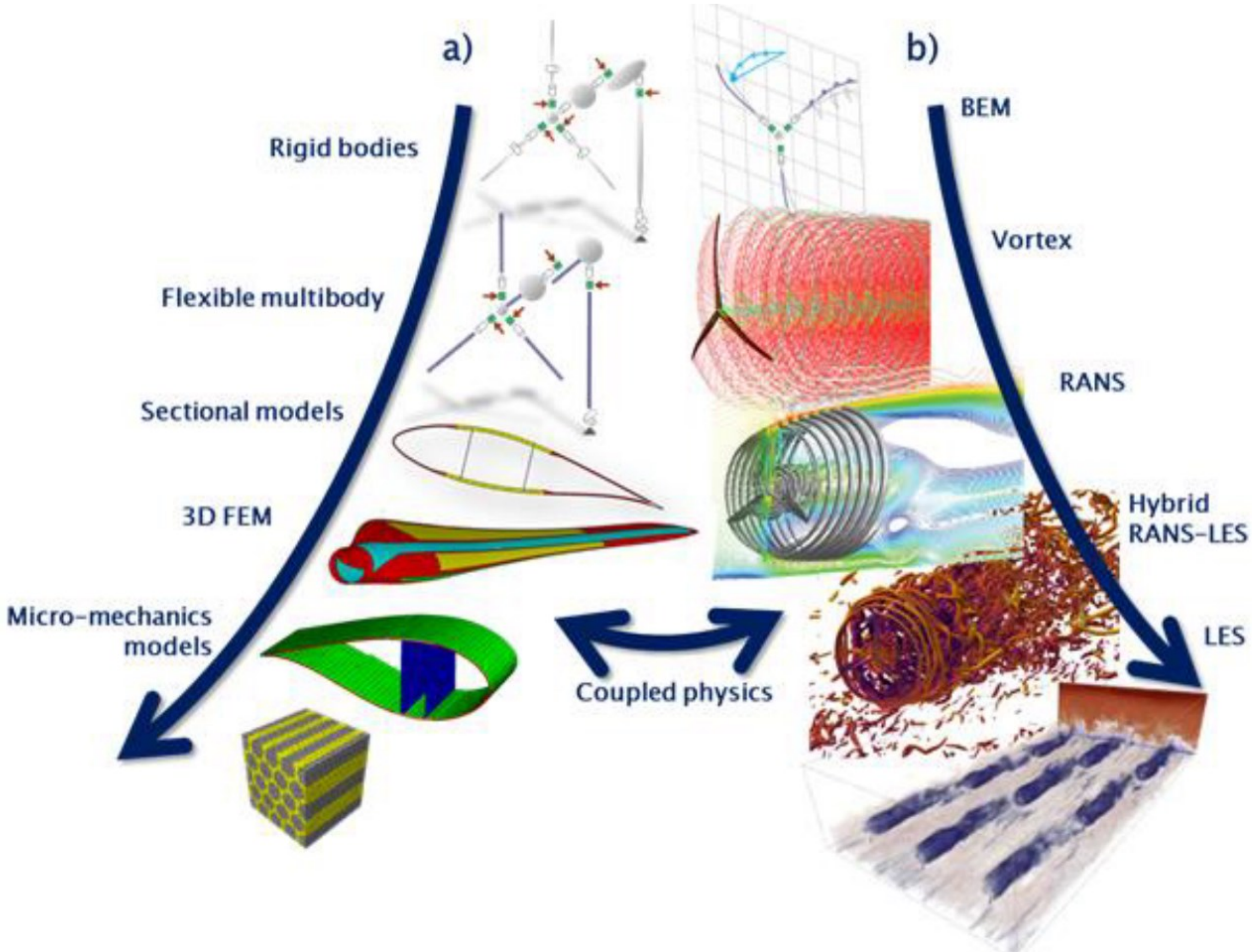
Imperial College London

Wind Energy: The Betz Limit

Rafael Palacios

Imperial College London

Modelling aeromechanics of WTs



Veers et al, WESC (2022)

Aerodynamic Analysis

Simple Approach

The simplest analysis of rotor power uses a 1-dimensional momentum and energy balance.

The energy balance is carried out along a stream-tube. The stream-tube comprises all the air flow which passes through the swept area of the rotor blades.

This analysis is also known as Betz analysis or 1-D Actuator Disk Analysis.

Industry Standard Approach

More detailed analysis is required to undertake rotor design such as choosing the blade geometry, the angle-of-attack etc.

Blade Element Momentum theory is the method most often used. It is an extension of the 1-dimensional combination of momentum and energy balance but applied separately in a series concentric annular rings.

High-Fidelity Approach

Computational Fluid Dynamic, CFD, analysis is possible but is computationally expensive and not yet used extensively in the industry.

Force and Velocity

Power is the scalar product of force and velocity.

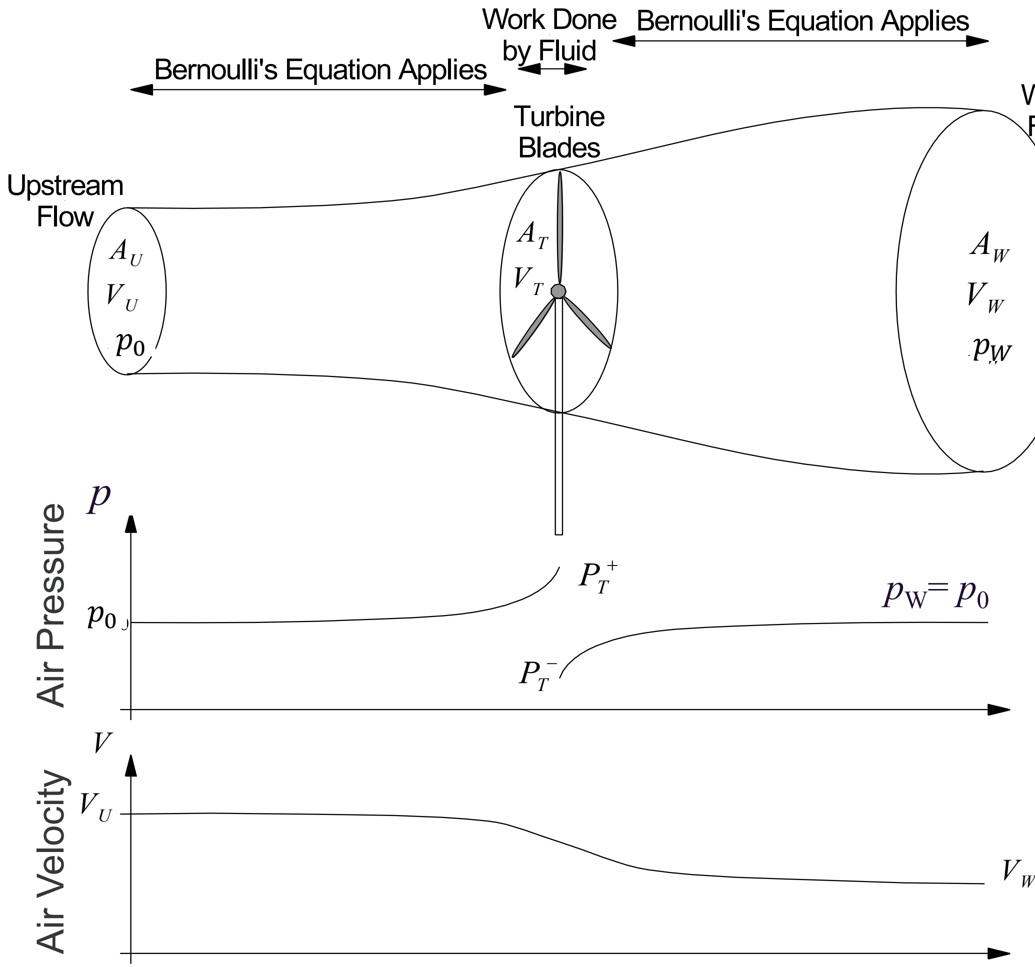
$$P = \bar{F} \cdot \bar{V}$$

When extracting energy from a fluid flow, these force and velocity are linked; they are not independent. The force acting on the airflow causes the velocity of the flow to decrease.

- If the force is too small, we will not generate much power
- If the force is too large, the velocity will become small, and again we will not generate much power
- There will be an optimal operating condition that maximises the product and the power, but what is that optimal condition?

The answer to the question on optimal power yield was published by Albert Betz in 1920 in "*Das Maximum der theoretisch möglichen Ausnutzung des Windes durch Windmotoren*" or "*Theoretical Limit for Best Utilization of Wind by Wind Motors*”.

An Idealised Wind Energy Converter



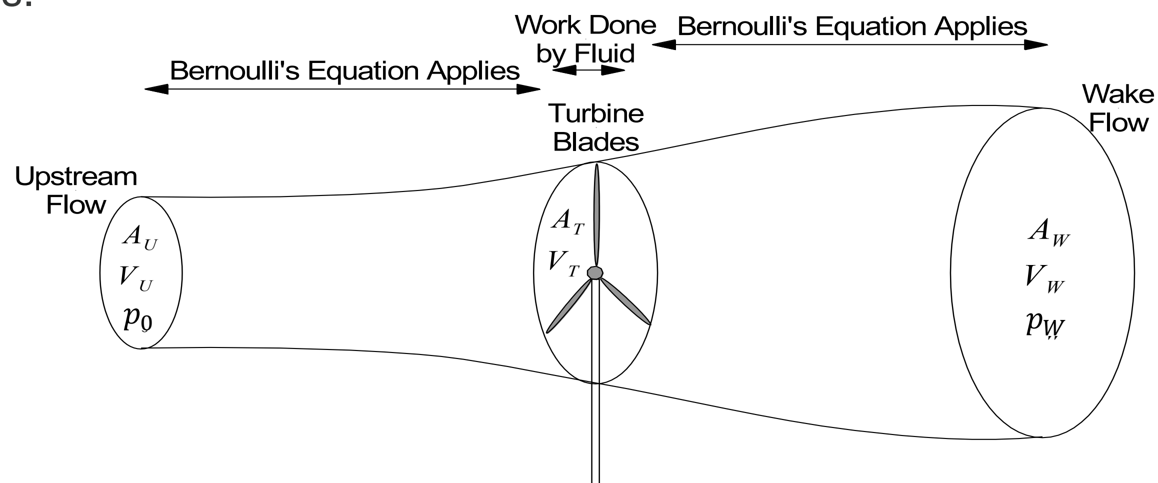
Note that here we have defined three different wind speeds:

- V_U is the speed upstream of the turbine. It is the general wind speed which is simply V elsewhere.
- V_T is the (average) speed as the air passes through the rotor.
- V_W is the speed of the air in the wake of the turbine.

- The turbine decelerates the airflow by applying a force on the air-mass and changing its momentum
- The force is applied through the build up of pressure ahead of the turbine but also a decrease in pressure behind the turbine.

Preservation of Mass Flow Rate

The "stream tube" shown here comprises all of the airflow that passes through the swept area of the blades.



Mass in a volume of air that travels down the stream tube is preserved:

$$m_U = m_T = m_W$$

$$\rho A_U x_U = \rho A_T x_T = \rho A_W x_W$$

x is the distance traveled by the air in unit time and depends on the flow velocity

Air density is taken to be constant $\rho_U = \rho_T = \rho_W = \rho$

Mass flow-rate is also preserved:

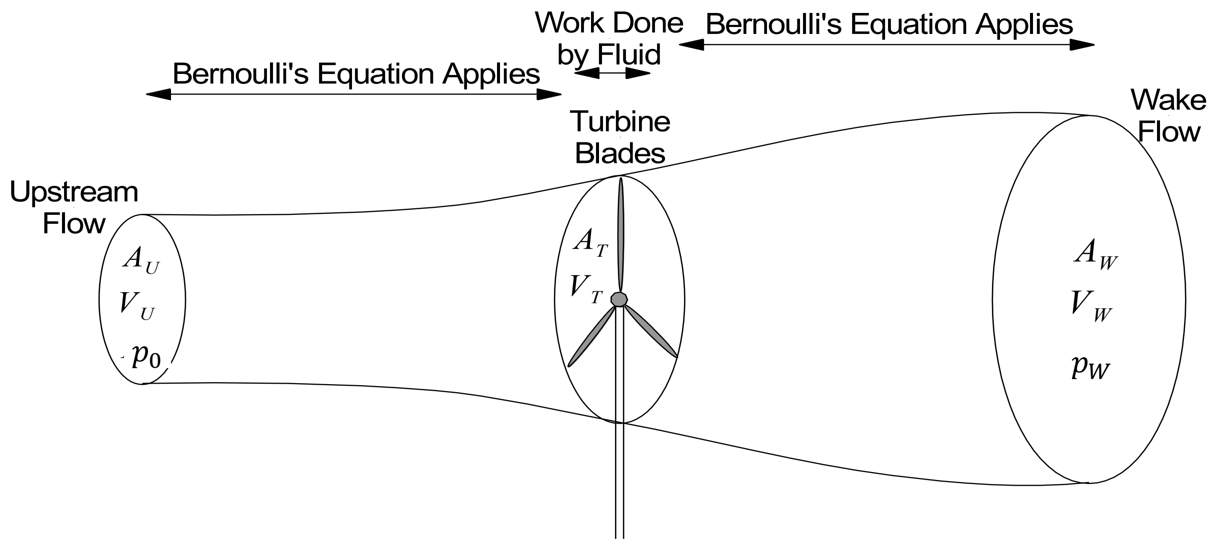
$$dm_U/_{dt} = dm_T/_{dt} = dm_W/_{dt}$$

$$\rho A_U V_U = \rho A_T V_T = \rho A_W V_W$$

The stream tube must become wider as the flow slows down because the mass-flow is preserved.

Regions of a Stream Tube

Bernoulli's principle states that an increase in the speed of a fluid occurs simultaneously with a decrease in static pressure (or a decrease in the fluid's potential energy).
This relates to the energy per unit volume being preserved in regions where no work is done on the fluid.



Three regions of the stream tube are defined,

(i) upstream where Bernoulli’s equation applies because the energy in the flow is constant,

(ii) the region close to the turbine where the flow does work on the turbine and looses energy

(iii) the wake where Bernoulli’s equation applies again.

Bernoulli's Principle

Bernoulli's principle can be expressed as:

$$\frac{1}{2}\rho V^2 + p + \rho g z = const$$

For a wind turbine, gravitational potential energy can be ignored since the height of the stream-tube does not change

When applying Bernoulli's principle to a wind turbine we must avoid the region of the turbine disc itself where work is done by the fluid on the rotor but we can apply it away from the turbine and express the exchange of energy between kinetic energy and pressure:

- Upstream $$\frac{1}{2}\rho V_U^2 + p_O = \frac{1}{2}\rho V_T^2 + p_T^+$$
- Downstream $$\frac{1}{2}\rho V_W^2 + p_O = \frac{1}{2}\rho V_T^2 + p_T^-$$

p_T^+ is the pressure immediately in front of the turbine and p_T^- is the pressure immediately behind the turbine .

Comparing Up and Down stream gives allows us to determine the pressure change over the turbine. The fluid does work as it flows through this pressure change:

$$p_T^+ - p_T^- = \frac{1}{2}\rho V_U^2 - \frac{1}{2}\rho V_W^2$$

The Energy in the Wind

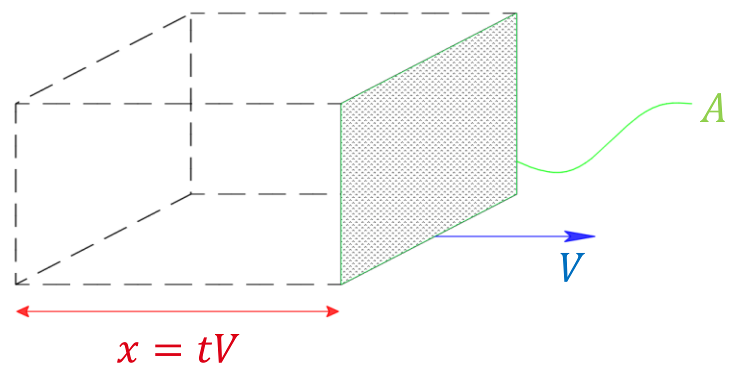
To extract this energy we need to decelerate the moving air. If we decelerated the air to standstill we could extract a power equal to the kinetic energy in the volume of air being processed per unit time.

$$E_W = \frac{1}{2}\rho A x V^2$$

$$P_W = \frac{d}{dt}E_W$$

$$= \frac{d}{dt}\left(\frac{1}{2}\rho A V t V^2\right)$$

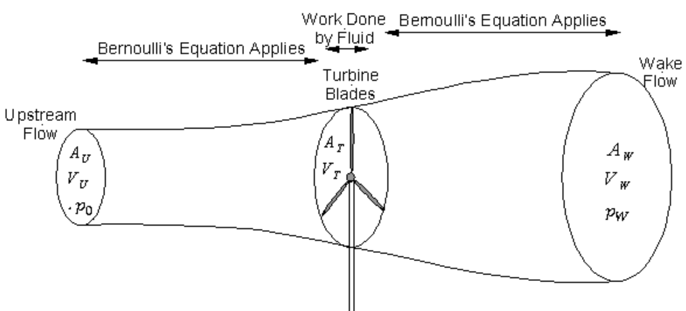
$$= \frac{1}{2}\rho A V^3$$



We define a “flow induction factor” that describes the fraction by which the air mass has slowed at the turbine with respect to the upstream flow.

$$a = \frac{V_U - V_T}{V_U}$$

We can also state $V_T = (1 - a)V_U$



Force and Change of Momentum

A force, F is exerted on the air-mass by the pressure difference across the turbine rotor disc is:

$$F = A_T(p_T^+ - p_T^-)$$

The force gives rise to a rate-of-change of momentum of the air mass given by:

$$F = \Delta\frac{d(mV)}{dt} = V_U(\rho A_U V_U) - V_W(\rho A_W V_W)$$

$$= \rho A_T V_T (V_U - V_W)$$

Equating the force due to pressure difference and the change in momentum

$$F = A_T(p_T^+ - p_T^-) = \rho A_T V_T (V_U - V_W)$$

Using the flow induction factor yields and equation for the pressure difference.

$$A_T(p_T^+ - p_T^-) = \rho A_T V_U (1 - a)(V_U - V_W)$$

$$p_T^+ - p_T^- = \rho V_U (1 - a)(V_U - V_W)$$

Combining Momentum Change and Bernoulli's Principle

- We now have two equations for pressure difference across the turbine: one from Bernoulli's Principle and one from the change in momentum.
- Equating these relationships allows the velocity in the wake, V_W, to be determined:

Momentum Change Bernoulli

$$p_T^+ - p_T^- = \rho V_U(1-a)(V_U - V_W) = \frac{1}{2}\rho(V_U^2 - V_W^2)$$

$$\rho V_U(1-a)(V_U - V_W) = \frac{1}{2}\rho(V_U - V_W)(V_U + V_W)$$

$$V_W = V_U(1-2a)$$

- The velocity in the wake, V_W can be then eliminated from the equation for pressure difference:

$$\begin{aligned} p_T^+ - p_T^- &= \big(V_U - V_U(1-2a)\big)\rho V_U(1-a) \\ &= 2\rho V_U^2 a(1-a) \end{aligned}$$

Power Extraction from a Moving Air-Mass

Here, the force and the velocity are aligned so can be simply multiplied without a scalar product

- Power can now be found in terms of flow induction factor:

$$P_T = F_T\, V_T$$
$$= A_T\,(p_T^+ - p_T^-)\, V_T$$
$$= 2\rho A_T V_U^3\ a(1-a)^2$$

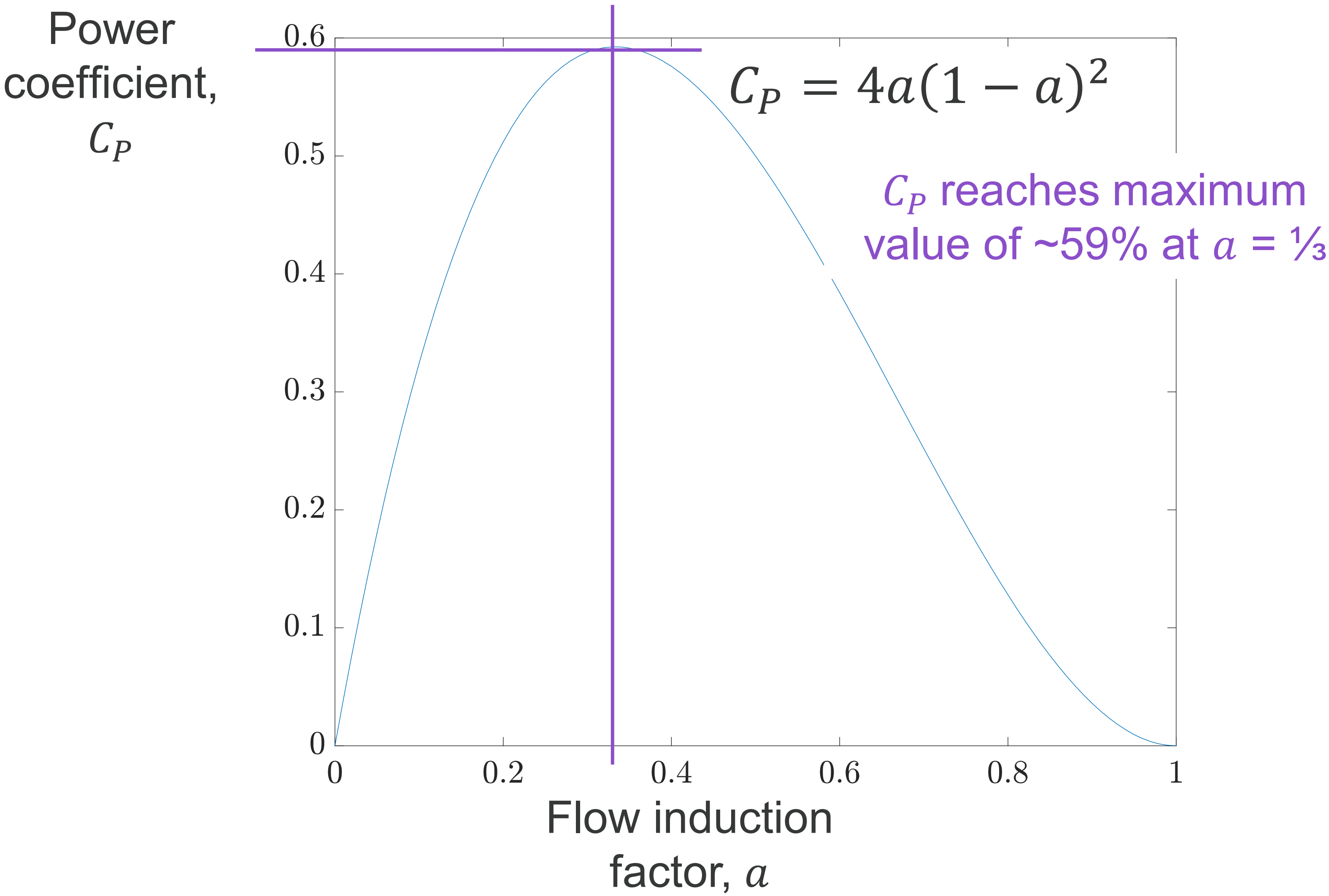
- The **Power Coefficient** is defined by comparing total power in kinetic energy of wind with the power extracted by the turbine:

$$C_P = {P_T}\Big/{½\rho A_T {V_U}^3}$$

- And the power coefficient expressed as a function of flow induction factor:

$$C_P = {2\rho A_T V_U^3\ a(1-a)^2}\Big/{½\rho A_T V_U^3}$$
$$= 4a(1-a)^2$$

Power coefficient as a function of flow induction



The Betz Limit on Power Extraction

- The maximum value of the power coefficient will occur at $\frac{d}{da}C_P = 0$

$$\frac{d}{da}C_P = \frac{d}{da}4\left(a - 2a^2 + a^3\right) = 4\left(1 - 4a + 3a^2\right) = 0$$

- Solving for a reveals that maximum power occurs at $a = ⅓$

- The maximum value of the power coefficient, which is the Betz Limit, is therefore:

$$\boxed{C_P^{Betz} = 4 \times \tfrac{1}{3}\left(1 - \tfrac{1}{3}\right)^2 = \tfrac{16}{27} \approx 59.3\%}$$

The power coefficient , C_P, as defined here accounts only for the aerodynamic efficiency of the actuator disk. We sometimes use $C_P$to refer to the power coefficient of a complete turbine accounting of various inefficiencies in the power train also.