

$$-\Delta P/g = \int_0^h (\rho_0 - \alpha h) dh = \rho_0 h - \frac{1}{2} \alpha h^2 \Rightarrow -\alpha h = -\frac{2\Delta P}{gh} - 2\rho_0$$

Therefore, we have:

$$-\frac{2\Delta P}{gh} - 2\rho = \frac{M}{R} \left( \frac{P_1}{T_1} - \frac{P_0}{T_0} \right)$$

Solving this, we get  $h = 1440$  m. Alternatively, we can write down the first line as:

$$\frac{P}{\rho T} = \text{constant}$$

and go from there.

(c) If there was no rain, the temperature would be given by:

$$T'_2 = \left( \frac{P_0}{P_2} \right)^{\frac{1-\gamma}{\gamma}} T_0 = 264.4 \text{ K.}$$

However, there is also condensation, which is an exothermic process and the condensation of the water vapor releases heat into the atmosphere. This is given by:

$$\frac{m_{\text{air}}}{M} c_V \Delta T = m_{\text{water}} L_V \Rightarrow \Delta T = \frac{m_{\text{water}}}{m_{\text{air}}} \frac{M L_V}{R \left( \frac{\gamma}{\gamma-1} \right)} = 6.1 \text{ K}$$

Therefore,

$$T_2 = T'_2 + \Delta T = 270.7 \text{ K.}$$

(d) Given a certain unit area, 2000 kg of moist air can travel up the mountain ridge in 1500 s, so the rate at which water condenses is:

$$r = \frac{2000 \cdot \frac{2.45}{1000}}{1500} = 0.00327 \text{ kg}/(\text{s} \cdot \text{m}^2)$$

The height of the water column after  $t = 3$  hours is hus:

$$\frac{rt}{\rho} = 3.5 \text{ cm}$$

(e) We use the same relationship:

$$T_3 = \left(\frac{P_3}{P_2}\right)^{\frac{1-\gamma}{\gamma}} T_2 = 300 \text{ K}$$

**pr 53.** **Lemma.** The molecular flux of molecules through the surface area  $A$  of ice is given by

$$\Phi = \frac{1}{4} n \langle v \rangle.$$

*Proof.* Let  $f(v)$  be the Maxwell-Boltzmann distribution of velocities of the particles. What this means is that the probability that a particle has velocity in  $[v_x, v_x + dv_x] \times [v_y, v_y + dv_y] \times [v_z, v_z + dv_z]$  is

$$f\left(\sqrt{v_x^2 + v_y^2 + v_z^2}\right) dv_x dv_y dv_z.$$

Set up spherical coordinates with origin at the hole. We will now count the number of particles that hit the hole in a time  $dt$  using a funny double counting argument, where we start by counting the number of particles that hit the hole with a certain velocity and then integrate over all velocities.

We will start by counting the number of particles that move with speed  $v$  (technically speed in  $[v, v + dv]$ , but from now on we'll be lazy about this) and spherical coordinate angles  $(\theta, \phi)$ . Here  $\theta = 0$  means pointing toward the hole, and  $\theta = \pi/2$  is parallel to the plane of the hole (the spherical coordinates for the velocity are flipped compared to those for space, since the  $\theta = 0$  rays are anti-parallel). In a given volume  $dV$ , the number of particles with this velocity is just

$$(ndV) \cdot f(v) \cdot v^2 \sin \theta dv d\theta d\phi.$$

For this given velocity, the volume in space that will allow such particles to hit the hole is a tilted cone object with base  $A$ , slant  $\theta$ , slant height  $vdt$ , and aligned in the proper  $\phi$  direction. In particular, its volume is  $A(vdt) \cos \theta$ , so the number of particles with velocity  $(v, \theta, \phi)$  hitting the hole in time  $dt$  is

$$(nAdt) \cdot f(v) \cdot v^3 \sin \theta \cos \theta dv d\theta d\phi.$$

Thus, the rate of particles leaving is

$$\alpha = nA \int_0^\infty v^3 f(v) dv \int_0^{\pi/2} \sin \theta \cos \theta d\theta \int_0^{2\pi} d\phi = \pi nA \int_0^\infty v^3 f(v) dv.$$

Note that

$$\langle v \rangle = \int_0^\infty \int_0^\pi \int_0^{2\pi} v \cdot f(v) \cdot v^2 \sin \theta dv d\theta d\phi = 4\pi \int_0^\infty v^3 f(v) dv,$$

which tells us that

$$\alpha = \frac{nA}{4} \langle v \rangle,$$

as desired. Note that proof didn't depend on the particular form of the Maxwell-Boltzmann speed distribution<sup>a</sup> for the proof<sup>b</sup>.  $\square$

This means that the mass flow rate of the molecules leaving the teacup is given by<sup>c</sup>

$$w = \frac{\rho_0 \langle v \rangle A}{4}.$$

The time  $t$  for the complete evaporation ice will then be given by

$$t \approx \frac{m}{w} = \frac{4m}{\rho \langle v \rangle A}$$

where  $m$  is the mass of the ice in the teacup. As we are to estimate the dimensions of the cup ourselves, let us do just that. Most conventional teacups hold up to 150 mL of water and since the density of ice is  $0.91 \text{ g/cm}^3$ , we can estimate the mass of the ice in the water to be  $m \approx 136 \text{ g}$ . The area of a cup of tea is  $A \approx \pi(4.5)^2 \approx 63 \text{ cm}^2$ .

Measurements were done with this teacup:



The average velocity is given by

$$\langle v \rangle = \int_0^\infty v f(v) dv = \sqrt{\frac{8RT}{\pi\mu}}$$

which means that

$$t = \frac{4m}{\rho \langle v \rangle A} = \frac{4m}{\rho_0 A} \sqrt{\frac{\pi\mu}{8RT}} = \frac{m}{\rho_0 A} \sqrt{\frac{2\pi\mu}{RT}}.$$

By the ideal gas law, the vapor pressure can be represented as  $\rho_0 = \frac{P\mu}{RT}$  which means that

$$t = \frac{m}{\rho_0 A} \sqrt{\frac{2\pi\mu}{RT}} = \frac{mRT}{P\mu A} \sqrt{\frac{2\pi\mu}{RT}} = \frac{m}{PA} \sqrt{\frac{2\pi RT}{\mu}}.$$

While the ice is in motion, the astronaut experiences a force given from problem 21 which implies that the astronaut's acceleration is (given that the astronaut's mass is  $M \approx 130 \text{ kg}$ ):

$$F = \frac{PA}{2} \implies a = \frac{PA}{2M}$$

This means that

$$\begin{aligned}
 L_{\text{acceleration}} &= \frac{1}{2}at^2 = \frac{1}{2} \left( \frac{PA}{2M} \right) \left( \frac{m}{PA} \sqrt{\frac{2\pi RT}{\mu}} \right)^2 \\
 &= \frac{1}{2} \left( \frac{PA}{2M} \right) \left( \frac{2\pi m^2 RT}{P^2 A^2 \mu} \right) \\
 &= \frac{m^2 \pi RT}{2M PA \mu} \\
 &= \frac{(0.136 \text{ kg})^2}{2 \cdot 130 \text{ kg}} \frac{\pi \cdot 8.3 \text{ J} \cdot \text{mol}^{-1} \cdot \text{K}^{-1} \cdot 272 \text{ K}}{550 \text{ Pa} \cdot 0.0063 \text{ m}^2 \cdot 0.018 \text{ kg/mol}} \approx 88 \text{ m}
 \end{aligned}$$

After the astronaut has stopped accelerating, it will be moving at a constant velocity of

$$v = at = \left( \frac{PA}{2M} \right) \left( \frac{m}{PA} \sqrt{\frac{2\pi RT}{\mu}} \right) = \frac{m}{2M} \sqrt{\frac{2\pi RT}{\mu}} \approx 0.46 \text{ m/s}.$$

The velocity is constant since after sublimation, there are no other external forces acting on the astronaut. Since this is happening in a vacuum, the astronaut could potentially move at a constant velocity until infinity. However, the aim of this problem is to estimate how realistic the method is. This means that we should aim to look at a realistic time that the astronaut travels the extra distance to the space station a distance of  $L = 100 \text{ m}$  away. The extra distance travelled will be

$$L_{\text{velocity}} = v(\tau - t) = \left( \tau - \frac{m}{PA} \sqrt{\frac{2\pi RT}{\mu}} \right) \frac{m}{2M} \sqrt{\frac{2\pi RT}{\mu}}$$

where  $\tau$  is the total time of travel. The total length travelled is then

$$L = L_{\text{acceleration}} + L_{\text{velocity}} = \frac{m^2 \pi RT}{2M PA \mu} + \left( \tau - \frac{m}{PA} \sqrt{\frac{2\pi RT}{\mu}} \right) \frac{m}{2M} \sqrt{\frac{2\pi RT}{\mu}}.$$

We now solve for  $\tau$ . By substituting values, we yield

$$100 \approx 88 + 0.46(\tau - 35) \implies \tau \approx 252 \text{ s}.$$

This is a realistic amount of time for the total travel, so yes, this method is realistic to use.

<sup>a</sup>Credits to [here](#) for the proof of this alternate thermodynamic system.

<sup>b</sup>Note that the flux here is only approximate as there will be an extra potential energy contribution to the vapors right after evaporation and since Maxwell's distribution is based on Boltzmann factors, we can't really calculate the extra energy that is added to the partition function. However, we can neglect these contributions as they are very small.

<sup>c</sup>This method is only approximate of the actual value of  $L$  because some of the molecules that escape the teacup will actually go back. However, the number of molecules that bounce back is very slim so we can neglect them.

**pr 54.** We use a model, according to which the already broken part of the soap film gathers into a single front and moves all together towards the still preserved part of the film. Let the surface area of the broken part of the soap film be  $S$ . It then follows that the surface energy of the soap film is

$$U = 2S\sigma$$

Using the facts that the surface energy will turn into all of kinetic energy and that the mass that was occupying the broken part of the soap film is  $2S\rho h$ , we can set an equation

$$\begin{aligned} 2S\sigma &= \frac{1}{2}mv^2 \\ 2S\sigma &= \frac{1}{2}S\rho hv^2 \\ v &= \sqrt{\frac{4\sigma}{\rho h}} \end{aligned}$$

Now that we have the velocity of the soap film, we can approximately find the distance the soap film travels by using  $v = dt$ ,

$$t = \frac{D}{v} \implies t = \frac{D}{\sqrt{\frac{4\sigma}{\rho h}}}.$$

**pr 55.** Bubble  $A$  has a radius  $r$  and the pressure difference between the inside and the air is  $\Delta p_{A,air} = 2 \cdot \frac{2\gamma}{r}$ . The extra factor of two is because a bubble film essentially has two surfaces. Bubble  $B$  has a radius  $2r$  and thus the pressure difference between the inside and the air is  $\Delta p_{B,air} = \frac{4\gamma}{2r}$ . We now look at the pressure difference between the two bubbles, and we get:

$$\Delta p_{B,A} = 4\gamma \left( \frac{1}{r} - \frac{1}{2r} \right)$$

Therefore, the radius of curvature of the surface in the middle is:

$$\frac{1}{R} = \frac{1}{2r} \implies R = 2r$$

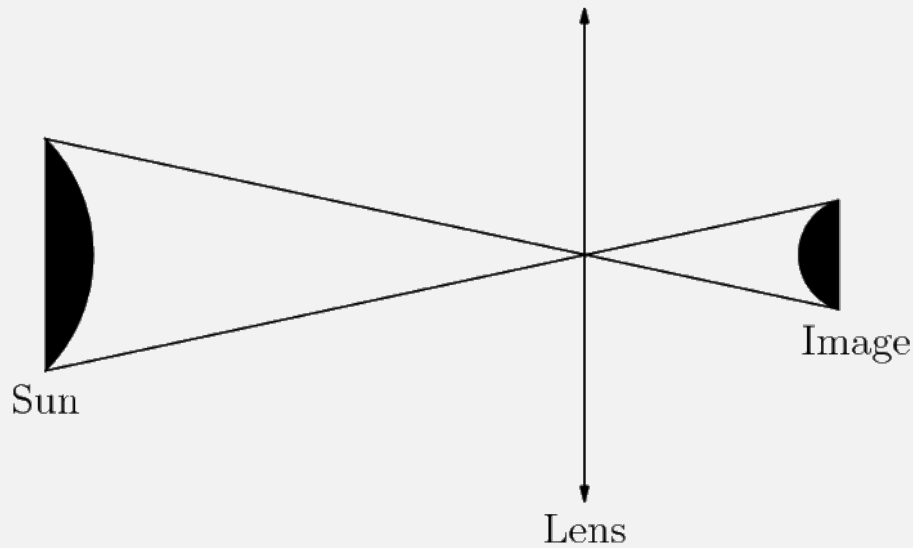
We can assume that the middle surface is in the form of a spherical cap so we can use the formula:

$$A = 2\pi R^2(1 - \cos \theta)$$

where  $\theta$  is the angle that the middle surface subtends. We can determine  $\theta$  by drawing a free body diagram. We look at the intersection of the middle film, the film from bubble  $A$  and the film from bubble  $B$ . Since all three films have the same surface tension, the three films must form an angle of  $120^\circ$  with each other. With a little bit of geometry, we can show that the angle the middle film subtends is also  $\theta = 120^\circ$ . Therefore, the surface area is:

$$A = 2\pi R^2 \left( \frac{3}{2} \right) = \boxed{12\pi r^2}$$

**pr 56.** (a) The power received by plate is equal to the power received by lens or  $\pi \left( \frac{d}{2} \right)^2 I$



With small angle approximations, the radius of the image is approximately  $f\alpha/2$ . Therefore, the power radiated by image  $\pi \left(\frac{f\alpha}{2}\right)^2 \sigma T^4$ . In thermodynamic equilibrium the power received is equal to the power radiated. This then implies that

$$\frac{\pi d^2}{4} I = \frac{\pi f^2 \alpha^2}{4} \sigma T^4 \implies T = \sqrt{\frac{d}{\alpha f}} \sqrt{\frac{I}{\sigma}}.$$

(b) At the sun's surface the intensity of the sun is given by

$$I_{\text{sun}} = \sigma T_{\text{sun}}^4 \implies 4I_{\text{Sun}} \pi R_{\text{Sun}}^2 = 4I \pi R_{\text{Orbit}}^2$$

where  $R_{\text{Orbit}}$  is the orbital distance of Earth from Sun. This means that

$$I = \frac{\sigma T_{\text{Sun}}^4 R_{\text{Sun}}^2}{R_{\text{Orbit}}^2}$$

By Second Law of Thermodynamics temperature of plate can't be more than temperature of Sun. Therefore,

$$T \leq T_{\text{Sun}} \implies \frac{d}{\alpha f} \sqrt{\frac{I}{\sigma}} \leq T_{\text{Sun}}^2 \implies \frac{d}{\alpha f} \times \frac{T_{\text{Sun}}^2 R_{\text{Sun}}^2}{R_{\text{Orbit}}^2} \leq T_{\text{Sun}}^2 \implies \frac{d}{\alpha f} \frac{\alpha}{2} \leq 1 \implies \boxed{d \leq 2f}.$$

**pr 57.** The smoke rises until it's density becomes the same as the density of the air around it. From problem 69 and 70, note that

$$\frac{dT}{T} = (1 - \gamma^{-1}) \frac{dp}{p} \implies \Delta T = T(1 - \gamma^{-1}) \frac{\Delta p}{p}.$$

The change in pressure is given by  $\Delta p = \rho gh$  and since  $\rho = p\mu/RT$  from the ideal gas law, we get

$$\Delta p = \frac{p\mu}{RT} gh$$

also note that

$$\gamma^{-1} = \frac{c_V}{c_p} = \frac{c_V}{c_V + R}$$

which means that

$$\Delta T = \frac{R}{c_V + R} \frac{\mu g h}{R}.$$

Rearranging to solve for  $h$  tells us that

$$h = \frac{\Delta T R (c_V + R)}{\mu g}.$$

**pr 58.** (a) Let us start out with the definition of the molar heat capacity and apply the first law of thermodynamics:

$$C = \frac{\partial Q}{\partial T} = C_V + P \left( \frac{\partial V}{\partial T} \right)_P.$$

Since the process takes place in a bubble, the thermodynamic process is polytropic. What this means is that

$$p^3 V = \text{const.}$$

Let us apply the ideal gas equation for one mole:

$$pV = RT \implies p = \frac{RT}{V}.$$

Substituting this relation of pressure into our polytropic equation gives us

$$\left( \frac{RT}{V} \right)^3 \cdot V = R^3 T^3 V^2 = \text{const.}$$

We can apply implicit differentiation (note that the derivative of a constant will become zero) to this relationship to yield

$$3R^3 T^2 V^2 dT + 2R^3 T^3 V dV = 0 \implies \frac{dV}{dT} = \frac{3V}{2T}.$$

The heat capacity of a diatomic gas at constant volume is  $C_V = \frac{5}{2}R$  so

$$C = \frac{5}{2}R + \frac{3PV}{2T} = \frac{5}{2}R + \frac{3}{2}R = \boxed{4R}.$$

(b) Consider a small expansion of the bubble. We can relate pressure and volume by:

$$\frac{dp}{p} = -\frac{dV}{V}.$$

The infinitesimal change in volume is  $dV = 4\pi r^2 dr$  and the volume is  $V = \frac{4}{3}\pi r^3$ . Also note that by Laplace's law, the pressure is given by  $p = \frac{4\sigma}{r}$ . This means that

$$dp_{\text{out}} = -\frac{pdV}{V} = -\frac{\frac{4\sigma}{r} \cdot 4\pi r^2 dr}{\frac{4}{3}\pi r^3} = -\frac{12\sigma dr}{r^2}.$$

Note that there is also a decrease in the pressure due to surface tension

$$dp_{\text{in}} = \Delta \left( \frac{4\sigma}{r} \right) = \frac{4\sigma dr}{r^2}.$$

This means that

$$dp_{\text{tot}} = \frac{4\sigma dr}{r^2} - \frac{12\sigma dr}{r^2} = -\frac{8\sigma dr}{r^2}.$$

Consider a small area element  $S$  that moves outward a small distance  $x$ . The mass of the bubble is given by  $m = \rho Sh$  which means that the force is

$$\rho h \ddot{x} = -\frac{8}{r^2}x \implies \omega = \sqrt{\frac{8\sigma}{\rho h r^2}}.$$

**pr 59.** (a) From  $E \rightarrow A$ , the gas is undergoing an isothermal expansion. Since  $PV = \text{constant}$  and  $V$  is increasing, then  $P$  must be decreasing. Therefore,  $p_E > p_A$ . From  $B \rightarrow D$  it is again an isothermal process. However, there is both a heating component and a cooling component to it. We need to determine what is the overall effect. This can be easily done by assuming that in one cycle, the net heat gain is zero (or else it would continuously cool down/warm up). Therefore, since energy is already gained from  $E \rightarrow A$ , it must be lost from  $B \rightarrow D$  so  $p_D > p_B$ . Finally, we can rank them based off of their relative heights:

$$p_E > p_A > p_D > p_B > p_C$$

(b) Note that we will use a different notation than the problem text, and let the subscript represent the lettered state. From  $D \rightarrow E$ , it is an isothermal process so:

$$P_D^{1-\gamma} T_D^\gamma = P_E^{1-\gamma} T_E^\gamma$$

where

$$C_P = C_V + R \implies 1 = \frac{C_V}{C_P} + \kappa \implies \gamma = \frac{1}{1 - \kappa} = \frac{7}{5}$$

Solving for  $T_D$  gives:

$$\begin{aligned} T_D &= T_E \left( \frac{P_E}{P_D} \right)^{(1-\gamma)/\gamma} \\ &= T_E \left( \frac{P_A + 20}{P_D} \right)^{(1-\gamma)/\gamma} \\ &= 194.8 \text{ K} \end{aligned}$$

(c) Again, the process from  $A \rightarrow B$  is adiabatic and we have  $T_B = T_D$ . Therefore:

$$\begin{aligned} P_B^{1-\gamma} T_B^\gamma &= P_A^{1-\gamma} T_A^\gamma \\ P_B &= P_A \left( \frac{T_A}{T_B} \right)^{\gamma/(1-\gamma)} \\ &= 220.6 \text{ hPa} \end{aligned}$$

Alternate Solution: Writing out the adiabatic compression and expansion equations, we get:

$$\begin{aligned} P_D^{1-\gamma} T_D^\gamma &= P_E^{1-\gamma} T_E^\gamma \\ P_B^{1-\gamma} T_B^\gamma &= P_A^{1-\gamma} T_A^\gamma \end{aligned}$$



By noting that  $T_A = T_E$  and  $T_B = T_D$ , we can divide the two equations and get:

$$\begin{aligned}\frac{P_D}{P_B} &= \frac{P_E}{P_A} \\ P_B &= \frac{P_D P_A}{P_E} \\ &= 220.6 \text{ hPa}\end{aligned}$$

(d) (i) There are four processes, an isothermal expansion, an adiabatic expansion, an isothermal cooling, and an adiabatic compression. Let us first deal with the adiabatic processes. For an adiabatic process, we have  $Q = 0$  so the net work done by the gas is equal to the change in energy:

$$W = -nC_V \Delta T$$

The net work due to the two adiabatic processes will be zero since  $\Delta T_1 + \Delta T_2 = 0$ . Therefore, we only need to focus on the isothermal processes. The work done at the top is:

$$\begin{aligned}W_{B \rightarrow D} &= \int P dV \\ &= -nRT \int_{P_B}^{P_D} \frac{1}{P} dP \\ &= nRT_B \ln \left( \frac{P_B}{P_D} \right)\end{aligned}$$

At the bottom, the work done is:

$$W_{E \rightarrow A} = nRT_E \ln \left( \frac{P_E}{P_A} \right)$$

so the net work per mole is:

$$\begin{aligned}W_{\text{net}}/n &= RT_B \ln \left( \frac{P_B}{P_D} \right) + RT_E \ln \left( \frac{P_E}{P_A} \right) \\ &= -RT_B \ln \left( \frac{P_E}{P_A} \right) + RT_E \ln \left( \frac{P_E}{P_A} \right) \\ &= R(T_E - T_B) \ln \left( \frac{P_E}{P_A} \right)\end{aligned}$$

where we have used:

$$\frac{P_D}{P_B} = \frac{P_E}{P_A}$$

from earlier.

(ii) From the first law of thermodynamics:

$$U = Q - W$$

Since  $U = 0$ , the heat loss must equal the work done. The heat loss at the top must be equal to the work done at the top, or:

$$Q_{\text{loss}}/n = -W_{C \rightarrow D} = RT_B \ln \left( \frac{P_D}{P_C} \right)$$

(e) The Hadley Circulation is just a modified Carnot Cycle so the efficiency is just:

$$\eta = 1 - \frac{194.8}{300} = 35.1\%$$

(f) The efficiency is:

$$\eta = \frac{W_{\text{net}}}{Q_{\text{gained}}} = \frac{W_{\text{net}}}{W_{\text{net}} + Q_{\text{loss}}}$$

Substituting in the expressions from part (D) gives:

$$\begin{aligned} \eta &= \frac{R(T_E - T_B) \ln\left(\frac{P_E}{P_A}\right)}{R(T_E - T_B) \ln\left(\frac{P_E}{P_A}\right) + RT_B \ln\left(\frac{P_D}{P_C}\right)} \\ &= \frac{(T_E - T_B) \ln\left(\frac{P_E}{P_A}\right)}{T_E \ln\left(\frac{P_E}{P_A}\right) - T_B \ln\left(\frac{P_E}{P_A}\right) + T_B \ln\left(\frac{P_B P_E}{P_A P_C}\right)} \\ &= \frac{(T_E - T_B) \ln\left(\frac{P_E}{P_A}\right)}{T_E \ln\left(\frac{P_E}{P_A}\right) - T_B \ln\left(\frac{P_C}{P_B}\right)} \end{aligned}$$

This is smaller than  $\eta_{\text{ideal}}$  if:

$$\begin{aligned} \frac{(T_E - T_B) \ln\left(\frac{P_E}{P_A}\right)}{T_E \ln\left(\frac{P_E}{P_A}\right) - T_B \ln\left(\frac{P_C}{P_B}\right)} &< \frac{T_E - T_B}{T_E} \\ T_E \ln\left(\frac{P_E}{P_A}\right) &< T_E \ln\left(\frac{P_E}{P_A}\right) - T_B \ln\left(\frac{P_C}{P_B}\right) \\ \ln\left(\frac{P_C}{P_B}\right) &> 0 \\ P_C &> P_B \end{aligned}$$

Since from part (A) we have stated that  $P_C > P_B$ , then that means even under ideal conditions, the system cannot reach Carnot efficiency. It can only reach Carnot efficiency if  $P_C = P_B$ , which will simply not be the case.

(g) (I) cannot be true. In an isothermal process  $W \neq 0$ . In fact, friction is even necessary to keep it at a constant temperature as the gas will be absorbing energy.

(II) By having condensation only occur at a higher temperature, this implies the process is irreversible. See (III) for why this lowers the efficiency.

(III) is true. An engine can only achieve Carnot efficiency when  $\Delta S = 0$ , that is the process is reversible. The evaporation is irreversible as the condensation only occurs at a much higher altitude. This means that the efficiency is lowered.

(IV) is not necessarily true. Although generally phase shifts involve a change in energy, if the transition is kept at constant volume, entropy can be kept constant.

Therefore the answer is (II)&(III)

**pr 60.** (a) In analogy to electrical circuits, the resistance of a single wire is given by:

$$R = \frac{L}{\kappa S}$$

Since we have four wires in parallel, the effective resistance is:

$$R = \frac{L}{4\kappa S}$$

(b) For a small period of time  $dt$ , the temperature is changed by  $\Delta T$  degrees. This means that

$$P \equiv \frac{dQ}{dt} = C\Delta\dot{T}.$$

Let us look at the electrical analogy between heat conduction and electric circuits. In this case,  $\dot{Q}$  acts as the current while the temperature difference  $\Delta T \equiv T - T_0$  acts as the voltage difference. From this perspective, we can also define

$$\dot{Q} = \frac{\Delta T}{R}.$$

This means that from adding these two individual expressions (as there is an extra  $\dot{Q}$  due to our "voltage difference") we have that

$$P \cos \omega t = C\Delta\dot{T} + \frac{\Delta T}{R}.$$

As power varies with time, we attempt to seek the solution in the form of

$$T = T_0 + \Delta T \sin(\omega t + \phi).$$

Taking the derivative of our sought solution and substituting into our differential equation gives us

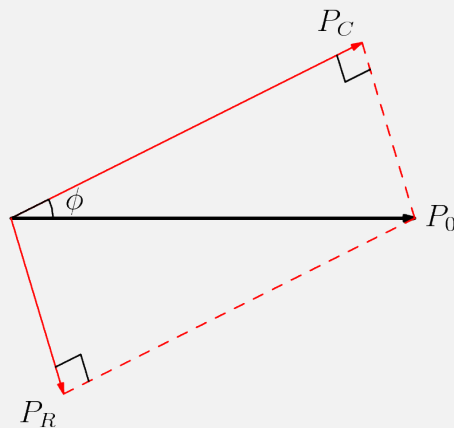
$$P \cos \omega t = C\Delta T \omega \cos(\omega t + \phi) + \frac{\Delta T}{R} \cos\left(\omega t + \phi - \frac{\pi}{2}\right)$$

which can be seen as the sum of multiple rotating vectors or:

$$\vec{P} = \vec{P}_C + \vec{P}_R.$$

Each of these vectors have an individual magnitude of

$$P_C = C\Delta T \omega, \quad P_R = \frac{\Delta T}{R}$$



Therefore, from pythagorean theorem, the amplitude of oscillations is

$$P_0^2 = (C\omega\Delta T)^2 + \left(\frac{\Delta T}{R}\right)^2 \implies \Delta T = \frac{P_0}{\sqrt{C^2\omega^2 + R^{-2}}}.$$

Our solution is in the form of

$$T = T_0 + \Delta T \sin(\omega t + \phi) = T_0 + \frac{P_0 \cos(\omega t + \phi)}{\sqrt{C^2\omega^2 + R^{-2}}}.$$

Let us estimate the phase difference. We can go back to our phase diagram and see that

$$\phi = \arcsin\left(\frac{P_C}{P_0}\right) = \arcsin\left(\frac{C\omega}{\sqrt{C^2\omega^2 + R^{-2}}}\right).$$

This means that our final solution is

$$T = T_0 + \frac{P_0 \cos\left(\omega t + \arcsin\left(C\omega/\sqrt{C^2\omega^2 + R^{-2}}\right)\right)}{\sqrt{C^2\omega^2 + R^{-2}}}.$$

(c) Remember from part b that our amplitude of oscillations of  $\Delta T$  is given by  $\Delta T = \frac{P_0}{\sqrt{C^2\omega^2 + R^{-2}}}$ . We want there to be as large a change as possible with a small change in  $C$ . This is equivalent to maximizing the derivative ie setting the double derivative to 0. Upon taking two derivatives, we get the function

$$\frac{3P_0C^2\omega^4}{(C^2\omega^2 + R^{-2})^{5/2}} - \frac{P_0\omega^2}{(C^2\omega^2 + R^{-2})^{3/2}} = 0.$$

Seperating and solving for  $\omega$  yields  $\omega = 1/\sqrt{2}CR$ .

(d) As the questions asks to estimate, we ignore all numerical prefactors. From part c), we know that  $C \approx 1/\omega R$ . This should be of the same order as the heat capacity of the bridges. Therefore, we have

$$\frac{1}{l/\kappa S \cdot \omega} = \rho l S c \implies \omega_c \approx \frac{\kappa}{c\rho L^2}.$$

**pr 61.** (a) Let the exitance and luminosities be  $M_1$  and  $J_1$  and  $M_2$  and  $J_2$  for the inner and outer sphere respectively. Assume that the spheres have different emmistivities  $\varepsilon_1$  and  $\varepsilon_2$  respectively. We then can write that

$$J_1 = M_1 + (1 - \varepsilon_1)F_{11}J_1 + (1 - \varepsilon_1)F_{12}J_2$$

and

$$J_2 = M_2 + (1 - \varepsilon_2)F_{21}J_2 + (1 - \varepsilon_2)F_{22}J_1$$

as the equations of heat radiation through each sphere respectively (note that  $F$  is the view factor). All the radiation emmited by one sphere will reach the other sphere so  $F_{11} = 0$  and  $F_{12} = 1$ . We also note that

$$A_1 = F_{21}A_2 \implies F_{21} = \left(\frac{R_1}{R_2}\right)^2.$$

This also means that

$$F_{22} = 1 - \left(\frac{R_1}{R_2}\right)^2.$$

Substituting these values of the view factor into our expressions of radiosities tells us:

$$\begin{aligned} J_1 &= M_1 + (1 - \varepsilon_1)J_2 \\ J_2 &= M_2 + (1 - \varepsilon_2) \left( \frac{R_1}{R_2} \right)^2 J_2 + (1 - \varepsilon_2) \left( 1 - \left( \frac{R_1}{R_2} \right)^2 \right) J_1 \end{aligned}$$

Using the fact that  $M_2 = \varepsilon_2 \sigma T_2^4$  we can solve for  $J_1$  in terms of  $J_2$  in the second expression:

$$J_1 = \frac{J_2 - \varepsilon_2 \sigma T_2^4 - (1 - \varepsilon_2) \left( \frac{R_1}{R_2} \right)^2 J_2}{(1 - \varepsilon_2) \left( 1 - \left( \frac{R_1}{R_2} \right)^2 \right)}$$

We can solve for  $J_2$  in terms of  $J_1$  in the first expression to get

$$J_1 = \frac{\frac{J_1 - \varepsilon_1 \sigma T_1^4}{1 - \varepsilon_1} - \varepsilon_2 \sigma T_2^4 - (1 - \varepsilon_2) \left( \frac{R_1}{R_2} \right)^2 \frac{J_1 - \varepsilon_1 \sigma T_1^4}{1 - \varepsilon_1}}{(1 - \varepsilon_2) \left( 1 - \left( \frac{R_1}{R_2} \right)^2 \right)}$$

Isolating for only  $J_1$  finally gives us

$$J_1 = \frac{\left( \varepsilon_1 \varepsilon_2 - \varepsilon_1 \left( \frac{R_1}{R_2} \right)^2 - \varepsilon_1 \varepsilon_2 \left( \frac{R_1}{R_2} \right)^2 \right) \sigma T_1^4 + (1 - \varepsilon_1) \varepsilon_2 \sigma T_2^4}{\varepsilon_2 + \varepsilon_1 \left( \frac{R_1}{R_2} \right)^2 - \varepsilon_1 \varepsilon_2 \left( \frac{R_1}{R_2} \right)^2}.$$

The net heat flux between the cylinders is the heat flux from the inner cylinder or

$$\dot{Q} = \frac{A_1 \varepsilon_1}{1 - \varepsilon_1} (\sigma T_1^4 - J_1) = \frac{4\pi R_1^2 \sigma (T_1^4 - T_2^4)}{\frac{1}{\varepsilon_1} + \frac{1 - \varepsilon_2}{\varepsilon_2} \left( \frac{R_1}{R_2} \right)^2} = \frac{\varepsilon \sigma 4\pi R_1^2 (T_2^4 - T_1^4)}{1 + (1 - \varepsilon) R_1^2 / R_2^2}.$$

The last step was done by noting  $\varepsilon = \varepsilon_1 = \varepsilon_2$ .

(b) Note that  $\tau P = \lambda m$  and  $m = \frac{4}{3} \pi \rho R^3$ . Therefore,

$$\tau = \frac{4}{3} \pi \rho R^3 \lambda \mu / P.$$

**pr 62. a)** After each pumping cycle, the amount of gas inside, reduces by a factor of  $1 - \alpha$ . This means that after  $N$  pumping cycles, the pressure inside the bulb decreases by a factor of  $\beta = (1 - \alpha)^N$ . When  $N \rightarrow \infty$ , the factor  $\beta$  can be approximated by  $\beta = e^{-N\alpha}$ . This means that

$$\beta = e^{-N\alpha} \implies N = \boxed{\frac{1}{\alpha} \ln \beta}.$$

**b)** We know that work is generally defined by

$$W = -pV.$$

However, since the volume in the cylinder is given by  $\alpha V$ , then after  $N$  cycles, the work is defined by

$$W = -N\alpha p_0 V \implies W = \boxed{p_0 V \ln \beta}.$$

c) We know that the adiabatic law gives us the proportion  $pV^\gamma \propto T^\gamma$  and the ideal gas law tells us  $pV \propto T \implies (pV)^\gamma \propto T^\gamma$ . Substituting these in, gives us  $p^{\gamma-1} \propto T^\gamma$ . When the pressure in the bulb becomes  $\beta p_0$ , the pressure in the cylinder, will in proportion, be increased by a factor of  $1/\beta$ . Therefore,

$$T = \boxed{T_0 \beta^{1/\gamma-1}}.$$

d) The trick to this part is that the only time energy is lost is in the heating up of the released air since everywhere else the work is "reused". There is two places work is done. Actually creating the vacuum and then heating the air. To actually create the vacuum takes  $PV$  energy and to heat up the air takes the sum of all the temperature heating. We calculate the second part by sum of

$$C_v \cdot \beta \cdot PV \alpha \frac{\beta^{(1/\gamma-1)} - 1}{1 + \alpha}.$$

Since  $\beta = 1/(1 + \alpha)^{(n/\gamma)}$  where  $n$  is the pump number we just use a geometric series to sum these up and replace  $C_v = 1/\gamma - 1$ . Thus, multiplying out everything cancels and you just end up with another  $PV$  so the answer is  $2PV$ .

**pr 63.** (a) By idea 1, note that

$$P \equiv \frac{dQ}{dt} = C \frac{dT}{dt}.$$

Therefore, by rearranging, we have that

$$T = \int_{T_0}^{T_1} dT = \int_0^\tau \frac{P}{C} dt \implies C = \frac{P\tau}{T_0 - T_1} = 360 \text{ J/}^\circ\text{C}.$$

(b) The delay in the change of temperature reading and the initial rise in temperature is due to the sensor's surroundings not initially heating up. By the definition of heat capacity, an infinitesimal amount of heat that is transferred to the system is

$$dQ = -Cd\bar{T}.$$

After the temperature of the sensor's surroundings and the plate become equal, the heat dissipated by the plate is given by

$$\kappa(T - T_{\text{amb}}) = -C \frac{dT}{dt} \implies \ln(T - T_{\text{amb}}) = -\frac{\kappa t}{C} + m$$

where  $m$  is an arbitrary constant. Before the sink receives heat  $Q$ , its temperature must be the ambient temperature as the electrical component has been switched off for a long time. Theoretically, we assume that the temperature of the whole plate and sensor is the same right after  $t = 0$  for our equation to be valid for the entire duration (although realistically it is valid only after some time). Let this temperature be  $T_2$ . Then we have that

$$Q = C(T_2 - T_{\text{amb}})$$

and therefore our goal is to determine the value of  $m$  from the graph given. Our solution of  $\Delta T \equiv T_2 - T_{\text{amb}}$  will be exponential or  $e^m$ . From the table, it is clear that  $T_{\text{amb}} = 20^\circ \text{C}$ . Therefore, we plot our graph with  $y \equiv \ln(T - T_{\text{amb}})$  and  $x \equiv t$ . From the graph drawn, we find  $m = 4.73$  and therefore,  $Q = 41 \text{ kJ}$ .

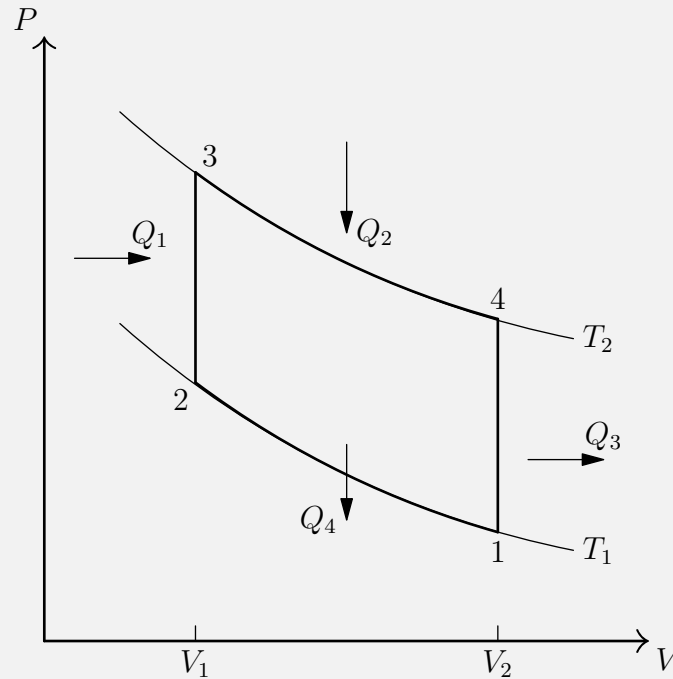
**pr 64.** Radiation occurs between the celestial body and its environment while conduction occurs inside the celestial body. When the body is in thermal equilibrium heat production is equal to heat emission. Let the surface temperature be  $T_S$  and the center's temperature be  $T_C$ . Then at the surface

$$(4\pi R^2) \sigma T_S^4 = \left( \frac{4}{3} \pi R^3 \rho \right) \Rightarrow T_S = \sqrt[4]{\frac{\rho P R}{3\sigma}}$$

Inside the celestial body consider a thin shell of radius  $x$  where  $x \leq R$  of thickness  $dx$  at a temperature  $T$  with a temperature difference  $dT$ . Then,

$$\left( \frac{4}{3} \pi x^3 \rho \right) P = \frac{-k(4\pi R^2) dT}{dx} \Rightarrow \int_{T_S}^{T_C} dT = -\frac{\rho P}{3k} \int_R^0 x dx \Rightarrow T_C = T_S + \frac{\rho P R^2}{6k} = 4689 \text{ K}$$

**pr 65.** An ideal-gas cycle consisting of two isotherms at temperatures  $T_1$  and  $T_2$ , and two isochores joining them will look like:



Since there are no adiabatic processes, heat will be lost on every step. For isochoric processes on  $2 \rightarrow 3$  and  $1 \rightarrow 4$ , the heat released will be

$$Q_1 = C_V R(T_2 - T_1)$$

$$Q_3 = C_V R(T_1 - T_2)$$

For isothermal processes on  $1 \rightarrow 2$  and  $3 \rightarrow 4$ , the heat released will be

$$Q_2 = RT_2 \ln \frac{V_2}{V_1}$$

$$Q_4 = -RT_1 \ln \frac{V_1}{V_2}$$

The heat released during cooling feeds the heating isochore. Therefore, the work done is the change in heat along both isotherms or:

$$W = Q_2 - Q_4.$$

The efficiency of the cycle will be given by

$$\eta = \frac{W}{Q_2} = \frac{Q_2 - Q_4}{Q_2} = 1 + \frac{T_1 \ln \frac{V_1}{V_2}}{T_2 \ln \frac{V_2}{V_1}} = \boxed{1 - \frac{T_1}{T_2}}.$$

**pr 66.** (a) The total heat energy will be given by  $Q = nq$  where  $n$  is the number of moles and  $q$  is the heat. Note that for a heat differential,  $dq = C_v dT$  holds true. By integrating over we find that

$$q = \int C_v dT \approx 550R.$$

The number of moles will be given by the mass over the molar mass or

$$n = \frac{a^3 \rho}{M} \approx 0.1 \text{ mol.}$$

Multiplying these two values together gives us approximately 520 J.

(b) Consider a photon radiated from the cube. It carries a momentum  $p = E_{\perp}/c$  where  $E = hf$ . Let us assume that the photon initially imparts at an angle  $\theta$ . The parallel momentum is given by

$$p_{\parallel} = \frac{E_{\perp}}{c} \cos \theta.$$

Note that the intensity of light is proportional to  $\cos \theta$  by Lambert's law. Since intensity is proportional to frequency, the momentum averaged over all directions is

$$\langle p_{\parallel} \rangle = \int_0^{\pi} \frac{E_{\perp}}{c} \cos^2 \theta d\Omega$$

where  $d\Omega = 2\pi \sin \theta d\theta$  is the solid angle. Integrating this tells us that

$$\langle p_{\parallel} \rangle = \frac{E_{\perp}}{3c} = \frac{Q}{3c}.$$

The mass of the cube is  $a^3 \rho$  so, the velocity is given by

$$v = \frac{Q}{3a^3 \rho c}.$$

(c) By Stefan-Boltzmann Law, we have that

$$\frac{P}{A} = \sigma T^4$$

and by idea 1:

$$P \equiv \frac{dQ}{dt} = C \frac{dT}{dt}.$$

This means that

$$CdT = \sigma T^4 dt \implies T^3 dT = \sigma T^4 dt.$$

Separating the differential equation tells us

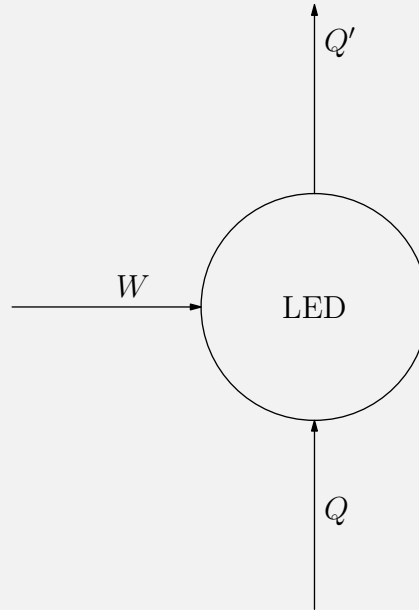
$$\frac{dT}{T} = -Bt \implies T = A \cdot e^{-Bt}.$$

(d) We can estimate the momentum imparted upon the cube will be equal to the heat energy over the average velocity of the hydrogen molecules. In other words,

$$a^3 \rho v = \frac{Q}{v_H} = Q \sqrt{\frac{M_H}{RT}} \implies v = \frac{Q}{a^3 \rho} \sqrt{\frac{M_H}{RT}}.$$



**pr 67.** The total light emission power of the LED is  $P_0 = 1\mu W$ .



Since we have

$$\frac{P_0}{A} = \int_{\nu_1}^{\nu_2} I(\nu, T) d\nu$$

We substitute the well known expression for  $I(\nu, T)$  from Planck's radiation law:

$$\frac{P_0}{A} = \int_{\lambda_1}^{\lambda_2} \frac{2\pi h c^2 d\lambda}{\lambda^5 (e^{\frac{hc}{\lambda kT}} - 1)} = \frac{2\pi h}{c^2} \int_{\nu_1}^{\nu_2} \frac{\nu^3}{e^{\frac{h\nu}{kT_0}} - 1} d\nu$$

We can use the well known integral

$$\int_0^\infty \frac{\eta^3 d\eta}{e^\eta - 1} = 6\xi(4) = \frac{\pi^4}{15}$$

and substitute the known values and using the problem condition. We find from evaluating the integral that  $T_1 \geq 1116K$ . Now, the maximum possible efficiency of the LED is just

$$\eta_{\max} = \frac{T_1}{T_1 - T_0} - 1 = 1 + 0.354 = \boxed{1.354}$$

**pr 68. Solution 1:** Note that the internal energy is given by  $c_V T$  and when this goes to the rocket nozzle, part of this turns into kinetic energy  $\mu v^2/2$ . Since the gas is monoatomic, part of it transfers into work  $pV = RT$ . Therefore, we have our conservation of energy equation to be

$$c_V T_0 + RT_0 = c_V T_1 + RT_1 + \frac{1}{2} \mu v_{\text{exit}}^2.$$

Using the fact that  $c_V = \frac{5}{2}R$ , we can simplify this to solve for the final exit velocity

$$\frac{5}{2}RT_0 + RT_0 = \frac{5}{2}RT_1 + RT_1 + \frac{1}{2}\mu v_{\text{exit}}^2 \implies v_{\text{exit}}^2 = \sqrt{\frac{7(T_0 - T_1)}{\mu}}.$$

The force is then given by the exit velocity or

$$F = mv = \rho_1 A v_{\text{exit}}^2 = \rho_1 A \frac{7(T_0 - T_1)}{\mu}.$$

We can attempt to use the approximation that  $T_0 \gg T_1$  to simplify this problem:

$$F = \frac{7A\rho T_1}{\mu} \left( \frac{T_0}{T_1} - 1 \right) \approx \frac{7Ap_1}{\mu} \frac{T_0}{T_1}.$$

**Solution 2:** Applying Bernoulli's principle for the gas between the nozzle and the chamber exit,

$$\frac{v^2}{2} + \frac{C_p T}{M} = \text{const.} \implies 0 + \frac{C_p T_0}{M} = \frac{v_{\text{exit}}^2}{2} + \frac{C_p T_1}{M}$$

Now, the thrust applied equals

$$F = \rho S v_{\text{exit}}^2$$

and  $\rho = \frac{P_1 M}{RT_1}$ ,  $v_{\text{exit}}^2 = \frac{2C_p(T_0 - T_1)}{M}$ . This means that

$$F = \frac{2C_p(T_0 - T_1)P_1 S}{RT_1}$$

Given that:  $T_0 \gg T_1 \implies T_0 - T_1 \approx T_0$  and  $C_p = C_v + R = \frac{7R}{2}$

$$\therefore F \approx \frac{7T_0 P_1 S}{T_1}$$

**pr 69.** Let us analyze a small section  $dz$  in the atmosphere with density  $\rho$ . The relation between pressure and density is then given by

$$dp = -\rho g dz.$$

We know that  $p = nk_B T$  and  $\rho = nm$  where  $m$  is the mass of a single molecule and therefore,

$$\frac{dp}{dz} = -\frac{mgp}{k_B T} \implies T \frac{dp}{p} = -\frac{mg}{k_B} dz.$$

Note that

$$p^{1-\gamma} T^\gamma = \text{const.}$$

and therefore,

$$(1 - \gamma) \frac{dp}{p} + \gamma \frac{dT}{T} = 0.$$

Now substituting gives us

$$\frac{dT}{dz} = -\left(\frac{\gamma - 1}{\gamma}\right) \frac{mg}{k_B} \implies dT = -(1 - \gamma^{-1}) \frac{\mu g}{R} dz.$$

Integrating from  $T_0$  to  $T$  gives us

$$\int_{T_0}^T dT = \int_0^z -(1 - \gamma^{-1}) \frac{\mu g}{R} dz \implies \boxed{T = T_0 - (1 - \gamma^{-1}) \frac{\mu g z}{R}}.$$

**pr 70.** (a) We essentially apply the same process as problem 69. Note that

$$p^{1-\gamma}T^\gamma = \text{const.}$$

and therefore,

$$(1-\gamma)\frac{dp}{p} + \gamma\frac{dT}{T} = 0.$$

This means that

$$\frac{dT}{T} = (1-\gamma^{-1})\frac{dp}{p}.$$

(b) Let us analyze a small section  $dz$  in the atmosphere with density  $\rho$ . The relation between pressure and density is then given by

$$dp = -\rho g dz.$$

We know that  $p = nk_B T$  and  $\rho = nm$  where  $m$  is the mass of a single molecule and therefore,

$$\frac{dp}{dz} = -\frac{mgp}{k_B T} \implies T \frac{dp}{p} = -\frac{mg}{k_B} dz.$$

(c) From parts a and b we can substitute to give us

$$\frac{dT}{dz} = -\left(\frac{\gamma-1}{\gamma}\right) \frac{mg}{k_B} \implies dT = -(1-\gamma^{-1}) \frac{\mu g}{R} dz.$$

Integrating from  $T_0$  to  $T$  gives us

$$\int_{T_0}^T dT = \int_0^z -(1-\gamma^{-1}) \frac{\mu g}{R} dz \implies T = T_0 - (1-\gamma^{-1}) \frac{\mu g z}{R} = \boxed{20.6 \text{ C}^\circ}.$$

**pr 71.** (a) The internal energy will be given by

$$U = \mu C V \Delta T = \mu C \left( \frac{4}{3} \pi R^3 \right) (T_c - T_0) \approx 16768 \text{ J.}$$

(b) By Fourier's Law,

$$J = \dot{Q} = \kappa \frac{\Delta T}{R} = \frac{\kappa (T_1 - T_0)}{R} \approx 2458 \text{ W/m}^2$$

(c) Heat is transferred to the egg via its surface

$$P = J \times 4\pi R^2 = [4\pi \kappa R (T_1 - T_0)] \approx 19.3 \text{ W}$$

(d) Power is equal to the change in internal energy over time or:

$$P = \frac{dU}{dt} \approx \frac{\Delta U}{\Delta t} \Rightarrow \Delta t = \frac{\Delta U}{P} = \frac{\mu c \left( \frac{4}{3} \pi R^3 \right) (T_c - T_0)}{4\pi \kappa R (T_1 - T_0)}.$$

This means that

$$\tau = \frac{\mu C R^2}{3K} \left( \frac{T_c - T_0}{T_1 - T_0} \right) \approx 869 \text{ s.}$$

**pr 72.**

1. The temperature dependence of the bubble volume is  $w = w_0 - \alpha(T - T_0)V$ . Therefore, by ideal gas law,

$$\frac{pw}{T} = \frac{p_0 w_0}{T_0} \implies p = \frac{p_0 T}{T_0} \frac{w_0}{w_0 - V\alpha(T - T_0)}.$$

2. The equality no longer holds when the volume is negative or

$$w_0 - V\alpha(T - T_0) \leq 0 \implies \frac{w_0}{V\alpha} + T_0 \leq T.$$

As  $T$  approaches  $T_{\max}$  pressure increases. Due to high pressure compressibility of glycerin and elasticity of walls must be considered.

**pr 73. A-i)** First, note that by the ideal gas law  $pV = nRT_0$  which means that  $\rho = \frac{PM_{\text{air}}}{RT_0}$ . The differential change in pressure for a differential change in height  $dz$  is  $dP = -\rho g dz$ . Then, after substituting the above expression for the density of air as a function of pressure, we get the following expression:

$$\int_{P_0}^P \frac{dP}{P} = -\frac{M_{\text{air}} g}{RT_0} \int_0^Z dz.$$

Evaluating, we find that  $\alpha = \frac{M_{\text{air}} g}{RT_0}$ .

**A-ii)** The density of air,  $\rho$ , can be taken as a constant; thus,

$$\int_{P_0}^P dP = -\rho g \int_0^Z dz,$$

and  $P(z) = P_0 - \rho gh$ .

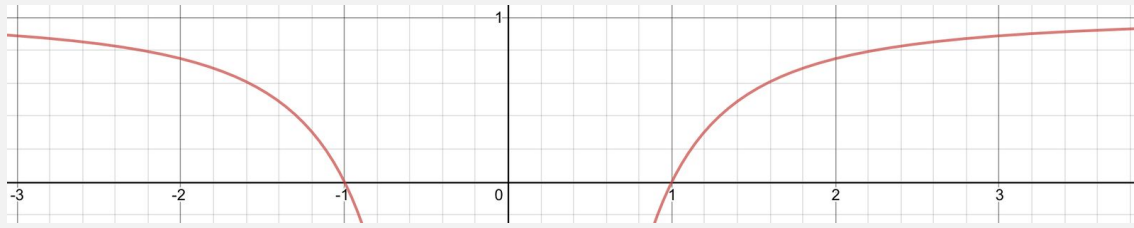
**A-iii)** After substituting the given values,  $P_B \approx 88.24 \text{ kPa}$ .

**B-i)** Consider an infinitesimally thin rectangular prism of air with an area  $A$  and a thickness  $dr$ . If the pressure goes from  $P$  to  $P + dP$  from one side of the piece of air to the other, the net force on it is  $AdP$ . This net force is the centripetal force, so  $AdP = \frac{mv^2}{r}$ . Noting that  $m = \rho V = \rho A dr$ , and solving algebraically,  $\frac{dP}{dr} = \rho_{\text{air}} \frac{v^2}{r}$ .

**B-ii)** Since angular momentum is conserved,  $mv_G r_G = mvr$ ; thus,  $v = \frac{v_G r_G}{r}$ .

**B-iii)** Point G is on the isobar boundary layer, and so is Point B; therefore, it can be said that  $P_G = P_B$ . Then, using the Bernoulli equation, approximating  $v_A \approx 0$ , and substituting  $P_G = P_B$ ,  $P_A = \frac{1}{2}\rho_{\text{air}} v_G^2 + P_B$ . Some algebraic manipulation yields  $v_G = \sqrt{\frac{2(P_A - P_B)}{\rho_{\text{air}}}} = \sqrt{2gh} \approx 141 \text{ m/s}$ .

**B-iv)** Points C and G are also on the isobar boundary layer, meaning that  $P_C = P_G$ . Again using Bernoulli's equation,  $P_G + \frac{1}{2}\rho_{\text{air}} v_G^2 = P_C + \rho_{\text{air}} g z + \frac{1}{2}\rho_{\text{air}} v_C^2$ . Cancelling out  $P_C$  and  $P_G$  and  $\rho_{\text{air}}$  as well as substituting the expression  $v = \frac{v_G r_G}{r}$  which was derived before,  $v_G^2(1 - \frac{r_G^2}{r^2}) = 2gz$ . As a result,  $z = \frac{v_G^2}{2g}(1 - \frac{r_G^2}{r^2})$ . However, through a simple rearrangement of  $v = \sqrt{2gh}$ ,  $h = \frac{v_G^2}{2g}$ . Thus,  $\frac{z}{h} = 1 - \frac{r_G^2}{r^2}$ , which enables us to graph  $\frac{z}{h}$  vs.  $\frac{r}{r_G}$ , as shown.



**B-v)** Since the term  $\frac{2gz}{v_G^2}$  will decrease as  $v_G$  increases, the radius will be less dependent on the height for high-speed tornadoes. As a result, tornadoes with relatively uniform diameter tend to have higher ground rotation speeds.

**C-i)** The angular velocity inside the core is constant since it behaves like a rigid body. Therefore,  $v = \omega r = v_G \frac{r}{r_G}$ .

**C-ii)** From before,  $\frac{dP}{dr} = \rho_{\text{air}} \frac{v^2}{r}$ . Multiplying both sides by  $dr$  and integrating the expression from the center to a far distance away, we have:

$$P_D - P_0 = \rho \left( \int_0^{r_G} \frac{v_G^2 r}{r_G^2} dr + \int_{r_G}^{\infty} \frac{v_G^2 r}{r^2} dr \right).$$

Evaluating and solving algebraically for  $P_D$ , we get  $P_D = P_0 - \rho v_G^2 \approx 76.48 \text{ kPa}$ .

**C-iii)** Assuming adiabatic behavior,  $P^{1-\gamma} \propto \frac{1}{T^\gamma}$ . Therefore,

$$T_G = T_0 \left( \frac{P_G}{P_0} \right)^{\frac{\gamma-1}{\gamma}} \approx 4.89^\circ \text{C}$$

and

$$T_D = T_0 \left( \frac{P_D}{P_0} \right)^{\frac{\gamma-1}{\gamma}} \approx -6.25^\circ \text{C}.$$

**C-iv)** The low temperature causes condensation of moisture that gets sucked into the core, which releases a lot of latent heat, a source of the tornado's energy.

**D-i)** We assume that the house is tightly enclosed with pressure  $P_0$  inside. By substituting  $v = v_G \frac{r}{r_G}$  into  $\frac{dP}{dr} = \rho_{\text{air}} \frac{v^2}{r}$ , the following integral can be set up:

$$\Delta P = \rho_{\text{air}} \int_{2r_G}^{\infty} \frac{v_G^2 r_G^2}{r^3} dr.$$

Evaluating,  $\Delta P = \frac{1}{8} \rho_{\text{air}} v_G^2$ . Since the lift force is  $F_L = \Delta P A$ , the ratio of the lift force to weight for the roof is  $\frac{F_L}{F_G} = \frac{\Delta P A}{\rho_{\text{roof}} A t g} = 3.75$ .

**D-ii)** The lift force is not much larger than the weight of the roof, most roof are mounted firmly to withstand forces many times it's weight. Therefore, chances are the pressure difference would not cause the house to explode so soon (unless the roof is very poorly mounted). Opening windows during a tornado isn't a good idea for another reason. Flying debris is responsible for most twister-related injuries, so standing next to an opening that could potentially blast you with shards of glass and other projectiles isn't a great idea.

**pr 74. A-i)** The buoyant force is going to be:

$$F = \rho_{\text{air}} V_{\text{balloon}} g$$

where  $\rho_{\text{air}} = \frac{PM_A}{RT}$  from the ideal gas law. Applying the ideal gas law to inside the balloon gives:

$$RT = \frac{(P + \Delta P)V}{n}$$

Therefore, the buoyant force is:

$$\frac{P}{P + \Delta P} n M_A g$$

**A-ii)** We know that  $\rho = \frac{PM_A}{RT} \implies \rho = \frac{PM_A z_0}{RT_0(z_0 - z)}$ . The differential change in pressure for a differential change in height  $dz$  is

$$dP = -\rho g dz \implies dP = -\frac{PM_A z_0 g}{RT_0(z_0 - z)} dz.$$

This means that by integrating,

$$\int_{P_0}^P \frac{dP}{P} = -\frac{M_A z_0 g}{RT_0} \int_0^z \frac{dz}{z_0 - z} \implies \ln\left(\frac{P}{P_0}\right) = \frac{M_A z_0 g}{RT_0} \ln\left(\frac{z_0 - z}{z_0}\right)$$

and

$$P = P_0 \left(1 - \frac{z}{z_0}\right)^{\frac{M_A z_0 g}{RT_0}}.$$

Also, note that  $\rho_0 = \frac{P_0 M_A}{RT_0} \implies \frac{M_A}{RT_0} = \frac{\rho_0}{P_0}$  and therefore:

$$\eta = \frac{\rho_0 z_0 g}{P_0}$$

**B-i)** We can apply the method of virtual work. The work needed to change the radius by  $dr$  is:

$$\Delta P (4\pi r^2) dr$$

which causes a change in energy of

$$dU = 4\pi r_0^2 \kappa RT (4\lambda - 4\lambda^{-5}) dr / r_0$$

Equating gives:

$$\Delta P (4\pi r^2) dr = 4\pi r_0 \kappa RT (4\lambda - 4\lambda^{-5}) \implies \Delta P = \frac{4\kappa RT}{r_0} (2\lambda^{-1} + \lambda^{-7})$$

The graph pretty much looks like:

$$\Delta P = \frac{8\kappa RT}{r_0 \lambda}$$

except for small values, at which it increases to infinity quickly.

We can alternatively treat the energy as:

$$U = 4\pi r^2 \gamma$$

where  $\gamma$  is a varying surface tension. Solving for  $\gamma$  and dropping the constant factor, we can apply Laplace's pressure  $\Delta P = 4\gamma/R$  and solve for  $\Delta P$ .

**B-ii)** Initially  $P_0 V_0 = n_0 R T_0$  where  $V_0$  is the unstretched volume. Finally

$$(P_0 + \Delta P) V_0 \lambda^3 = n R T_0$$

as  $V \propto r^3$  using the result from part B(i) we get  $\Delta P = \frac{4\kappa R T}{r_0} (\lambda^{-1} - \lambda^{-7})$ . This gives us

$$P_0 V_0 \lambda^3 (1 + a(\lambda^{-1} - \lambda^{-7})) = n R T_0 \implies a = \frac{\left(\frac{n}{n_0}\right) \frac{1}{\lambda^3} - 1}{(\lambda^{-1} - \lambda^{-7})} = 0.11$$

**pr 75.** We follow the same sign notation as in problem 6, since they are parts of the same problem.

Since we clearly have the constant corresponding to the cutoff frequency  $\eta_0 = \frac{1200 \text{ K}}{6000 \text{ K}}$ , thus

$$\gamma = \frac{\int_0^{\eta_0} \frac{\eta^3 d\eta}{e^\eta - 1}}{\int_0^\infty \frac{\eta^3 d\eta}{e^\eta - 1}}$$

follows from Planck's radiation law. To simplify this result, we use the approximation for small  $\eta$

$$e^\eta \approx 1 + \eta \quad \forall \quad \eta \ll 1$$

Also since we have

$$\int_0^\infty \frac{\eta^3 d\eta}{e^\eta - 1} = 6\xi(4) = \frac{\pi^4}{15}$$

$\gamma$  is simplified to

$$\gamma = \frac{\int_0^{\eta_0} \frac{\eta^3 d\eta}{e^\eta - 1}}{\frac{\pi^4}{15}} \approx 4.106 \times 10^{-4}$$

Now by definition of  $\gamma$ , we have the relation  $P_{\text{in}} = \gamma P_{\text{out}}$ , which gives us

$$P_{\text{output}} = P_\odot \gamma_{\text{eff}}$$

$$\epsilon_\times 4\pi R_\oplus^2 \sigma T^4 = a \times \sigma T_\odot^4 (4\pi R_\odot^2) \left(\frac{R_\oplus}{2L}\right)^2$$

$$\not\epsilon_\times 4\pi R_\oplus^2 \sigma T^4 = \not{a} \times \sigma T_\odot^4 (4\pi R_\odot^2) \left(\frac{R_\oplus}{2L}\right)^2$$

$$\pi R_\odot^2 \times \left(\frac{\sigma T_\odot^4 (4\pi R_\odot^2)}{4\pi L^2}\right) \times \gamma = 4\pi R_\oplus^2 (\sigma T_S^4)$$

$$T_S = \left(\frac{\gamma T_\odot^4 R_\odot^2}{4L^2}\right)^{\frac{1}{4}} = T_\odot \sqrt{\frac{\sqrt{\gamma} R_\odot}{2L}}$$

Note that the temperature of the satellite is independent of the size of the satellite, as long as the size is not large. Thus, substituting the value of  $\gamma$ , we get

$$T_S = T_\odot \sqrt{\frac{\sqrt{\gamma} R_\odot}{2L}} \approx \boxed{41 \text{ K}}$$

**pr 76.** Since an ideal heat engine has practically zero heat and friction losses, it takes some amount of heat from one object and delivers heat to another object having the same thermal capacity. Say  $Q_A$  amount of heat was initially withdrawn from A and  $Q_B$  heat given to B, and let  $T_B < T_C < T_A$ , where  $T_C$  is the common temperature reached. By second law of thermodynamics, entropy is equal or:

$$\frac{Q_A}{T_1} = \frac{Q_B}{T_2}$$

Also, by definition

$$\begin{aligned} dQ_A &= -CdT_1 \\ dQ_B &= CdT_2 \end{aligned}$$

From these we have

$$-C \frac{dT_1}{T_1} = C \frac{dT_2}{T_2} \Rightarrow -C \int_{T_A}^{T_C} \frac{dT_1}{T_1} = C \int_{T_B}^{T_C} \frac{dT_2}{T_2} \Rightarrow \ln \frac{T_C}{T_B} = \ln \frac{T_A}{T_C} \Rightarrow T_C = \sqrt{T_A T_B}$$

We can also apply relations for differential quantities of  $Q_A$  and  $Q_B$  as we wrote earlier in the solution. For  $dQ_A$ , we have:

$$dQ_A = -CdT_1 \Rightarrow Q_A = C \int_{T_A}^{T_C} dT_1 = C(T_A - T_C) = C(T_A - \sqrt{T_A T_B}).$$

For  $dQ_B$ :

$$dQ_B = -CdT_2 \Rightarrow Q_B = C \int_{T_C}^{T_B} dT_2 = C(T_C - T_B) = C(\sqrt{T_A T_B} - T_B)$$

Now clearly by the first law of thermodynamics, we have

$$\Delta W = Q_A - Q_B = C(T_A - T_B - 2\sqrt{T_A T_B}) = C(\sqrt{T_A} - \sqrt{T_B})^2.$$

**pr 77.** Define  $T_0$  to be the air temperature,  $T_1$  to be temperature supplied to the colder joint, and  $T_2$  to be the temperature supplied to the warmer joint.

Let temperature of colder joint be  $T_0 - T_1$ , and warmer joint be  $T_0 + T_2$ . The thermocouple acts as a reversible carnot engine in reverse, with electrical energy supplying the necessary work; therefore we can relate work done to heat in/heat out. The Carnot Cycle usually involves work/heat, but we can also look at the work per second (or power) and obtain an equivalent result. Note that the amount of power done is

$$P = IV = I\alpha(T_1 + T_2).$$

The amount of heat that is supplied to the cold side of the thermocouple is  $\kappa T_1$ . The amount of work done is then

$$W = \frac{(T_0 + T_2) - (T_0 - T_1)}{T_0 - T_1} \kappa T_1$$

Similarly the amount of heat supplied to the warm part of the thermocouple is  $\kappa T_2$ . This tells us the amount of work done is

$$W = \frac{(T_0 + T_2) - (T_0 - T_1)}{T_0 + T_2} \kappa T_2.$$



We can now equate these together to get the equality of

$$I\alpha(T_1 + T_2) = \frac{(T_0 + T_2) - (T_0 - T_1)}{T_0 - T_1} \kappa T_1 = \frac{(T_0 + T_2) - (T_0 - T_1)}{T_0 + T_2} \kappa T_2$$

simplifying gives us

$$I\alpha = \frac{\kappa T_1}{T_0 - T_1} = \frac{\kappa T_2}{T_0 + T_2}.$$

We can now easily solve for  $T_1$  to get

$$\begin{aligned} I\alpha(T_0 - T_1) &= \kappa T_1 \\ I\alpha T_0 &= T_1(\kappa + I\alpha) \\ T_1 &= \boxed{T_0 \frac{I\alpha}{\kappa + I\alpha}} \end{aligned}$$

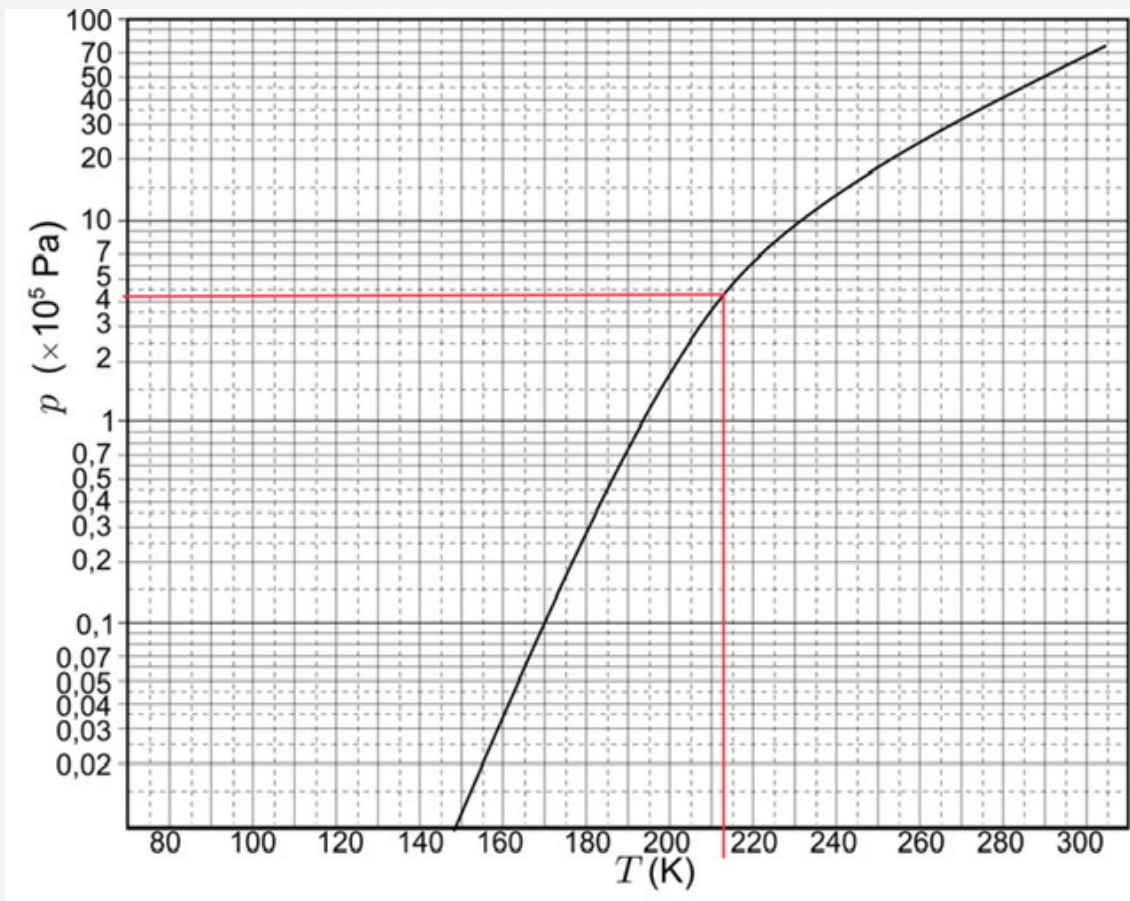
**pr 78.** The friction in the disk disappears because the  $\text{CO}_2$  evaporates. The created pressure due to the force applied to the disk is given by

$$P = \frac{F}{A} = \frac{F}{\pi r^2}.$$

Dry ice begins to sublime only when vapor pressure exceeds ambient pressure. Therefore, the minimum vapour pressure needed is given by

$$P_{\text{vap}} = P_{\text{air}} + \frac{F}{\pi r^2} = 419.3 \text{ Pa}.$$

From the graph, we can look at the points to see that  $T \sim \boxed{213 \text{ K}}$



**pr 79.** The heat exchange through walls is equal to  $P \frac{t-t_0}{t_1-t_0}$  while the heat exchanged with incoming air  $= \frac{v}{V} C_p (t - t_0)$  where  $V$  is molar volume. Therefore

$$P = P \frac{t - t_0}{t_1 - t_0} + \frac{v}{V} C_p (t - t_0)$$

which implies that

$$t = t_0 + \frac{P}{\frac{P}{t_1 - t_0} + \frac{v C_p}{V}} \Rightarrow t = 13.16^\circ \text{C}.$$

**pr 80.** The density of some matter of mass  $m$  can be given by

$$\rho = N \frac{m}{V}$$

where  $N$  is the number density of the substance. This means that for dry and humid air on each respective air can be given as

$$\rho_d = N_d \frac{m_d}{V}, \quad N_h \frac{m_h}{V}.$$

The number density of the dry air is given by

$$N_d = \frac{M}{M_a} = \frac{M}{28.8}$$

while the number density of the humid air will be

$$N_h = 0.02 \cdot \frac{M'}{M_w} + 0.98 \cdot \frac{M'}{M_a} = 0.02 \cdot \frac{M'}{28.8} + 0.98 \cdot \frac{M'}{18}.$$

We can now compare the ratio of densities since we require the number density to be the same or in other words,

$$\frac{\rho_d}{\rho_h} = \frac{N_d m_d}{N_h m_h} = \frac{\frac{M}{28.8} \cdot M'}{M' \cdot (0.02 \cdot \frac{M'}{28.8} + 0.98 \cdot \frac{M'}{18})} = \frac{1}{28.8 \left( \frac{0.02}{18} + \frac{0.98}{18} \right)} = 0.9881.$$

This means that the moist air is then

$$\rho_m = \rho_h = 0.9881 \rho_d = 1.2352 \text{ kg/m}^3.$$

**pr 81.** (a) Since the process happens in a pipe, we apply bernoulli's law:

$$P + \frac{1}{2} \rho v^2 + \rho g h = \text{const.}$$

The two places we apply Bernoulli's law is at the bottom of the furnace and at a height  $h$ . Using the assumption that smoke temperature can be assumed to be constant throughout the entire length of the chimney we find that

$$P_h + \frac{1}{2} \rho_{\text{smoke}} v^2 + \rho_{\text{smoke}} g h = P_{\text{atm}}$$

from taking the reference point to be at the bottom of the furnace. We write that

$$P_h = P_{\text{atm}} + P_{\text{air}} \implies P_{\text{atm}} - P_h = -P_{\text{air}} = \rho_{\text{air}} g h.$$

This then means that

$$\frac{1}{2} \rho_{\text{smoke}} v^2 = g h (\rho_{\text{air}} - \rho_{\text{smoke}}) \implies v = \sqrt{2 g h \left( \frac{\rho_{\text{air}}}{\rho_{\text{smoke}}} - 1 \right)}.$$

If we want efficient withdrawal of all the gas, we then need for  $v \geq \frac{B}{A}$  so all the gas can leave. In other word, we can create an inequality:

$$\sqrt{2 g h \left( \frac{\rho_{\text{air}}}{\rho_{\text{smoke}}} - 1 \right)} \geq \frac{B}{A} \implies h \geq \frac{B^2}{2 g A^2} \left( \frac{\rho_{\text{air}}}{\rho_{\text{smoke}}} - 1 \right)^{-1}$$

and by the ideal gas law

$$\frac{\rho_{\text{air}}}{\rho_{\text{smoke}}} = \frac{T_{\text{smoke}}}{T_{\text{air}}}$$

which gives us

$$\begin{aligned} h &\geq \frac{B^2}{2 g A^2} \left( \frac{T_{\text{air}}}{T_{\text{smoke}}} - 1 \right)^{-1} \\ &\geq \frac{B^2}{A^2} \frac{1}{2 g} \frac{T_{\text{air}}}{\Delta T}. \end{aligned}$$

(b) We simply analyze the ratios of  $h(30)$  and  $h(-30)$ :

$$\frac{h(30)}{h(-30)} = \frac{\frac{T(30)}{T_{\text{smoke}} - T(30)}}{\frac{T(-30)}{T_{\text{smoke}} - T(-30)}} \implies \frac{h(30)}{100} = 1.45 \implies h(30) = 145 \text{ m.}$$

(c) Note that

$$v = \sqrt{2gh \left( \frac{\rho_{\text{air}}}{\rho_{\text{smoke}}} - 1 \right)} = \sqrt{2gh \left( \frac{T_{\text{air}}}{T_{\text{smoke}}} - 1 \right)} = \sqrt{2gh \frac{\Delta T}{T_{\text{air}}}}.$$

This means that the velocity is constant.

(d) Consider the gas at a height  $z$ . From the bernoulli equation in part (a), we find that the pressure is given by

$$P_z + \frac{1}{2}\rho_{\text{smoke}}v^2 + \rho_{\text{smoke}}gz = P_{\text{atm}}.$$

Using the fact that

$$\frac{1}{2}\rho_{\text{smoke}}v^2 = gh(\rho_{\text{air}} - \rho_{\text{smoke}}),$$

we can then say that

$$P_z = P_{\text{atm}} - gh(\rho_{\text{air}} - \rho_{\text{smoke}}) - \rho_{\text{smoke}}gz.$$

(e) In a time  $\Delta T$ , the mass of air that is released is given by  $m = \rho \cdot Av\Delta t$  which means the kinetic energy of the air is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}(\rho \cdot Av\Delta t)v^2.$$

Substituting our expression of  $v^2$  from before yields,

$$T = \rho av\Delta t gh \frac{\Delta T}{T_{\text{air}}}.$$

Note that  $P = \Delta E/\Delta t$  which means that

$$P_{\text{air}} = \rho Avgh \frac{\Delta T}{T_{\text{air}}}.$$

The power produced by the sun is equal to (note the mass here is per unit time)

$$P_{\text{sun}} = mc\Delta T = \rho Av c \Delta T$$

which means that we can then divide both expressions to result in the efficiency of the system or

$$\eta = \frac{P_{\text{air}}}{P_{\text{sun}}} = \frac{gh}{cT_{\text{air}}}.$$

(f) Since  $\eta \propto h$ , the relationship is linear.

(g) By plugging in numbers to our efficiency in (e), we find that  $\eta = 0.64\%$ .

(h) The power produced can be given by

$$P_{\text{sun}} = \eta GS \approx 45 \text{ kW.}$$

- (i) If there are 8 sunny hours per each day, we simply multiply our result from (h) to get 360 kW.  
 (j, k) The mass flow rate will be given by

$$w = \rho A v = \rho A \sqrt{2gh \frac{\Delta T}{T_{\text{air}}}}.$$

We also know that

$$P = w c \Delta T = G S \implies w = \frac{G S}{c \Delta T}.$$

Therefore, by equating both expressions we have the equation

$$\rho A \sqrt{2gh \frac{\Delta T}{T_{\text{air}}}} = \frac{G S}{c \Delta T} \implies \Delta T = \left( \frac{G^2 S^2 T_{\text{atm}}}{A^2 c^2 \rho^2 2gh} \right)^{1/3}.$$

**pr 82.** At a temperature of 77.4 K (i.e. at the boiling point of nitrogen), the pressure of saturated nitrogen vapour is 1 atm, while the saturated pressure of oxygen becomes 1 atm at a higher temperature of 90.2 K.

The molar ratio of oxygen and nitrogen is 21 : 79. The ratio of the partial pressures of the two components will also be very close to molar ratio, because, until the start of liquefaction, the behaviour of each gas constituent is very close to that of an ideal gas. When the partial pressure of oxygen is 0.2226 atm, liquefaction of oxygen starts. The partial pressure of oxygen does not increase further as temperature is constant. The partial pressure of  $N_2$  at the onset of oxygen liquefaction is  $0.2226 \times \frac{79}{21} \text{ atm} = 0.8374 \text{ atm}$ . This is less than the saturated vapour pressure of nitrogen at this temperature, which, since 77.4 K is nitrogen's boiling point, has a value of 1 atm. Consequently, nitrogen does not liquefy at this pressure. Therefore total pressure at this point  $P_1$  is  $0.8374 + 0.2226 = 1.06 \text{ atm}$ . As the compression is isothermal, volume at this point  $V_1$  is  $1.001 \times \frac{0.500}{1.06} \text{ l} = 0.4721 \text{ l}$ . During the subsequent compression, the partial pressure of the oxygen, already in two phases, does not change, while the nitrogen pressure increases from 0.8374 atm to 1.00 atm. This latter pressure will be reached when the volume has been reduced by a factor of  $(0.8374/1.00) = 0.8374$ . Therefore volume at this point  $V_2$  is  $0.8374 \times 0.4721 \text{ l} = 0.3953 \text{ l}$ . After that, the total pressure remains constant (at  $0.2226 + 1.00 = 1.22 \text{ atm}$ ) until the liquefaction is complete, just the volume is decreased now.

**pr 83.** (a) Consider a cross-section with width  $dh$ . The pressure is then given by

$$dP = -\rho g dh.$$

From the ideal gas law, we have that

$$\rho = \frac{P M}{R T}$$

which means that

$$\int_0^H -\frac{Mg}{RT} dh = \int_P^{P/e} \frac{dP}{P} \implies -\frac{MgH}{RT} = \ln\left(\frac{1}{e}\right) = -1.$$

This means that

$$MgH = RT \implies \bar{M} = \frac{RT}{gH} = 14.5 \text{ g/mol}.$$

(b) By the ideal gas law

$$\frac{n}{N_A}RT = P_0 \implies n = \frac{N_A P_0}{RT}.$$

The mean free path  $\lambda$  is given by

$$\lambda = (\sigma n)^{-1} \implies \lambda = \frac{RT}{N_A \sigma P_0} = 3286.67 \text{ m}.$$

(c) We are given in part (a) that

$$\lambda_{EB} = H \implies n_{EB} = \frac{1}{\lambda_{EB} \sigma} = \frac{1}{\sigma H}.$$

This means that from the ideal gas law:

$$P_{EB} = \frac{n_{EB} RT}{N_A}.$$

Now, once again integrating and doing the same steps as in part (a), we have that

$$-\int_{h_0}^{h_{EB}} \frac{Mg dh}{RT} = \int_{P_0}^{P_{EB}} \frac{dP}{P} \implies -(h_{EB} - h) \frac{Mg}{RT} = \ln \left( \frac{P_{EB}}{P_0} \right) \implies h_{EB} = 425 \text{ km}.$$

(d) Note that Maxwell's distribution in spherical form is given by

$$f(v)dv = v^2 \left( \frac{m}{2\pi k_B T} \right)^{3/2} \exp \left( -\frac{mv^2}{2k_B T} \right) \sin \theta d\theta d\varphi.$$

The probability of the particle having a velocity more than the escape velocity will then be given by  $\int_{v_{esc}}^{\infty} f(v)dv$  or in other words, integrating over all spaces gives us

$$\begin{aligned} P(v) &= \left( \frac{m}{2\pi k_B T} \right)^{3/2} \int_{v_{esc}}^{\infty} v^2 \exp \left( -\frac{mv^2}{2k_B T} \right) \int_0^{\pi/2} \sin \theta d\theta \int_0^{2\pi} d\varphi \\ &= \sqrt{\frac{2m^3}{\pi k_B^3 T^3}} \int_{v_{esc}}^{\infty} v^2 \exp \left( -\frac{mv^2}{2k_B T} \right) \end{aligned}$$

We need to calculate the escape velocity at the exobase which can be done with the equation

$$v_{esc} = \sqrt{\frac{2GM}{R+h}} = 1.08 \times 10^4 \text{ m/s}$$

and therefore, the integral gives us

$$P(v) = \sqrt{\frac{2m^3}{\pi k_B^3 T^3}} \int_{1.08 \times 10^4}^{\infty} v^2 \exp \left( -\frac{mv^2}{2k_B T} \right) = \boxed{2.6 \cdot 10^{-3}}.$$

(e) Note that flux is given by  $\Phi = \frac{dN}{dA \cdot dt}$ . We can calculate  $dN$  first using the spherical Maxwell's distribution and using the fact that the number of molecules coming in at a certain time is given by  $nv \cos \theta dA dt$  where  $dA$  is a surface element. Therefore, expanding  $N$  gives us

$$dN = \sum n_H v \cos \theta dA dt \cdot f(v, \theta, \varphi) dv d\theta d\varphi$$

which tells us

$$dN = n_H \left( \frac{m}{2\pi k_B T} \right)^{3/2} \exp \left( -\frac{mv^2}{2k_B T} \right) v^3 dv \cdot \sin \theta \cos \theta d\theta \cdot d\varphi \cdot dAdt.$$

Integrating over  $\theta$  and  $\varphi$  tells us

$$dN = n_H \left( \frac{m}{2\pi k_B T} \right)^{3/2} \exp \left( -\frac{mv^2}{2k_B T} \right) v^3 dAdt$$

and therefore,

$$d\Phi = \frac{dN}{dAdt} = n_H \left( \frac{m}{2\pi k_B T} \right)^{3/2} \exp \left( -\frac{mv^2}{2k_B T} \right) v^3 dv.$$

The flux of the escaping atoms are then given as

$$\Phi = \int_{v_{\text{esc}}}^{\infty} n_H \left( \frac{m}{2\pi k_B T} \right)^{3/2} \exp \left( -\frac{mv^2}{2k_B T} \right) v^3 dv = \boxed{7.5 \times 10^{11} \text{ 1/m}^2}.$$

(f) Note that the atmosphere produces a force of  $F = PA = 4\pi R_{\odot}^2 P$ . Equating this force to the mass of the atmosphere tells us that

$$mg = 4\pi R_{\odot}^2 P \implies m = \frac{4\pi R_{\odot}^2 P}{g}.$$

We then write

$$N = \frac{N_A m}{M_{\text{Atm}}} = \frac{4\pi N_A R_{\odot}^2 P}{M_{\text{Atm}} g}.$$

Then note that since Nitrogen is diatomic, we have  $N_H = 2\chi_H N = 1.2 \times 10^{38}$  which gives us our answer.

(g) The number of molecules escaping after a unit time is

$$\dot{N}_H = \Phi(R_{\odot} + h_{EB})^2 = 4.35 \times 10^{26} \text{ 1/s}.$$

The total time required for half the atmosphere to evaporate is then given by the characteristic time interval:

$$\tau = \frac{N_H/2}{\dot{N}_H} \approx \boxed{4500 \text{ years}}.$$

(h) Redoing the calculations from the other parts for hydrogen gives us our new answer of  $9.6 \times 10^{11}$  years.

(i) One possible reason of why there is currently still some hydrogen in the Earth's atmosphere is that the hydrogen inside the Earth's water gets cycled through the hydrologic cycle.

**pr 84.** (a) Since the temperature is constant, we have an isothermal compression where:

$$P_0 V_0 = P_1 V_1 \implies P_0 r_0^3 = P_1 r_1^3$$

Since  $r_1 = \frac{1}{2}r_0$ , we have  $P_1 = 8P_0$ .

(b) The heat generated is equal to the negative work done:

$$Q = -W = - \int P dV$$

Using  $PV = nRT$ , we get:

$$Q = -nRT \ln \left( \frac{V_0}{V_3} \right) = \frac{3mRT_0}{\mu} \ln \left( \frac{r_0}{r_3} \right)$$

(c) We now have an adiabatic compression since not heat is exiting the system:

$$PV^\gamma = \text{constant} \implies TV^{\gamma-1} = \text{constant} \implies T_3 = T_0 \left( \frac{r_3}{r_0} \right)^{3\gamma-3}.$$

(d) The final radius refers to the point in which the gravitational potential energy is comparable to the thermal energy:

$$\frac{GM\mu}{r_4} = RT_0 \left( \frac{r_3}{r_4} \right)^{3\gamma-3} \implies r_4 = r_3 \left( \frac{RT_0 r_3}{\mu Gm} \right)^{\frac{1}{4-3\gamma}}$$

and using the relationship from part (c), we get:

$$T_4 = T_0 \left( \frac{RT_0 r_3}{\mu Gm} \right)^{\frac{3\gamma-3}{4-3\gamma}}$$

**pr 85.** (a) First, we assume that the process is reversible (even though this is not very likely). Then, the work done on the liquid is:

$$dW = (P - P_0)4\pi r^2 dr = dE_k \implies (P - P_0)R^2 dr = \frac{\rho}{2} d(r^3 \dot{r}^2)$$

The initial pressure is given by:

$$P_i = \frac{P_0 R_0^3}{R_i^3}$$

so using  $PV^\gamma$ , we get:

$$P = P_0 \left( \frac{R_0^3}{R_i^3} \right) \left( \frac{R_i}{r} \right)^{3\gamma}$$

Since  $\gamma = 5/3$ , this simplifies to:

$$P_0 \left( 49 \left( \frac{R_0}{r} \right)^5 - 1 \right) r^2 dr = \frac{\rho}{2} d(r^3 \dot{r}^2)$$

Integrating the left hand side, we can first make the substitution  $\beta = \frac{r}{R_0}$  to simplify it to:

$$P_0 R_0^3 \int_7^\alpha \left( \frac{49}{\beta^3} - \beta^2 \right) d\beta = P_0 R_0^3 \left( \frac{1}{2} - \frac{49}{2\alpha^2} + \frac{7^3}{3} - \frac{\alpha^3}{3} \right)$$

The right hand side evaluates to zero since it starts and ends off at rest. Thus, setting this to zero, we get the equation:

$$6\alpha^5 + 147 - 689\alpha^2$$

Making the assumption that  $6\alpha^5 \ll 1$ , we get a quadratic:

$$\alpha = \sqrt{\frac{147}{689}} \implies R_{\min} = 0.462 R_0 = 2.31 \mu\text{m}$$



We also know from  $TV^{\gamma-1}$  that the maximum temperature is thus:

$$T_{\max} = 6.86 \times 10^4 \text{ K.}$$

**(b)** We can apply the same differential equation. The LHS stays the same, but the RHS no longer becomes zero. The RHS can be evaluated to:

$$\int_0^{\alpha^3 \dot{\beta}^2} \frac{\rho R_0^5}{2} d(\beta^3 \dot{\beta}^2) = \frac{\rho R_0^5}{2} (\alpha^3 \dot{\beta}^2)$$

Setting it equal, we see that:

$$P_0 R_0^3 \left( \frac{1}{2} - \frac{49}{2\alpha^2} + \frac{7^3}{3} - \frac{\alpha^3}{3} \right) = \frac{\rho R_0^5}{2} (\alpha^3 \dot{\beta}^2) \implies \dot{\beta}^2 \propto \frac{689}{6\alpha^3} - \frac{49}{2\alpha^5} - \frac{1}{3}$$

This is at a maximum when:

$$\alpha = \sqrt{\frac{6 \cdot 5 \cdot 49}{2 \cdot 3 \cdot 689}} = 0.596 \implies R_f = 2.98 R_0$$

**(c)** We make the assumption that between these two times, the speed is roughly the same. The average radius is:

$$\langle R \rangle = 2.645 \mu\text{m}$$

and thus plugging in this into our expression for  $\dot{r}$  gives  $\dot{r} = 192.77 \text{ m/s}$  such that the total time is:

$$\Delta t = 3.48 \times 10^{-9} \text{ s}$$

**(d)** By Stefan-Boltzmann Law, we have that

$$\dot{Q} = a\sigma \cdot 4\pi r^2 T^4.$$

Substituting

$$T = T_0 \left( \frac{R_i}{r} \right)^2,$$

we result in the expression of

$$\dot{Q} = 4\pi a\sigma R_i^8 T_0^4 / r^6.$$

We require that

$$Q \leq \frac{1}{5} U \implies \left| \dot{Q} \right| \leq \left| \frac{1}{5} \dot{U} \right|$$

and therefore, we attempt to find  $\dot{U}$  as well. Note that

$$\dot{U} = -P_i \dot{V} = -P_i \left( \frac{V_i}{V} \right)^\gamma \dot{V} = -4\pi P_i R_i^5 \dot{r} / r^3.$$

We now can set our expression to be

$$\frac{4\pi a\sigma R_i^8 T_0^4}{r^6} \leq \frac{1}{5} \cdot \frac{4\pi P_i R_i^5 \dot{r}}{r^3} \implies a \leq \frac{P_i r^3 \dot{r}}{5 R_i^3 \sigma T_0^4} \approx 0.0107$$

**pr 86.** (a) We know that

$$C_v = \frac{dE_{\text{avg}}}{dT}$$

A free electron has 3 degrees of freedom. Therefore:

$$E_{\text{avg}} = \frac{3}{2}k_B T \implies C_v = \frac{3}{2}k_B$$

(b) The total energy is

$$U = \int_0^S E f(E) dS = CV \int_0^\infty E^{3/2} f(E) dE.$$

From the graph

$$f(E) = \begin{cases} 1 & 0 \leq E \leq E_F - 2k_B T \\ \frac{E}{4k_B T} + \frac{E_F + 2k_B T}{4k_B T} & E_F - 2k_B T \leq E \leq E_F + 2k_B T \\ 0 & E \geq E_F + 2k_B T \end{cases}$$

Therefore, by integrating from all cases we have:

$$\int_0^\infty E^{3/2} f(E) dE = \int_0^{E_F - 2k_B T} E^{3/2} \cdot 1 \cdot dE + \int_{E_F - 2k_B T}^{E_F + 2k_B T} E^{3/2} \frac{E_F - 2k_B T - E}{4k_B T} dE + \int_{E_F + 2k_B T}^\infty E^{3/2} \cdot 0 \cdot dE$$

Simplifying and evaluating the integral tells us

$$\begin{aligned} \int_0^\infty E^{3/2} f(E) dE &= \frac{2(E_F - 2k_B T)^{5/2}}{5} + \frac{2(E_F + 2k_B T) \left( (E_F + 2k_B T)^{5/2} - (E_F - 2k_B T)^{5/2} \right)}{5 \times 4k_B T} \\ &\quad - \frac{2 \left( (E_F + 2k_B T)^{7/2} - (E_F - 2k_B T)^{7/2} \right)}{4k_B T}. \end{aligned}$$

As  $k_B T \ll E_F$

$$\int_0^\infty E^{3/2} f(E) dE = \frac{2E_F^{5/2}}{5} \left( 1 - \frac{5k_B T}{E_F} + \frac{15}{2} \left( \frac{k_B T}{E_F} \right)^2 \right) + E_F^{5/2} \left( 1 + \frac{2k_B T}{E_F} \right) - E_F^{5/2} = \frac{2E_F^{5/2}}{5} + 3 \left( \frac{k_B T}{E_F} \right)^2 E_F^{5/2}.$$

Therefore:

$$U = CV E_F^{5/2} \left( \frac{2}{5} + 3 \left( \frac{k_B T}{E_F} \right)^2 \right)$$

and

$$E_{\text{avg}} = \frac{U}{N}$$

where  $N = \int dS = CV \int_0^{E_F^0} E^{1/2} dE$  as electrons are below energy below  $E_F^0$  at  $T = 0\text{K}$ . It is given that fermi level is not dependent on temperature. Finally, we note that  $E_F^0 = E_F \implies N = \frac{2CV E_F^{3/2}}{3}$  which means

$$E_{\text{avg}} = \frac{3}{2} E_F \left( \frac{2}{5} + 3 \left( \frac{k_B T}{E_F} \right)^2 \right).$$

The heat capacity is then:

$$C_v = \frac{dE_{\text{avg}}}{dT} = \frac{9k_B^2 T}{E_F}.$$

**pr 87.** (a) The energy for spin up dipole is given by  $\mu_B B$  while the energy for spin down dipole  $-\mu_B B$ . Note that by Boltzmann's law

$$\text{Probability: } p = A e^{-E/k_B T} \text{ where } A \text{ is a constant.}$$

As there are only 2 possible states for the dipoles. The total sum of both states must add up to 1 by the rule of probability.

$$p_{\mu_B} + p_{-\mu_B} = 1 \implies A = \frac{1}{e^{-\mu_B B/k_B T} + e^{\mu_B B/k_B T}}$$

Therefore,  $p_{\mu_B} = \frac{e^{-\mu_B B/k_B T}}{e^{-\mu_B B/k_B T} + e^{\mu_B B/k_B T}}$  and  $p_{-\mu_B} = \frac{e^{\mu_B B/k_B T}}{e^{-\mu_B B/k_B T} + e^{\mu_B B/k_B T}}$ . The average energy will be the weighted average of both states which is the multiplied by  $N$  or in other words:

$$E_s = \left( \frac{\mu_B B p_{\mu_B} + (-\mu_B B p_{-\mu_B})}{p_{\mu_B} + p_{-\mu_B}} \right) N$$

Therefore,

$$E_s = N \mu_B B \left( \frac{e^{-\mu_B B/k_B T} - e^{\mu_B B/k_B T}}{e^{-\mu_B B/k_B T} + e^{\mu_B B/k_B T}} \right) = -N \mu_B B \tanh \left( \frac{\mu_B B}{k_B T} \right)$$

,

(b) For  $1 \gg \frac{\mu_B B}{k_B T}$ , we can approximate the energy to be

$$E_s \approx \frac{-N (\mu_B B)^2}{k_B T}$$

We then find that  $C = \frac{dE_s}{dT} = \frac{N (\mu_B B)^2}{k_B T^2}$

Here is a purely statistical analysis of this problem. Suppose there are  $N_{\uparrow}$  states and  $N_{\downarrow}$  states such that  $N_{\uparrow} + N_{\downarrow} = N$  and the total energy is  $U = \mu_B B (N - 2N_{\uparrow})$ . Then, the multiplicity of having  $N_{\uparrow}$  states is given by:

$$\Omega = \frac{N!}{N_{\uparrow}! N_{\downarrow}!}$$

The entropy of this is thus:

$$S = k \ln \Omega = k [\ln(N!) - \ln(N_{\uparrow}!) - \ln((N - N_{\uparrow})!)]$$

Applying Stirling's approximation, we get:

$$S/k = N \ln N - N_{\uparrow} \ln(N_{\uparrow}) - (N - N_{\uparrow}) \ln(N - N_{\uparrow})$$

The definition of temperature is given by:

$$\frac{1}{T} = \frac{\partial S}{\partial U} = \frac{\partial N_{\uparrow}}{\partial U} \frac{\partial S}{\partial N_{\uparrow}} = -\frac{1}{2\mu_B} \frac{\partial S}{\partial N_{\uparrow}}$$

where

$$\frac{\partial S}{\partial N_{\uparrow}} = -\ln \left( \frac{N_{\uparrow}}{N - N_{\uparrow}} \right) = \ln \left( \frac{N - U/\mu_B}{N + U/\mu_B} \right)$$

where we have substituted  $N_{\uparrow} = \frac{N}{2} - \frac{U}{2\mu_B}$ . Solving for  $U$ , we get the desired:

$$U = -N \mu_B B \tanh \left( \frac{\mu_B B}{k_B T} \right)$$

**pr 88.** We consider the vessel to be spherically symmetric. Let the mass of vessel be  $m$  and the radius of vessel be  $r$  and thickness of steel be  $d$  with  $d \ll r$

$$m = 4\pi r^2 d \rho \Rightarrow 2\pi r^2 d = \frac{m}{2\rho}$$

For the hydrogen stored

$$PV = \nu RT \Rightarrow P \left( \frac{4}{3} \pi r^3 \right) = \nu RT \Rightarrow P = \frac{\nu RT}{\frac{4\pi r^3}{3}}$$

For the vessel to not break

$$P \cdot \pi r^2 \leq \sigma \cdot 2\pi r d \Rightarrow \frac{\nu RT}{\frac{4\pi r^3}{3}} \pi r^2 \leq \sigma \cdot 2\pi r d \Rightarrow \frac{3RT\nu}{4} \leq \sigma \cdot 2\pi r^2 d.$$

This implies that

$$\frac{3RT\nu}{4} \leq \sigma \cdot \frac{m}{2\rho} \Rightarrow \frac{3RT\nu\rho}{2\sigma} \leq m \Rightarrow m_{\min} = \frac{3RT\nu\rho}{2\sigma}.$$