REFLECTION GROUPS AND CONES OF SUMS OF SQUARES

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Abstract. We consider cones of real forms which are sums of squares forms and invariant by a (finite) reflection group. We show how the representation theory of these groups allows to use the symmetry inherent in these cones to give more efficient descriptions. We focus especially on the A_n, B_n, and D_n case where we use so called higher Specht polynomials [2] to give a uniform description of these cones. These descriptions allow us, for example, to study the connection of these cones to non-negative forms. In particular, we give a new proof of a result by Harris [20] who showed that every non-negative ternary even symmetric octic form is a sum of squares.

1. Introduction

A real form (homogeneous polynomial) $f \in \mathbb{R}[X_1, \ldots, X_n]$ is called a *sum of squares* if it admits a representation in the form $f = f_1^2 + \ldots + f_m^2$ for some real forms $f_1, \ldots, f_m \in \mathbb{R}[X_1, \ldots, X_n]$ and it is called *positive semi-definite* or *non-negative* if it assumes only non-negative values on $\mathbb{R}^n$. We will denote by $\Sigma_{n,2d}$ the cone of sums of squares forms in n variables of degree $2d$ and by $\mathcal{P}_{n,2d}$ the corresponding cone of non-negative forms. Clearly, every sum of squares is also non-negative, and we therefore have the inclusion $\Sigma_{n,2d} \subset \mathcal{P}_{n,2d}$. Hilbert [22] addressed and solved the question to characterise the cases, when the two cones coincide. As it turns out this only happens seldom, namely only in the case of bivariate forms ($n = 2$), quadratic forms ($2d = 2$), and ternary quartics ($n = 3, 2d = 4$). Sums of squares play a fundamental role in real algebraic geometry and have in the last two decades become also a very important tool for polynomial optimisation (see for example [36]). Several authors have considered situations in which one supposes that the forms are invariant by the action of a group: For a group $G \subset \mathrm{Gl}_n(\mathbb{R})$ we denote by $\mathcal{P}^G_{n,2d}$ and $\Sigma^G_{n,2d}$ the invariant forms in the respective cones. Since this additional requirement can shrink the dimensions of the cones their study may become more tractable. Furthermore, as presented in [18], representation theory of groups can be particularly used to simplify the sums of squares decomposition. Building on this, it was found in [35, 32] that sums of squares invariant by the symmetric group are highly structured and the complexity of a sum of squares decomposition in this case stabilised with $n > 2d$. Furthermore, symmetric sums of squares appear quite naturally in various contexts (for example [31]). This makes these cones an interesting object of study. Choi and Lam [10] initiated a systematic study of Hilbert's classification restricted to the case of symmetric forms and in a collaboration with Reznick they further provided a complete study of the cone of even symmetric sextics [11]. Whereas they could show that in the sextic case there exists a form which is non-negative but not a sum of squares Harris [20], who studied the case of even symmetric octics, was able to show that the cones of even symmetric octics that are sums of squares coincides with the non-negative cone. Recently, Goel, Kuhlmann and Reznick [19] constructed even symmetric polynomials of every degree $2d > 8$ and every number of variables $n > 3$ which are non-negative but not a sum of squares, so for even symmetric forms Harris' example remains the only exceptional case compared to Hilbert's classification. Despite the classical case analysis done by Hilbert it can also be interesting to study the quantitative comparison of sums of squares on non-negative polynomials in an asymptotic situation, i.e., when the number of variables grows to infinity. In contrary to the general situation, where for large numbers of variables almost every non-negative

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form is not a sum of squares (see [4]) a detailed analysize of the symmetric sum of squares cone and symmetric non-negative cone in [7] showed that this is not the case in the symmetric case and that in particular in the quartic case the two cones coincide in the limit.

In this article we study further the previously mentioned lines of research by focusing on the situation of sums of squares invariant by some families of finite irreducible real reflection groups $G \subset \mathrm{Gl}_n(\mathbb{R})$. Such groups are generated by a set of orthogonal reflections across hyperplanes passing through the origin. The invariant theory of these groups is well understood and generalizes the theory of symmetric polynomials. Therefore, our setup provides a natural unification and extensions to the previously mentioned works on symmetric and even symmetric forms.

Outline of the article and contributions: The beginning of the next section gives a short general introduction into the machinery of symmetry reduction for sums of squares based on linear representation theory. In the case of finite reflection groups these techniques combined with result from invariant theory, and in particular the covariant algebra and harmonic polynomials, allow for a concrete description of the qudratic module of invariant sums of squares in Theorem 2.22. The results we give in this second section are similar to previous works, notable [7, 16, 18, 42]. Section 3 then turns to the special situation of the three infinite families A_n, B_n and D_n of irreducible reflection groups for which we can integrate the notion of the higher Specht polynomials [2] with the previously mentioned techniques. These polynomial allow for a convenient way to combinatorially describe an isotypic decomposition of the coinvariant algebra in the case of finite reflection groups whose irreducible components fall to the classes A_n, B_n, D_n (see Theorem 3.9). As we show in Theorem 3.12 this combinatorial description then in turn implies a concrete characterization of the quadratic module of invariant sums of squares the cone of invariant sums of squares. In particular, we show in Theorem 3.21 that if the degree $2d$ is fixed and the number of variables n is growing, a stabilization of the isotypic decomposition and a resulting combinatorial stabilisation of the structure of the cone of invariant sums of squares is happening in the case of all three families.

Building on these general results we study the cone of even symmetric (i.e., B_n-invariant) forms of degree 8 in more detail in section 3.2. In Theorem 3.24 we obtain an explicit description of the dual cone of even symmetric octics, which we can use to we revisit the remarkable finding of Harris, which follow immediately from our description. Furthermore, we provide a complete description of the cone of even symmetric quartic sums of squares for all number of variables in Theorem 3.38. Following our discussion of even symetric forms we turn to forms that are D_n-invariant in subsection 3.3. We first show that Harris' remarkable equality for even symmetric ternary quartics remains valid for forms invariant by the slightly smaller group D_3 (see Theorem 3.42). We then examine the dual cone of D_4 invariant quartic sums of squares in Theorem 3.46, which turns out to be simpllicial. Similarly to our approach in the even-symmetric case this yields in particular that every D_4-invariant quarternary quartic non-negative form is a sum of squares. These results allow us to conclude complete charaterization of the cases in which for D_n invariant forms we have an equality between the cones of sums of squares and non-negative forms (see Theorem 3.50). To conclude our considerations we highlight some connections to non-negativity testing of forms with the help of semi-definite programming in the last subsection. Although it follows from recent works of Scheiderer [37] the cone of non-negative forms in general is not a so called spectrahedral shadow, i.e., it can in general not be represented by projections of feasibility sets of semi-definite programming, we can observe here that additionally to the cases where the cone of invariant sums of squares coincides with the corresponding cone of sums of squares, there are cases where we can represent the cone of non-negative forms by LMIs.

We remark that parts of the results presented here are also included in the Master thesis [15] written by the first author at Universität Wien under the supervision of the second author.

2. Invariant sums of squares

2.1. **General symmetry reduction.** Let $\underline{X} := (X_1, \ldots, X_n)$ always denote a tuple of variables and write $\mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \ldots, X_n] = \bigoplus_{d \in \mathbb{N}_0} H_{n,d}$ for the polynomial ring in these variables, where $H_{n,d}$ denotes the subspace of forms of degree d. Let $G \subset \mathrm{Gl}_n(\mathbb{R})$ be a reductive group acting linearly on $\mathbb{R}^n$. This action then naturally gives rise to an action of G on the polynomial ring $\mathbb{R}[X_1, \ldots X_n]$ and thus we can view this $\mathbb{R}$-vector space as a G-module. It follows from Maschke's theorem that this G-module is completely reducible and thus for any degree d there exists a decomposition of the form

$$H_{n,d} \;=\; V^{(1)} \oplus V^{(2)} \oplus \cdots \oplus V^{(h)} \tag{2.1}$$

with $V^{(j)} = \theta_1^{(j)} \oplus \cdots \oplus \theta_{\eta_j}^{(j)}$ and $\vartheta_j := \dim \theta_i^{(j)}$, i.e., we denote by η_j the *multiplicity* of a G-module and by ϑ_j its *dimension*. Here, the $\theta_i^{(j)}$ are the *irreducible components* and the $V^{(j)}$ are the *isotypic components*, i.e., the direct sum of isomorphic irreducible components. The component with respect to the trivial irreducible representation in $\mathbb{R}[\underline{X}]$ is the invariant ring $\mathbb{R}[\underline{X}]^G$. In general, an irreducible representation $\theta_i^{(j)}$ can occur with infinite multiplicity in $\mathbb{R}[\underline{X}]$. For $f \in \mathbb{R}[\underline{X}]$ we write $\langle f \rangle_G$ for the G-submodule that is G-generated by f. Any irreducible representation θ occurs $\dim \theta$ many times in the regular representation $\mathbb{R}[G]$ of G, i.e., $\vartheta = \eta$ for a representation θ in $\mathbb{R}[G]$.

It is classically known that $\mathbb{R}[\underline{X}]^G$ is a finitely generated $\mathbb{R}$-algebra and furthermore each isotypic component in $\mathbb{R}[\underline{X}]$ is a finitely generated $\mathbb{R}[\underline{X}]^G$-module (see [39, Theorem 1.3]). These properties follow for reductive groups from the existence of a linear projection onto $\mathbb{R}[\underline{X}]^G$, called the *Reynolds-Operator*. For finite groups this operator is simply given by normalized summation over the group elements:

Definition 2.1. *For a finite group G the linear map*

$$\begin{array}{rcl} \mathcal{R}_G : H_{n,d} & \longrightarrow & H_{n,d}^G \\ f & \longmapsto & \frac{1}{|G|} \sum_{\sigma \in G} \sigma(f) \end{array}$$

is called the Reynolds operator of G.

More general for compact groups the above definition generalizes naturally by using the *Haar measure* on G.

An important tool for the study of invariant sums of squares is Schur's lemma, which we include for the convenience of the reader.

Lemma 2.2 (Schur's lemma)**.** *Let $\mathbb{K}$ be field which is algebraically closed and V a G-module defined over $\mathbb{K}$, i.e., a linear representation of G given in $\mathrm{GL}_{\mathbb{K}}(V)$. Further let $\mathcal{V}, \mathcal{W} \subset V$ denote two irreducible G-submodules of V. Then the G-module $\mathrm{Hom}_G(\mathcal{V}, \mathcal{W})$ of G-homomorphisms between $\mathcal{V}$ and $\mathcal{W}$ satisfies*

$$\mathrm{Hom}_G(\mathcal{V}, \mathcal{W}) \cong \mathbb{K}$$

if and only if $\mathcal{V}$ and $\mathcal{W}$ are G-isomorphic. Otherwise $\mathrm{Hom}_G(\mathcal{V}, \mathcal{W}) = 0$.

Given be an irreducible G-module $\mathcal{V}$ G-generated by one element $f_1 \in \mathcal{V}$ and an irreducible G-module $\mathcal{W}$ G-isomorphic to $\mathcal{V}$. Then any G-homomorphism $\phi : \mathcal{V} \to \mathcal{W}$ is uniquely defined by $f_2 := \phi(f_1)$ and if further $\phi \neq 0$ then for any $\psi \in \mathrm{Hom}_G(\mathcal{V}, \mathcal{W})$ it is $\psi = \lambda\phi, \lambda \in \mathbb{K}$. This observation motivates the following.

Definition 2.3. *Let V be a finite dimensional G-module such that*

$$V = \bigoplus_{j=1}^{l} \bigoplus_{i=1}^{\eta_j} \mathcal{W}_{j,i},$$

where $\mathcal{W}_{j_1,i_1} \simeq \mathcal{W}_{j_2,i_2}$ if and only if $j_1 = j_2$. We call a G-basis $(f_{11}, \dots, f_{1\eta_1}, f_{21}, \dots, f_{l\eta_l})$ of V a symmetry adapted basis of the G-module V if for every j each f_{ji} is the image of one fixed f_{j1} under a G-isomorphism (which is unique up to scalar multiplication).

Remark 2.4. *In the sequel we will mostly work with G-modules defined over the real numbers. In this setup one devotes some care to the fact that irreducible representations defined over the reals may be reducible over the complex numbers. This additional difficulty is in fact not hard to overcome and, in particular, in the case of real reflection group, which are the main focus of this work, all the irreducible representations are constructible over the reals. Therefore all complexifications of real irreducible G-modules remain irreducible.*

By integrating the idea of a symmetry adapted basis together with Schur's lemma one arrives at the following observation more or less directly (see also [7, 13, 18, 35] for more details on the following statement).

Theorem 2.5. *Let $\{f_{11}, \dots, f_{1\eta_1}, f_{21}, \dots, f_{l\eta_l}\}$ be a symmetry adapted basis for the space $H_{n,d}$ of forms of degree d. Then any G-invariant sum of squares form $g \in H^G_{n,2d}$ can be written as*

$$g = \mathcal{R}_G \left(\sum \mathbb{R}\{f_{11}, \dots, f_{1\eta_1}\}^2 + \dots + \sum \mathbb{R}\{f_{l1}, \dots, f_{l\eta_l}\}^2 \right),$$

where $\mathbb{R}\{g_1, \dots, g_m\}^2$ denotes an element of the form $\sum_{j=1}^{l} c_j \left(\sum_{i=1}^{m} c_{ji} g_i\right)^2$ for some $c_j \in \mathbb{R}_{\geq 0}$ and $c_{ji} \in \mathbb{R}$.

In some situations it is convenient to formulate Theorem 2.5 in terms of matrix polynomials, i.e., matrices with polynomial entries. Given two $k \times k$ symmetric matrices A and B define their inner product as $\langle A, B \rangle = \mathrm{trace}(AB)$. We define a block-diagonal symmetric matrix B with j blocks $B^{(1)}, \dots, B^{(j)}$ with the entries of each block given by:

$$B^{(j)} = (\mathcal{R}_G(f_{jk} f_{jl}))_{k,l}. \tag{2.2}$$

Then Theorem 2.5 is equivalent to the following statement:

Corollary 2.6. *Let $g \in H^G_{n,2d}$. Then $g \in \Sigma^G_{n,2d}$ if and only if*

$$g = \mathrm{Tr}\left(A_1 \cdot B^{(1)}\right) + \dots + \mathrm{Tr}\left(A_l \cdot B^{(l)}\right),$$

where $A_j \in \mathrm{Mat}_{\eta_j \times \eta_j}(\mathbb{R})$ are symmetric and positive semidefinite.

2.2. **Representation theory of finite reflection groups.** The aim of this subsection is to provide an introduction in the representation theory of finite real reflection groups and how their symmetry can be exploited to reduce complexity in calculations. The presented material is mainly based on work in [18, 16, 7].

Definition 2.7. *A real reflection group is a pair (G, ρ), where G is a finite group, V a finite dimensional $\mathbb{R}$-vector space and $\rho : G \to \mathrm{Gl}(V)$ a linear representation of G such that $\rho(G)$ is generated by a set of reflections, i.e., linear maps of the form*

$$\begin{array}{rccc} \sigma_a : & V & \longrightarrow & V \\ & v & \longmapsto & v - 2\frac{\langle v, a\rangle}{\langle a, a\rangle} a \end{array}$$

for some $a \in V \setminus \{0\}$. A reflection group is called essential, if the action of G on V does not contain a non-trivial G-submodule.

Usually, we will just say that a group G is a reflection group and the relevant linear map ρ should be clear from the context. An action of G on $\mathbb{R}^n$ induces naturally an action on the polynomial ring in n variables via $\sigma f(\underline{X}) = f(\sigma^{-1}\underline{X})$ for $\sigma \in G$ and $f \in \mathbb{R}[\underline{X}]$.

Example 2.8. (i) The symmetric group $\mathfrak{S}_n$ on n letters is a reflection group acting via coordinate permutation on $\mathbb{R}^n$. The action of $\mathfrak{S}_n$ on $\mathbb{R}^n$ is not essential, as the linear subspace $\mathbb{R}\cdot(1,\ldots,1)$ is fixed point wise. The induced action of $\mathfrak{S}_n$ on $\mathbb{R}^n/\mathbb{R}\cdot(1,\ldots,1)$ is known as the reflection group of type A_{n-1} and is essential.

(ii) The symmetry group of the regular m-gon is a reflection group, called the dihedral group and denoted by $I_2(m)$.

Remark 2.9. *Any real reflection group can be identified with a direct product of essential reflection groups. The essential real reflection groups have been classified and are precisely the infinite series $A_{n-1}, B_n, D_n, I_2(m)$ and the six exceptional reflection groups $E_6, E_7, E_8, F_4, H_3, H_4$ (see e.g. [23]).*

The reflection group of type B_n can be identified with the hyperoctahedral group $\mathfrak{S}_2 \wr \mathfrak{S}_n$ acting on $\mathbb{R}^n$ via sign changing and permutation of coordinates. Then B_n is generated by the reflections at $\{X_i = \pm X_j\}$, for $1 \le i \le j \le n$. Furthermore, D_n can be identified with the subgroup of B_n of index 2, generated by the reflections at $\{X_i = \pm X_j\}$, for $1 \le i < j \le n$.

Theorem 2.10 (Chevalley-Shephard-Todd theorem)**.** *Let G be a finite group and let G act linear on $\mathbb{R}^n$. Then the invariant ring $\mathbb{R}[\underline{X}]^G$ is as $\mathbb{R}$-algebra isomorphic to a polynomial ring if and only if G is a real reflection group. Moreover, in this case $\mathbb{R}[\underline{X}]^G$ is generated by n algebraically independent forms $\psi_1,\ldots,\psi_n$.*

While the generators are not unique but well explored (e.g. the elementary symmetric polynomials or the power sums are generators for the symmetric group), the sets containing their degrees $\{d_1,\ldots,d_n\}$ are unique and $\prod_i d_i = |G|$ (consult e.g. [23] for further details).

Definition 2.11. *Let G be a finite reflection group and $(d_1,\ldots,d_n)$ the sequence of degrees of the fundamental invariants. Then, we define*

$$N_G(k) := |\{(\alpha_1,\ldots,\alpha_n) \in \mathbb{N}_0^n \,:\, \alpha_1 d_1 + \ldots + \alpha_n d_n = k\}|.$$

With this definition the following is a direct consequence of Theorem 2.10.

Corollary 2.12. *Let G be a finite reflection group. Then the dimension of the vector space of G-invariant forms of degree d equals $N_G(d)$, i.e., $\dim H^G_{n,d} = N_G(d)$.*

Example 2.13. (i) $\mathbb{R}[\underline{X}]^{\mathfrak{S}_n} = \mathbb{R}[e_1,e_2,\ldots,e_n] = \mathbb{R}[p_1,p_2,\ldots,p_n]$, where $e_j(\underline{X}) := \sum_{I\subset[n]:|I|=j}\prod_{i\in I} X_i$ are the elementary symmetric and $p_j(\underline{X}) := \sum_{i=1}^n X_i^j$ are the power sum polynomials.

(ii) $\mathbb{R}[\underline{X}]^{B_n} = \mathbb{R}[e_1(\underline{X}^2), e_2(\underline{X}^2),\ldots,e_n(\underline{X}^2)] = \mathbb{R}[p_2,p_4,\ldots,p_{2n}]$, where $\underline{X}^2 := (X_1^2,\ldots,X_n^2)$.

(iii) $\mathbb{R}[\underline{X}]^{D_n} = \mathbb{R}[p_2,p_4,\ldots,p_{2n-2},e_n]$.

(iv) $\mathbb{R}[\underline{X}]^{I_2(m)} = \mathbb{R}[X_1^2+X_2^2, (X_1+\sqrt{-1}X_2)^m + (X_1-\sqrt{-1}X_2)^m]$.

Remark 2.14. *For $\lambda := (\lambda_1,\ldots,\lambda_l) \in \mathbb{N}^l$ we often write $p_\lambda := p_{(\lambda_1,\ldots,\lambda_l)} := p_{\lambda_1}\cdots p_{\lambda_l}$ for the l products of the power sums p_{λ_i}.*

From an computational perspective, invariant theory as outlined above can be used to reduce computations for polynomials in $\mathbb{R}[\underline{X}]$ to the smaller ring $\mathbb{R}[\underline{X}]^G$. Note however, that an invariant polynomial which can be expressed as a sum of squares in the ring $\mathbb{R}[\underline{X}]$ will not necessarily have a sum of squares decomposition in invariant polynomials, i.e.,

$$\mathbb{R}[\underline{X}]^G \bigcap \sum \mathbb{R}[\underline{X}]^2 \neq \sum (\mathbb{R}[\underline{X}]^G)^2.$$

However, since $\mathbb{R}[\underline{X}]$ is in general a finite $\mathbb{R}[\underline{X}]^G$- module, the quadratic module $\mathbb{R}[\underline{X}]^G \bigcap \sum \mathbb{R}[\underline{X}]^2$ can be described quite conveniently. We outline this in the case of reflection groups below, using the convariant algebra and a Theorem of Chevalley. Although we only focus on this more restrictive setup, it is worth mentioning that this can be done in a more general setup for all compact groups.

Definition 2.15. *Let G be a reflection group acting linear on $\mathbb{R}^n$ and $\mathbb{R}[\underline{X}]^G = \mathbb{R}[\psi_1, \ldots, \psi_n]$. We call the forms $\psi_1, \ldots, \psi_n$ the fundamental invariants of G. The quotient $\mathbb{R}$-algebra of the polynomial ring modulo the ideal generated by the non constant elements of the invariant ring is called the coinvariant algebra of G and denoted by $\mathbb{R}[\underline{X}]_G$, i.e.,*

$$\mathbb{R}[\underline{X}]_G := \mathbb{R}[\underline{X}]/\left(\psi_1, \ldots, \psi_n\right)_{\mathbb{R}[\underline{X}]}.$$

In particular, the coinvariant algebra of G has also the structure of a finite dimensional real vector space and G-module.

Theorem 2.16. [24] *Let G be a real reflection group acting linear on $\mathbb{R}^n$. Then the coinvariant algebra $\mathbb{R}[\underline{X}]_G$ is as G-module isomorphic to the regular representation $\mathbb{R}[G]$ and*

$$\mathbb{R}[\underline{X}] \cong \mathbb{R}[\underline{X}]^G \otimes_{\mathbb{R}} \mathbb{R}[\underline{X}]_G$$

as graded $\mathbb{R}$-algebras.

When we speak in the following about elements in $\mathbb{R}[\underline{X}]_G$ or a basis of the coinvariant algebra, we mean representing elements in $\mathbb{R}[\underline{X}]$ that are reduced with respect to the ideal $(\psi_1, \ldots, \psi_n)_{\mathbb{R}[\underline{X}]}$.

Corollary 2.17. *Let $\mathbb{R}[\underline{X}]^G = \mathbb{R}[\psi_1, \ldots, \psi_n]$ be a polynomial ring in the fundamental invariants $\psi_1, \ldots, \psi_n$. Let $\mathbb{R}[\underline{X}]_G = \bigoplus_{j=1}^{l} \eta_j \theta_j$ be the isotypic decomposition of the coinvariant algebra. Then there exists a symmetry adapted basis $f_{11}, \ldots, f_{l\eta_l} \in \mathbb{R}[\underline{X}]$ of $\mathbb{R}[\underline{X}]_G$ such that any $f \in \mathbb{R}[\underline{X}]$ can be written as*

$$f = \sum_{j=1}^{l} \sum_{i=1}^{\eta_j} \sum_{\sigma \in G} g_{ji,\sigma} \sigma f_{ji},$$

where $g_{ji,\sigma} \in \mathbb{R}[\underline{X}]^G$.

Proof. By Schur's lemma 2.2, the use of a symmetry adapted basis 2.3 and the graded tensor decomposition of the polynomial ring into the invariant ring and the coinvariant algebra by Theorem 2.16. □

The second sum in the representation of a polynomial in Corollary 2.17 goes up to η_j. We point out that the multiplicity η_j of an irreducible representation $\mathcal{V}_j$ in the coinvariant algebra equals its dimension ϑ_j. This is due to the fact that the coinvariant algebra is as G-module G-isomorphic to the regular representation of G by Theorem 2.16 for a reflection group G.

Remark 2.18. *For a reflection group G the calculation of one symmetry adapted basis of the coinvariant algebra allows easily the computation of the isotypic composition of the G-module $H_{n,d}$ for any degree. One computes all the products of elements from the symmetric adapted basis with fundamental invariants of G, such that the degree of the obtained homogeneous polynomial equals d.*

Definition 2.19. *Let $S := \{s_1, \ldots, s_{|G|}\}$ be a basis of $\mathbb{R}[\underline{X}]_G$. Then we define the matrix polynomial $H^S(z_1, \ldots, z_n) \in \mathbb{R}[z]^{|G| \times |G|}$ to be*

$$H^S_{i,j} := R_G(s_i \cdot s_j),$$

where we express each entry $R_G(s_i \cdot s_j)$ in terms of the fundamental invariants $\psi_1, \ldots, \psi_n$.

Lemma 2.20. *Let $f \in \mathbb{R}[\underline{X}]$ be G-invariant and let $\gamma \in \mathbb{R}[z]$ with $\gamma(\psi_1, \dots, \psi_n) = f$ then f is a sum of squares if and only if $\gamma(\psi_1, \dots, \psi_n)$ admits a representation of the form*

$$\gamma = \mathrm{Tr}(G(z) \cdot H^S(z)),$$

where $G(z)$ is a sum of squares matrix polynomial, i.e., $G(z) = L(z)^t L(z)$ for some $L(z) \in \mathbb{R}[z_1, \dots, z_n]^{n \times m}$ for some $1 \leq m \leq n$.

Proof. This follows from the decomposition $\mathbb{R}[\underline{X}] \cong \mathbb{R}[\underline{X}]^G \otimes \mathbb{R}[\underline{X}]_G$ in Theorem 2.16. $\square$

Now, since in the case of finite reflection groups we have that $\mathbb{R}[\underline{X}]_G$ is G-isomorphic to the regular representation, it follows that we can find a basis which is symmetry adapted. This allows the following

Definition 2.21. *For every irreducible representation θ_j of G we can construct a matrix polynomial $H^{\vartheta_j}(z) \in \mathbb{R}[z]^{\eta_j \times \eta_j}$ in the following way: Let $\mathbb{R}[\underline{X}]_G = \bigoplus_{i=1}^{l} \mathbb{R}[\underline{X}]_G^{\vartheta_j}$ be the isotypic decomoposition of the coinvariant algebra and further $\{s_{1,1}, \dots, s_{1,\eta_1}, s_{2,1}, \dots, s_{l,\eta_l}\}$ be a symmetry adapted basis of $\mathbb{R}[\underline{X}]_G$. Then we define*

$$H_{u,v}^{\vartheta_j} = R_G(s_{j,v} \cdot s_{j,u}).$$

Combining above definition and lemma, and the results from Schur's lemma we immediately get

Theorem 2.22. *Let G be a finite reflection group with $\mathbb{R}[\underline{X}]^G = \mathbb{R}[\psi_1, \dots, \psi_n]$, then we have*

$$\Sigma\mathbb{R}[\underline{X}]^2 \cap \mathbb{R}[\underline{X}]^G = \left\{ g \in \mathbb{R}[\psi_1, \dots, \psi_n] \ : \ g = \sum_j \mathrm{Tr}(H^{\vartheta_j} \cdot A_j) \right\},$$

where $A_j \in \mathbb{R}[\psi_1, \dots, \psi_n]^{\eta_j \times \eta_j}$ is a sums of squares matrix polynomial.

Example 2.23. Let $f \in \mathbb{R}[X_1, X_2]$ be a homogeneous polynomial of degree $2d$ which is invariant by a dihedral group $I_2(k)$. The dihedral group $I_2(k)$ has only irreducible representations of dimension 1 or 2. In fact, if k is odd (resp. even), then 2 (resp. 4) representations of dimension one and $\frac{k-1}{2}$ (resp. $\frac{k-2}{2}$) representations of dimension two. By block-diagonalisation we end up with $H^S(z)$ and $G(z)$ having 2 (resp. 4) 1×1 blocks $H^{\theta_1}, H^{\theta_2}$ (resp. $H^{\theta_1}, \dots, H^{\theta_4}$) and $\frac{k-1}{2}$ (resp. $\frac{k-2}{2}$) 2×2 blocks $H^{\theta_3}, \dots, H^{\theta_{\frac{k+3}{2}}}$ (resp. $H^{\theta_5}, \dots, H^{\theta_{\frac{k+6}{2}}}$). Then for n odd (resp. even) $f \geq 0$ if and only if there exist sums of squares matrix polynomials $A_i \in \mathbb{R}[X_1^2 + X_2^2, (X_1 + \sqrt{-1}X_2)^k + (X_1 - \sqrt{-1}X_2)^k]^{\dim \theta_j \times \dim \theta_j}$ such that

$$f = \sum_{j=1}^{m} \mathrm{Tr}\left(H^{\theta_j} \cdot A_j\right),$$

where $m = \frac{k+3}{2}$ (resp. $m = \frac{k+6}{2}$).

Definition 2.24. *Let G be a finite reflection group and θ an irreducible representation. We write h_k^{ϑ} for the multiplicity of θ in $(\mathbb{R}[\underline{X}]_G^{\theta})_k$, i.e., the multiplicity of θ in the isotypic decomposition of the subspace of forms of degree k in the coinvariant algebra.*

Corollary 2.25. *Let G be a finite reflection group and θ be an irreducible representation. Then the multiplicity of the corresponding irreducible representation in the G-module $H_{n,d}$ equals*

$$\sum_{k=0}^{d} N(d-k) \cdot h_k^{\vartheta}.$$

2.3. **G-harmonic polynomials.** In this subsection we present a specific basis of the coinvariant algebra for reflection groups which can be simply computed.

Definition 2.26. *For a polynomial $f = \sum_\alpha c_\alpha \underline{X}^\alpha \in \mathbb{R}[\underline{X}]$ we denote by $f(\partial)$ the linear operator*

$$\begin{array}{rccc} f(\partial): & \mathbb{R}[\underline{X}] & \longrightarrow & \mathbb{R}[\underline{X}] \\ & g & \longmapsto & \sum_\alpha c_\alpha \frac{\partial^\alpha}{(\partial \underline{X})^\alpha} g \end{array},$$

i.e., $f(\partial)$ is the formal sum of scaled partial derivatives considered as a linear map.

Example 2.27. Let $f = X_1^2 + X_1X_2 \in \mathbb{R}[X_1, X_2, X_3]$, then $f(\partial) = \frac{\partial^2}{\partial X_1 \partial X_1} + \frac{\partial^2}{\partial X_1 \partial X_2}$ and

$$f(\partial)\left(X_1^2 + X_2^2 + X_3^2 + X_1X_2X_3\right) = 1 + X_3.$$

Definition 2.28. *Let G be a real reflection group and $\mathbb{R}[\underline{X}]^G = \mathbb{R}[\psi_1, \psi_2, \ldots, \psi_n]$. We define the $\mathbb{R}$-vector space of harmonic polynomials $\mathcal{H}_G := \left(\mathbb{R}[\underline{X}]^G\right)^\perp$, with respect to the scalar product on $\mathbb{R}[\underline{X}]$ given by*

$$\begin{array}{rccc} \langle \cdot, \cdot \rangle: & \mathbb{R}[\underline{X}] \times \mathbb{R}[\underline{X}] & \longrightarrow & \mathbb{R}[\underline{X}] \\ & (f, g) & \longmapsto & \mathrm{ev}_{(0,\ldots,0)}\left(f(\partial) g(\underline{X})\right) \end{array}.$$

Theorem 2.29. [3] *Let G be a real reflection group and $\Delta := \prod L_i$, be the product of the linear polynomials defining the reflection hyperplanes. Then, the vector space of G-harmonic polynomials $\mathcal{H}_G$ is generated by all partial derivatives of Δ, i.e., $\mathcal{H}_G = \langle \frac{\partial^\alpha}{\partial x^\alpha} \Delta : \alpha \in \mathbb{N}_0^n \rangle_\mathbb{R}$. Furthermore, $\mathcal{H}_G$ is as G-module isomorphic to the regular representation of G and $\mathbb{R}[\underline{X}] = \mathbb{R}[\psi_1, \ldots, \psi_n] \otimes_\mathbb{R} \mathcal{H}_G$.*

Remark 2.30. *Let G be a finite reflection group and $\psi_1, \ldots, \psi_n$ generators of the invariant ring. Consider the map*

$$\begin{array}{rccc} \Psi: & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & \underline{X} & \longmapsto & (\psi_1(\underline{X}), \ldots, \psi_n(\underline{X})) \end{array}.$$

Then, thanks to a statement of Steinberg in [40] we have

$$\Delta = c \cdot \mathrm{Jac}\ \Psi,$$

where $c \in \mathbb{R} \setminus \{0\}$ and $\mathrm{Jac}\ \Psi$ denotes the Jacobian matrix of Ψ. The choice of fundamental invariants $\psi_1, \ldots, \psi_n$ does not matter.

Example 2.31. For $\mathfrak{S}_n$ the symmetric group acting on $\mathbb{R}^n$ via coordinate permutation and $\psi_i := p_i = \sum_{j=1}^n X_j^i$ the power sums, we obtain $\Delta = \prod_{i<j}(x_i - x_j)$ equals the determinant of the Vandermonde matrix. Δ is the Jacobian of Ψ, which is precisely the product over all reflections $\{X_i = X_j\}$ of $\mathfrak{S}_n$.

Remark 2.32. *Computing a basis of the coinvariant algebra $\mathbb{R}[\underline{X}]_G = \mathbb{R}[\underline{X}]/\mathbb{R}[\underline{X}]_{>0}^G$, that is defined as a quotient space, is highly complex and involves the calculation of a Groebner basis. However, the approach using harmonic polynomials is more efficient because it is based on linear algebra for given fundamental invariants. As the fundamental invariants of real reflection groups are well-known one can calculate the polynomial Δ and all its partial derivatives explicitly.*

2.4. **Convex geometric properties of $\Sigma^G, \mathcal{P}^G$.** An interesting and highly useful feature of $\Sigma_{n,2d}^G$ and $\mathcal{P}_{n,2d}^G$ is their *convex geometry*, which enables the use of convex geometric techniques to study these sets. In the research on non-negativity versus sums of squares have the convex cones and their dual cones been studied intensively (see e.g. Blekherman's work in [5] on Hilbert's inequality cases or [6]). The methods in this paper make it possible to computationally verify equality or non-equality between $\Sigma_{n,2d}^G$ and $\mathcal{P}_{n,2d}^G$, and to calculate explicitly the boundary of the invariant sums of squares cone. In this subsection we present known and adapted knowledge on the convex geometric properties of $\Sigma_{n,2d}^G$ and $\mathcal{P}_{n,2d}^G$.

Remark 2.33. *We always identify two maps $\ell_1, \ell_2 : V \to W$, if $\ell_1(f) = \ell_2(f)$ for all $f \in V$.*

Highly relevant linear functionals on $H_{n,2d}^G$ are the point-evaluations:

Definition 2.34. *Let $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ be a point. We write $\mathrm{ev}_a \in \mathbb{R}[\underline{X}]^*$ for the point-evaluation of polynomials in a, i.e.,*

$$\begin{array}{rcl} \mathrm{ev}_a : \mathbb{R}[\underline{X}] & \longrightarrow & \mathbb{R} \\ f(\underline{X}) & \longmapsto & f(a) \end{array}.$$

The sets $\Sigma_{n,2d}^G$ and $\mathcal{P}_{n,2d}^G$ are *convex cones*, i.e., they are convex sets which are closed under scalar multiplication by non-negative scalars. Moreover, these sets are closed and *pointed* (i.e. they do not contain a non-trivial linear subspace) (see e.g. [4]). Such convex cones are called *proper*.

Definition 2.35. *Let $K \subset \mathbb{R}^N$ be a proper convex cone. The dual cone K^* is defined as*

$$K^* := \left\{ \ell \in \mathrm{Hom}\left(\mathbb{R}^N, \mathbb{R}\right) : \ell(P) \subseteq \mathbb{R}_{\geq 0} \right\}.$$

A facet F of K is a maximal face of K.
Let $\mathcal{A}$ be the vector space of real quadratic forms on $H_{n,d}$. An element $Q \in \mathcal{A}$ is said to be G-invariant if $Q(f) = Q(\sigma f)$ for all $\sigma \in G$. We denote by $\mathcal{A}^G$ the space of G-invariant quadratic forms on $H_{n,d}$. We identify a linear functional $\ell \in H_{n,2d}^G$ with an associated quadratic form Q_ℓ defined by

$$\begin{array}{rcl} Q_\ell : H_{n,d} & \longrightarrow & \mathbb{R} \\ f & \longmapsto & \ell\left(\mathcal{R}_G(f^2)\right) \end{array}.$$

We write $\mathcal{A}^{G,+}$ for the cone of positive semidefinite forms in $\mathcal{A}^G$, i.e.,

$$\mathcal{A}^{G,+} := \{Q \in \mathcal{A}^G \,:\, Q(f) \geq 0 \text{ for all } f \in H_{n,d}\}.$$

Any facet F of a proper convex cone K is the intersection of the kernel of an extremal element (an element which spans an extremal ray) in K^* with K, i.e., $F = K \cap \ker \ell$ for an $\ell \in K^*$ contained in an extremal ray.

The following lemma is the dual version of Theorem 2.5.

Lemma 2.36. *Let $\ell \in \left(H_{n,2d}^G\right)^*$ and $\{f_{11}, \ldots, f_{1\eta_1}, f_{21}, \ldots, f_{l\eta_l}\}$ be a symmetry adapted basis for the space $H_{n,d}$ of forms of degree d and $B^{(j)} = (\mathcal{R}_G(f_{jk}f_{jl}))_{k,l}$. Then ℓ is contained in $\left(\Sigma_{n,2d}^G\right)^*$ if and only if $\ell(B_j)$ is positive semidefinite for all $j = 1, \ldots, l$.*

Proof. Assume that for a $j \in \{1, \ldots, l\}$ the matrix $\ell(B_j)$ is not positive semidefinite, i.e., for some $w \in \mathbb{R}^m$ it is $\mathrm{Tr}(\ell(B_j)ww^t) = w^t\ell(B_j)w < 0$. The form $f := \mathrm{Tr}(ww^tB_j) \in H_{n,2d}$ is an invariant sum of squares contained in $\Sigma_{n,2d}^G$ by Theorem 2.5. However, $\ell(f) < 0$ and thus $\ell \notin \left(\Sigma_{n,2d}^G\right)^*$.
Conversely, we assume that $\ell \notin \left(\Sigma_{n,2d}^G\right)^*$, i.e., $\ell(f) < 0$ for some $f = \sum_{j=1}^l \mathrm{Tr}(A^{(j)}B^{(j)})$ and positive semidefinite matrices $A^{(j)}$. After relabelling we can assume that $\ell\left(\mathrm{Tr}(A^{(1)}B^{(1)})\right) < 0$. Since $A^{(1)}$ is positive semidefinite there exist $w_1, \ldots, w_k \in \mathbb{R}^m$ such that $A^{(1)} = \sum_{j=1}^k w_j^t w_j$. This implies that $0 > \sum_{j=1}^m w_j^t \ell(B^{(1)}) w_j$. Thus $\ell(B^{(1)})$ cannot be positive semidefinite. □

The following lemma enables the characterisation of extremal elements via its kernels.

Lemma 2.37. [5, Lemma 2.2] *Let V be a $\mathbb{R}$-vector space, $\mathcal{A}$ the vector space of quadratic forms on V and $\mathcal{A}^+ \subset \mathcal{A}$ the cone of positive semidefinite quadratic forms. Let L be a linear subspace of $\mathcal{A}$ and K be the section of $\mathcal{A}^+$ with L, i.e., $K := \mathcal{A}^+ \cap L$. Then a quadratic form $Q \in K$ spans an extreme ray of K if and only if its kernel is maximal for all forms in L, i.e. if $\ker Q \subseteq \ker P$ for a $P \in L$, it is $P = \lambda Q$ for some $\lambda \in \mathbb{R}$.*

In order to examine the kernels of quadratic forms we use the following construction. For a linear subspace $W \subset H_{n,d}$ we define its quadratic symmetrization w.r.t. G as

$$W^{<2>} := \left\{ h \in H^G_{n,2d} \,:\, h = \mathcal{R}_G \left(\sum f_i g_i \right) \text{ with } f_i \in W \text{ and } g_i \in H_{n,d} \right\}.$$

With the notations from above the following straightforward proposition can be used to characterize the elements in the kernels of the quadratic forms that we consider.

Proposition 2.38. *Let Q_ℓ be a quadratic form in $(\Sigma^G_{n,2d})^*$ and let $W_\ell \subset H_{n,d}$ be the kernel of Q_ℓ. Then $p \in W_\ell$ if and only if $\ell\left(\mathcal{R}_G(pq)\right) = 0$ for all $q \in H_{n,d}$. In particular $\ell(f) = 0$ for all $f \in {W_\ell}^{<2>}$.*

Proof. This follows from the positive semidefiniteness of the quadratic form Q_ℓ. $\square$

Since we are in the homogeneous case we have the following description of the dual cones:

Proposition 2.39. [4] *The dual cone of the non-negative invariant forms is the convex cone that is generated by all point-evaluations, i.e.,*

$$\left(\mathcal{P}^G_{n,2d}\right)^* = \operatorname{cone}\{\operatorname{ev}_a : a \in \mathbb{S}^{n-1}\}$$

where $\operatorname{ev}_a(f) = f(a)$. By duality any $f \in \mathcal{P}^G_{n,2d}$ contained in the boundary has a real projective zero.

In order to characterize the extreme rays of $\left(\Sigma^G_{n,2d}\right)^*$ we use Lemma 2.36 to identify the dual cone $\left(\Sigma^G_{n,2d}\right)^*$ with a linear section of the cone of positive semidefinite forms with the subspace $\mathcal{A}^G$ of G-invariant quadratic forms on $H_{n,d}$.

Lemma 2.40. *Let*

$$\begin{array}{rccc} \iota: & \left(H^G_{n,2d}\right)^* & \longrightarrow & \mathcal{A}^G \\ & \ell & \longmapsto & Q_\ell := (f \mapsto \ell(\mathcal{R}_G(f^2))) \end{array},$$

where $\mathcal{A}^G$ (resp. $\mathcal{A}^{G,+}$) denotes the $\mathbb{R}$-vector space of G-invariant (resp. positive semidefinite) quadratic forms on $H_{n,d}$. Then $\ker \iota = \{\ell \in H^G_{n,2d} : \ell(\Sigma^G_{n,2d}) = 0\}$ and ι induces an isomorphism

$$\left(\Sigma^G_{n,2d}\right)^* \cong \mathcal{A}^{G,+} / \ker \iota.$$

We can identify $\left(\Sigma^G_{n,2d}\right)^*$ with $\iota\left(\left(\Sigma^G_{n,2d}\right)^*\right) = \iota\left(\left(H^G_{n,2d}\right)^*\right) \cap \mathcal{A}^{G,+}$ which is the intersection of the cone of positive semidefinite quadratic forms with an affine subspace and hence a spectrahedral cone.

Proof. Since any $\ell \in \ker \iota$ does not give any relevant information on $\Sigma^G_{n,2d}$, we can neglect that ι is not an isomorphism. $\square$

We point out that Q_ℓ is G-invariant but defined on the whole space $H_{n,d}$. This is important since an invariant sum of squares is not always a sum of squares of invariant forms. In particular the kernel of Q_ℓ is a G-submodule of $H_{n,d}$ and one can use complexity reduction by symmetry to describe elements in the kernel of Q_ℓ or ℓ.

Proposition 2.41. *An element $\ell \in \left(\Sigma_{n,2d}^G\right)^*$ is extremal if and only if $\ker \ell$ is a hyperplane in $H_{n,2d}^G$. Let $W := \ker Q_\ell$. Then $W^{\langle 2\rangle} := \{\mathcal{R}_G(fg) : f \in W, g \in H_{n,d}\}$ is equal to the kernel of ℓ. Moreover, if $(f_{11}, \dots, f_{l\eta_l})$ is a symmetry adapted basis of $H_{n,d}$ and $\left(g_{11}, \dots, g_{l\eta_l'}\right)$ is a symmetry adapted basis of W such that g_{ji_1} and f_{ji_2} span G-isomorphic irreducible G-modules, and $f_{ji_2} \mapsto g_{ji_1}$ defines the unique G-isomorphism, then*

$$W^{\langle 2\rangle} = \langle \mathcal{R}_G(g_{ji_1} \cdot f_{ji_2}) : 1 \le j \le l, 1 \le i_2 \le \eta_j, 1 \le i_1 \le \eta_j' \rangle_{\mathbb{R}}.$$

Proof. The first claim follows from Lemma 2.37. The second claim follows from the positive semidefiniteness of the quadratic form Q_ℓ and Proposition 2.38. The complexity reduction gives the above description of $W^{\langle 2\rangle}$ according to the use of a symmetry adapted basis and applying Schur's lemma. □

To prove equality or inequality of $\Sigma_{n,2d}^G$ and $\mathcal{P}_{n,2d}^G$ we propose a dual approach. Both sets are full dimensional, pointed, closed and convex cones in $H_{n,2d}^G$. By Minkowski's theorem any element in a proper convex cone can be written as a conic combination of extremal elements. We conclude the subsection with the following verification method of the non-negativity versus sums of squares question:

Corollary 2.42. *The sets of G-invariant n-ary non-negative and sums of squares forms of degree $2d$ are equal if and only if any extremal ray in $\left(\Sigma_{n,2d}^G\right)^*$ is generated by a point-evaluation.*

Proof. The primal cones $\mathcal{P}_{n,2d}^G$ and $\Sigma_{n,2d}^G$ are equal if and only if the dual cones are equal. By Minkowski's theorem any $\ell \in \left(\Sigma_{n,2d}^G\right)^*$ can be written as a sum of extremal elements. If any extremal ray in $\left(\Sigma_{n,2d}^G\right)^*$ is generated by a point-evaluation, then there exists a set $M \subset \mathbb{R}^n$ such that

$$\left(\mathcal{P}_{n,2d}^G\right)^* \subseteq \left(\Sigma_{n,2d}^G\right)^* = \operatorname{cone}\{\operatorname{ev}_a : a \in M \subset \mathbb{R}^n\} \subset \operatorname{cone}\{\operatorname{ev}_a : a \in \mathbb{S}^{n-1}\} = \left(\mathcal{P}_{n,2d}^G\right)^*,$$

where the last equality follows by Proposition 2.39. Conversely, if $\Sigma_{n,2d}^G = \mathcal{P}_{n,2d}^G$ then also the dual cones are equal. However, $\left(\mathcal{P}_{n,2d}^G\right)^*$ is the convex cone that is generated by all point-evaluations. Hence, any extremal ray in $\left(\Sigma_{n,2d}^G\right)^*$ is generated by a point-evaluation. □

3. Sums of squares invariant under A_{n-1}, B_n and D_n

In this section we demonstrate how the presented techniques from section 2 can be used to solve non-negativity versus sums of squares questions by providing explicit applications. We start with presenting an algorithmic construction of a symmetry adapted basis of the coinvariant algebra for reflection groups of type A_{n-1}, B_n and D_n. The benefit is a stabilization of the symmetry adapted basis for fixed degree and large enough number of variables. In contrast to the non-equivariant case the B_n-invariant forms have a non-trivial equality of the sets of even symmetric sums of squares and non-negatives in 3 variables and degree 8. This was proven by Harris [20]. In fact, it turns out that this is the only non-trivial equality case [19]. We will present a characterization of the dual and primal cones of B_3-invariant sum of squares ternary octics and obtain a new elementary proof of Harris' theorem. Moreover, we study D_n-invariant forms, prove that $\mathcal{P}_{4,4}^{D_4}$ is a simplicial cone and answer the non-negativity versus sums of squares question there. We conclude this section by showing that some of the considered sets of invariant non-negative forms are spectrahedral shadows.

In general testing non-negativity of a polynomial in more than two variables is already for quartics an NP-hard problem (see e.g. [8] or [27]). In equivariant situations it is therefore of interest to exploit the symmetry of invariant polynomials to reduce this complexity. The work

in [20, 41, 33, 34, 1, 17, 26] focuses on providing *test sets* for verification of non-negativity of invariant polynomials. A test set is a lower dimensional subspace of $\mathbb{R}^n$, so that one obtains equivalence of a polynomial being non-negative and several polynomial, obtained by linear transformations with lower degree or a fewer number of variables, being non-negative. It was shown that for several reflection groups these test sets can be chosen to be subspaces of the hyperplane arrangement associated to a reflection group (see [17, 1] for details).
We remark that for the infinite series $I_2(m)$ of dihedral groups, there is only an essential action on $\mathbb{R}^2$, i.e., the non-negativity versus sums of squares question for $I_2(m)$-invariant forms is therefore answered.

3.1. **Higher Specht polynomials.** We present an algorithmic approach for calculating a symmetry adapted basis of the coinvariant algebra for reflection groups of type A_{n-1}, B_n or D_n. A well known classical construction of the irreducible $\mathfrak{S}_n$-modules in the real polynomial ring is due to Specht [38]. The $\mathfrak{S}_n$-generators of these representations are called *Specht polynomials.* However, we need a complete decomposition into irrreducibles of the coinvariant algebra in order to apply the methods described in section 2. Since the coinvariant algebra is isomorphic to the regular representation, the multiplicities with which each of the irreducible representations is occurring equals their dimension. A nice combinatorial algorithm to decompose the coinvariant algebra into irreducibles was given in [25]. This approach can be used to explicitly calculate a decomposition for all complex reflection groups of type $G(m, n, p)$. In the following we briefly present their construction method.

We start by recalling some basic definitions from combinatorics.

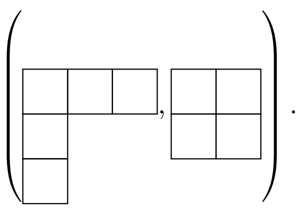
Definition 3.1. *A non-increasing sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_l)$ is called a partition and l is the length of λ. We denote by $|\lambda| = \sum_{i=1}^{l} \lambda_i$ the value of λ and say that λ is a partition of n if $|\lambda| = n$ and write $\lambda \vdash n$. For partitions λ^1 and λ^2 we call the pair $\Lambda = (\lambda^1, \lambda^2)$ a multipartition (here we also allow that either $\lambda^1 = \emptyset$ or $\lambda^2 = \emptyset$). We say that $|\Lambda| = |\lambda^1| + |\lambda^2| = n$ is the length of Λ and write $\Lambda \vdash n$ for Λ being a multipartition of n.*

We always denote multipartitions by capital letters and partitions by small letters. However, sometimes we write $(\lambda, \emptyset)$ instead of λ for a partition λ.

Definition 3.2. *The Ferres diagram associated to a partition $\lambda \vdash n$ is a sequence of ordered boxes which i-th line contains λ_i boxes (ordered from the left to the right). If one fills the boxes with all the integers in $[n]$, one calls the obtained object a Young tableaux of shape λ. If the numbers in all columns and rows are increasing we call it a standard Young tableaux. We associate multipartitions with pairs of Ferres diagrams. A Young tableaux is then a filling of both Ferres diagrams with all the numbers in $[n]$ and we call it standard if both Ferres diagrams are standard.*
We denote by $\mathrm{YT}(\Lambda)$ *the set of Young tableau of shape* Λ *and by* $\mathrm{SYT}(\Lambda)$ *the subset of standard Young tableau.*

The group $\mathfrak{S}_n$ acts naturally on a Young tableaux by replacing the entry i with $\sigma(i)$ for an element $\sigma \in \mathfrak{S}_n$.

Example 3.3. A Ferres diagram of shape $((3, 1, 1), (2, 2))$ is given by



The famous Robinson-Schensted correspondence gives a bijection between the standard Young tableau of shape λ and the elements in the conjugacy class of $\mathfrak{S}_n$ which are labelled by λ. Hence this number equals the multiplicity of the Specht module S^λ in the coinvariant algebra. The Robinson-Schensted correspondence has been adapted to complex reflection groups of type $G(m,n,p)$ and in particular for the contained series of reflection groups of types $B_n = G(2,1,n)$ and $D_n = G(2,2,n)$ (see e.g. section 10 in [9]).
Young tableau of multipartitions can be used to generate irreducible representations of B_n and D_n in the sense of Specht's construction [38] for $\mathfrak{S}_n$, resp. A_{n-1}. However, we need all representations in the coinvariant algebra. An answer to this are the higher Specht polynomials in [2] for the symmetric group and [25] for complex reflection groups of type $G(m,n,p)$.
For other essential real reflection groups the Garnir relations can be used to provide for any irreducible module one explicit representation.

We introduce the objects arising in the construction of a symmetry adapted basis of the coinvariant algebra by following [2]:

Definition 3.4. *Let T be a Young tableaux of shape $\lambda \vdash n$. The $\mathfrak{S}_n$-subgroups*

$$\mathcal{C}_T := \{\sigma \in \mathfrak{S}_n : \sigma T \text{ is obtained by permutation of the columns of } T\}$$
$$\mathcal{R}_T := \{\sigma \in \mathfrak{S}_n : \sigma T \text{ is obtained by permutation of the rows of } T\}$$

are called the column, resp. the row stabilizer of T. Furthermore, we define for T the formal linear combination

$$\epsilon_T := \frac{f^\lambda}{n!} \sum_{\sigma \in \mathcal{C}_T, \tau \in \mathcal{R}_T} \mathrm{sgn}(\sigma)\sigma\tau \in \mathbb{R}[\mathfrak{S}_n],$$

where f^λ is the number of standard Young tableau of shape λ. For $T = (T^1, T^2)$ a multi-Young tableaux, we define $\epsilon_{T^1}, \epsilon_{T^2} \in \mathbb{R}[\mathfrak{S}_n]$ analogously and set $\epsilon_T := \epsilon_{T^1} \cdot \epsilon_{T^2}$.

Example 3.5. Let $T = (T^1, T^2) = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1} \end{array}, \begin{array}{|c|}\hline 3 \\ \hline \end{array} \right)$ be a standard Young tableaux of shape $((2,1),(1))$. The column stabilizer of T^1 (resp. T^2) equals $C_{T^1} = \mathfrak{S}_{\{1,4\}}$ (resp. $C_{T^2} = \{\mathrm{id}\}$), and the row stabilizer is $R_{T^1} = \mathfrak{S}_{\{1,2\}}$ (resp. $R_{T^2} = \{\mathrm{id}\}$), where $\mathfrak{S}_{\{a_1,\ldots,a_l\}}$ denotes the symmetric group acting on the letters $a_1, \ldots, a_l$. The numbers of standard Young tableau are $f^{(2,1)} = 2$ and $f^{(1)} = 1$. We calculate the formal linear combinations $\epsilon_{T^1} = \frac{2}{3!}(\mathrm{id} - (14))(\mathrm{id} + (12)) = \frac{1}{3}(\mathrm{id} - (14) + (12) - (14)(12)) \in \mathbb{R}[\mathfrak{S}_4]$.

We associate (pairs of) Young tableau with sequences, monomials and polynomials:

Definition 3.6. *Let $T \in \mathrm{YT}(\lambda)$. The word of T is the sequence $w(T) \in \mathbb{N}^{|\lambda|}$ obtained via reading and notating as entries in $w(T)$ all the entries of the tableaux T from the left-most column from below to the top and proceeding iteratively by continuing with the next columns rightward. For a multipartition Λ and a Young tableaux $T = (T^1, T^2)$ of shape Λ we define the word $w(T) \in \mathbb{N}^{|\Lambda|}$ as the concatenation of $w(T^1)$ and $w(T^2)$.*

For a permutation $w \in \mathbb{N}^n$ of the numbers $1, \ldots, n$ we define its index, denoted by $i(w)$, as the sequence in $\mathbb{N}^n$ obtained in the following way: We write 0 in the place in $i(w)$ where 1 is located in w. Then we write 0 in the place in $i(w)$ where 2 is located in w if 2 is located on the right side of 1 in w. Otherwise, we write a 1 in $i(w)$ and proceed iteratively by writing $0, 1, 2, \ldots, n$ in the next places. We call the sum of the entries of $i(w)$ the charge of T and write $\mathrm{ch}(T)$.

We associate to a tuple of (possibly multi-) Young tableau (T, S) of same shape $\lambda \vdash n$ (resp. $\Lambda \vdash n$) a monomial in n variables $\underline{X}_T^S := X_{w(T)_1}^{i(w(S))_1} \cdots X_{w(T)_{|\lambda|}}^{i(w(S))_{|\lambda|}}$. Moreover, we define the polynomials assoicated to the pair (T, S)

$$F_T^S := \epsilon_T \cdot \underline{X}_T^S \in \mathbb{R}[\underline{X}] \text{ and } \widehat{F}_T^S := F_T^S(\underline{X}^2) \cdot \prod_{j \in T^2} X_j,$$

where $\underline{X}^2 := (X_1^2, \dots, X_n^2)$.

Example 3.7. Let $\Lambda = ((2,1),(1)) \vdash 4$ be a multipartition. We consider the standard Young tableau $S = \left(\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 3 \\ \hline\end{array}\right)$ and $T = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 3 \\ \hline\end{array}\right)$ of shape Λ. The word of S is $w(S) = (2,1,4,3)$ and the word of T is $w(T) = (4,1,2,3)$. We calculate the indices $i(w(S)) = (1,0,2,1)$ and $i(w(T)) = (1,0,0,0)$. We compute $\underline{X}_T^S = X_4^1 X_1^0 X_2^2 X_3^1 = X_2^2 X_3 X_4$ and $F_T^S = X_1^2 X_3 X_4 + X_2^2 X_3 X_4 - X_1 X_2^2 X_3 - X_1 X_3 X_4^2$.

The authors in [25] used the following definition for the G-generators of irreducible modules referring to Specht's polynomial representation of irreducible $\mathfrak{S}_n$-modules:

Definition 3.8. *Let* $G \in \{A_{n-1}, B_n, D_n\}$. *We call the* G*-generators of the coinvariant algebra* $\mathbb{R}[\underline{X}]_G$ *higher Specht polynomials.*

In the following we will denote an irreducible representation labelled by a (multi-) partition Λ by S^Λ. The underlying group should be clear from the context.

Theorem 3.9. [25] *For reflection groups of type* A_{n-1}, B_n *or* D_n *the higher Specht polynomials can be calculated as follows:*

a) For A_{n-1} *the higher Specht polynomials are*

$$\left\{F_T^S : \lambda \vdash n, (T,S) \in \mathrm{SYT}(\lambda) \times \mathrm{SYT}(\lambda)\right\}.$$

For standard Young tableau T *and* S *of shape* λ *the* A_{n-1}*-module*

$$\langle F_T^S \rangle_{A_{n-1}} = \langle F_{\widetilde{T}}^S : \widetilde{T} \in \mathrm{YT}(\lambda)\rangle_{\mathbb{R}}$$

is irreducible and A_{n-1}*-isomorphic to the Specht module* S^λ.

b) For B_n *the higher Specht polynomials are*

$$\left\{\widehat{F}_T^S : \Lambda \vdash n, (T,S) \in \mathrm{SYT}(\Lambda) \times \mathrm{SYT}(\Lambda)\right\}.$$

For standard Young tableau T *and* S *of shape* Λ *the* B_n*-module*

$$\langle \widehat{F}_T^S \rangle_{B_n} = \langle \widehat{F}_{\widetilde{T}}^S : \widetilde{T} \in \mathrm{YT}(\Lambda)\rangle_{\mathbb{R}}$$

is irreducible and B_n*-isomorphic to the irreducible representation* S^Λ *labelled by* Λ.

c) Let $\mathcal{L} := \{\Lambda = (\lambda,\mu) \vdash n : \lambda \neq \mu, |\lambda| \geq |\mu|\}$. *For* D_n *the higher Specht polynomials are the union of the two sets*

$$\left\{\widehat{F}_T^S : \Lambda \in \mathcal{L}, (T,S) \in \mathrm{SYT}(\Lambda) \times \mathrm{SYT}(\Lambda)\right\}, \textit{ and}$$
$$\left\{\widehat{F}_{(T^1,T^2)}^S \pm \widehat{F}_{(T^2,T^1)}^S : (\lambda,\lambda) \vdash n, ((T^1,T^2),S) \in \mathrm{SYT}((\lambda,\lambda)) \times \mathrm{SYT}((\lambda,\lambda))\right\}.$$

For (λ,μ) *and* (μ,λ) *the associated irreducible* B_n*-representations remain* D_n*-irreducible, but are* D_n*-isomorphic. For* $\Lambda = (\lambda,\lambda) \vdash n$ *the irreducible* B_n*-module* $S^{(\lambda,\lambda)}$ *splits into two irreducible, non-isomorphic* D_n*-modules. For a pair* $((T^1,T^2),S)$ *of standard Young tableau of same shape* $(\lambda,\lambda) \vdash n$ *it is*

$$\langle \widehat{F}_{(T^1,T^2)}^S \rangle_{D_n} = \langle \widehat{F}_{(T^1,T^2)}^S + \widehat{F}_{(T^2,T^1)}^S \rangle_{D_n} \oplus \langle \widehat{F}_T^S - \widehat{F}_{(T^2,T^1)}^S \rangle_{D_n},$$

where the D_n*-modules* $\langle \widehat{F}_{(T^1,T^2)}^S + \widehat{F}_{(T^2,T^1)}^S \rangle_{D_n}$ *and* $\langle \widehat{F}_{(T^1,T^2)}^S - \widehat{F}_{(T^2,T^1)}^S \rangle_{D_n}$ *are irreducible and non-isomorphic.*

In particular, by Schur's lemma 2.2, for the groups B_n and D_n (resp. A_{n-1}) and $T = (T^1, T^2), S_1, S_2$ standard Young tableau of shape Λ (resp. λ) the maps

$$\widehat{F}_T^{S_1} \mapsto \widehat{F}_T^{S_2} \text{ (resp. } F_T^{S_1} \mapsto F_T^{S_2})$$

define the (up to scalar) unique G-module isomorhism. In the case that Λ has the form (λ, λ), the unique D_n-isomorphism are then

$$\widehat{F}^{S_1}_{(T^1,T^2)} \pm \widehat{F}^{S_1}_{(T^2,T^1)} \mapsto \widehat{F}^{S_2}_{(T^1,T^2)} \pm \widehat{F}^{S_2}_{(T^2,T^1)}.$$

Let $G \in \{A_{n-1}, B_n, D_n\}$. We recall that the multiplicity of an irreducible G-module S^Λ (resp. of the D_n-modules $S_1^{(\lambda,\lambda)}$ and $S_2^{(\lambda,\lambda)}$) in the coinvariant algebra $\mathbb{R}[\underline{X}]$ equals the dimension of S^Λ, which is the number of standard Young tableau of shape Λ by the generalized Robinson-Schensted correspondence.

Definition 3.10. *Let $G \in \{A_{n-1}, B_n, D_n\}$ and $\Lambda \vdash n$ a multipartition, resp. a partition when $G = A_{n-1}$. We write q_d^Λ for the multiplicity of the G-module S^Λ in $H_{n,d}$.*

Remark 3.11. *From Theorem 3.9 we obtain a combinatorial description of h_k^θ, i.e., of the multiplicity of an irreducible representation θ in the subspace of the coinvariant algebra of forms of degree k. Namely, in the case of A_{n-1} θ is labelled by a partition $\lambda \vdash n$ and*

$$h_k^\lambda = |\{T \in \mathrm{SYT}(\lambda) : \mathrm{ch}(T) = k\}|.$$

While for B_n and D_n θ is labelled by a multipartition $\Lambda = (\lambda, \mu) \vdash n$ and

$$h_k^\Lambda = |\{(T, S) \in \mathrm{SYT}(\Lambda) : 2\,\mathrm{ch}(T, S) + |\mu| = k\}|.$$

In particular, the multiplicity of S^Λ in $H_{n,d}$ can be described combinatorially via the number of standard Young tableau and the degrees of G

$$q_d^\Lambda = \sum_{k=0}^{d} N_G(d-k) \cdot h_k^\Lambda.$$

The degrees of the considered reflection groups and the standard Young tableau combinatorially encode the following information about sums of squares:

Theorem 3.12. *Let $G \in \{A_{n-1}, B_n\}$.*

(1) The isotypic decomposition of $H_{n,d}$ is

$$\bigoplus_{\Lambda \vdash n} q_d^\Lambda \cdot S^\Lambda,$$

where Λ ranges over partitions for A_{n-1} and otherwise multipartitions.

(2) There exists a symmetry adapted basis of the coinvariant algebra $\mathbb{R}[\underline{X}]_G$ consisting of higher Specht polynomials $(s_1^\Lambda, \ldots, s_{\vartheta_\Lambda}^\Lambda)_{\Lambda \vdash n}$, where ϑ_Λ denotes the dimension of S^Λ. By defining symmetric matrix polynomials $H^\Lambda \in \mathbb{R}[\underline{X}]^{\vartheta_\Lambda \times \vartheta_\Lambda}$ via $H^\Lambda_{v,u} := \mathcal{R}_G(s_v^\Lambda \cdot s_u^\Lambda)$ we have

$$\Sigma\mathbb{R}[\underline{X}]^2 \cap \mathbb{R}[\underline{X}]^G = \left\{ g \in \mathbb{R}[\psi_1, \ldots, \psi_n] \,:\, g = \sum_{\Lambda \vdash n} \mathrm{Tr}(H^{\vartheta_j} \cdot A_\Lambda) \right\},$$

where $A_\Lambda \in \mathbb{R}[\psi_1, \ldots, \psi_n]^{\vartheta_\Lambda \times \vartheta_\Lambda}$ is a sums of squares matrix polynomial.

(3) There exists a symmetry adapted basis of $H_{n,d} = \bigoplus_{\Lambda \vdash n} q_d^\Lambda \cdot S^\Lambda$, where the elements $\left(s_1^\Lambda, \ldots, s_{q_d^\Lambda}^\Lambda\right)$ belonging to the isotypic component $q_d^\Lambda \cdot S^\Lambda$ are products each of one higher Specht polynomial and a monomial in $\psi_1, \ldots, \psi_n$. By defining matrix polynomials

$B^\Lambda \in \left(\mathbb{R}[\underline{X}]^G\right)^{q_d^\Lambda \times q_d^\Lambda}$ *via* $B^\Lambda_{v,u} := \mathcal{R}_G(s^\Lambda_v \cdot s^\Lambda_u)$ *a form* $f \in H^G_{n,2d}$ *is a sum of squares if and only if*

$$f = \sum_\Lambda \mathrm{Tr}(B^\Lambda \cdot A_\Lambda)$$

for some positive semidefinite matrices $A_\Lambda \in \mathbb{R}^{q_d^\Lambda \times q_d^\Lambda}$*.*

For D_n *differ the isotypic decomposition in* (1) *and the sizes of the matrices in* (2) *and* (3) *slightly, since then the* D_n*-module* $S^{(\lambda,\lambda)}$ *decomposes into two irreducible* D_n*-modules and* $S^{(\lambda,\mu)}$ *is* D_n*-isomorphic to* $S^{(\mu,\lambda)}$*.*

Proof. The isotypic decomposition of $H_{n,d}$ can be realized through calculating the higher Specht polynomials of G of degree $\leq d$ and multiplying them with products of fundamental invariants by Theorem 3.9 and 2.16. For every k the multiplicity of G-modules isomorphic to S^Λ in the subspace of the coinvariant algebra of degree k is precisely h_k^Λ, while $N_G(d-k)$ gives the dimension of $H^G_{n,d-k}$. (2) and (3) follow now from Theorem 2.22 and Corollary 2.6. □

We present how the isotypic decomposition of the D_4-module $H_{4,2}$ can be calculated via the higher Specht polynomial approach:

Example 3.13. The D_4 fundamental invariants are

$$p_2 = X_1^2 + X_2^2 + X_3^2 + X_4^2,\ \ p_4 = X_1^4 + X_2^4 + X_3^4 + X_4^4,$$
$$p_6 = X_1^6 + X_2^6 + X_3^6 + X_4^6,\ \ e_4 = X_1X_2X_3X_4,$$

i.e., we have $\mathbb{R}[\underline{X}]^{D_4} = \mathbb{R}[p_2, p_4, p_6, e_4]$. By Corollary 2.17 the symmetry adapted basis for $H_{4,2}$ can be obtained by multiplication of the fundamental invariants with higher Specht polynomials (such that the degree equals 2).
We apply Theorem 3.9 to calculate the D_4 higher Specht polynomials. For a multipartition $\Lambda \vdash 4$ the minimal degree of a higher Specht polynomial associated with Λ is given by the smallest number, that is $2\,\mathrm{ch}(T) + |\lambda^2|$ for $T \in \mathrm{SYT}(\Lambda)$.
Since the degrees of the fundamental invariants are at least 2, we need to compute all higher Specht polynomials of degree 0 and 2. Therefore, we only need to consider partitions $(\lambda^1, \lambda^2) \vdash 4$ where $\lambda^2 \vdash m$, $m \in \{0, 2\}$. In the case $\lambda^2 \vdash 2$ it must be $\mathrm{ch}(T) = 0$. This can only occur if $w(T) = (1, 2, 3, 4)$. Which forces $\Lambda = ((2), (2))$. The possible remaining cases are $\Lambda^1 = ((4), \emptyset), \Lambda^2 = ((3, 1), \emptyset), \Lambda^3 = ((2, 2), \emptyset), \Lambda^4 = ((2, 1, 1), \emptyset), \Lambda^5 = ((1, 1, 1, 1), \emptyset)$. We are looking for a standard Young tableaux T of shape $\Lambda^j, j \in \{1, 2, 3, 4, 5\}$ such that $\mathrm{ch}(T) \in \{0, 1\}$. The case that it equals 0 is only possible for Λ^1. In the remaining cases $\mathrm{ch}(T) = 1$ if and only if $T = \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}\ ,\ \emptyset\right)$. The D_4-module $S^{((2),(2))}$ decomposes by Theorem 3.9 into two irreducible, non-isomorphic modules $S_1^{((2),(2))}$ and $S_2^{((2),(2))}$. Hence the D_4-module $H_{4,2}$ has the isotypic decomposition

$$H_{4,2} = S^{((4),\emptyset)} \oplus S^{((3,1),\emptyset)} \oplus S_1^{((2),(2))} \oplus S_2^{((2),(2))}.$$

The relevant higher Specht polynomials are 1 (for $S^{((4),\emptyset)}$), $X_4^2 - X_1^2$ (for $S^{((3,1),\emptyset)}$) and $X_1X_2 \pm X_3X_4$ (for $S_i^{((2),(2)}$, $i \in \{1, 2\}$).

In the following we aim to prove a stabilization of the isotypic decompositions for the Z_n-modules $H_{n,d}$, for large n and $(Z_n)_n \in \{(A_{n-1})_n, (B_n)_n, (D_n)_n\}$.

Definition 3.14. *For a partition* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) \vdash n$ *we write* $\lambda + 1 := (\lambda_1 + 1, \lambda_2, \dots, \lambda_l)$ *for the partition of* $n + 1$ *of same length as* λ*, whose first entry is* 1 *plus the first entry of* λ *and the other entries are the same as in* λ*. Further, for a multipartition* $\Lambda \vdash n$ *we write* $\Lambda + 1 := (\lambda + 1, \mu) \vdash n + 1$*.*

Lemma 3.15. *Let $\lambda \vdash n = d + k$ be a partition. Then, if the first row of a standard Young tableaux T does not begin with $1, 2, \ldots, k$ it is $\operatorname{ch}(T) > d$.*

Proof. We assume that a standard Young tableaux T of shape λ does not contain $1, 2, \ldots, k$ in the first row. Then the first non-zero entry in $i(T)$ lies before or at the position of k in $w(T)$. In particular, $i(T)$ does contain at least $n - k + 1$ entries which are larger than or equal to 1 and therefore $\operatorname{ch}(T) \geq n - k + 1 = d + 1$. □

We rephrase Lemma 3.15 for multipartitions.

Lemma 3.16. *Let $(\lambda, \mu) \vdash n$ be a multipartition, where $|\mu| \leq d$ and $|\lambda| \geq \frac{d-1}{2} + j$. Let (T, S) be a standard Young tableaux of shape (λ, μ) where $\alpha_1 < \ldots < \alpha_{|\lambda|}$ are all the entries in T. Assume that the first row of T does not begin with $\alpha_1, \ldots, \alpha_j$, then $\deg \widehat{F}^{(T,S)}_{(T,S)} > d$.*

Proof. Assume that for some $i \leq j$ the i-th entry in the first row of T is not α_i and let i be minimal with this property. Then α_i must be the first entry in the second row and $|\lambda| - i + 1$ entries in $i(T, S)$ are at least 1. Hence

$$\deg \widehat{F}^{(T,S)}_{(T,S)} = 2 \operatorname{ch}(T, S) + |\mu| \geq 2(|\lambda| - i + 1) \geq 2\left(\frac{d-1}{2} + j - j + 1\right) \geq d + 1.$$

□

We write $T = (\alpha_{ij})$ for a standard Young tableaux T of shape λ, where α_{ij} denotes the entry in the i-th row and j-th coloumn of T, counted from the left to the right and the top to the bottom. Analogously, we write $(T, S) = ((\alpha_{ij}), (\beta_{ij}))$ for a standard Young tableaux of a multipartition.

Definition 3.17. *For a partition $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n = d + k$ with $\lambda_1 \geq k$, we define*

$$\Pi^\lambda_k := \{T = (\alpha_{ij}) \in \operatorname{SYT}(\lambda) : \alpha_{1j} = j, 1 \leq j \leq k\}.$$

Further, for a multipartition $\Lambda \vdash n = d + k$ we define

$\Pi^\Lambda_k := \{(T, S) = ((\alpha_{ij}), (\beta_{ij})) \in \operatorname{SYT}(\Lambda) : T_1$ *starts with the first k smallest integers in* $\{\alpha_{ij}\}\}$,

where T_1 denotes the first row of T.

Lemma 3.18. *Let $n = d + k$, $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$ a partition and $\rho^\lambda_{n,n+1}$ a map defined by*

$$\begin{array}{cccc} \rho^\lambda_{n,n+1}: & \Pi^\lambda_k & \longrightarrow & \Pi^{\lambda+1}_{k+1} \\ & S = (\alpha_{ij}) & \longmapsto & \widetilde{S} = (\widetilde{\alpha}_{ij}) \end{array},$$

where $\widetilde{\alpha}_{1i} = i$ for $1 \leq i \leq \alpha_{21}$. Further, $\widetilde{\alpha}_{ji} = \alpha_{ji-1} + 1$ for $j = 1, i \geq \alpha_{21} + 1$ and all (j, i) with $j \geq 2$.
Then $\rho^\lambda_{n,n+1}$ is injective and $i(S), i(\widetilde{S})$ differ only by a zero, i.e., any non-zero entry in $i(S)$ occurs with the same multiplicity in $i(\widetilde{S})$, while 0 occurs one more time. Furthermore, if $k > \min\{d - 1, \sqrt{2d + 25/4} - 5/2\}$ then for any $\widetilde{S} \in \Pi^{\lambda+1}_{k+1} \setminus \rho^\lambda_{n,n+1}(\Pi^\lambda_k)$ it is $\operatorname{ch}(\widetilde{S}) > d$.

Proof. Since $S \in \Pi^\lambda_k$ is standard, we observe that α_{21} is the smallest integer i for which $\alpha_{1i} \neq i$. For $S \in \Pi^\lambda_k$ the Young tableaux $\widetilde{S}$ of shape $\lambda + 1$ is indeed standard: $\widetilde{S}$ is filled with $1, \ldots, n+1$. Increasing rows and columns are inherited from S, as $\alpha_{1\alpha_{21}} > \alpha_{21}$. $\widetilde{S}$ is clearly increasing in any column from the second row onward. But also from the first row to the second. For $1 \leq i \leq \alpha_{21}$ this is clear from S. For $i > \alpha_{21}$ this follows because $\widetilde{\alpha}_{1i} = \alpha_{1,i-1} + 1 < \alpha_{2,i-1} + 1 < \alpha_{2,i} + 1 = \widetilde{\alpha}_{2i}$. The smallest j which is written left of $j - 1$ in $w(S)$ (resp. $w(\widetilde{S})$) is α_{21} (resp. $\widetilde{\alpha}_{21} = \alpha_{21} + 1$). From there any $j > \alpha_{21}$ is left of $j - 1$ in $w(S)$ if and only if $j + 1$ is left of j in $w(\widetilde{S})$. Hence $i(w(S))$ and $i(w(\widetilde{S}))$ differ only by a zero.
Consider $\psi^{\lambda+1}_{n+1,n} : \Pi^{\lambda+1}_{k+1} \to \operatorname{YT}(\lambda)$ which maps a standard Young tableaux $\widetilde{S}$ to a Young tableaux

S by removing the box of the first entry $\widetilde{\alpha}_{1j}$ in the first row of $\widetilde{S}$, that is strictly smaller than $\widetilde{\alpha}_{1j+1} - 1$ (otherwise the last entry). The entries to the right are shifted to the left. Any entry that was to the right of $\widetilde{\alpha}_{1j}$ or in a lower row is decreased by one. If $\psi^{\lambda+1}_{n+1,n}(\widetilde{S}) =: S$ is again standard, then $\psi^{\lambda+1}_{n+1,n} \circ \rho^{\lambda}_{n,n+1}(S) = S$. This shows the injectivity of $\rho^{\lambda}_{n,n+1}$.
If S is not standard, then one entry in the first column must be smaller than the entry below. Assume that this happens at S's entry α_{1j}. By assumption $j > k$, but this means $\lambda_2 \geq j > k$. However, for $1 \leq i \leq k$ the entry in $i(\widetilde{S})$ at the position of $\widetilde{\alpha}_{2i}$ in $w(\widetilde{S})$ is $\geq i$ and at least k for $\widetilde{\alpha}_{2,k+1}, \widetilde{\alpha}_{1,j+1}$, i.e., for $k > \sqrt{2d + 25/4} - 5/2$ it is

$$\operatorname{ch}(\widetilde{S}) \geq \sum_{i=1}^{k} i + 2k = \frac{k^2 + k}{2} + 2k > d.$$

For the other bound notice that $\operatorname{ch}(\widetilde{S}) \geq \lambda_2 + 1 \geq k + 2 \geq d + 1$.

□

Lemma 3.19. *Let $n = d + k$, $\Lambda = (\lambda, \mu) \vdash n$ a multipartition, where $\lambda = (\lambda_1, \ldots, \lambda_l)$, and*

$$\begin{array}{rccc} \rho^{\Lambda}_{n,n+1} : & \Pi^{\Lambda}_{k} & \longrightarrow & \Pi^{\Lambda+1}_{k+1} \\ & (T, S) = ((\alpha_{ij}), (\beta_{ij})) & \longmapsto & (\widetilde{T}, \widetilde{S}) = ((\widetilde{\alpha_{ij}}), (\widetilde{\beta_{ij}})) \end{array},$$

where $(\widetilde{T}, \widetilde{S})$ is defined by: Let i be minimal with $\alpha_{1i} \neq i$, then $\widetilde{\alpha}_{1j} = j$, $1 \leq j \leq i$ and $\widetilde{\alpha}_{1k} = \alpha_{1k-1} + 1$, for $i + 1 \leq k \leq \lambda_1 + 1$, $\widetilde{\alpha}_{jk} = \alpha_{jk} + 1$, when $j \geq 2$, and $\widetilde{\beta}_{jk} = \beta_{jk} + 1$. If such an i does not exist, then $\widetilde{\alpha}_{1k} = k$, $\widetilde{\alpha}_{ji} = \alpha_{ji} + 1, j \geq 2$ and $\widetilde{\beta}_{ji} = \beta_{ji} + 1$.
Then $\rho^{\Lambda}_{n,n+1}$ is injective, $i(S,T), i(\widetilde{S}, \widetilde{T})$ differ only by a zero, i.e., any non-zero entry in $i(S,T)$ occurs with the same multiplicity in $i(\widetilde{S}, \widetilde{T})$, and 0 occurs one more time. Furthermore, if $k > \min\{\frac{d}{2} - 2, \sqrt{d + 25/4} - 5/2\}$ then for any $(\widetilde{T}, \widetilde{S}) \in \Pi^{\Lambda+1}_{k+1} \setminus \rho^{\Lambda}_{n,n+1}(\Pi^{\Lambda}_{k})$ it is $2\operatorname{ch}(\widetilde{T}, \widetilde{S}) > d$.

Proof. For $(T, S) \in \Pi^{\Lambda}_{k}$ $(\widetilde{T}, \widetilde{S})$ is indeed a standard Young tableaux of shape $\Lambda + 1$, since increasing entries in every row and column is inherited from (T, S). An integer j occurs left of $j - 1$ in $w(T, S)$ if and only if $j + 1$ occurs left of j in $w(\widetilde{T}, \widetilde{S})$. In particular $i(T, S)$ and $i(\widetilde{T}, \widetilde{S})$ differ only by an additional zero entry and hence their charges are equal. Consider $f : \Pi^{\Lambda+1}_{k+1} \to \mathrm{YT}(\Lambda)$ which maps an element $(\widetilde{T}, \widetilde{S}) \in \Pi^{\Lambda+1}_{k+1}$ to a Young tableaux of shape Λ by removing $\widetilde{\alpha}_{11}$, if $\widetilde{\alpha}_{11} \neq 1$ and otherwise the box containing the largest entry in the first row of $\widetilde{T}$ that is not the predecessor of the following number, and subtracting 1 from any larger entry $\widetilde{\alpha}_{ji}, \widetilde{\beta}_{ji}$. Then f is the inverse of $\rho^{\Lambda}_{n,n+1}$ and therefore $\rho^{\Lambda}_{n,n+1}$ is injective.
If $f(\widetilde{T}, \widetilde{S})$ is not standard, then $\lambda_2 \geq k + 1$. For $k > \frac{d}{2} - 2$ it is

$$2\operatorname{ch}(\widetilde{T}, \widetilde{S}) \geq 2(k + 2) > d.$$

If $k \geq \sqrt{d + 25/4} - 5/2$ then any $\widetilde{\alpha}_{2i}$ gives value $\geq i$ in $i(\widetilde{T}, \widetilde{S})$ for $1 \leq i \leq k$ and $\widetilde{\alpha}_{2,k+1}, \widetilde{\alpha}_{1,k+1}$ values $\geq k$. In particular,

$$2\operatorname{ch}(\widetilde{T}, \widetilde{S}) \geq 2(\sum_{i=1}^{k} i + 2k) > d.$$

□

Definition 3.20. *For $m > n \geq d$ and partitions $\Lambda, \lambda \vdash n$ we write $\rho^{\lambda}_{n,m} := \rho^{\lambda+m-n-1}_{m-1,m} \circ \cdots \circ \rho^{\lambda}_{n,n+1}$ and $\rho^{\Lambda}_{n,m} := \rho^{\Lambda+m-n-1}_{m-1,m} \circ \cdots \circ \rho^{\Lambda}_{n,n+1}$.*

The following stabilization was already proven in [32] for the symmetric group.

Theorem 3.21. *Let $n \in \mathbb{N}, \Lambda \vdash n$ and $Z_n \in \{A_{n-1}, B_n, D_n\}$. For large enough n the Z_n- and Z_{n+1}-isotypic decompositions remain stable, in the sense that $S^{(\lambda,\mu)}$ occurs with the same multiplicity in $H_{n,d}$ as $S^{(\lambda+1,\mu)}$ in $H_{n+1,d}$, where $\lambda+1 := (\lambda_1+1, \lambda_2, \ldots, \lambda_l)$. The stabilization of the isotypic decomposition of $H_{n,d}$ occurs at least from $n = 2d$ for A_{n-1}, B_n and $n > 2d$ in the case of D_n.*

Proof. By iteration it is sufficient to consider the isotypic decompositions of $H_{n,d}$ and $H_{n+1,d}$ for $n \geq 2d$.
Let $n \geq 2d$ and $\Lambda = (\lambda, \mu) \vdash n$ be a multipartition with $|\mu| \leq d$ (resp. $\lambda \vdash n$ a partition in the case of A_{n-1}). Further be $f_1, \ldots, f_m$ a symmetry adapted basis for the higher Specht polynomials of the Z_n-module $\bigoplus_{i=1}^m S^\Lambda$ (resp. $\bigoplus_{i=1}^m S^\lambda$) obtained from Theorem 3.9, i.e., there exist m many standard Young tableau $T := T_1, T_2 \ldots, T_m$ of shape Λ (resp. λ) and $f_j = \pi \widehat{F}_T^{T_j}$ (resp. $f_j = \pi F_T^{T_j}$), for $\pi \in \mathbb{R}[\underline{X}]^{Z_n}$. Since πf_j must be homogeneous, π can be chosen as a product of fundamental invariants of Z_n. The degree of a polynomial f_j is determined by $d_1, \ldots, d_n$, the charge of a standard Young tableaux T_j and $|\mu|$.

We proceed with the case $Z_n = A_{n-1}$.The relevant degrees of fundamental invariants are the same for n and $n+1$, namely $1, \ldots, d$. Hence any higher Specht polynomial associated with a standard Young tableaux of shape λ with charge $\leq d$ occurs at least once in $H_{n,d}$. By Lemma 3.15 $T_1, \ldots, T_m \in \Pi_{n-d}^\lambda$, since otherwise $\deg F_T^{T_j} > d$. By Lemma 3.18 for any i $\rho_{n,n+1}^\lambda(T_i)$ is a standard Young tableaux with same charge. Furthermore the map $\rho_{n,n+1}^\lambda$ is injective and any standard Young tableaux that is not contained in the image has too large charge. The claim follows as only standard Young tableau in $\rho_{k,k+1}^\lambda(\Pi_k^\lambda)$ are possible options for higher Specht polynomials in $H_{n+1,d}$ by Lemma 3.15.

We continue with $Z_n = B_n$. By the Lemmas 3.16 and 3.19 the standard Young tableau (T, S) of shape Λ with $2\operatorname{ch}(T, S) \leq d$ are in bijection with the standard Young tableau $(\widetilde{T}, \widetilde{S})$ of shape $\Lambda + 1$ with $2\operatorname{ch}(\widetilde{T}, \widetilde{S}) \leq d$ and the bijection preserves the charge. Furthermore, our bijection adds a zero to the index of the image tableaux and preserves the other entries. Which proves already the claim.

For $n > 2d$ the relevant fundamental invariants of degree $\leq d$ are equal for B_n and D_n. The same argument as in the B_n case applies. $\square$

We note that in the case of D_n and $n = d$, an additional fundamental invariant of degree d occurs which does not occur for $n > d$ anymore. Thus at least the trivial representation occurs with larger multiplicity in $H_{d,d}$ than in $H_{d+1,d}$. However, Example 3.22 shows that already for the symmetric group the stabilization does not occur in the step from d to $d+1$ in general.

Example 3.22. Consider the standard Young tableaux $T = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}$ of shape $\lambda + 1 = (3, 2) \vdash 5$. It is $\operatorname{ch}(T) = 3$, i.e., $p_1 F_T^T \in H_{5,4}$. However, $\operatorname{SYT}(\lambda) = \left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} \right\}$ with charges 2 and 4. For any $S \in \operatorname{SYT}(\lambda)$ we can construct a tableaux $\widetilde{S} \in \operatorname{SYT}(\lambda)$ with same charge and T cannot be one of these.
In particular the A_3-module S^λ has smaller multiplicity in $H_{4,4}$ than the A_4-module $S^{\lambda+1}$ in $H_{5,4}$.

Corollary 3.23. *For a fixed degree $d \in \mathbb{N}$ and a sequence $(Z_n)_n$ of reflection groups $(A_{n-1})_n$ or$(B_n)_n$ the sums of squares decomposition in $H_{n,2d}^{Z_n}$ for $n \geq 2d$ are equal up to the map $\rho_{n,m}^\Lambda$, i.e., up to $\rho_{n,m}^\Lambda$ the same matrix polynomials can be used in a sums of squares representation. The same is true for the sequence $(D_n)_n$ and $n > 2d$.*

Proof. This follows from Theorem 3.21 and Lemmas 3.15, 3.16, 3.18, 3.19. $\square$

The case $n = 2d$ is the last, where $\Lambda \vdash n$ can be of the form (λ, λ), i.e., the D_{2d}-module S^Λ is not irreducible in $H_{n,d}$ but the D_{2d+1}-module $S^{\Lambda+1}$ is irreducible in $H_{n+1,d}$ (see Theorem 3.9). Nevertheless, the multiplicities in $H_{2d,d}$ and $H_{n,d}$ are equal for $n \geq 2d$.
The bounds on the stabilization in Theorem 3.21 and Corollary 3.23 can be improved by using the sharper bounds from Lemmas 3.18 and 3.19. Moreover, whenever $n \geq d$ for B_n, or $n > d$ in case of D_n one can use that if $S^\Lambda \subset H_{n,d}$, for $\Lambda = (\lambda, \mu) \vdash n$, and d even (odd), then $|\mu|$ must also be even (odd).

3.2. **Even symmetric octics.** One of the well known and rare cases of equality of sums of squares and non-negative forms in equivariant situations was proven by Harris in [20]. Harris' proof is quite analytical. In this subsection we derive a lower dimensional test set for non-negativity of even symmetric ternary octics and as a byproduct we give a new proof of equality. Furthermore, we present an uniform description of the cones of n-ary even symmetric sums of squares octics. We apply the presented techniques from section 2.

We recall that we identify functions $f, g : V \to W$, if $f(v) = g(v)$ for all $v \in V$.

Theorem 3.24. *The dual cone of even symmetric ternary octics sums of squares has the following description*

$$\left(\Sigma_{3,8}^{B_3}\right)^* = \left\{ \mathrm{ev}_{(a,\sqrt{1-a^2},0)}, \mathrm{ev}_{(b,c,c)} : \frac{1}{2} \leq a \leq 1, 0 \leq b \leq 1, c = \sqrt{\frac{(1-b^2)}{2}} \right\}.$$

As a consequence of Theorem 3.24 we can give a new proof for Harris' result on even symmetric ternary octics.

Corollary 3.25. [20, Theorem 4.1] *The sets of non-negative even symmetric ternary octics and sums of squares are equal, i.e.,* $\Sigma_{3,8}^{B_3} = \mathcal{P}_{3,8}^{B_3}$.

Proof. By Theorem 3.24 the cone $\left(\Sigma_{n,2d}^{G}\right)^*$ is generated by point-evaluations. The claim follows now from Corollary 2.42. ☐

In the following we provide a study of the even symmetric sums of squares ternary octics.

Lemma 3.26. *The B_3-module $H_{3,4}$ has the isotpyic decomposition*

$$H_{3,4} = 2 \cdot S^{((3),\emptyset)} \oplus 2 \cdot S^{((2,1),\emptyset)} \oplus 2 \cdot S^{((1),(2))} \oplus \mathcal{S}^{((1),(1,1))}.$$

A symmetry adapted basis for $H_{3,4}$ realising the B_3-isotypic decomposition is generated by the following polynomials:

$$S^{((3),\emptyset)} : \left\{ e_1(\underline{X}^2)^2, e_2(\underline{X}^2) \right\}, \qquad S^{((2,1),\emptyset)} : \left\{ e_1(\underline{X}^2)(X_3^2 - X_1^2), X_2^2X_3^2 - X_1^2X_2^2 \right\},$$
$$S^{((1),(2))} : \left\{ e_1(\underline{X}^2)X_2X_3, X_1^2X_2X_3 \right\}, \qquad S^{((1),(1,1))} : \left\{ (X_3^2 - X_2^2)X_2X_3 \right\}.$$

Proof. We need to determine the multiplicity of the irreducible B_3-modules $S^{(\lambda,\mu)}$ in $H_{3,4}$ for any multipartition $(\lambda, \mu) \vdash 3$. We can immediately exclude some multipartitions: Since we need only higher Specht polynomials of degree $0, 2$ or 4, the degree - which equals 2 times the charge of a standard Young tableaux of shape (λ, μ) plus $|\mu|$ - must be $0, 2$ or 4. However, this implies that only multipartitions with $\mu \in \{\emptyset, (2), (1, 1)\}$ are feasible. By going through all the remaining cases one obtains precisely the following higher Specht polynomials of degree $0, 2$ and 4:

$$\left\{ 1, X_3^2 - X_1^2, X_2^2X_3^2 - X_1^2X_2^2, X_2X_3, X_1^2X_2X_3, (X_3^2 - X_2^2)X_2X_3. \right\}$$

Multiplying by the invariants $1, e_1(\underline{X}^2)^2$ and $e_2(\underline{X}^2)$ results accordingly in the above mentioned symmetry adapted basis. ☐

Corollary 3.27. *An even symmetric ternary octic $f \in H_{3,8}^{B_3}$ is a sum of squares if and only if there exist positive semidefinite matrices $A^{(1)}, A^{(2)}, A^{(3)} \in \mathbb{R}^{2\times 2}$ and $A^{(4)} \in \mathbb{R}^{1\times 1}$ such that*

$$f = \operatorname{Tr}\left(A^{(1)}B^{(1)}\right) + \operatorname{Tr}\left(A^{(2)}B^{(2)}\right) + \operatorname{Tr}\left(A^{(3)}B^{(3)}\right) + \operatorname{Tr}\left(A^{(4)}B^{(4)}\right),$$

where $B^{(j)}$ are the following matrix polynomials corresponding to the B_3-modules in $H_{3,4}$

$$\begin{aligned}
B^{(1)} &:= \begin{pmatrix} e_1(\underline{X}^2)^4 & e_1(\underline{X}^2)^2 e_2(\underline{X}^2) \\ e_1(\underline{X}^2)^2 e_2(\underline{X}^2) & e_2(\underline{X}^2)^2 \end{pmatrix}, \\
B^{(2)} &:= \begin{pmatrix} \frac{2}{3}e_1(\underline{X}^2)^4 - 2e_1(\underline{X}^2)^2 e_2(\underline{X}^2) & -3e_1(\underline{X}^2)e_3(\underline{X}^2) + \frac{1}{3}e_1(\underline{X}^2)^2 e_2(\underline{X}^2) \\ -3e_1(\underline{X}^2)e_3(\underline{X}^2) + \frac{1}{3}e_1(\underline{X}^2)^2 e_2(\underline{X}^2) & \frac{2}{3}e_2(\underline{X}^2)^2 - 2e_1(\underline{X}^2)e_3(\underline{X}^2) \end{pmatrix}, \\
B^{(3)} &:= \begin{pmatrix} \frac{1}{3}e_1(\underline{X}^2)^2 e_2(\underline{X}^2) & e_1(\underline{X}^2)e_3(\underline{X}^2) \\ e_1(\underline{X}^2)e_3(\underline{X}^2) & \frac{1}{3}e_1(\underline{X}^2)e_3(\underline{X}^2) \end{pmatrix}, \\
B^{(4)} &:= \left(e_1(\underline{X}^2)e_3(\underline{X}^2) - \frac{4}{3}e_2(\underline{X}^2)^2 + \frac{1}{3}e_1(\underline{X}^2)^2 e_2(\underline{X}^2)\right).
\end{aligned}$$

Proof. The matrices $B^{(1)}, \dots, B^{(4)}$ are the symmetrizations of the products of the symmetry adapted basis from Lemma 3.26. By Theorem 2.5 any invariant sum of squares form has such a representation. □

Corollary 3.28. *A linear form $\ell \in \left(H_{3,8}^{B_3}\right)^*$ is contained in $\left(\Sigma_{3,8}^{B_3}\right)^*$ if and only if the following matrices are positive semidefinite*

$$\begin{pmatrix} m_{(1^4)} & m_{(2,1^2)} \\ m_{(2,1^2)} & m_{(2^2)} \end{pmatrix}, \begin{pmatrix} \frac{2}{3}m_{(1^4)} - 2m_{(2,1^2)} & \frac{1}{3}m_{(2,1^2)} - 3m_{(3,1)} \\ \frac{1}{3}m_{(2,1^2)} - 3m_{(3,1)} & \frac{2}{3}m_{(2^2)} - 2m_{(3,1)} \end{pmatrix}, \begin{pmatrix} \frac{1}{3}m_{(2,1^2)} & m_{(3,1)} \\ m_{(3,1)} & \frac{1}{3}m_{(3,1)} \end{pmatrix}, \left(\frac{1}{3}m_{(2,1^2)} - \frac{4}{3}m_{(2^2)} + m_{(3,1)}\right),$$

where we write $m_{(1^4)} := \ell(e_1(\underline{X}^2)^4), m_{(3,1)} := \ell(e_1(\underline{X}^2)e_3(\underline{X}^2)), m_{(2,1^2)} := \ell(e_1(\underline{X}^2)^2 e_2(\underline{X}^2))$ and $m_{(2^2)} := \ell(e_2(\underline{X}^2)^2)$.

Proof. By Lemma 2.36 this is precisely the dual statement to Corollary 3.27. □

Remark 3.29. *We observe that*

$$H_{3,8}^{B_3} = \langle p_2^4, p_2^2 p_4, p_2 p_6, p_4^2 \rangle_{\mathbb{R}} = \langle e_1(\underline{X}^2)^4, e_1(\underline{X}^2)e_3(\underline{X}^2), e_1(\underline{X}^2)^2 e_2(\underline{X}^2), e_2(\underline{X}^2)^2 \rangle_{\mathbb{R}}$$

is a 4-dimensional $\mathbb{R}$-vector space. We pick as the fundamental invariants the elementary symmetric polynomials evaluated in $\underline{X}^2 = (X_1^2, X_2^2, X_3^2)$ and work with the $\mathbb{R}$-basis

$$\left(e_1(\underline{X}^2)^4, e_1(\underline{X}^2)e_3(\underline{X}^2), e_1(\underline{X}^2)^2 e_2(\underline{X}^2), e_2(\underline{X}^2)^2\right)$$

of $H_{3,8}^{B_3}$. We study explicitly the extremal elements in $\left(\Sigma_{3,8}^{B_3}\right)^*$ and show that all of them are point-evaluations which is then used to prove Theorem 3.24. In the remaining part of this subsection we will always use the following notation for an extremal element $\ell \in \left(\Sigma_{3,8}^{B_3}\right)^*$. $\mathcal{Q}_\ell$ denotes the associated B_3-invariant quadratic form on $H_{3,4}$, $W_\ell := \ker \mathcal{Q}_\ell$ its kernel and

$$W_\ell^{\langle 2\rangle} := \ker \ell = \left\{h \in H_{n,2d}^G \, : \, h = \mathcal{R}_G\left(\sum f_i g_i\right) \quad \text{with } f_i \in W \text{ and } g_i \in H_{n,d}\right\}$$

(see Proposition 2.41). A hyperplane in $H_{3,8}^{B_3}$ is of dimension 3, hence from Lemma 2.37 we know that $\dim W_\ell^{\langle 2\rangle} = 3$. By Lemma 3.26 the isotypic decomposition of the B_3-submodule W_ℓ of $H_{3,4}$ has the form

$$W_\ell = \ker \mathcal{Q}_\ell = \alpha \cdot S^{((3),\emptyset)} \oplus \beta \cdot S^{((2,1),\emptyset)} \oplus \gamma \cdot S^{((1),(2))} \oplus \delta \cdot S^{((1),(1,1))},$$

where $\alpha, \beta, \gamma \in \{0,1,2\}$ and $\delta \in \{0,1\}$.
We make frequently use of the fact that $\ker \ell$ is maximal among any kernel of elements in

$\left(\Sigma_{3,8}^{G}\right)^*$, *i.e., when* $\ker \ell$ *contains a non trivial zero then* ℓ *must be a scalar of the point-evaluation at this point (see Lemma 2.37).*

In the following lemmas are we doing case distinctions on α, β, γ and δ to obtain a classification of all extremal elements in the dual cone $\left(\Sigma_{3,8}^{B_3}\right)^*$.

Lemma 3.30. *Let* $\ell \in \left(\Sigma_{3,8}^{B_3}\right)^*$ *be an extremal element. Then* $\alpha < 2$, *i.e., the multipilcity of the trivial representation in* W_ℓ *is smaller than 2.*

Proof. If $\alpha = 2$ then $e_1(\underline{X}^2)^2 \in W_\ell$ and hence $e_1(\underline{X}^2)^4 \in W_\ell^{\langle 2\rangle} = \ker \ell$. However, any monomial of degree 8 that is a square occurs with positive coefficients in $e_1(\underline{X}^2)^4$, which implies $\ell = 0$ must be the zero map. $\square$

Lemma 3.31. *Let* $\ell \in \left(\Sigma_{3,8}^{B_3}\right)^*$ *be an extremal element and* $\alpha = 0$. *Then* ℓ *is a scalar of the point-evaluation* ev_z, *where* $z \in \{(1,1,1),(1,0,0),(1,1,0)\}$.

Proof. In the case $\beta = 2$ we know by dimension reasons on $W_\ell^{\langle 2\rangle}$ that any other B_3-module occurring in W_ℓ must already be contained in $2 \cdot S^{((2,1),\emptyset)}$. However, the forms in the module $2 \cdot S^{((2,1),\emptyset)}$ have the common zero $(1,1,1)$.
If $\beta = 1$, then it must be $\gamma \geq 1$ or $\delta = 1$ such that $W_\ell^{\langle 2\rangle}$ is a hyperplane. For $\delta = 1$ the elements in W_ℓ have the common root $(1,1,1)$. Now, we consider the case $\beta = 1, \gamma \geq 1$. Thus for some pairs $(a,b),(c,d) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$ae_1(\underline{X}^2)(X_3^2 - X_1^2) + b(X_2^2X_3^2 - X_1^2X_2^2),\, ce_1(\underline{X}^2)X_2X_3 + dX_1^2X_2X_3 \in W_\ell,$$

and their symmetrized products with elements in $H_{3,4}$ are contained in $W_\ell^{\langle 2\rangle}$, i.e.,

$$\begin{aligned}
0 =& a\left(\frac{2}{3}m_{(1^4)} - 2m_{(2,1^2)}\right) + b\left(\frac{1}{3}m_{(2,1^2)} - 3m_{(3,1)}\right),\\
0 =& a\left(\frac{1}{3}m_{(2,1^2)} - 3m_{(3,1)}\right) + b\left(\frac{2}{3}m_{(2^2)} - 2m_{(3,1)}\right),\\
0 =& \frac{c}{3}m_{(2,1^2)} + dm_{(3,1)},\\
0 =& cm_{(3,1)} + \frac{d}{3}m_{(3,1)}.
\end{aligned}$$

We now distinguish between $m_{(3,1)}$ equals or not equals zero:

i) In the case that $m_{(3,1)} \neq 0$ we have that $c + \frac{d}{3} = 0$. Since W_ℓ is a linear space we can set $c = 1$ and $d = -3$. However, then the B_3-module W_ℓ has the common zero $(1,1,1)$. Thus ℓ is a scalar of the point-evaluation $\mathrm{ev}_{(1,1,1)}$.

ii) Let $m_{(3,1)} = 0$. We first assume that $c \neq 0$. Then $m_{(2,1^2)} = 0$ and since $m_{(1^4)} > 0$ it is $a = 0$. Hence, $b \neq 0$ and $m_{(2^2)} = 0$ which implies that the elements in W_ℓ all vanish at $(1,0,0)$ and ℓ is a scalar of $\mathrm{ev}_{(1,0,0)}$.
If $c = 0$ we have

$$\begin{aligned}
0 =& a\left(\frac{2}{3}m_{(1^4)} - 2m_{(2,1^2)}\right) + b\left(\frac{1}{3}m_{(2,1^2)}\right),\\
0 =& a\left(\frac{1}{3}m_{(2,1^2)}\right) + b\left(\frac{2}{3}m_{(2^2)}\right).
\end{aligned}$$

If $a=0$ then ℓ is a scalar of $\mathrm{ev}_{(1,0,0)}$, since any form in $W_\ell^{\langle 2\rangle}$ has the zero $(1,0,0)$. Otherwise, we may assume that $a=1$ since $W_\ell^{\langle 2\rangle}$ is a linear space. It is

$$0 = \frac{2}{3}m_{(1^4)} + (-2+\frac{b}{3})m_{(2,1^2)},$$
$$0 = \frac{1}{3}m_{(2,1^2)} + \frac{2b}{3}m_{(2^2)}.$$

Through scaling of ℓ and $m_{(1^4)}>0$, we can assume that $m_{(1^4)}=1$. If $b=0$, then $0=m_{(1^4)}=1$ which cannot be true. So $b\neq 0$ and $m_{(2,1^2)}=\frac{2}{6-b}, m_{(2^2)}=\frac{1}{-6b+b^2}$, for a non zero $b\neq 6$. From the positive semidefiniteness conditions in Corollary 3.28 we obtain from the first matrix

$$\det\begin{pmatrix} 1 & m_{(2,1^2)} \\ m_{(2,1^2)} & m_{(2^2)} \end{pmatrix} \geq 0,$$

which implies that $-2\leq b<0$. And the positive semidefiniteness of the last matrix in 3.28

$$\frac{1}{3}m_{(2,1^2)} - \frac{4}{3}m_{(2^2)} + m_{(3,1)} \geq 0$$

implies that $b\leq -2$ or $0<b<6$. Thus $b=-2$ and ℓ is the point-evaluation $\mathrm{ev}_{(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0)}$.

Finally, if $\gamma\geq 1$, then $\beta=1$ or $\delta=1$. However, we have already examinied the case $\beta=1$. For $\delta=1$ the elements in W_ℓ have the common zero $(1,0,0)$. Thus ℓ is a scalar of $\mathrm{ev}_{(1,0,0)}$. □

Therefore we proceed with the cases where $\alpha=1$, which implies that $ae_1(\underline{X}^2)^2+e_2(\underline{X}^2)\in W_\ell$ for an $a\in\mathbb{R}$, since $e_1(\underline{X}^2)^4\notin W_\ell$. This means for $\ker\ell$

$$am_{(1^4)} + m_{(2,1^2)} = 0,$$
$$am_{(2,1^2)} + m_{(2^2)} = 0.$$

Moreover, since $m_{(1^4)}>0$ and ℓ is a linear form we can set without loss of generality $m_{(1^4)}=1$, as ℓ is then just a positive scalar. The positive semidefinitness conditions with the reductions $m_{(2,1^2)}=-am_{(1^4)}, m_{(2^2)}=a^2m_{(1^4)}$ and $m_{(1^4)}=1$ become to

$$\begin{pmatrix} 1 & -a \\ -a & a^2 \end{pmatrix}, \begin{pmatrix} \frac{2}{3}+2a & -\frac{1}{3}a-3m_{(3,1)} \\ -\frac{1}{3}a-3m_{(3,1)} & \frac{2}{3}a^2-2m_{(3,1)} \end{pmatrix}, \begin{pmatrix} -\frac{1}{3}a & m_{(3,1)} \\ m_{(3,1)} & \frac{1}{3}m_{(3,1)} \end{pmatrix}, \left(\frac{-a}{3}-\frac{4a^2}{3}+m_{(3,1)} \right) \succeq 0. \tag{3.1}$$

From the positive semidefiniteness of the second matrix and $-a=m_{(2,1^2)}\geq 0$ we obtain $a\in[\frac{-1}{3},0]$.

We now proceed with a case distinction on the paramaters β,γ,δ:

Lemma 3.32. *Let $\ell\in\left(\Sigma_{3,8}^{B_3}\right)^*$ be an extremal element. If $\alpha=\delta=1$, then ℓ is a scalar of a point-evaluation in $(1,1,0)$.*

Proof. $\delta=1$ means that $S^{((1),(1,1))}\subset W_\ell$ which implies $(X_3^2-X_2^2)X_2X_3\in W_\ell$ and

$$-\frac{a}{3}-\frac{4a^2}{3}+m_{(3,1)}=0.$$

Positiveness yields $0\leq m_{(3,1)}=\frac{1}{3}(a+4a^2)$ and therefore that $a\leq -\frac{1}{4}$. We use that the determinant of the second matrix in (3.1) is non-negative, i.e.,

$$0\leq\left(\frac{2}{3}+2a\right)\left(\frac{2}{3}a^2-2m_{(3,1)}\right)-\left(-\frac{1}{3}a-3m_{(3,1)}\right)^2=-\frac{4}{9}a(1+3a)^2(1+4a).$$

This is not satisfied for $a<-\frac{1}{4}$. Hence $a=-\frac{1}{4}, m_{(1^4)}=1, m_{(3,1)}=0, m_{(2,1^2)}=\frac{1}{4}, m_{(2^2)}=\frac{1}{16}$ and ℓ is a scalar of $\mathrm{ev}_{\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)}$. □

Lemma 3.33. *Let $\ell \in \left(\Sigma_{3,8}^{B_3}\right)^*$ be an extremal element. If $\alpha = 1, \gamma \geq 1$, then ℓ is a scalar of a point-evaluation in* $(1,0,0), (1,1,1)$ *or* $\left(\sqrt{\frac{1}{2} + \sqrt{a + \frac{1}{4}}}, \sqrt{\frac{1}{2} - \sqrt{a + \frac{1}{4}}}, 0\right)$*, for* $-\frac{1}{4} \leq a \leq 0$.

Proof. It is $S^{((1),(2))} \subset W_\ell$, i.e., for a pair $(b,c) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$be_1(\underline{X}^2)X_2X_3 + cX_1^2X_2X_3 \in W_\ell$$

and their symmetrized products with elements in $H_{3,4}$ are contained in $W_\ell^{\langle 2\rangle}$, i.e.,

$$\begin{aligned} 0 =& b\frac{-a}{3} + cm_{(3,1)}, \\ 0 =& bm_{(3,1)} + \frac{c}{3}m_{(3,1)}. \end{aligned}$$

Inserting $\frac{ab}{3} = cm_{(3,1)}$ in the second equation gives $b\left(\frac{a}{9} + m_{(3,1)}\right) = 0$.

a) We first assume that $b \neq 0$. Then $m_{(3,1)} = -\frac{a}{9}$. In this case we obtain from the positive semidefiniteness of the second matrix in (3.1) that

$$0 \leq \frac{2}{3}a^2 - 2m_{(3,1)} = \frac{2}{3}a(a + \frac{1}{3}).$$

Thus $a \in \{0, -\frac{1}{3}\}$. If $a = 0$ then $m_{(3,1)} = m_{(2,1^2)} = m_{(2^2)} = 0$ and $\ell = \mathrm{ev}_{(1,0,0)}$. For $a = -\frac{1}{3}$ it is $m_{(3,1)} = \frac{1}{27}, m_{(2,1^2)} = \frac{1}{3}, m_{(2^2)} = \frac{1}{9}$ and $\ell = \mathrm{ev}_{\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)}$.

b) In the remaining case $b = 0$ we can assume by linearity of W_ℓ that $c = 1$, which implies $m_{(3,1)} = 0$. By the non-negativity of the last 1×1 matrix in (3.1), i.e.,

$$0 \leq -\frac{a}{3} - \frac{4a^2}{3} + m_{(3,1)}$$

we obtain $-\frac{1}{4} \leq a \leq 0$. However, for any such $-\frac{1}{4} \leq a \leq 0$ it is $m_{(1^4)} = 1, m_{(3,1)} = -a, m_{(2,1^2)} = a^2, m_{(2^2)} = 0$ and $\ell = \mathrm{ev}_{\left(\sqrt{\frac{1}{2}+\sqrt{a+\frac{1}{4}}},\sqrt{\frac{1}{2}-\sqrt{a+\frac{1}{4}}},0\right)}$.

□

Lemma 3.34. *Let $\ell \in \left(\Sigma_{3,8}^{B_3}\right)^*$ be an extremal element. If $\alpha = \beta = 1$, then ℓ is a scalar of a point-evaluation in* $\left(\sqrt{\frac{1+2\sqrt{1+3a}}{3}}, \sqrt{\frac{1-\sqrt{1+3a}}{3}}, \sqrt{\frac{1-\sqrt{1+3a}}{3}}\right)$*, for* $-\frac{1}{3} \leq a \leq 0$*, or at* $\left(\sqrt{\frac{1-2\sqrt{1+3b}}{3}}, \frac{\sqrt{1+\sqrt{1+3b}}}{3}, \frac{\sqrt{1+\sqrt{1+3b}}}{3}\right)$*, for* $-\frac{1}{3} \leq b \leq -\frac{1}{4}$.

Proof. If $\beta = 1$ then $S^{((2,1),\emptyset)} \subset W_\ell$, i.e., for a pair $(b,c) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$be_1(\underline{X}^2)(X_3^2 - X_1^2) + c(X_2^2X_3^2 - X_1^2X_2^2) \in W_\ell$$

and their symmetrized products with elements in $H_{3,4}$ are contained in $W_\ell^{\langle 2\rangle}$, i.e.,

$$\begin{aligned} 0 =& b\left(\frac{2}{3} + 2a\right) + c\left(-\frac{1}{3}a - 3m_{(3,1)}\right), \\ 0 =& b\left(-\frac{1}{3}a - 3m_{(3,1)}\right) + c\left(\frac{2}{3}a^2 - 2m_{(3,1)}\right). \end{aligned}$$

We distinguish two cases:

i) If $b = 0, c = 1$ or if $b = 1, c = 0$ then $-\frac{1}{3} = a, m_{(3,1)} = \frac{1}{27}$ and $\ell = \mathrm{ev}_{\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)}$.

ii) We continue with the remaining case $b \neq 0$ and $c \neq 0$. Since W_ℓ is a vector space we assume without loss of generality that $b=1$ and obtain $m_{(3,1)} = \frac{2}{9c} + \frac{2a}{3c} - \frac{a}{9}$ and $\frac{2(1+3a)(-3-2c+ac^2)}{9c} = 0$. Hence $a = \frac{-1}{3}$ (then $\ell = \mathrm{ev}_{(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})}$) or $-3-2c+ac^2 = 0$. If $a = 0$ then $c = -\frac{3}{2}$ and $m_{(3,1)} = -\frac{4}{27}$ which does not satisfy the positive semidefiniteness conditions. If $-\frac{1}{3} < a < 0$ then either $c = \frac{1}{a} - \sqrt{\frac{1+3a}{a^2}}$ or $c = \frac{1}{a} + \sqrt{\frac{1+3a}{a^2}}$.

In the first case it is $m_{(1^4)} = 1, m_{(3,1)} = \frac{a\left(1+a\left(6+\sqrt{\frac{1+3a}{a^2}}\right)\right)}{9-9a\sqrt{\frac{1+3a}{a^2}}}, m_{(2,1^2)} = -a, m_{(2^2)} = a^2$.

For any $-\frac{1}{3} < a < 0$ ℓ is the point-evaluation at $\left(\sqrt{\frac{1+2\sqrt{1+3a}}{3}}, \sqrt{\frac{1-\sqrt{1+3a}}{3}}, \sqrt{\frac{1-\sqrt{1+3a}}{3}}\right)$.

In the second case it is $m_{(1^4)} = 1, m_{(3,1)} = \frac{a\left(1-a\left(-6+\sqrt{\frac{1+3a}{a^2}}\right)\right)}{9+9a\sqrt{\frac{1+3a}{a^2}}}, m_{(2,1^2)} = -a, m_{(2^2)} = a^2$.

However, $m_{(3,1)} \geq 0$ is equivalent to $-\frac{1}{3} < a \leq -\frac{1}{4}$. For any $-\frac{1}{3} < a \leq -\frac{1}{4}$ ℓ is the point-evaluation at $\left(\sqrt{\frac{1-2\sqrt{1+3a}}{3}}, \frac{\sqrt{1+\sqrt{1+3a}}}{3}, \frac{\sqrt{1+\sqrt{1+3a}}}{3}\right)$.

□

Now, we can easily prove the characterisation of the dual cone of non-negative B_3-invariant octics in Theorem 3.24.

Proof of Theorem 3.24. In Lemmas 3.30, 3.31, 3.32, 3.33 and 3.34 we have seen that the extremal rays in $\left(\Sigma_{3,8}^{B_3}\right)^*$ are all generated by point-evaluations in points, which are precisely the elements in the set

$$\left\{\left(a, \sqrt{1-a^2}, 0\right), (b,c,c) : \frac{1}{2} \leq a \leq 1, 0 \leq b \leq 1, c = \frac{1}{\sqrt{2}}\sqrt{(1-b^2)}\right\}.$$

□

Corollary 3.35. *The set of non-negative even symmetric ternary octics $\mathcal{P}_{3,8}^{B_3}$ is the convex cone generated by the following six forms*

$$e_1(\underline{X}^2)^4 - 3e_1(\underline{X}^2)^2 e_2(\underline{X}^2), -9e_1(\underline{X}^2)e_3(\underline{X}^2) + e_1(\underline{X}^2)^2 e_2(\underline{X}^2), e_2(\underline{X}^2)^2 - 3e_1(\underline{X}^2)e_3(\underline{X}^2),$$
$$e_1(\underline{X}^2)^2 e_2(\underline{X}^2), e_1(\underline{X}^2)e_3(\underline{X}^2), 3e_1(\underline{X}^2)e_3(\underline{X}^2) - 4e_2(\underline{X}^2)^2 + e_1(\underline{X}^2)^2 e_2(\underline{X}^2)$$

and the following two families of forms

$$\left(\ ae_1(\underline{X}^2)^4 + e_1(\underline{X}^2)e_2(\underline{X}^2), ae_1(\underline{X}^2)e_2(\underline{X}^2) + e_2(\underline{X}^2)^2 : -\tfrac{1}{3} \leq a \leq 0\ \right)$$

Proof. These are precisely the sums of squares elements contained in the kernels of extremal rays of $\left(\Sigma_{3,8}^{B_3}\right)^*$. Since by Corollary 3.25 $\Sigma_{3,8}^{B_3} = \mathcal{P}_{3,8}^{B_3}$, these are also precisely the elements in the boundary of the pointed convex cone $\mathcal{P}_{3,8}^{B_3}$. The claim follows from Minkowski's theorem. □

Remark 3.36. *In [20] Harris showed that $\Omega := \{(a,a,b),(0,a,b) : a,b \in \mathbb{R}_{\geq 0}\}$ is a test set for even symmetric ternary octics and used this as main ingredient in his proof of equality. In fact, our description in Theorem 3.24 provides the subset of Ω consisting of all points of norm 1, which was derived by describing $\left(\Sigma_{3,8}^{B_3}\right)^*$.*

It is worth to point out that Harris result does not follow from Hilbert's equality case $\Sigma_{3,4}^{\mathfrak{S}_3} = \mathcal{P}_{3,4}^{\mathfrak{S}_3}$ for the symmetric group under natural identification via the $\mathfrak{S}_3$-isomorphism

$$\begin{array}{cccc} \Phi: & H_{3,8}^{B_3} & \longrightarrow & H_{3,4}^{\mathfrak{S}_3} \\ & \sum_{\alpha\in 2\mathbb{N}_0^3} c_\alpha \underline{X}^\alpha & \longmapsto & \sum_{\alpha \in 2\mathbb{N}_0^3} c_\alpha \underline{X}^{\frac{1}{2}\alpha} \end{array}.$$

For $g \in H_{3,4}^{\mathfrak{S}_3}$ it is $\Phi^{-1}(g) = g(X_1^2, X_2^2, X_3^2)$. Then g is non-negative on the first orthant if and only if $\Phi^{-1}(g)$ is non-negative. However, the example

$$f := e_1(\underline{X}^2)e_3(\underline{X}^2) = (X_1^2 + X_2^2 + X_3^2)(X_1^2X_2^2X_3^2) \in \mathcal{P}_{3,8}^{B_3}, \text{ with } \Phi(f)(-1,-1,1) = -1 < 0$$

shows $\mathcal{P}_{3,4}^{\mathfrak{S}_3} \subsetneq \Phi(\mathcal{P}_{3,8}^{B_3})$.

Now, we explicitly point out the stabilizing process from Theorem 3.21 of B_n-Specht modules in $H_{n,d}$ for a fixed degree and large enough number of variables for the example of even symmetric octics. This allows a uniform description of the sums of squares sets $\Sigma_{n,8}^{B_n}$, as stated in Corollary 3.23.
We work with power means $p_i^{(n)} := \frac{1}{n}\sum_{j=1}^n X_j^i \in \mathbb{R}[\underline{X}]^{\mathfrak{S}_n}$ instead of power sums. The upper index n denotes that $p_i^{(n)}$ is a power mean in n variables. Furthermore, for a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ we write $p_\lambda^{(n)} := p_{\lambda_1}^{(n)} \cdot \ldots \cdot p_{\lambda_l}^{(n)}$. A reason for working with power means is that they are nicely weighted, i.e., for any i, n it is $p_i^{(n)}(1,1,\ldots,1) = 1, p_i^{(n)}(1,0,\ldots,0) = \frac{1}{n}$.

Lemma 3.37. *The B_n-isotypic decomposition of $H_{n,4}$ for $n \geq 4$ is*

$$2 \cdot S^{((n),\emptyset)} \oplus 2 \cdot S^{((n-1,1),\emptyset)} \oplus S^{((n-2,2),\emptyset)} \oplus 2 \cdot S^{((n-2),(2))} \oplus S^{((n-2),(1,1))} \oplus S^{((n-3,1),(2))} \oplus S^{((n-4),(4))}.$$

A symmetry adapted basis for $H_{n,4}$ realising the B_n-isotypic decomposition is generated by the following seven sets of polynomials

$$\begin{aligned}
S^{((n),\emptyset)} &: \left\{p_{(4)}^{(n)}, p_{(2^2)}^{(n)}\right\}, & S^{((n-1,1),\emptyset)} &: \left\{(X_n^2 - X_1^2)p_{(2)}^{(n)}, X_n^4 - X_1^4\right\}, \\
S^{((n-2,2),\emptyset)} &: \left\{(X_1^2 - X_3^2)(X_2^2 - X_4^2)\right\}, & S^{((n-2),(2))} &: \left\{X_{n-1}X_n p_{(2)}^{(n)}, (X_{n-1}^2 + X_n^2)X_{n-1}X_n\right\}, \\
S^{((n-2),(1,1))} &: \left\{(X_n^2 - X_{n-1}^2)X_{n-1}X_n\right\}, & S^{((n-4),(4))} &: \{X_1X_2X_3X_4\}, \\
S^{((n-3,1),(2))} &: \left\{(X_n^2 - X_1^2)X_{n-2}X_{n-1}\right\}.
\end{aligned}$$

Proof. We determine the multiplicity of an irreducible B_n-module $S^{(\lambda,\mu)}$ in $H_{n,4}$ for a multipartition $(\lambda,\mu) \vdash n$ using Theorem 3.9. We can exclude some Λ immediately. The fundamental invariants of degree ≤ 4 are of degree 2 and 4. Only (λ,μ) such that $\mu \vdash n_2$, with $n_2 \leq 4$ can occur, since a corresponding higher Specht polynomial has as a factor the monomial consisting of all products of the X_i's, where i ranges over the entries of the second Young tableaux. Furthermore, we only need to consider partitions (λ,μ) such that $|\mu|$ is even because a factor of the higher Specht polynomial is of degree $|\mu|$, while any additional occurring factor has even degree. We can restrict us to multipartitions (λ,μ) such that there exist $(T,S) \in \mathrm{SYT}(\lambda,\mu)$ with $2\,\mathrm{ch}(T,S) + |\mu| \leq 4$. Therefore a charge ≤ 2 is necessary. We calculated all relevant higher Specht polynomials for $n \geq 4$:

$$\begin{aligned}
S^{((n),\emptyset)} &: \{1\}, & S^{((n-1,1),\emptyset)} &: \left\{X_n^2 - X_1^2, \frac{1}{n}\sum_{i=2}^{n-1} X_i^2(X_n^2 - X_1^2)\right\}, \\
S^{((n-2,2),\emptyset)} &: \left\{(X_1^2 - X_3^2)(X_2^2 - X_4^2)\right\}, & S^{((n-2),(2))} &: \left\{X_{n-1}X_n, \frac{1}{n-2}(X_1^2 + \ldots + X_{n-2}^2)X_{n-1}X_n\right\}, \\
S^{((n-2),(1,1))} &: \left\{(X_n^2 - X_{n-1}^2)X_{n-1}X_n\right\}, & S^{((n-4),(4))} &: \{X_1X_2X_3X_4\}, \\
S^{((n-3,1),(2))} &: \left\{(X_n^2 - X_1^2)X_{n-2}X_{n-1}\right\}.
\end{aligned}$$

Multiplying them with the weighted power sums gives the B_n-symmetry adapted basis of $H_{n,4}$. However, since

$$X_n^4 - X_1^4 \in \langle p_2^{(n)}(X_n^2 - X_1^2), \frac{1}{n}\sum_{i=2}^{n-1} X_i^2(X_n^2 - X_1^2)\rangle_{\mathbb{R}},$$

$$(X_{n-1}^2 + X_n^2)X_{n-1}X_n \in \langle p_2^{(n)} X_{n-1}X_n, \frac{1}{n-2}(X_1^2 + \ldots + X_{n-2}^2)X_{n-1}X_n\rangle_{\mathbb{R}},$$

we can work with the above mentioned symmetry adapted basis. □

Theorem 3.38. *Let $n \geq 4$. An even symmetric n-ary octic $f \in H_{n,8}^{B_n}$ is a sum of squares if and only if there exist positive semidefinite matrices $A^{((n),\emptyset)}, A^{((n-1,1),\emptyset)}, A^{((n-2,2),\emptyset)}, A^{((n-2),(2))} \in \mathbb{R}^{2\times 2}$ and $A^{((n-2),(1,1))}, A^{((n-4),(4))}, A^{((n-3,1),(2))} \in \mathbb{R}^{1\times 1}$ such that*

$$\begin{aligned}\mathfrak{f} = {} & \operatorname{Tr}\left(A^{((n),\emptyset)}B^{((n),\emptyset)}\right) + \operatorname{Tr}\left(A^{((n-1,1),\emptyset)}B^{((n-1,1),\emptyset)}\right) + \operatorname{Tr}\left(A^{((n-2,2),\emptyset)}B^{((n-2,2),\emptyset)}\right)\\ & + \operatorname{Tr}\left(A^{((n-2),(2))}B^{((n-2),(2))}\right) + A^{((n-2),(1,1))}B^{((n-2),(1,1))} + A^{((n-4),(4))}B^{((n-4),(4))}\\ & + A^{((n-3,1),(2))}B^{((n-3,1),(2))},\end{aligned}$$

where

$$\begin{aligned}
B^{((n),\emptyset)} &:= \begin{pmatrix} p_{(4^2)}^{(n)} & p_{(4,2^2)}^{(n)} \\ p_{(4,2^2)}^{(n)} & p_{(2^4)}^{(n)} \end{pmatrix},\\
B^{((n-1,1),\emptyset)} &:= \begin{pmatrix} p_{(4,2^2)}^{(n)} - p_{(2^4)}^{(n)} & p_{(6,2)}^{(n)} - p_{(4,2^2)}^{(n)} \\ p_{(6,2)}^{(n)} - p_{(4,2^2)}^{(n)} & p_{(8)}^{(n)} - p_{(4^2)}^{(n)} \end{pmatrix},\\
B^{((n-2,2),\emptyset)} &:= \left(\frac{-n+1}{n^2}p_{(8)}^{(n)} + \frac{4n-4}{n^2}p_{(6,2)}^{(n)} + \frac{n^2-3n+3}{n^2}p_{(4^2)}^{(n)} - 2p_{(4,2^2)}^{(n)} + p_{(2^4)}^{(n)}\right),\\
B^{((n-2),(2))} &:= \begin{pmatrix} p_{(2^4)}^{(n)} - \frac{1}{n}p_{(4,2^2)}^{(n)} & 2p_{(4,2^2)}^{(n)} - \frac{2}{n}p_{(6,2)}^{(n)} \\ 2p_{(4,2^2)}^{(n)} - \frac{2}{n}p_{(6,2)}^{(n)} & 2p_{(6,2)}^{(n)} + 2p_{(4^2)}^{(n)} - \frac{4}{n}p_8^{(n)} \end{pmatrix},\\
B^{((n-2),(1,1))} &:= \left(p_{(6,2)}^{(n)} - p_{(4^2)}^{(n)} \right),\\
B^{((n-4),(4))} &:= \left(p_{(2^4)}^{(n)} - \frac{6}{n}p_{(4,2^2)}^{(n)} + \frac{3}{n^2}p_{(4^2)}^{(n)} + \frac{8}{n^2}p_{(6,2)}^{(n)} - \frac{6}{n^3}p_{(8)}^{(n)}) \right),\\
B^{((n-3,1),(2))} &:= \left(\frac{2}{n^2}p_{(8)}^{(n)} - \frac{2n+2}{n^2}p_{(6,2)}^{(n)} - \frac{1}{n}p_{(4^2)}^{(n)} + \frac{n+3}{n}p_{(4,2^2)}^{(n)} - p_{(2^4)}^{(n)} \right).
\end{aligned}$$

Proof. The matrices $B^{(i)}$ are the matrices containing the symmetrized products of the symmetry adapted basis of the B_n-module $H_{n,4}$ from Lemma 3.37. By Theorem 2.5 any invariant sums of squares form has such a representation. □

We observe that for $n \geq 4$ the $\mathbb{R}$-vector spaces

$$H_{n,8}^{B_n} = \langle p_{(2^4)}^{(n)}, p_{(4,2^2)}^{(n)}, p_{(4^2)}^{(n)}, p_{(4,2)}^{(n)}, p_{(6,2)}^{(n)}, p_8^{(n)}\rangle_{\mathbb{R}}$$

are of the same dimension and thus can be identified. We identify the vector spaces with respect to the isomorphisms

$$p_\lambda^{(n)} \mapsto p_\lambda^{(m)}$$

for $n, m \in \mathbb{N}_{\geq 4}$. In [7] Blekherman and the second author studied symmetric quartic forms and defined a limit set consisting of all $\left(p_\lambda^{(n)}\right)_{n\geq 4}$. They showed that for symmetric quartics the limit sets of the cones of symmetric sums of squares and non-negative quartics are equal. As a first step towards a similar result in the B_n case we provide a classification of the limit set of the cones of even symmetric octics that are sums of squares.

Remark 3.39. *The matrices in Theorem 3.38 have the following limits for $n \to \infty$*

$$\begin{aligned}
\mathcal{B}^{((n),\emptyset)} &:= \begin{pmatrix} \mathfrak{p}_{(4^2)} & \mathfrak{p}_{(4,2^2)} \\ \mathfrak{p}_{(4,2^2)} & \mathfrak{p}_{(2^4)} \end{pmatrix} \\
\mathcal{B}^{((n-1,1),\emptyset)} &:= \begin{pmatrix} \mathfrak{p}_{(4,2^2)} - \mathfrak{p}_{(2^4)} & \mathfrak{p}_{(6,2)} - \mathfrak{p}_{(4,2^2)} \\ \mathfrak{p}_{(6,2)} - \mathfrak{p}_{(4,2^2)} & \mathfrak{p}_{(8)} - \mathfrak{p}_{(4^2)} \end{pmatrix}, \\
\mathcal{B}^{((n-2,2),\emptyset)} &:= \begin{pmatrix} \mathfrak{p}_{(4^2)} - 2\mathfrak{p}_{(4,2^2)} + \mathfrak{p}_{(2^4)} \end{pmatrix}, \\
\mathcal{B}^{((n-2),(2))} &:= \begin{pmatrix} \mathfrak{p}_{(2^4)} & 2\mathfrak{p}_{(4,2^2)} \\ 2\mathfrak{p}_{(4,2^2)} & 2\mathfrak{p}_{(6,2)} + 2\mathfrak{p}_{(4^2)} \end{pmatrix}, \\
\mathcal{B}^{((n-2),(1,1))} &:= \begin{pmatrix} \mathfrak{p}_{(6,2)} - \mathfrak{p}_{(4^2)} \end{pmatrix}, \\
\mathcal{B}^{((n-4),(4))} &:= \begin{pmatrix} \mathfrak{p}_{(2^4)} \end{pmatrix}, \\
\mathcal{B}^{((n-3,1),(2))} &:= \begin{pmatrix} \mathfrak{p}_{(4,2^2)} - \mathfrak{p}_{(2^4)} \end{pmatrix}.
\end{aligned}$$

Corollary 3.40. *An even symmetric homogeneous octic limit sum of squares inequality $\mathfrak{f}$ has the form*

$$\begin{aligned}
\mathfrak{f} = {} & \alpha_1 \mathfrak{p}_{(4^2)} + 2\alpha_2 \mathfrak{p}_{(4,2^2)} + \alpha_3 \mathfrak{p}_{(2^4)} \\
& + \beta_1(\mathfrak{p}_{(4,2^2)} - \mathfrak{p}_{(2^4)}) + 2\beta_2(\mathfrak{p}_{(6,2)} - \mathfrak{p}_{(4,2^2)}) + \beta_3(\mathfrak{p}_{(8)} - \mathfrak{p}_{(4^2)}) \\
& + \delta_1 \mathfrak{p}_{(2^4)} + 4\delta_2 \mathfrak{p}_{(4,2^2)} + \delta_3(2\mathfrak{p}_{(6,2)} + 2\mathfrak{p}_{(4,2^2)}) \\
& + \epsilon(\mathfrak{p}_{(6,2)} - \mathfrak{p}_{(4^2)}),
\end{aligned}$$

where $\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{pmatrix}, \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_2 & \beta_3 \end{pmatrix}, \begin{pmatrix} \delta_1 & \delta_2 \\ \delta_2 & \delta_3 \end{pmatrix}, (\epsilon)$ *are positive semidefinite real matrices.*

Proof. We observe that an invariant limit sum of squares coming from the irreducible representation $S^{((n-2,2),\emptyset)}$, i.e., $\mathfrak{p}_{(4^2)} - 2\mathfrak{p}_{(4,2^2)} + \mathfrak{p}_{(2^4)}$, is contained in the first line. The limit sum of squares $\mathfrak{p}_{(2^4)}$ from $S^{((n-4),(4))}$ is contained in the first line, while the limit form from $S^{((n-3,1),(2))}$, i.e., $\mathfrak{p}_{(4,2^2)} - \mathfrak{p}_{(2^4)}$ is contained in the second line for $\beta_1 = 1$. □

Remark 3.41. *We compare the cone $\Sigma^{B_\infty}_{\infty,8}$ (consisting of all limit forms from Corollary 3.40) with the cone $\Sigma^{\mathfrak{S}_\infty}_{\infty,4}$ characterized in [7] via the $\mathfrak{S}_\infty$-homomorphism $\Phi : H^{B_\infty}_{\infty,8} \to H^{\mathfrak{S}_\infty}_{\infty,4}$. It was shown in [5] that a limit symmetric quartic $\mathfrak{f}$ is a sum of squares if and only if*

$$\mathfrak{f} = \alpha_1 \mathfrak{p}_{(2^2)} + 2\alpha_2 \mathfrak{p}_{(2,1^2)} + \alpha_3 \mathfrak{p}_{(1^4)} + \beta_1(\mathfrak{p}_{(2,1^2)} - \mathfrak{p}_{(1^4)}) + 2\beta_2(\mathfrak{p}_{(3,1)} - \mathfrak{p}_{(2,1^2)}) + \beta_3(\mathfrak{p}_{(4)} - \mathfrak{p}_{(2^2)})$$

for positive semidefinite matrices $\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{pmatrix}$ *and* $\begin{pmatrix} \beta_1 & \beta_2 \\ \beta_2 & \beta_3 \end{pmatrix}$. *However,* $\Phi(\mathfrak{p}_{(6,2)} - \mathfrak{p}_{(4^2)}) = \mathfrak{p}_{(3,1)} - \mathfrak{p}_{(2^2)} \notin \Sigma^{S}_{\infty,4} = \mathcal{P}^{S}_{\infty,4}$. *Thus, we can conclude that the limit cones $\Phi\left(\Sigma^{B_\infty}_{\infty,8}\right)$ and $\Sigma^{\mathfrak{S}_\infty}_{\infty,4} = \mathcal{P}^{\mathfrak{S}_\infty}_{\infty,4}$ are different. This is not surprising, since the cone $\Phi\left(\mathcal{P}^{B_\infty}_{\infty,8}\right)$ can be identified with the limit of all symmetric forms that are non-negative on the positive orthant (compare with Polya's Nichtnegativenstellensatz [28]).*

It is a question for further studies to determine the relation between the limit cones of even symmetric sums of squares and non-negatives octics.

3.3. **Forms invariant under D_n.** It is a natural question to wonder, to what extend Harris' result on ternary forms invariant under B_3 carries over to the slightly smaller group D_3. As is shown in the following theorem we obtain equality between the sets $\Sigma^{D_3}_{3,8}$ and $\mathcal{P}^{D_3}_{3,8}$. Furthermore, we prove that $\mathcal{P}^{D_4}_{4,4}$ is a simplicial cone which gives a test set for non-negativity consisting of three points. Moreover, we prove that for quaternary quartics invariant by D_4 we also have that non-negativity implies a sums of squares representation. We conclude this subsection with a full characterization of the non-negativity versus sums of squares question for forms invariant under D_n.

Theorem 3.42. *The sets of non-negative and sums of squares ternary octics invariant by D_3 are equal, i.e., $\Sigma_{3,8}^{D_3} = \mathcal{P}_{3,8}^{D_3}$.*

Proof. The invariant ring $\mathbb{R}[X_1, X_2, X_3]^{D_3} = \mathbb{R}[p_2, e_3, p_4]$ is a polynomial ring in the symmetric polynomials p_2, e_3 and p_4. A vector space basis of $H_{3,8}^{D_3}$ is given by $\left(p_{(2^4)}, p_{(4,2^2)}, p_{(4^2)}, p_2 e_3^2\right)$. In Remark 3.29 we have seen that $H_{3,8}^{B_3} = \langle p_{(2^4)}, p_{(4,2^2)}, p_{(4^2)}, p_{(6,2)}\rangle_{\mathbb{R}}$. Since in the following identity for symmetric functions in three variables

$$p_{(2^3)} - 3p_{(4,2)} + 2p_6 - 6e_3^2 = 0$$

the functions p_6 and e_3^2 occur linearly, we deduce that $H_{3,8}^{D_3} = H_{3,8}^{B_3}$. Now the claim follows by Corollary 3.25. □

Remark 3.43. *In particular we have the same conical generators and test set as in the B_3 case for non-negative ternary octics invariant under D_3, i.e., a form $f \in H_{3,8}^{D_3}$ is non-negative if and only if $f(y) \geq 0$ for all $y \in \{(a, a, b), (0, a, b) : a, b \in \mathbb{R}_{\geq 0}\}$.*

In the following we study quaternary quartics invariant by D_4.

Lemma 3.44. *The D_4-module $H_{4,2}$ has the isotypic decomposition*

$$H_{4,2} = S^{((4),\emptyset)} \oplus S^{((3,1),\emptyset)} \oplus S_1^{((2),(2))} \oplus S_2^{((2),(2))}.$$

The symmetry adapted basis which realizes the D_4-module decomposition of $H_{4,2}$ is the following:

$$S^{((4),\emptyset)} : \left\{p_{(2)}\right\}, \qquad S^{((3,1),\emptyset)} : \left\{X_4^2 - X_1^2\right\},$$
$$S_1^{((2),(2))} : \{X_1X_2 + X_3X_4\}, \qquad S_2^{((2),(2))} : \{X_1X_2 - X_3X_4\}.$$

Proof. By Theorem 3.9 we have to determine the multiplicity of the irreducible D_4-modules labelled by multipartitions $(\lambda, \mu) \vdash 4$ of the form $|\lambda| \geq |\mu|$. We are only interested in higher Specht polynomials of degree 0 or 2, since the only D_4 fundamental invariant of degree ≤ 2 is p_2. Thus it must be $|\mu| \in \{0, 2\}$. If $\mu = \emptyset$, then only the partitions $((4), \emptyset), ((3, 1), \emptyset)$ have precisely one standard Young tableaux whose charge is at most 1, i.e., they occur precisely once in $H_{4,2}$. Any occurring module labelled by (λ, μ) with $|\mu| = 2$ must have a standard Young tableaux with index $(0, 0, 0, 0)$. This can only occur if the word equals $(1, 2, 3, 4)$. Thus only the multipartition $((2), (2))$ has a standard Young tableaux with zero charge. By Theorem 3.9 the module $S^{((2),(2))}$ decomposes into two irreducible D_4-modules $S_1^{((2),(2))}$ and $S_2^{((2),(2))}$. We calculated the relevant higher Specht polynomials according to Theorem 3.9

$$\left\{1,\ X_4^2 - X_1^2,\ X_1X_2 + X_3X_4,\ X_1X_2 - X_3X_4\right\},$$

and find accordingly the above mentioned polynomials. □

Corollary 3.45. *A D_4-invariant quaternary quartic $f \in H_{4,4}^{D_4}$ is a sum of squares if and only if there exist positive numbers $A^{(1)}, A^{(2)}, A^{(3)}, A^{(4)} \in \mathbb{R}_{\geq 0}$ such that*

$$f = A^{(1)}B^{(1)} + A^{(2)}B^{(2)} + A^{(3)}B^{(3)} + A^{(4)}B^{(4)},$$

where

$$B^{((4),\emptyset)} := \left(p_{(2^2)}\right), \qquad B^{((3,1),\emptyset)} := \left(\frac{2}{3}p_{(4)} - \frac{1}{6}p_{(2^2)}\right),$$
$$B_1^{((2),(2))} := \left(\frac{1}{6}p_{(2^2)} - \frac{1}{6}p_{(4)} + 2e_4\right), \qquad B_2^{((2),(2))} := \left(\frac{1}{6}p_{(2^2)} - \frac{1}{6}p_{(4)} - 2e_4\right).$$

Proof. The matrices $B^{(i)}$ are obtained by calculating the Reynolds operator evaluated at squares of the symmetry adapted basis of the irreducible D_4-modules from Lemma 3.44. By Theorem 2.5 any invariant sum of squares form has such a representation. $\square$

Theorem 3.46. *The dual cone of D_4-invariant sum of squares quartics is a simplicial cone with the following description* $\left(\Sigma_{4,4}^{D_4}\right)^* = \operatorname{cone}\left\{\operatorname{ev}_{(1,0,0,0)}, \operatorname{ev}_{(1,1,1,-1)}, \operatorname{ev}_{(1,1,1,1)}\right\}$.

Proof. Let $\ell \in \left(\Sigma_{4,4}^{D_4}\right)^*$ denote an extremal element. Let

$$W_\ell := \alpha \cdot S^{((4),\emptyset)} \oplus \beta \cdot S^{((3,1),\emptyset)} \oplus \gamma \cdot S_1^{((2),(2))} \oplus \delta \cdot S_2^{((2),(2))}$$

denote the D_4-submodule of $H_{4,2}$ which is the kernel of the associated quadratic form, for $\alpha, \beta, \gamma, \delta \in \{0,1\}$. Now, we show that ℓ must be a scalar of one of the three point-evaluations above, respectively that $W_\ell^{\langle 2\rangle}$ must have one of the points as a zero.
Since $p_{(2^2)}$ is not contained in the boundary of $\Sigma_{4,4}^{D_4}$ it must be $\alpha = 0$. Furthermore, $\dim_{\mathbb{R}} W_\ell^{\langle 2\rangle} = 2$ and therefore we have that precisely two of the parameters are non-zero, because the symmetrized squares of the symmetry adapted basis elements belonging to the D_4-modules $S^{((3,1),\emptyset)}$, $S_1^{((2),(2))}$ and $S_2^{((2),(2))}$ are linearly independent.

i) We start by examining the case $\gamma = \delta = 1$. Then $\ell(e_4) = 0, \ell(p_{(2^2)}) = \ell(p_{(4)})$ and

$$W_\ell^{\langle 2\rangle} = \langle e_4, p_{(2^2)} - p_{(4)}\rangle_{\mathbb{R}}.$$

However, $W_\ell^{\langle 2\rangle}$ has the root $(1,0,0,0)$.

Therefore we can proceed with the cases $\gamma = \beta = 1$ or $\beta = \delta = 1$.

ii) We notice that if $\gamma = \beta = 1$ then

$$W_\ell = \langle X_4^2 - X_1^2, X_1X_2 + X_3X_4\rangle_{D_4},$$

but all elements in W_ℓ have the common root $(1,1,1,-1)$.

iii) If $\beta = \delta = 1$ then

$$W_\ell = \langle X_4^2 - X_1^2, X_1X_2 - X_3X_4\rangle_{D_4}$$

with the common root $(1,1,1,1)$.

$\square$

Corollary 3.47. *The set of non-negative and sums of squares quaternary quartics invariant by D_4 are equal, i.e., it is $\Sigma_{4,4}^{D_4} = \mathcal{P}_{4,4}^{D_4}$.*

This does not already follow from $\Sigma_{4,4}^{B_4} = \mathcal{P}_{4,4}^{B_4}$ in [19], because $D_4 \subsetneq B_4$ is a proper subgroup and $H_{4,4}^{D_4} \setminus H_{4,4}^{B_4} \neq \emptyset$. It might be possible that there exist non B_4-invariant, but D_4-invariant positive forms which are not sums of squares.

Proof. By Theorem 3.46 the cone $\left(\Sigma_{4,4}^{D_4}\right)^*$ is generated by point-evaluations. Hence any extremal ray in $\left(\Sigma_{4,4}^{D_4}\right)^*$ is spanned by a point-evaluation. The claim follows from Corollary 2.42. $\square$

By reformulating Theorem 3.46 we obtain the following very simple test set for D_4-quartics:

Corollary 3.48. *A form $f = a(X_1^2+X_2^2+X_3^2+X_4^2)^2+b(X_1^4+X_2^4+X_3^4+X_4^4)+cX_1X_2X_3X_4$, with $a,b,c \in \mathbb{R}$, is non-negative if and only if $f(z) \geq 0$ for all $z \in \{(1,0,0,0),(1,1,1,-1),(1,1,1,1)\}$.*

Proof. An invariant form $f \in H_{4,4}^{D_4}$ is non-negative if and only if $\ell(f) \geq 0$ for any ℓ in $\left(\mathcal{P}_{4,4}^{D_4}\right)^*$. By Corollary 3.47 it is $\left(\mathcal{P}_{4,4}^{D_4}\right)^* = \left(\Sigma_{4,4}^{D_4}\right)^*$. The claim follows from Theorem 3.46, since the cone is generated by the point-evaluations in $(1,0,0,0),(1,1,1,-1),(1,1,1,1)$. □

Corollary 3.49. *The convex cone $\mathcal{P}_{4,4}^{D_4}$ of non-negative D_4-quartics is a simplicial cone generated by*

$$4p_{(4)} - p_{(2^2)}, p_{(2^2)} - p_{(4)} + 12e_4, \text{ and } p_{(2^2)} - p_{(4)} - 12e_4.$$

Proof. The sets $\mathcal{P}_{4,4}^{D_4}$ and $\Sigma_{4,4}^{D_4}$ are equal by Corollary 3.47. The boundary of $\Sigma_{4,4}^{D_4}$ is equal to the union of all kernels of extremal elements in $\left(\Sigma_{4,4}^{D_4}\right)^*$ intersected with $\Sigma_{4,4}^{D_4}$. The above generators are precisely the invariant sums of squares contained in the kernels of the three extremal rays in Theorem 3.46. □

The results from the previous two subsections allow us to conclude the following classification for the equivariant non-negativity versus sums of squares question in case of the reflection group of type D_n.

Theorem 3.50. $\Sigma_{n,2d}^{D_n} = \mathcal{P}_{n,2d}^{D_n}$ *if and only if* $(n, 2d) \in \{(n,2),(n,4),(3,8)\}$.

Proof. Suppose that there exists $f \in \mathcal{P}_{n,2d}^{B_n} \setminus \Sigma_{n,2d}^{B_n}$. This implies $f \in \mathcal{P}_{n,2d}^{D_n} \setminus \Sigma_{n,2d}^{D_n}$. Therefore, we can directly rely on the classification carried out in [19] and only need to consider those cases specifically, where all even symmetric positive semidefinite forms are sums of squares. These are only the following non-trivial cases: $(n,2d) \in \{(3,8),(n,4)\}$. But we have shown in Theorem 3.42 that in the case $(3,8)$ the equality does survive, and while following Corollary 3.47 it does also for $(4,4)$. Furthermore, if $n > 4$ then the invariant quartics with respect to B_n are precisely the invariant quartics with respect to D_n as $H_{n,4}^{B_n} = \langle p_{(2^2)}, p_{(4)} \rangle_{\mathbb{R}} = H_{n,4}^{D_n}$ for $n \geq 5$, which finishes the proof. □

3.4. **LMIs and non-negativity testing.** In general testing non-negativity of a polynomial in more than two variables is already for quartics an NP-hard problem (see e.g. [8] or [27]). On the other hand, certifying that a given polynomial is a sum of squares can be done with so called semi-definite programming. Although the complexity status of this procedure in the Turing or in the real numbers model is not yet known (see [30]) SDPs can be solved numerically in polynomial time to a given accuracy via the ellipsoid algorithm and this approach generally provides a tractable way to certify that a polynomial is non-negative, if it is a sum of squares. The feasible region of a semi-definite program is given by a linear matrix inequality (LMI), i.e., an inequality of the form $A_0 + x_1A_1 + x_2A_2 + \ldots + x_nA_n \succeq 0$, where $A_0, \ldots, A_n$ are real symmetric matrices all of the same size and $x_1, \ldots, x_n$ are supposed to be real scalars. The set of all $x \in \mathbb{R}^n$ satisfying a given LMI is called a *spectrahedron*. For every $f \in H_{n,2d}$ one can construct an LMI ([29]) which possesses a solution if and only if f is a sum of squares. The corresponding spectrahedron is called the *Gram spectrahedron* of f[12], and it represents in fact all possible ways to decompose f into sums of squares. Accordingly it is non-empty if and only if f is a sum of squares. The results presented in the article can be directly transferred into the setup of symmetry adapted Gram-spectrahedra, which were, for example, recently studied by [21].

Theorem 3.51. *Let G be a finite reflection group and consider $f \in H_{n,2d}^G$ and $\theta_1, \ldots, \theta_l$ be all non G-isomorphic irreducible representations. Then the Gram spectrahedron of f can be defined by a block diagonal matrix, consisting of l blocks $B_1, \ldots, B_l$ and the size of the block B_i equals*

$$\sum_{k=0}^{d} N(d-k) \cdot h_k^{\vartheta_i}.$$

In particular, in the case $G \in \{A_{n-1}, B_n, D_n\}$ the size of the matrix is independent of n, for large n

Proof. This follows from choosing a symmetry adapted basis of $H_{n,d}$ and Corollary 2.25. When $G \in \{A_{n-1}, B_n, D_n\}$ the stabilization follows from Corollary 3.23 $\square$

A convex set which is not a spectrahedron but can be obtained as the projection of higher dimensional spectrahedron is called *spectrahedral shadow*. Following a question by Nemirovski, which convex sets can be represented as projections of spectrahedra, Scheiderer [37] showed that the cones of non-negative forms in general are not spectrahedral shadows. In the next theorem we give some examples of invariant non-negative forms, which form spectrahedral shadows.

Theorem 3.52. *For all n the families of cones $\mathcal{P}_{n,4}^{\mathfrak{S}_n}$, $\mathcal{P}_{n,6}^{B_n}$, $\mathcal{P}_{n,8}^{B_n}$ and $\mathcal{P}_{n,10}^{B_n}$ are spectrehedral shadows. Moreover, for forms in any of these families there exists an LMI of size $O(n^3)$ certifying the non-negativity.*

Proof. For $n \leq 2$ this is trivial and in the case $n = 3$ this follows either from Hilbert's Theorem in the $\mathfrak{S}_3$ case or from Harris' result 3.25 in the B_3 case. So we assume $n \geq 4$. By the half-degree principle an element $f \in H_{n,4}^{\mathfrak{S}_n}$ is non-negative on $\mathbb{R}^n$ if and only if for any partition $\lambda \vdash n$ of length 2 the form $f^\lambda \in H_{2,4}$ is non-negative on $\mathbb{R}^2$, where $f^\lambda(x, y) := f(x, \ldots, x, y, \ldots, y)$ and x occurs precisely λ_1 times and y λ_2 times. Notice that each f^λ is non-negative if and only if it is a sum of squares, i.e., if we have $f^\lambda \in \Sigma_{2,4}$. If we denote by Φ^λ the linear map $f \mapsto \tilde{f}^\lambda(x, y)$ and if $\lambda^1, \ldots, \lambda^m$ are all partitions of n with length 2 then

$$\mathcal{P}_{n,4}^{\mathfrak{S}_n} = \bigcap_{i=1}^{m} \left(\Phi^{\lambda^i}\right)^{-1} (\Sigma_{2,4})$$

which proves the claim in the $\mathfrak{S}_n$ case. Using the half-degree principle [34, Theorem 3.1] for B_n and considering instead of $f(\underline{X}) \in \mathbb{R}[\underline{X}]^{B_n}$ the form $f(\sqrt{|X_1|}, \ldots, \sqrt{|X_n|}) \in \mathbb{R}[\underline{X}]^{\mathfrak{S}_n}$, one can argue analogously with slight modifications. $\square$

Remark 3.53. *In the case of symmetric polynomials, the above statement was implicitly already stated in* [35, Theorem 5.5] *for symmetric quartic forms, albeit without mentioning of the term spectrahedral shadow.*

The core of the proof above is the reduction to bivariate forms via test sets.

Theorem 3.54. *For the families of cones $\mathcal{P}_{n,6}^{\mathfrak{S}_n}$, $\mathcal{P}_{n,12}^{B_n}$ and $\mathcal{P}_{n,14}^{B_n}$ membership can be decided with $O(n^3)$ many LMIs, each of which has size bounded independent of n.*

Proof. Using the half-degree principle [34, Theorem 3.1] one finds that the membership in each of the above mentioned cones can be decided by reducing the $O(n^3)$ many ternary forms, similarly to the proof above. For each of these ternary forms one can decide non-negativity individually. De Klerk and Pasechnik [14] provided a construction to decide non-negativity of a ternary from of degree $2d$ by means of $d/4$ LMIs each of which is polynomial in d. Combining their construction with the arguments above thus yields an LMI of the announced size. $\square$

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