On Lévy and Markov additive friendships

Stochastics Seminar Aarhus

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Positive and negative definite functions

- a function $\varphi\colon \mathbb{R} \to \mathbb{C}$ is called positive definite, if

$$\forall n \in \mathbb{N}, (x_i)_{i=1,\dots,n} \in \mathbb{R}^n, (c_i)_{i=1,\dots,n} \in \mathbb{C}^n : \quad \sum_{i,j=1}^{n} \varphi(x_i - x_j) c_i \overline{c_j} \geqslant 0$$

- a function $\psi\colon \mathbb{R} \to \mathbb{C}$ is called negative definite if
  1. $\psi(0) \geqslant 0$
  2. $\psi(-\cdot) = \overline{\psi}$
  3. $$\forall n \in \mathbb{N}, (x_i)_{i=1,\dots,n} \in \mathbb{R}^n, (c_i)_{i=1,\dots,n} \in \mathbb{C}^n \text{ s.t. } \sum_{i=1}^{n} c_i = 0 : \quad \sum_{i,j=1}^{n} \psi(x_i - x_j) c_i \overline{c_j} \leqslant 0$$

Relation to probability theory

$$
\begin{aligned}
\psi \text{ is cont. negative definite} &\overset{\text{Schoenberg}}{\iff} \psi(0) \geqslant 0 \text{ and } x \mapsto \mathrm{e}^{-\psi(x)} \text{ is cont. positive definite} \\
&\overset{\text{Bochner}}{\iff} \mathrm{e}^{-\psi(x)} = \widehat{\mu}(x) \text{ for some finite measure } \mu \text{ w. } \mu(\mathbb{R}) \leqslant 1 \\
&\overset{\text{LK}}{\iff} \mu/\mu(\mathbb{R}) \text{ is infinitely divisible with char. exponent } \psi - \psi(0) \\
&\iff \exists \text{ killed Lévy process } (X_t)_{t \geqslant 0} \text{ with char. exponent } \psi,
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$\rightsquigarrow$ $1:1$-relation between cont. negative definite functions and Lévy processes. In particular,

$$\psi(\theta) = q - \mathrm{i}a\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} \left(1 - e^{\mathrm{i}\theta x} + \mathrm{i}\theta x \mathbf{1}_{[-1,1]}(x)\right) \Pi(\mathrm{d}x), \quad \theta \in \mathbb{R},$$

where the Lévy measure Π is a σ-finite measure s.t. $\Pi(\{0\}) = 0$ and $\int_{\mathbb{R}} (1 \wedge x^2)\, \Pi(\mathrm{d}x) < \infty$

Operations on negative definite functions in terms of Lévy processes

cont. negative definite functions	Lévy interpretation
$(\psi, \phi) \mapsto \psi + \phi$	Summation of independent Lévy processes $\rightsquigarrow$ $(X, Y) \mapsto X + Y$
$\psi \mapsto a \cdot \psi$ for some $a > 0$	speed up Lévy process X by factor a $\rightsquigarrow$ $(X_t)_{t \geqslant 0} \mapsto (X_{at})_{t \geqslant 0}$
$\psi \mapsto \psi(-\cdot)$	reflection of Lévy process (dual) $\rightsquigarrow X \mapsto -X$
$(\psi, \phi) \mapsto \phi(i\psi)$ with ϕ being analytical extension from $\mathbb{R}$ to $\{z \in \mathbb{C} : \Im z \geqslant 0\}$ of char. exponent of a subordinator	subordination of Lévy process X by independent subordinator $S \rightsquigarrow (X, S) \mapsto X_S$.

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Question

For which types of LK-exponents ψ_1, ψ_1 is $\psi := \psi_1 \cdot \psi_2$ a LK-exponent and how can we interpret this probabilistically?

Vigon's équations amicales and théorème des amis

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- We say that H^+ and H^- are compatible if $d^{\mp} > 0$ implies that $\Pi^{\pm} \ll \text{Leb}$ with densities $\partial\Pi^+(x) = \nu^+(x, \infty)$ and $\partial\Pi^-(x) = \nu^-(-\infty, -x)$ for some signed measures $\nu^{\pm}$, resp.

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Theorem [Vigon (2002)]

Two subordinators H^+ and H^- are friends if and only they are compatible and the function

$$\Upsilon(x) = \begin{cases} \int_{(0,\infty)} (\Pi^-(y,\infty) + \psi^-(0))\,\Pi^+(x+\mathrm{d}y) + d^-\partial\Pi^+(x), & x > 0, \\ \int_{(0,\infty)} (\Pi^+(y,\infty) + \psi^+(0))\,\Pi^-(-x+\mathrm{d}y) + d^+\partial\Pi^-(-x), & x < 0, \end{cases}$$

is a.e. increasing on $(0,\infty)$ and a.e. decreasing on $(-\infty,0)$. Moreover, if H^+ and H^- are friends, then the tails of the Lévy measure Π of the bonding process are given by $\Upsilon(x) = \Pi(x,\infty)\mathbf{1}_{(0,\infty)}(x) + \Pi(-\infty,x)\mathbf{1}_{(-\infty,0)}(x)$ for a.e. $x \neq 0$.

Vigon's philanthropists

Vigon calls a subordinator with decreasing Lévy density on $(0, \infty)$ a philanthropist.

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Example: hypergeometric Lévy processes with LK-exponent

$$\psi(\theta) = \underbrace{\frac{\Gamma(1 - a^+ + \gamma^+ - \mathrm{i}\theta)}{\Gamma(1 - a^+ - \mathrm{i}\theta)}}_{=\psi^+(\theta)} \times \underbrace{\frac{\Gamma(a^- + \gamma^- + \mathrm{i}\theta)}{\Gamma(a^- + \mathrm{i}\theta)}}_{=\psi^-(-\theta)}, \quad \theta \in \mathbb{R},$$

for appropriate choices of $a^\pm, \gamma^\pm$, where $\psi^\pm$ are LK-exponents of β-subordinators.

Probabilistic interpretation

Wiener–Hopf factorization

Let H^+ and H^- be the ascending/descending ladder height processes of a given Lévy process ξ, i.e., $H_t^+ = \xi_{L_t^{-1}}$ and $H_t^- = \widehat{\xi}_{\widehat{L}_t^{-1}}$ for some versions of local time at the maximum $L, \widehat{L}$ of ξ and its dual $\widehat{\xi}$, resp. Then,

$$\psi(\theta) = c\psi^+(\theta)\psi^-(-\theta), \quad \theta \in \mathbb{R},$$

where the constant $c > 0$ depends on the scaling of local times $L, \widehat{L}$.

Uniqueness of the Wiener–Hopf factorization

Theorem [Döring, Watson, T. (2021+)]

The Wiener–Hopf factorization of ξ is unique if one of the following conditions is satisfied:

1. ξ is killed
2. $\mathbb{E}[\exp(\lambda X_1)] < \infty$ for some $\lambda \neq 0$
3. ξ has non-trivial Brownian component
4. Both $\Pi|_{(-\infty,0)}$ and $\Pi|_{(0,\infty)}$ are not purely singular

In particular, a pair of philanthropists is always equal in law to the ascending resp. descending ladder height processes of their bonding process.

Markov additive processes

- Let (ξ, J) be a MAP on $\mathbb{R} \times \{1, \dots, n\}$
- J Markov chain with transition rate matrix Q
- On $[\sigma(i), \sigma(i+1))$, ξ has the same law as a Lévy process $\xi^{(J_{\sigma(i)})}$ started in $\xi_{\sigma(i)}$
- jumps from i to j trigger jump $\Delta_{i,j}$ of ξ with distribution $F_{i,j}$
- characteristic exponent $\boldsymbol{\Psi}$ given by

$$\boldsymbol{\Psi}(\theta) = \operatorname{diag}\big((\psi_i(\theta))_{i \in [n]}\big) - \boldsymbol{Q} \odot \big(\widehat{F_{i,j}}(\theta)\big)_{i,j \in [n]}, \quad \theta \in \mathbb{R},$$

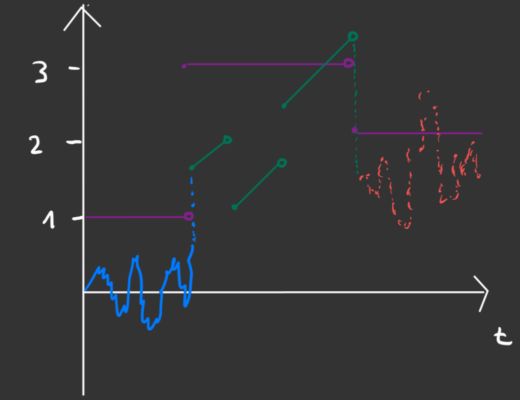
  is such that

$$\mathbb{E}^{0,i}[\exp(\mathrm{i}\theta\xi_t)\mathbf{1}_{\{J_t = j\}}] = \big(\mathrm{e}^{-t\boldsymbol{\Psi}(\theta)}\big)_{i,j}$$

- Lévy system $\boldsymbol{\Pi} = (\Pi_{i,j})_{i,j \in [n]}$ defined by

$$\Pi_{i,i} = \Pi_i, \quad \Pi_{i,j} = q_{i,j} F_{i,j}$$

MAPs in a nutshell



$\xi^{(1)}$ = Brownian motion

$\xi^{(2)}$ = α-stable Lévy process

$\xi^{(3)}$ = compound Poisson with drift

J = cont. time Markov chain on $\{1,2,3\}$

$(\xi_t, J_t)_{t\geq 0}$ = Markov additive process

Matrix Wiener–Hopf factorization for MAPs

MAP Wiener–Hopf factorization [Dereich, Döring, Kyprianou (2018)]

Let (H^+, J^+) and (H^-, J^-) be the ascending/descending ladder height processes of a given MAP (ξ, J), i.e., $(H_t^+, J_t^+) = (\xi_{L_t^{-1}}, J_{L_t^{-1}})$ and $H_t^- = (\widehat{\xi}_{\widehat{L}_t^{-1}}, \widehat{J}_{\widehat{L}_t^{-1}})$ for some versions of local time at the maximum $L, \widehat{L}$ of ξ and $\widehat{\xi}$, resp. Then, for an appropriate scaling of local time,

$$\boldsymbol{\Psi}(\theta) = \boldsymbol{\Delta}_\pi^{-1} \boldsymbol{\Psi}^-(-\theta)^\top \boldsymbol{\Delta}_\pi \boldsymbol{\Psi}^+(\theta), \quad \theta \in \mathbb{R}. \qquad \text{(MAP-WH)}$$

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- $\rightsquigarrow$ (H^+, J^+) is a $\boldsymbol{\pi}$-friend of (H^-, J^-) :$\Leftrightarrow$ LHS of (MAP-WH) is matrix exponent of a MAP (ξ, J) (bonding MAP)
- $\rightsquigarrow$ if WH-factorization of the bonding MAP is unique and $\boldsymbol{\pi}$ is stationary distribution of bonding modulator J, then $(H^\pm, J^\pm)$ are equal in law to ascending/descending ladder height MAP, resp.

Uniqueness of MAP-WH factorization I

Classical result: For a random variable X we have

$$\exists\theta \neq 0 : \mathbb{E}[\exp(i\theta X)] = 1 \iff \exists\theta \neq 0 : \operatorname{supp}(\mathbb{P}_X) \subset \frac{2\pi}{\theta}\mathbb{Z}$$

In particular, for a Lévy process ξ it holds that

$$\exists\theta \neq 0 : \psi(\theta) = 0 \iff \xi_1 \text{ has lattice support}$$

Proposition [Döring, Watson, T. (2021+)]

If J is irreducible and none of the Lévy components $\xi^{(i)}$ has lattice support, then $\Psi(\theta) \in \mathrm{GL}_n(\mathbb{C})$ for any $\theta \neq 0$.

Uniqueness of MAP-WH factorization II

- Let $\mathcal{A}_0$: MAP subordinator exponents s.t. the modulator J is irreducible and the Lévy components and transitional jumps have finite mean
- $\mathcal{A}_\infty$: MAP subordinator exponents s.t. $\lim_{|\theta|\to\infty}|\psi_i(\theta)| = \infty$ for all $i \in [n]$.
- $\mathcal{A}_\ll$: MAP subordinator exponents s.t. Lévy system Π is absolutely continuous

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- $\mathcal{A}_\ll$: MAP subordinator exponents s.t. Lévy system $\boldsymbol{\Pi}$ is absolutely continuous

Theorem [Döring, Watson, T. (2021+)]

If (H^+, J^+) is a $\boldsymbol{\pi}$-friend of (H^-, J^-) and either $(H^\pm, J^\pm) \in (\mathcal{A}_0 \cap \mathcal{A}_\infty)^2$ or $(H^\pm, J^\pm) \in (\mathcal{A}_0 \cap \mathcal{A}_\ll)^2$, then the WH-factorization of the bonding MAP is unique within these classes.

Theorem of friends for MAPs

We introduce a notion of $\boldsymbol{\pi}$-compatibility between two MAP subordinators $(H^{\pm}, J^{\pm})$ which

- generalizes the notion of compatibility of Lévy subordinators
- is necessary for off-diagonal entries to be the negative of characteristic functions
- e.g., $\boldsymbol{\pi}$-compatibility implies that

$$\boldsymbol{\Delta}_{\boldsymbol{d}}^{-}\boldsymbol{\Pi}^{+}(\{0\}) = \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\left(\boldsymbol{\Delta}_{\boldsymbol{d}}^{+}\boldsymbol{\Pi}^{-}(\{0\})\right)^{\top}\boldsymbol{\Delta}_{\boldsymbol{\pi}}.$$

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Theorem [Döring, Watson, T. (2021+)]

(H^+, J^+) is a $\boldsymbol{\pi}$-friend of (H^-, J^-) if and only if (H^+, J^+) is $\boldsymbol{\pi}$-compatible with (H^-, J^-) and the matrix valued function

$$\boldsymbol{\Upsilon}(x) = \begin{cases} \int_{0+}^{\infty} \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1} \Big(\boldsymbol{\Pi}^{-}(y, \infty) + \boldsymbol{\Psi}^{-}(0) \Big)^{\top} \boldsymbol{\Delta}_{\boldsymbol{\pi}} \boldsymbol{\Pi}^{+}(x + \mathrm{d}y) + \boldsymbol{\Delta}_{\boldsymbol{d}}^{-} \partial \boldsymbol{\Pi}^{+}(x), & x > 0, \\ \int_{0+}^{\infty} \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1} (\boldsymbol{\Pi}^{-}(-x + \mathrm{d}y))^{\top} \boldsymbol{\Delta}_{\boldsymbol{\pi}} \left(\boldsymbol{\Pi}^{+}(y, \infty) + \boldsymbol{\Psi}^{+}(0) \right) + \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1} (\boldsymbol{\Delta}_{\boldsymbol{d}}^{+} \partial \boldsymbol{\Pi}^{-}(-x))^{\top} \boldsymbol{\Delta}_{\boldsymbol{\pi}}, & x < 0, \end{cases}$$

is a.e. decreasing on $(0, \infty)$ and a.e. increasing on $(-\infty, 0)$. Moreover, if (H^+, J^+) is a $\boldsymbol{\pi}$-friend of (H^-, J^-), then for a.e. $x \neq 0$, $\boldsymbol{\Upsilon}(x) = \boldsymbol{\Pi}(x, \infty) \mathbf{1}_{(0,\infty)}(x) + \boldsymbol{\Pi}(-\infty, x) \mathbf{1}_{(-\infty,0)}(x)$.

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$$\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\boldsymbol{\Psi}^-(0)^\top \boldsymbol{\Delta}_{\boldsymbol{\pi}}\boldsymbol{\Pi}^+(x, \infty) + \boldsymbol{\Delta}_{\boldsymbol{d}}^- \partial\boldsymbol{\Pi}^+(x),$$

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Theorem [Döring, Watson, T. (2021+)]

Two $\boldsymbol{\pi}$-philanthropists that are $\boldsymbol{\pi}$-fellows of each other are $\boldsymbol{\pi}$-friends.

Application

- So far, the only explicit MAP Wiener–Hopf factorization is the deep factorization of the stable process

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& \alpha\text{-self similar MPs} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow, "\text{Lamperti–Kiu}"] & \\
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- Using our MAP-philanthropy result we construct
    1. spectrally positive MAPs whose ascending ladder height's Lévy components have completely monotone Lévy densities, i.e.,

$$\Pi_i^+(\mathrm{d}x) = \int_{\mathbb{R}_+} \mathrm{e}^{-xy}\, \mu_i^+(\mathrm{d}y)\, \mathrm{d}x, \quad i = 1, 2, x > 0$$

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  2. two-sided MAPs, whose ladder height processes have exponentially distributed jump structure