Kochanek–Bartels Cubic Splines

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Given a sequence of positions $\{P_n\}_{n=0}^{N-1}$, the Kochanek–Bartels splines [1] provide a cubic interpolation between each pair P_n and P_{n+1} with varying properties specified at the endpoints. These properties are *tension* τ which controls how sharply the curve bends at a control point, *continuity* γ which provides a smooth **visual** variation in the continuity at a control point ($\gamma = 0$ yields derivative continuity, but $\gamma \neq 0$ gives discontinuities), and *bias* which controls how the direction of the path at a contrl point by taking weighted combination of one–sided derivatives at that control point.

Using a Hermite interpolation basis $H_0(t) = 2t^3 - 3t^2 + 1$, $H_1(t) = -2t^3 + 3t^2$, $H_2(t) = t^3 - 2t^2 + t$, and $H_3(t) = t^3 - t^2$, a parametric cubic curve passing through points P_n and P_{n+1} with tangent vectors T_n and T_{n+1} respectively is

$$P(t) = H_0(t)P_n + H_1(t)P_{n+1} + H_2(t)T_n + H_3(t)T_{n+1} \tag{1}$$

where $0 \leq t \leq 1$. Catmull–Rom interpolation is the case when $T_n = (P_{n+1} - P_{n-1})/2$, a centered finite difference.

The equation (1) may be modified to allow specification of an "outgoing" tangent T_n^0 at $t = 0$ and an "incoming" tangent T_{n+1}^1 at $t = 1$,

$$P(t) = H_0(t)P_n + H_1(t)P_{n+1} + H_2(t)T_n^0 + H_3(t)T_{n+1}^1. \tag{2}$$

The paper [1] discusses various choices for the ingoing and outgoing tangents.

Tension $\tau \in [-1, 1]$ can be introduced by using

$$T_n^0 = T_n^1 = \frac{(1-\tau)}{2}\left((P_{n+1} - P_n) + (P_n - P_{n-1})\right).$$

If $\tau = 0$, then we have the Catmull–Rom spline. For τ near 1 the curve is "tightened" at the control point while τ near -1 produces "slack" at the control point. Varying τ changes the length of the tangent at the control point, a smaller tangent leading to a tightening and a larger tangent leading to a slackening.

Continuity $\gamma \in [-1, 1]$ can be introduced by using

$$T_n^0 = \left(\frac{1-\gamma}{2}(P_{n+1} - P_n) + \frac{1+\gamma}{2}(P_n - P_{n-1})\right)$$

and

$$T_n^1 = \left(\frac{1+\gamma}{2}(P_{n+1} - P_n) + \frac{1-\gamma}{2}(P_n - P_{n-1})\right).$$

When $\gamma = 0$, the curve has a continuous tangent vector at the control point. As $|\gamma|$ increases, the resulting curve has a "corner" at the control point, the direction of the corner depending on the sign of γ.

Bias $\beta \in [-1, 1]$ can be introduced by using

$$T_n^0 = T_n^1 = \left(\frac{1-\beta}{2}(P_{n+1} - P_n) + \frac{1+\beta}{2}(P_n - P_{n-1}) \right).$$

When $\beta = 0$ the left and right one–sided tangents are equally weighted (producing the Catmull–Rom spline). For β near -1, the outgoing tangent dominates the direction of the path of the curve through the control point (undershooting). For β near 1, the incoming tangent dominates (overshooting).

The three effects may be combined into a single set of equations

$$T_n^0 = \frac{(1-\tau)(1-\gamma)(1-\beta)}{2}(P_{n+1} - P_n) + \frac{(1-\tau)(1+\gamma)(1+\beta)}{2}(P_n - P_{n-1}) \tag{3}$$

and

$$T_n^1 = \frac{(1-\tau)(1+\gamma)(1-\beta)}{2}(P_{n+1} - P_n) + \frac{(1-\tau)(1-\gamma)(1+\beta)}{2}(P_n - P_{n-1}). \tag{4}$$

These formulas assume a uniform spacing in time of the position samples. An adjustment can be made for nonuniform spacing. For equation (4) the multiplier is $2\Delta_n/(\Delta_{n-1} + \Delta_n)$ and for equation (3) the multiplier is $2\Delta_{n-1}/(\Delta_{n-1} + \Delta_n)$ where $\Delta_n = s_{n+1} - s_n$ and s_n is the sample time for position P_n.

References

[1] Doris H. U. Kochanek and Richard H. Bartels, *Interpolating splines with local tension, continuity, and bias control*, ACM SIGGRAPH 1986, Course Notes 22, Advanced Computer Animation.