Relativistic correction term for GPS system time

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Abstract

The GPS Interface Specification Document [1] in chapter 20.3.3.3.3.1 "User Algorithm for SV Clock Correction" provides equation to calculate GPS system time. This equation requires calculation of so called "relativistic correction term" which is given by the following formula

$$\Delta t_r = Fe\sqrt{a}\sin E_k \tag{0.1}$$

where

$F = \dfrac{-2\sqrt{\mu}}{c^2}$

$\mu = 3.986005 * 10^{14}[m^3/s^2]$ - WGS 84 value of the earth's gravitational constant for GPS user

$c = 299792458[m/s]$ - speed of light

e - satellite orbit eccentricity

a - satellite orbit semi-major axis

E_k - satellite orbit eccentric anomaly/angle

This "relativistic correction term" can also be expressed by alternative but equivalent expression

$$\Delta t_r = -\frac{2\boldsymbol{R}\boldsymbol{v}}{c^2} \tag{0.2}$$

where

$\boldsymbol{R}$ - is the instantaneous position vector of the satellite

$\boldsymbol{v}$ - is the instantaneous velocity vector of the satellite

This article is yet another in the series "I want to understand how those expressions were derived". It aims to provide explanations and derivations leading to formulas for "relativistic correction term".

1 Ellipse

To start this process of derivations and explanations we have to begin with the understanding of the equations describing the ellipse.

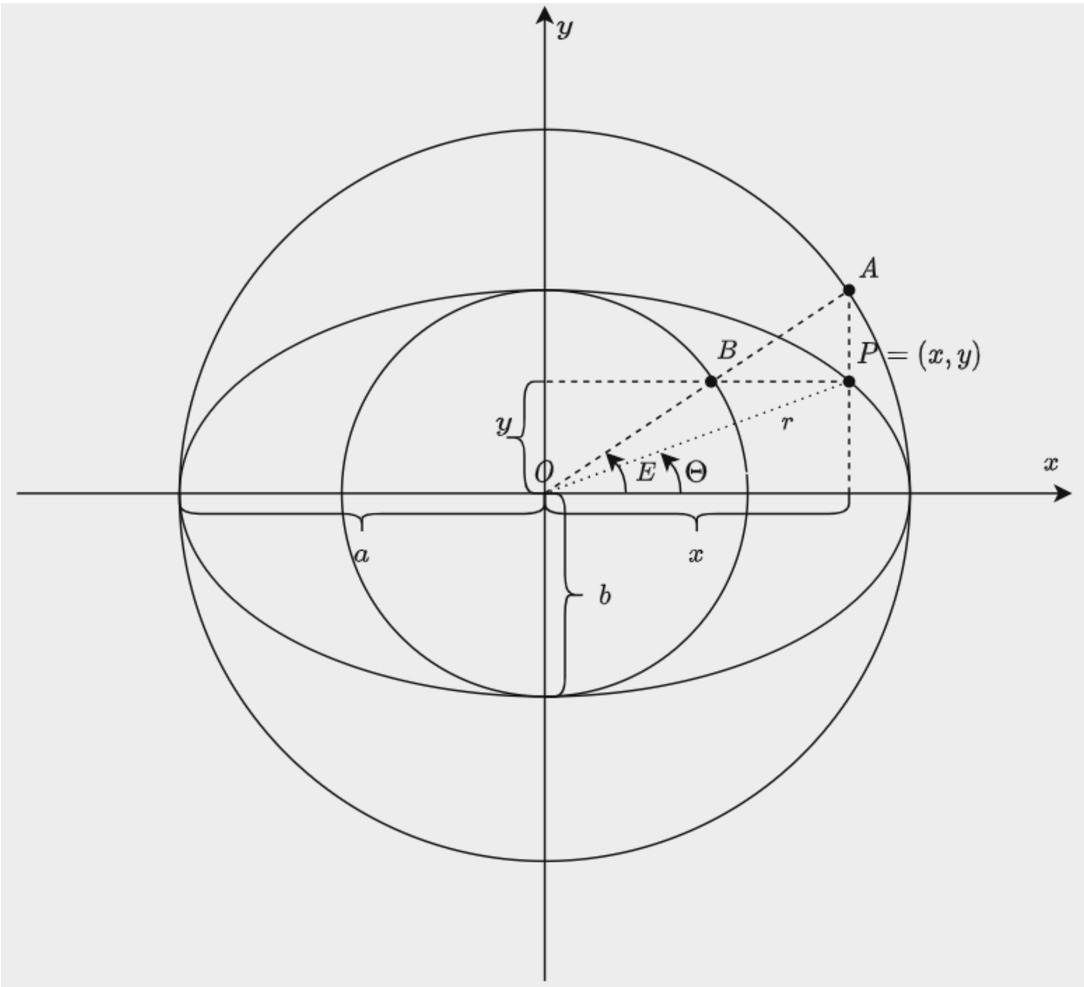

Figure 1: Centre based ellipse

1.1 Equation for centre based ellipse in Cartesian coordinates

We start our derivations from the equation of the ellipse centred at the origin of the Cartesian coordinate system whose x axis coincide with major and y axis coincide with minor axis of the ellipse. If we denote semi-major axis of the ellipse as a and semi-minor axis as b, then the equation of the ellipse gains the following analytical form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{1.1}$$

1.2 Equation for centre based ellipse in terms of eccentric angle

From figure 1 we read that

$$x = a\cos E$$
$$y = b\sin E$$

where E is the angle called eccentric anomaly. E is subtended at the centre O of the ellipse between semi-major axis and OA (or OB). For the radius of the ellipse from the origin O to the arbitrary point P, we have

$$r^2 = x^2 + y^2 = a^2\cos^2 E + b^2\sin^2 E$$

which further gives

$$r^2 = a^2\left(1 - \sin^2 E\right) + b^2\sin^2 E = a^2 - \left(a^2 - b^2\right)\sin^2 E.$$

Eccentricity of the ellipse may be defined as

$$e = \sqrt{1 - \frac{a^2}{b^2}}. \tag{1.2}$$

Using that definition we gain following formula for ellipse radius expressed as a function of eccentric angle

$$r(E) = a\sqrt{1 - e^2\sin^2 E}. \tag{1.3}$$

1.3 Equation for centre based ellipse in polar coordinates

From figure 1 we read that

$$x = r\cos\Theta$$
$$y = r\sin\Theta$$

where Θ is the angle subtended at the centre O of the ellipse between semi-major axis and radius r. Using (1.1) we get

$$\frac{r^2\cos^2\Theta}{a^2}+\frac{r^2\sin^2\Theta}{b^2}=1.$$

When multiplying both sides of the equation by b^2 we get

$$r^2\left(\frac{b^2}{a^2}\cos^2\Theta+\sin^2\Theta\right)=b^2.$$

Transforming (1.2) we get $b^2=a^2\left(1-e^2\right)$. Now using that relation we gain

$$r^2\left(\frac{a^2\left(1-e^2\right)}{a^2}\cos^2\Theta+\sin^2\Theta\right)=a^2\left(1-e^2\right),$$

and further

$$r^2\left(1-e^2\cos^2\Theta\right)=a^2\left(1-e^2\right).$$

Finally we get formula for ellipse radius expressed as a function of Θ angle

$$r(\Theta)=a\sqrt{\frac{1-e^2}{1-e^2\cos^2\Theta}}.$$

1.4 Equation for focus based ellipse in Cartesian coordinates

Till that time, we were discussing equations for the ellipse whose "centre" coincides with the origin of the coordinate system. Let us now place one of the foci of the ellipse in the origin. We do so by following transformations

$$\begin{aligned}x&=x^{'}+ae\\ y&=y^{'}\end{aligned}$$

Then equation (1.1) gains following shape

$$\frac{\left(x^{'}+ae\right)^2}{a^2}+\frac{y^{'2}}{b^2}=1. \tag{1.4}$$

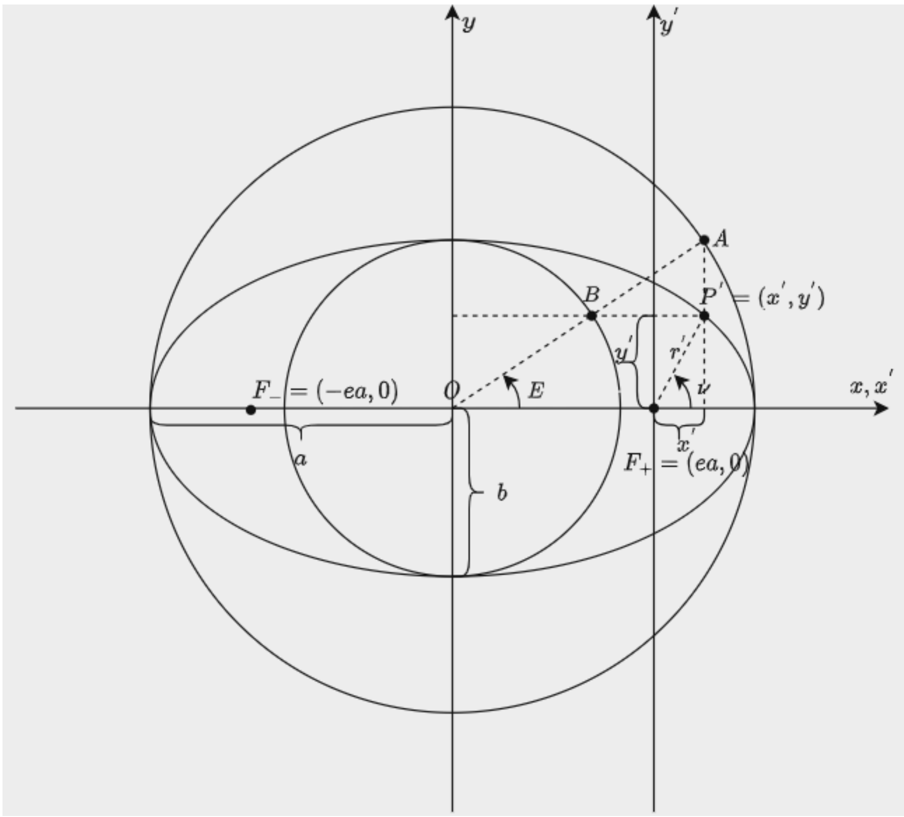

Figure 2: Focus based ellipse

1.5 Equation for focus based ellipse in terms of eccentric angle

To express relation between r' and eccentric angle E we start from the following formula

$$
\begin{aligned}
r'^2 &= x'^2 + y'^2 \\
&= (x - ae)^2 + y^2 \\
&= x^2 + 2xae + a^2e^2 + y^2 \\
&= r^2 + 2xae + a^2e^2.
\end{aligned}
$$

Now noting that $x = a \cos E$ and using r from (1.3) we may write

$$
\begin{aligned}
r'^2 &= a^2\left(1-e^2\sin^2 E\right)-2a^2e\cos(E)+a^2e^2\\
&= a^2\left(1-e^2\sin^2 E-2e\cos(E)+e^2\right)\\
&= a^2\left(1-e^2+e^2\cos^2 E-2e\cos(E)+e^2\right)\\
&= a^2\left(1-e\cos E\right)^2.
\end{aligned}
$$

Which finally gives us equation for the r' expressed as a function of eccentric angle

$$
r'(E) = a\left(1-e\cos E\right). \tag{1.5}
$$

1.6 Equation for focus based ellipse in polar coordinates

To express r' as a function of angle ν (true anomaly), we start from (1.4)

$$
\frac{\left(x'+ae\right)^2}{a^2}+\frac{y'^2}{b^2}=1.
$$

Using (1.2) and replacing b^2 by $a^2\left(1-e^2\right)$ we write

$$
\frac{\left(x'+ae\right)^2}{a^2}+\frac{y'^2}{a^2\left(1-e^2\right)}=1.
$$

Now multiplying both sides of the equation by $a^2\left(1-e^2\right)$ we receive

$$
\left(x'^2+2x'ae+a^2e^2\right)\left(1-e^2\right)+y'^2=a^2\left(1-e^2\right).
$$

Rearranging a little

$$
x'^2-x'^2e^2+\left(2x'ae+a^2e^2\right)\left(1-e^2\right)+y'^2=a^2-a^2e^2
$$

and further

$$
\begin{aligned}
x'^2+y'^2 &= a^2-a^2e^2-\left(2x'ae+a^2e^2\right)\left(1-e^2\right)+x'^2e^2\\
&= a^2-a^2e^2-2x'ae+a^2e^2+2x'ae^3+a^2e^4+x'^2e^2\\
&= a^2-2a^2e^2+a^2e^4-2x'ae+2x'ae^3+x'^2e^2\\
&= a^2\left(1-e^2\right)^2-2x'ae\left(1-e^2\right)+x'^2e^2
\end{aligned}
$$

which brings us to

$$x'^2 + y'^2 = \left[a\left(1 - e^2\right) - x'e\right]^2.$$

Using now polar coordinates

$$\begin{aligned} x' &= r' \cos\nu \\ y' &= r' \sin\nu \end{aligned}$$

we may write

$$r' = a\left(1 - e^2\right) - er' \cos\nu$$

which finally gives us equation for the r' expressed as a function of true anomaly ν

$$r'(\nu) = \frac{a\left(1 - e^2\right)}{1 + e\cos\nu}. \tag{1.6}$$

1.7 Summary

Let us now place all those equations in one table. We may have noticed that we have 3 equations for centre based ellipse and 3 for focus based ellipse, so let's keep that arrangement.

	Centre based ellipse	Focus based ellipse
Cartesian coordinates	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{\left(x' + ae\right)^2}{a^2} + \frac{y'^2}{b^2} = 1$
Radius and eccentric angle	$r(E) = a\sqrt{1 - e^2 \sin^2 E}$	$r'(E) = a\left(1 - e\cos E\right)$
Polar coordinates	$r(\Theta) = a\sqrt{\frac{1 - e^2}{1 - e^2\cos^2\Theta}}$	$r'(\nu) = \frac{a\left(1 - e^2\right)}{1 + e\cos\nu}$

1.8 Bonus - true anomaly and eccentric anomaly

Having got all those relations for the ellipse we may now derive relationship between true anomaly and eccentric anomaly. To do so, let us combine equations (1.6) and (1.5)

$$\frac{a\left(1-e^2\right)}{1+e\cos\nu} = a\left(1-e\cos E\right). \tag{1.7}$$

Thus

$$\cos\nu = \frac{\cos E - e}{1-e\cos E} \tag{1.8}$$

and further using $\sin^2\nu + \cos^2\nu = 1$ identity

$$\sin\nu = \frac{\sqrt{1-e^2}\sin E}{1-e\cos E}. \tag{1.9}$$

Now we may write

$$\begin{aligned} 2\cos^2\frac{\nu}{2} = 1+\cos\nu = 1+\frac{\cos E - e}{1-e\cos E} \\ = \frac{\left(1-e\right)\left(1+\cos E\right)}{1-e\cos E} = \frac{2\left(1-e\right)\cos^2\frac{E}{2}}{1-e\cos E} \end{aligned} \tag{1.10}$$

$$\begin{aligned} 2\sin^2\frac{\nu}{2} = 1-\cos\nu = 1-\frac{\cos E - e}{1-e\cos E} \\ = \frac{\left(1+e\right)\left(1-\cos E\right)}{1-e\cos E} = \frac{2\left(1+e\right)\sin^2\frac{E}{2}}{1-e\cos E}. \end{aligned} \tag{1.11}$$

Dividing (1.11) by (1.10) yields

$$\tan^2\frac{\nu}{2} = \frac{1+e}{1-e}\tan^2\frac{E}{2} \tag{1.12}$$

and

$$\tan^2\frac{E}{2} = \frac{1-e}{1+e}\tan^2\frac{\nu}{2}. \tag{1.13}$$

2 Kepler problem

Yet again we will start from the basics. Force F acting on a satellite in the Earth gravitational field is given by

$$\boldsymbol{F} = -G\frac{mM}{r^2}\hat{\boldsymbol{e}}_r. \tag{2.1}$$

G is the universal gravitational constant, M is the mass of the Earth and m is the mass of a satellite. Oh, well (credit to Peter Green in that exact moment). It's not so easy. It would be true if the Earth were perfectly spherical and of uniform density and if we omitted gravitation from third bodies like Sun or Moon. That's not all. There is solar radiation pressure, out-gassing and probably some other forces which I am not aware of. But we may omit them in our analysis. Let's stick with (2.1) for now. This is conservative, spherically symmetric central force. And because of that it has associated scalar function called potential. Relation between gravitational force F and gravitational potential V is given below.

$$\frac{\boldsymbol{F}}{m} = -\nabla V \tag{2.2}$$

So that equation of motion of satellite in the Earth gravitational field can be described as follows

$$\frac{d^2\boldsymbol{r}}{dt^2} = -\frac{GM}{r^2}\hat{\boldsymbol{e}}_r = -\nabla V. \tag{2.3}$$

Product GM is called Earth's gravitational constant and is given μ symbol. GPS specification [1] uses $\mu = 3.986005 * 10^{14}[m^3/s^2]$. From now on, we will use μ wherever GM shall be used. If F is given by (2.1) then potential also gains simple form

$$V = -\frac{\mu}{r}. \tag{2.4}$$

But as mentioned earlier neither force (2.1) nor associated potential (2.4) take into account that Earth is not spherical and has uneven distribution of mass. When we take those factors into consideration the Earth's gravitational potential will not be as simple as (2.4). In the case of GPS, the Earth's gravitational potential V is modelled by a spherical harmonic series. Please see [3]. At point P defined by spherical coordinates (r, Θ, ϕ) it is given by the following formula

$$V(r,\Theta,\phi) = -\frac{\mu}{r}\left[1 + \sum_{l=1}\sum_{m=0}^{m=l}\left(\frac{a}{r}\right)^l P_{lm}(\cos\Theta)(C_{lm}\cos m\phi + S_{lm}\sin m\phi)\right] \tag{2.5}$$

where

r - distance of P from the origin

Θ - angle between $\boldsymbol{r}$ and +z-axis (normal to the xy-plane)

ϕ - geocentric longitude of P

a - mean equatorial radius of the Earth (6378137 m)

P_{lm} - Legendre function of degree l and order m

C_{lm} - spherical harmonic cosine coefficient of degree l and order m

S_{lm} - spherical harmonic sine coefficient of degree l and order m

What is worth noting in the above potential is that its first term still describes two-body potential function (2.4). The second thing to be noticed is that it is defined using spherical coordinate system. Thus in the next chapter I will bring some refresher topics regarding spherical coordinate system.

3 Spherical coordinate system

Spherical coordinate system is defined by following basis vectors

$$\begin{aligned}\hat{\boldsymbol{e}}_r &= (\sin\Theta\cos\phi, \sin\Theta\sin\phi, \cos\Theta)\\ \hat{\boldsymbol{e}}_\Theta &= (\cos\Theta\cos\phi, \cos\Theta\sin\phi, -\sin\Theta)\\ \hat{\boldsymbol{e}}_\phi &= (-\sin\phi, \cos\phi, 0)\end{aligned} \tag{3.1}$$

The line element is

$$d\boldsymbol{l} = dr\hat{\boldsymbol{e}}_r + rd\Theta\hat{\boldsymbol{e}}_\Theta + r\sin\Theta d\phi\hat{\boldsymbol{e}}_\phi. \tag{3.2}$$

The surface element dS_r is

$$dS_r = r^2\sin\Theta d\Theta d\phi. \tag{3.3}$$

The surface element dS_Θ is

$$dS_\Theta = r\sin\Theta d\phi dr. \tag{3.4}$$

The surface element dS_ϕ is

$$dS_\phi = rdrd\Theta. \tag{3.5}$$

The volume element is

$$dV = r^2 \sin\Theta dr d\Theta d\phi. \tag{3.6}$$

Nabla operator is defined as

$$\boldsymbol{\nabla} = \frac{\partial}{\partial r}\hat{\boldsymbol{e}}_r + \frac{1}{r}\frac{\partial}{\partial \Theta}\hat{\boldsymbol{e}}_\Theta + \frac{1}{r\sin\Theta}\frac{\partial}{\partial \phi}\hat{\boldsymbol{e}}_\phi. \tag{3.7}$$

Thus divergence of vector $\boldsymbol{A} = A_r\hat{\boldsymbol{e}}_r + A_\Theta\hat{\boldsymbol{e}}_\Theta + A_\phi\hat{\boldsymbol{e}}_\phi$ is

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right)\hat{\boldsymbol{e}}_r + \frac{1}{r\sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta A_\Theta\right)\hat{\boldsymbol{e}}_\Theta + \frac{1}{r\sin\Theta}\frac{\partial}{\partial\phi}\left(A_\phi\right)\hat{\boldsymbol{e}}_\phi. \tag{3.8}$$

Curl of $\boldsymbol{A}$ is

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{A} = {} & \frac{1}{r\sin\Theta}\left(\frac{\partial}{\partial\Theta}\left(A_\phi\sin\Theta\right) - \frac{\partial}{\partial\phi}\left(A_\Theta\right)\right)\hat{\boldsymbol{e}}_r \\ & + \frac{1}{r}\left(\frac{1}{\sin\Theta}\frac{\partial}{\partial\phi}\left(A_r\right) - \frac{\partial}{\partial r}\left(rA_\phi\right)\right)\hat{\boldsymbol{e}}_\Theta \\ & + \frac{1}{r}\left(\frac{\partial}{\partial r}\left(rA_\Theta\right) - \frac{\partial}{\partial\Theta}\left(A_r\right)\right)\hat{\boldsymbol{e}}_\phi.\end{aligned} \tag{3.9}$$

And the Laplacian

$$\boldsymbol{\nabla}^2\Psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial t}\right) + \frac{1}{r^2\sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta\frac{\partial\Psi}{\partial\Theta}\right) + \frac{1}{r^2\sin^2\Theta}\frac{\partial^2\Psi}{\partial\phi^2}. \tag{3.10}$$

The position vector is

$$\boldsymbol{r} = r\hat{\boldsymbol{e}}_r. \tag{3.11}$$

The time derivatives of the unit vectors are

$$\frac{d\hat{\boldsymbol{e}}_r}{dt} = \frac{d\Theta}{dt}\hat{\boldsymbol{e}}_\Theta + \sin\Theta\frac{d\phi}{dt}\hat{\boldsymbol{e}}_\phi \tag{3.12}$$

$$\frac{d\hat{\boldsymbol{e}}_\Theta}{dt} = -\frac{d\Theta}{dt}\hat{\boldsymbol{e}}_r + \frac{d\phi}{dt}\cos\Theta\hat{\boldsymbol{e}}_\phi \tag{3.13}$$

$$\frac{d\hat{\boldsymbol{e}}_\phi}{dt} = -\sin\Theta\frac{d\phi}{dt}\hat{\boldsymbol{e}}_r - \frac{d\phi}{dt}\cos\Theta\hat{\boldsymbol{e}}_\Theta. \tag{3.14}$$

The velocity vector is

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{dr}{dt}\hat{\boldsymbol{e}}_r + r\frac{d\hat{\boldsymbol{e}}_r}{dt} = \frac{dr}{dt}\hat{\boldsymbol{e}}_r + r\frac{d\Theta}{dt}\hat{\boldsymbol{e}}_\Theta + r\sin\Theta\frac{d\phi}{dt}\hat{\boldsymbol{e}}_\phi. \tag{3.15}$$

The acceleration vector is

$$\begin{aligned}
\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = &\left(\frac{d^2r}{dt^2} - r\left(\frac{d\Theta}{dt}\right)^2 - r\left(\frac{d\phi}{dt}\right)^2\sin^2\Theta\right)\hat{\boldsymbol{e}}_r \\
&+ \left(r\frac{d^2\Theta}{dt^2} + 2\frac{dr}{dt}\frac{d\Theta}{dt} - r\left(\frac{d\phi}{dt}\right)^2\sin\Theta\cos\Theta\right)\hat{\boldsymbol{e}}_\Theta \\
&+ \left(r\frac{d^2\phi}{dt^2}\sin\Theta + 2\frac{dr}{dt}\frac{d\phi}{dt}\sin\Theta + 2r\frac{d\Theta}{dt}\frac{d\phi}{dt}\cos\Theta\right)\hat{\boldsymbol{e}}_\phi.
\end{aligned} \tag{3.16}$$

The angular momentum per unit mass is

$$\begin{aligned}
\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{v} = r\hat{\boldsymbol{e}}_r \times &\left(\frac{dr}{dt}\hat{\boldsymbol{e}}_r + r\frac{d\Theta}{dt}\hat{\boldsymbol{e}}_\Theta + r\sin\Theta\frac{d\phi}{dt}\hat{\boldsymbol{e}}_\phi\right) \\
= &\, r^2\frac{d\Theta}{dt}\hat{\boldsymbol{e}}_\phi - r^2\sin\Theta\frac{d\phi}{dt}\hat{\boldsymbol{e}}_\Theta.
\end{aligned} \tag{3.17}$$

The squared magnitude of the angular momentum is (see (3.22))

$$\begin{aligned}
L^2 &= \boldsymbol{L}\cdot\boldsymbol{L} = (\boldsymbol{r}\times\boldsymbol{v})\cdot(\boldsymbol{r}\times\boldsymbol{v}) \\
&= \left(r^2\frac{d\Theta}{dt}\hat{\boldsymbol{e}}_\phi - r^2\sin\Theta\frac{d\phi}{dt}\hat{\boldsymbol{e}}_\Theta\right)\cdot\left(r^2\frac{d\Theta}{dt}\hat{\boldsymbol{e}}_\phi - r^2\sin\Theta\frac{d\phi}{dt}\hat{\boldsymbol{e}}_\Theta\right) \\
&= r^4\left(\frac{d\Theta}{dt}\right)^2 + r^4\sin^2\Theta\left(\frac{d\phi}{dt}\right)^2 \\
&= r^2\left(\left(r\frac{d\Theta}{dt}\right)^2 + \left(r\sin\Theta\frac{d\phi}{dt}\right)^2\right) \\
&= r^2\left(v^2 - \left(\frac{dr}{dt}\right)^2\right).
\end{aligned} \tag{3.18}$$

Because angular momentum $\boldsymbol{L}$ is defined as the cross product of $\boldsymbol{r}$ and $\boldsymbol{v}$, both $\boldsymbol{r}$ and $\boldsymbol{v}$ are perpendicular to $\boldsymbol{L}$. This can be shown analytically

$$\boldsymbol{r}\cdot\boldsymbol{L} = r\hat{\boldsymbol{e}}_r\cdot\left(r^2\frac{d\Theta}{dt}\hat{\boldsymbol{e}}_\phi - r^2\sin\Theta\frac{d\phi}{dt}\hat{\boldsymbol{e}}_\Theta\right) = 0, \tag{3.19}$$

$$
\begin{aligned}
\boldsymbol{v}\cdot\boldsymbol{L} &= \left(\frac{dr}{dt}\hat{\boldsymbol{e}}_r + r\frac{d\Theta}{dt}\hat{\boldsymbol{e}}_\Theta + r\sin\Theta\frac{d\phi}{dt}\hat{\boldsymbol{e}}_\phi\right)\cdot\left(r^2\frac{d\Theta}{dt}\hat{\boldsymbol{e}}_\phi - r^2\sin\Theta\frac{d\phi}{dt}\hat{\boldsymbol{e}}_\Theta\right)\\
&= r^3\sin\Theta\frac{d\phi}{dt}\frac{d\Theta}{dt} - r^3\frac{d\Theta}{dt}\sin\Theta\frac{d\phi}{dt} = 0.
\end{aligned}
\tag{3.20}
$$

Let us also note that

$$
\boldsymbol{r}\cdot\boldsymbol{v} = r\hat{\boldsymbol{e}}_r\cdot\left(\frac{dr}{dt}\hat{\boldsymbol{e}}_r + r\frac{d\Theta}{dt}\hat{\boldsymbol{e}}_\Theta + r\sin\Theta\frac{d\phi}{dt}\hat{\boldsymbol{e}}_\phi\right) = r\frac{dr}{dt},
\tag{3.21}
$$

$$
\begin{aligned}
\boldsymbol{v}\cdot\boldsymbol{v} &= \left(\frac{dr}{dt}\hat{\boldsymbol{e}}_r + r\frac{d\Theta}{dt}\hat{\boldsymbol{e}}_\Theta + r\sin\Theta\frac{d\phi}{dt}\hat{\boldsymbol{e}}_\phi\right)\cdot\left(\frac{dr}{dt}\hat{\boldsymbol{e}}_r + r\frac{d\Theta}{dt}\hat{\boldsymbol{e}}_\Theta + r\sin\Theta\frac{d\phi}{dt}\hat{\boldsymbol{e}}_\phi\right)\\
&= \left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\Theta}{dt}\right)^2 + \left(r\sin\Theta\frac{d\phi}{dt}\right)^2 = v^2.
\end{aligned}
\tag{3.22}
$$

4 Solution to the Kepler problem

To solve equation

$$
\frac{d^2\boldsymbol{r}}{dt^2} = -\frac{\mu}{r^2}\hat{\boldsymbol{e}}_r
\tag{4.1}
$$

we will start from derivation of time derivative of angular momentum (3.17).

4.1 Angular momentum analysis

$$
\frac{d\boldsymbol{L}}{dt} = \frac{d}{dt}(\boldsymbol{r}\times\boldsymbol{v}) = \frac{d\boldsymbol{r}}{dt}\times\boldsymbol{v} + \boldsymbol{r}\times\frac{d\boldsymbol{v}}{dt} = 0 + \boldsymbol{r}\times\frac{d^2\boldsymbol{r}}{dt^2} = 0 - \boldsymbol{r}\times\frac{\mu}{r^2}\hat{\boldsymbol{e}}_r = 0.
\tag{4.2}
$$

Thus in the central gravitational field, the angular momentum is constant. Furthermore, using following identity

$$
\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}\cdot(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}\cdot(\boldsymbol{a}\cdot\boldsymbol{b})
\tag{4.3}
$$

and relation (3.21) we notice that

$$
\begin{aligned}
\frac{d\hat{\boldsymbol{e}}_r}{dt} &= \frac{d}{dt}\left(\frac{\boldsymbol{r}}{r}\right) = \frac{d\boldsymbol{r}}{dt}\frac{1}{r} + \boldsymbol{r}\frac{d}{dt}\left(\frac{1}{r}\right) = \frac{r\frac{d\boldsymbol{r}}{dt} - \boldsymbol{r}\frac{dr}{dt}}{r^2} = \frac{r^2\frac{d\boldsymbol{r}}{dt} - r\frac{dr}{dt}\boldsymbol{r}}{r^3}\\
&= \frac{(\boldsymbol{r}\cdot\boldsymbol{r})\,\boldsymbol{v} - (\boldsymbol{r}\cdot\boldsymbol{v})\,\boldsymbol{r}}{r^3} = \frac{\boldsymbol{r}\times(\boldsymbol{v}\times\boldsymbol{r})}{r^3} = -\frac{\boldsymbol{r}\times\boldsymbol{L}}{r^3} = \frac{\boldsymbol{L}\times\boldsymbol{r}}{r^3}.
\end{aligned}
\tag{4.4}
$$

We already know that $\boldsymbol{L}$ is constant. We also know that $\hat{\boldsymbol{e}}_r$ is normal (see (3.19)) to the constant angular momentum $\boldsymbol{L}$. If $\boldsymbol{L} = 0$, then $\hat{\boldsymbol{e}}_r$ is also constant meaning that movement is in a straight line towards the origin. If $\boldsymbol{L} \neq 0$ then motion takes place in fixed/constant plane with $\boldsymbol{L}$ being normal to that plane.

Now, noting that $\frac{d\boldsymbol{L}}{dt} = 0$ and using equation of motion in gravitational field (4.1) we may further write

$$\frac{d\hat{\boldsymbol{e}}_r}{dt} = \frac{\boldsymbol{L} \times \boldsymbol{r}}{r^3} = -\frac{\boldsymbol{L} \times \frac{d^2\boldsymbol{r}}{dt^2}}{\mu} = -\frac{1}{\mu}\frac{d}{dt}\left(\boldsymbol{L} \times \boldsymbol{v}\right) = \frac{1}{\mu}\frac{d}{dt}\left(\boldsymbol{v} \times \boldsymbol{L}\right), \tag{4.5}$$

which brings us to the following relation

$$\frac{d}{dt}\left(\mu\hat{\boldsymbol{e}}_r - \boldsymbol{v} \times \boldsymbol{L}\right) = 0. \tag{4.6}$$

If we choose vector $\mu\boldsymbol{e}$ to denote the constant of integration, then (4.6) can be written in the form

$$\mu\hat{\boldsymbol{e}}_r - \boldsymbol{v} \times \boldsymbol{L} = \mu\boldsymbol{e} \tag{4.7}$$

or

$$\mu\left(\hat{\boldsymbol{e}}_r - \boldsymbol{e}\right) = \boldsymbol{v} \times \boldsymbol{L}. \tag{4.8}$$

We know that $\hat{\boldsymbol{e}}_r$ is normal to the constant angular momentum $\boldsymbol{L} \neq 0$. Let's check how situation looks for vector $\boldsymbol{e}$. To do so we take scalar product of $\boldsymbol{e}$ with $\boldsymbol{L}$, and noting that $\boldsymbol{L}$ is normal to $\boldsymbol{r}$ and $\boldsymbol{v}$, we find that

$$\boldsymbol{e} \cdot \boldsymbol{L} = \left(\hat{\boldsymbol{e}}_r - \frac{1}{\mu}\boldsymbol{v} \times \boldsymbol{L}\right) \cdot \boldsymbol{L} = 0. \tag{4.9}$$

Noting that (we are again using identity (4.3) and (3.21))

$$\boldsymbol{v} \times \boldsymbol{L} = \boldsymbol{v} \times \left(\boldsymbol{r} \times \boldsymbol{v}\right) = \left(\boldsymbol{v} \cdot \boldsymbol{v}\right) \cdot \boldsymbol{r} - \left(\boldsymbol{r} \cdot \boldsymbol{v}\right) \cdot \boldsymbol{v} = v^2\boldsymbol{r} - r\frac{dr}{dt}\boldsymbol{v} \tag{4.10}$$

equation (4.8) can be written as

$$\mu\left(\hat{\boldsymbol{e}}_r - \boldsymbol{e}\right) = v^2\boldsymbol{r} - r\frac{dr}{dt}\boldsymbol{v}. \tag{4.11}$$

Now taking the scalar product of (4.11) with $\boldsymbol{r}$

$$\mu\left(\hat{\boldsymbol{e}}_r - \boldsymbol{e}\right) \cdot \boldsymbol{r} = \left(v^2\boldsymbol{r} - r\frac{dr}{dt}\boldsymbol{v}\right) \cdot \boldsymbol{r} \tag{4.12}$$

we receive (see (3.18))

$$\mu\left(r-\boldsymbol{e}\boldsymbol{r}\right)=v^{2}r^{2}-r^{2}\left(\frac{dr}{dt}\right)^{2}=r^{2}\left(v^{2}-\left(\frac{dr}{dt}\right)^{2}\right)=L^{2}. \tag{4.13}$$

If we now use constant vector $\boldsymbol{e}$ to define a base axis and if we introduce ν to define the angle between the base axis vector $\boldsymbol{e}$ and vector $\boldsymbol{r}$, then pair (r,ν) constitutes polar coordinates in so defined coordinate system. Equation (4.13) can gain following form

$$r\left(1+e\cos\nu\right)=\frac{L^{2}}{\mu} \tag{4.14}$$

or

$$r=\frac{\frac{L^{2}}{\mu}}{1+e\cos\nu}. \tag{4.15}$$

In polar coordinates it is an equation of conic section with the focus at the origin, where e is the eccentricity and $\frac{L^{2}}{\mu}$ is the semi-latus rectum. If $e=0$, the conic is a circle, for $0<e<1$, the conic is an ellipse, for $e=1$ a parabola and for $e>1$ a hyperbola. Let us also note, that the equation (4.15) is very similar to equation (1.6) derived in chapter (1) describing an ellipse using polar coordinates based in the focus. The difference is in the numerator (semi-latus rectum). To clarify this difference, we first need to derive energy in motion described by gravitational equation (4.1).

4.2 Energy analysis

To derive formula for total energy in gravitational motion we start by taking scalar product of (4.1) with the velocity vector $\boldsymbol{v}$

$$\frac{d^{2}\boldsymbol{r}}{dt^{2}}\cdot\boldsymbol{v}=-\frac{\mu}{r^{2}}\hat{\boldsymbol{e}}_{r}\cdot\boldsymbol{v}=-\frac{\mu}{r^{2}}\frac{\boldsymbol{r}}{r}\cdot\frac{d\boldsymbol{r}}{dt}. \tag{4.16}$$

Noting that $d\left(\boldsymbol{a}\cdot\boldsymbol{a}\right)=2\boldsymbol{a}d\boldsymbol{a}$, we may further write

$$\frac{d^{2}\boldsymbol{r}}{dt^{2}}\cdot\boldsymbol{v}=\frac{d\boldsymbol{v}}{dt}\cdot\boldsymbol{v}=\frac{d\left(\boldsymbol{v}\cdot\boldsymbol{v}\right)}{2dt}=-\frac{\mu}{2r^{3}}\frac{d\left(\boldsymbol{r}\cdot\boldsymbol{r}\right)}{dt}=-\frac{\mu}{2r^{3}}\frac{d\left(r^{2}\right)}{dt}=-\frac{\mu}{r^{2}}\frac{dr}{dt}. \tag{4.17}$$

Thus

$$\frac{d\left(\boldsymbol{v}\cdot\boldsymbol{v}\right)}{2dt} = \frac{1}{2}\frac{d\left(v^2\right)}{dt} = -\frac{\mu}{r^2}\frac{dr}{dt} = \frac{d}{dt}\left(\frac{\mu}{r}\right) \tag{4.18}$$

and finally

$$\frac{d}{dt}\left(\frac{v^2}{2} - \frac{\mu}{r}\right) = 0. \tag{4.19}$$

If we choose symbol $\mathcal{E}$ to denote the constant of integration, then (4.19) can be written in the form

$$\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r}. \tag{4.20}$$

The term $\frac{v^2}{2}$ denotes kinetic energy and the term $-\frac{\mu}{r}$ potential energy of a body of unit mass. If we consider the case of negative total enery, we can write

$$\mathcal{E} = -\alpha^2. \tag{4.21}$$

Additionally if we introduce symbol a by the following definition

$$a = \frac{\mu}{2\alpha^2} \tag{4.22}$$

and note that kinetic energy is always positive, we can write

$$\frac{\mu}{r} - \alpha^2 \geq 0,$$

which finally gives us relation describing upper limit on r

$$r \leq \frac{\mu}{\alpha^2} = 2a.$$

Thus negative total energy implies that orbit is bound with r being less or equal to $2a$. Finally we may write the formula for total energy in case of bound orbits, which is

$$\mathcal{E} = -\frac{\mu}{2a}. \tag{4.23}$$

4.3 To the shore

If we come back to relation (4.8) and square both sides of it we get

$$
\begin{aligned}
v^2L^2 = (\boldsymbol{v} \times \boldsymbol{L})^2 &= \mu^2 \left(\hat{\boldsymbol{e}}_r - \boldsymbol{e}\right)^2 = \mu^2 \left(1 + 2\hat{\boldsymbol{e}}_r\boldsymbol{e} + e^2\right) \\
&= \mu^2 \left(1 + 2\left(\frac{L^2}{\mu r} - 1\right) + e^2\right) \\
&= \mu^2 \left(e^2 - 1\right) + 2\mu\frac{L^2}{r}.
\end{aligned} \tag{4.24}
$$

Thus

$$
\mu^2 \left(e^2 - 1\right) = 2L^2 \left(\frac{v^2}{2} - \frac{\mu}{r}\right) \tag{4.25}
$$

and now, introducing (4.20) we obtain

$$
\mu^2 \left(1 - e^2\right) = -2L^2\mathcal{E} = 2L^2\alpha^2 = L^2\frac{\mu}{a}. \tag{4.26}
$$

After some rearrangements

$$
a \left(1 - e^2\right) = \frac{L^2}{\mu}. \tag{4.27}
$$

Recalling equation (4.15) and replacing $\frac{L^2}{\mu}$ by $a\,(1 - e^2)$ we receive solution of gravitational equation (4.1) in polar coordinates

$$
\boxed{r(\nu) = \frac{a\,(1 - e^2)}{1 + e\cos\nu}} \tag{4.28}
$$

where e is the ellipse eccentricity ($e < 1$) and a is the semi-major of such ellipse. e and a depend on initial conditions which are given to body moving in the gravitational field.

4.4 Time

In the previous section we found the solution of r as a function of true anomaly ν, which gives us the shape of the orbit. In this section we will find the solution of ν as a function of time. Let's start from bringing definition for terms *perigee* and *apogee*. Although in the literature we may find many other synonyms (perihelion/aphelion, pericenter/apocenter, periapsis/apoapsis) for those terms, I decided to stick with the naming perigee and apogee as the same terminology is used in GPS specification to which I am referring in the abstract and which was motivation was this article. Thus

- The perigee is the point in the orbit where the orbiting body is the closest to its focus-based counterpart body in the two body system.
- The apogee is the point in the orbit where the orbiting body is the farthest to its focus-based counterpart body in the two body system.

To find relation in polar coordinates for perigee and apogee we take ν derivative from (4.28)

$$\frac{dr}{d\nu} = \frac{ae\left(1-e^2\right)\sin\nu}{\left(1+e\cos\nu\right)^2}. \tag{4.29}$$

Thus $\dfrac{dr}{d\nu} = 0$ for $\nu = 0, \pm\pi, ..., \pm n\pi$. For $\nu = 2n\pi$, $r(\nu)$ takes minimum values, hence this is our perigee

$$r(2n\pi) = \frac{a\left(1-e\right)\left(1+e\right)}{1+e\cos 2n\nu} = a\left(1-e\right). \tag{4.30}$$

For $\nu = (2n+1)\,\pi$, $r(\nu)$ takes maximum values, hence this is our apogee

$$r((2n+1)\,\pi) = \frac{a\left(1-e\right)\left(1+e\right)}{1+e\cos\left(2n+1\right)\pi} = a\left(1+e\right). \tag{4.31}$$

Now, let us assume that orbiting body passes through its perigee $r_0(0) = a\left(1-e\right)$ at $t_0 = 0$. We will try to find time which is needed for an orbiting body to move from the perigee to an arbitrary point r on the orbit. To do so, let us recall equation (4.20) for total energy of an orbiting body. Using (3.18) and (4.27) we may write

$$\mathcal{E} = \frac{1}{2}v^2 - \frac{\mu}{r} = \frac{1}{2}\left(\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{r^2}\right) - \frac{\mu}{r} = \frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{\mu a\left(1-e^2\right)}{2r^2} - \frac{\mu}{r}. \tag{4.32}$$

After rearrangement and noting that $\mathcal{E} = -\dfrac{\mu}{2a}$ we get

$$\left(\frac{dr}{dt}\right)^2 = \frac{2\mu}{r} - \frac{\mu}{a} - \frac{\mu a\left(1-e^2\right)}{r^2} = \frac{\mu}{r^2}\left(2r - \frac{r^2}{a} - a\left(1-e^2\right)\right). \tag{4.33}$$

Taking the square root and integrating over the range from perigee r_0 till arbitrary r, we get

$$\int_{r_0}^{r} \frac{r}{\sqrt{2r - \dfrac{r^2}{a} - a\left(1-e^2\right)}}\,dr = \sqrt{\mu}t. \tag{4.34}$$

Now we have to recall equation (1.5) derived in section 1 describing a focus based ellipse in terms of eccentric angle. Let us repeat it here

$$r(E) = a\left(1 - e\cos E\right) = a - ae\cos E. \tag{4.35}$$

Thus

$$a - r = ae\cos E. \tag{4.36}$$

Now, squaring (4.36) we get

$$(a - r)^2 = a^2 + r^2 - 2ar = a^2e^2\cos^2 E. \tag{4.37}$$

Dividing both sides of the (4.37) by $-a$ and then adding to both sides term ae^2 we reach

$$2r - \frac{r^2}{a} - a + ae^2 = ae^2 - ae^2\cos^2 E \tag{4.38}$$

or, after yet another rearrangement

$$2r - \frac{r^2}{a} - a\left(1 - e^2\right) = ae^2\left(1 - \cos^2 E\right) = ae^2 sin^2 E. \tag{4.39}$$

Now it shall be clear that left side of the (4.39) corresponds to radical in (4.34). Moreover, differentiating (4.35) over E, we write

$$dr = ae\sin E dE. \tag{4.40}$$

Finally, using (4.35), (4.39) and (4.40) our integral (4.34) can be written in the form

$$\int_{r_0}^{r} \frac{a\left(1 - e\cos E\right)ae\sin E}{\sqrt{ae^2 sin^2 E}}\, dE = \sqrt{\mu}t \tag{4.41}$$

which after small rearrangement gives (please note that $E = 0$ for $r_0 = a\left(1 - e\right)$)

$$\int_{0}^{E} \left(1 - e\cos E\right)\, dE = \sqrt{\frac{\mu}{a^3}}t. \tag{4.42}$$

Solving integral (4.42) we obtain

$$\boxed{E - e\sin E = \sqrt{\frac{\mu}{a^3}}t} \tag{4.43}$$

which is known as the Kepler's equation. The quantity $\sqrt{\frac{\mu}{a^3}}$ is given a symbol n and expresses mean motion in the ellipsoidal movement. Mean motion is defined as the quotient of the angle traced in one complete orbit (2π) to the time needed for such rotation. Thus

$$n = \frac{2\pi}{T}. \tag{4.44}$$

To derive mean motion for ellipsoidal orbits ($0 < e < 1$), we will introduce yet another definition, this time *areal velocity*.

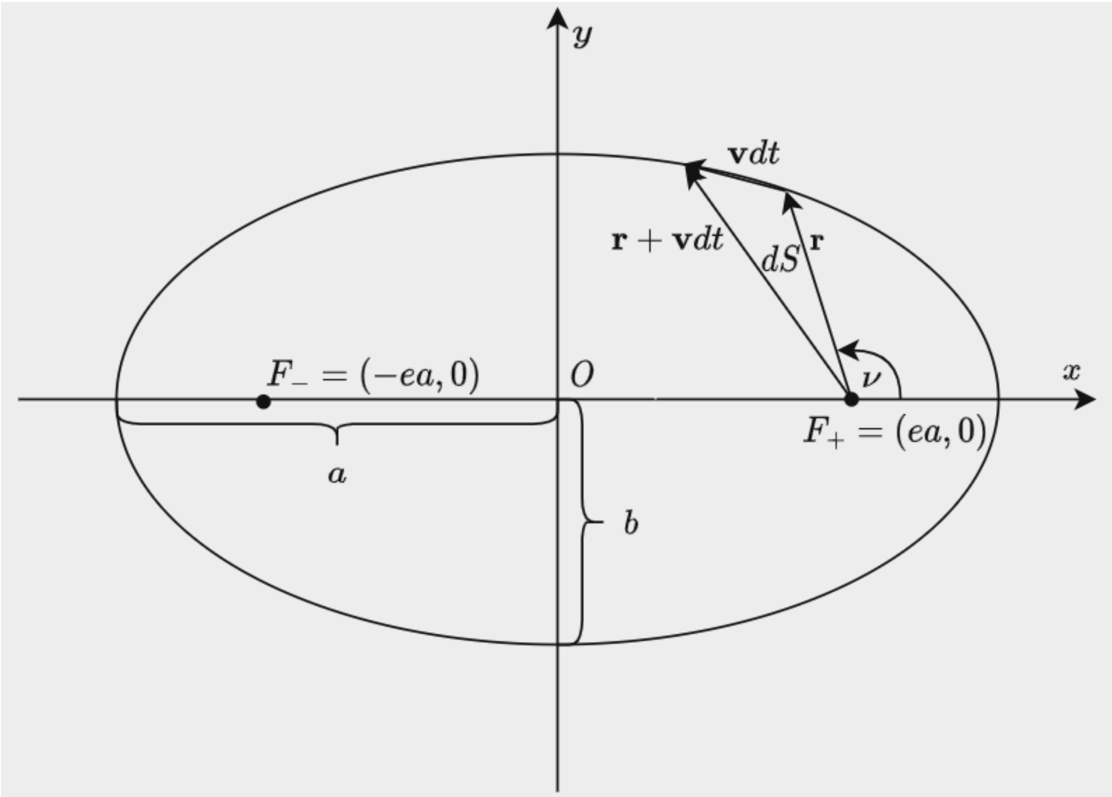

Figure 3: Areal velocity

During time dt, the position vector $\boldsymbol{r}$ traces out the area dS shown in the figure 3. This area is equal to the half of the area of the parallelogram spanned over vectors $\boldsymbol{r}$ and $\boldsymbol{v}dt$. This latter is on the other hand equal to the magnitude of the cross product $\boldsymbol{r} \times \boldsymbol{v}dt$. The area dS of the shown triangle is hence equal

$$dS = \frac{1}{2} |\boldsymbol{r} \times \boldsymbol{v}| \, dt = \frac{1}{2} L dt. \tag{4.45}$$

We divide both sides of the equation (4.45) by dt and get

$$\frac{dS}{dt} = \frac{L}{2}. \tag{4.46}$$

The quantity $\frac{dS}{dt}$ is called areal velocity. In the central fields, $L = const$, so does the areal velocity. Because areal velocity is constant it will be equal to the mean areal velocity which is on the other hand equal to the area (πab) traced out in one orbit over an orbit period T

$$\frac{L}{2} = \frac{\pi ab}{T} = \pi ab \frac{n}{2\pi} = \frac{1}{2} nab = \frac{1}{2} na^2 \sqrt{1-e^2}. \tag{4.47}$$

Using relation (4.27) for semi latus rectum and after cancellation of the common factor $\sqrt{1-e^2}$ we finally get formula for mean motion

$$n = \sqrt{\frac{\mu}{a^3}}. \tag{4.48}$$

At the end of this section let us recall that the quantity $E - \sin E$ is called mean anomaly and is given the symbol M. Thus

$$M = E - e \sin E = nt. \tag{4.49}$$

We see that the mean anomaly changes linearly with time.

5 The relativistic correction term

Now, when we are familiar with equations governing satellite motion we could have though that we are close to finally show derivations of so called "relativistic correction term" (sometimes also called "eccentricity correction term"). But unfortunately, at this exact moment we have to bring out our powerful and quite complex weapon which is Einstein's general theory of relativity. Uhhh. To the work.

5.1 Schwarzschild metric and gravitational time dilation

We will start from presenting Schwarzschild metric, which is the solution to the Einstein's field equations for a mass that is spherically symmetric, non-rotating and non-charged. Although Schwarzschild metric is derived for a non-rotating mass it can be used to describe the curvature of space-time near slowly rotating planets like Earth. Thus the Schwarzschild metric is

$$\begin{bmatrix} \left(1+\frac{2V}{c^2}\right) & & & \\ & -\left(1+\frac{2V}{c^2}\right)^{-1} & & \\ & & -r^2 & \\ & & & -r^2\sin^2\Theta \end{bmatrix} \tag{5.1}$$

where $V = -\frac{\mu}{r}$ is the Newtonian gravitational potential for a spherically symmetric distribution of mass (please see equation (2.4)). Please also note that I am using mostly minuses convention $(+,-,-,-)$ when presenting the Schwarzschild metric. Replacing V by $-\frac{\mu}{r}$ and using "line element" notation (which is more compact) we may rewrite metric (5.1) as

$$ds^2 = \left(1-\frac{2\mu}{c^2r}\right)c^2dt^2 - \left(1-\frac{2\mu}{c^2r}\right)^{-1}dr^2 - r^2d\Theta^2 - r^2\sin^2\Theta d\phi^2. \tag{5.2}$$

Now I need to highlight that in our process of derivation of "eccentricity correction term" we will be using lots of approximations. Many of them will be related with rejection of terms of order higher than $\frac{1}{c^2}$. And this is time for the first of them. The term $\left(1-\frac{2\mu}{c^2r}\right)^{-1}$ can be approximated to the first order in $\frac{1}{c^2}$ as follows

$$\left(1-\frac{2\mu}{c^2r}\right)^{-1} = \frac{\left(1+\frac{2\mu}{c^2r}\right)}{\left(1-\frac{2\mu}{c^2r}\right)\left(1+\frac{2\mu}{c^2r}\right)} = \frac{\left(1+\frac{2\mu}{c^2r}\right)}{1-\left(\frac{2\mu}{c^2r}\right)^2} \approx 1+\frac{2\mu}{c^2r}. \tag{5.3}$$

Thus metric (5.2) can be written as

$$\begin{aligned} ds^2 &\approx \left(1-\frac{2\mu}{c^2r}\right)c^2dt^2 - \left(1+\frac{2\mu}{c^2r}\right)dr^2 - r^2d\Theta^2 - r^2\sin^2\Theta d\phi^2 \\ &= c^2dt^2\left[\left(1-\frac{2\mu}{c^2r}\right) - \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2 - \frac{1}{c^2}\left(r\frac{d\Theta}{dt}\right)^2 - \frac{1}{c^2}\left(r\sin\Theta\frac{d\phi}{dt}\right)^2 - \frac{2\mu}{c^4r}\left(\frac{dr}{dt}\right)^2\right]. \end{aligned} \tag{5.4}$$

Recalling equation (3.22) we may further write

$$ds^2 = c^2dt^2 \left[\left(1 - \frac{2\mu}{c^2r}\right) - \frac{v^2}{c^2} - \frac{2\mu}{c^4r}\left(\frac{dr}{dt}\right)^2 \right]. \tag{5.5}$$

Because $\frac{v^2}{c^2} \gg \frac{2\mu}{c^4r}\left(\frac{dr}{dt}\right)^2$ the last term in (5.5) can be neglated and equation (5.5) can be written as

$$ds^2 = c^2dt^2 \left[\left(1 - \frac{2\mu}{c^2r}\right) - \frac{v^2}{c^2} \right]. \tag{5.6}$$

Taking square root of (5.6) and replacing $-\frac{\mu}{r}$ by $V(r)$ we obtain

$$ds = cdt\sqrt{\left(1 + \frac{2V(r)}{c^2}\right) - \frac{v^2}{c^2}}. \tag{5.7}$$

Proper time $d\tau$ is the time interval measured by an observer at rest in any coordinate system and is related with ds by the following formula

$$ds = cd\tau. \tag{5.8}$$

The quantity dt is called coordinate time and is related with proper time by equation (5.9)

$$d\tau = \sqrt{\left(1 + \frac{2V(r)}{c^2}\right) - \frac{v^2}{c^2}}\,dt. \tag{5.9}$$

Thus an observer at rest in a system that is moving relative to "coordinate" system with velocity v, in a gravitational potential $V(r)$ will measure a proper time interval $d\tau$ which is related with coordinate time dt by equation (5.9).

5.2 Relativistic correction term - formula 1

Finally we have reached the point where we can derive the formula for the relativistic correction term as presented in the abstract of this article. Let me remind. The GPS specification [1] provides following equation for the relativistic correction term

$$\Delta t_r = Fe\sqrt{a}\sin E_k. \tag{5.10}$$

We will start this process of derivations from assigning dedicated subscripts to equation (5.9). Let subscript S denote any satellite and subscript R denote any receiver (or any ground/earth station). Then

$$d\tau_S = \sqrt{\left(1 + \frac{2V(r_S)}{c^2}\right) - \frac{v_S^2}{c^2}}\, dt \tag{5.11}$$

$$d\tau_R = \sqrt{\left(1 + \frac{2V(r_R)}{c^2}\right) - \frac{v_R^2}{c^2}}\, dt \tag{5.12}$$

where τ_S is the proper time interval as measured by a satellite clock and τ_R is the proper time interval as measured by a receiver clock. Please also note, that there is no subscript on the coordinate time interval as we are only comparing clock rates.

To compare the ratio of clock rates for satellite and receiver we may divide equation (5.11) by (5.12)

$$\frac{d\tau_S}{d\tau_R} = \left[1 + \frac{2V(r_S)}{c^2} - \frac{v_S^2}{c^2}\right]^{\frac{1}{2}} \cdot \left[1 + \frac{2V(r_R)}{c^2} - \frac{v_R^2}{c^2}\right]^{-\frac{1}{2}}. \tag{5.13}$$

I mentioned earlier that we will be using lots of approximations. And now it's time for another ones. We will expand radicals in (5.13) as Taylor series in terms of $\frac{2V(r)}{c^2} - \frac{v^2}{c^2}$ and we will drop all terms of order higher than $\frac{1}{c^2}$. Thus

$$\frac{d\tau_S}{d\tau_R} \approx \left[1 + \frac{1}{c^2}\left(V(r_S) - \frac{v_S^2}{2}\right) + \ldots\right] \cdot \left[1 - \frac{1}{c^2}\left(V(r_R) - \frac{v_R^2}{2}\right) + \ldots\right] \tag{5.14}$$

which after multiplication and yet again rejecting terms higher than $\frac{1}{c^2}$ gives

$$\frac{d\tau_S}{d\tau_R} \approx 1 + \frac{1}{c^2}\left(V(r_S) - \frac{v_S^2}{2}\right) - \frac{1}{c^2}\left(V(r_R) - \frac{v_R^2}{2}\right). \tag{5.15}$$

Recalling equations (4.20) and (4.23) describing the total satellite energy (elliptical/bound orbits) we my write

$$\begin{aligned} V(r_S) - \frac{v_S^2}{2} &= -\left(\frac{v_S^2}{2} + V(r_S) - 2V(r_S)\right) \\ &= -\left(\mathcal{E}_S - 2V(r_S)\right) \\ &= \frac{\mu}{2a} - 2\frac{\mu}{r}. \end{aligned} \tag{5.16}$$

Thus

$$\frac{d\tau_S}{d\tau_R} = 1 + \frac{2\mu}{c^2}\left(\frac{1}{4a_S} - \frac{1}{r_S}\right) + \frac{1}{c^2}\left(\frac{\mu}{r_R} + \frac{v_R^2}{2}\right). \tag{5.17}$$

Now we have to recall equation (1.5) describing focus based ellipse in terms of eccentric angle E. Let me rewrite it here using appropriate subscripts

$$r_S(E_S) = a_S\left(1 - e_S \cos E_S(\tau_R)\right). \tag{5.18}$$

Please note, that eccentric angle is expressed as a function of τ_R since the "ground station clock is used as a standard". e_S is the eccentricity of the orbit and a_S is the orbit's semi-major axis.

If we use r_S from (5.18) and substitute to term $\dfrac{1}{4a_S} - \dfrac{1}{r_S}$ we get

$$\begin{aligned} \frac{1}{4a_S} - \frac{1}{r_S} &= \frac{1}{4a_S}\left(\frac{r_S - 4a_S}{r_S}\right) \\ &= \frac{1}{4a_S}\left(\frac{a_S\left(1 - e_S\cos E_S(\tau_R)\right) - 4a_S}{a_S\left(1 - e_S\cos E_S(\tau_R)\right)}\right) \\ &= \frac{-1}{4a_S}\left(\frac{3 + e_S\cos E_S(\tau_R)}{1 - e_S\cos E_S(\tau_R)}\right) \end{aligned} \tag{5.19}$$

and noting that

$$\frac{3 + e\cos E}{1 - e\cos E} = \frac{3 - 3e\cos E + 4e\cos E}{1 - e\cos E} = 3 + \frac{4e\cos E}{1 - e\cos E} \tag{5.20}$$

equation (5.17) can be rewritten as

$$\frac{d\tau_S}{d\tau_R} = 1 + \frac{1}{c^2}\left(\frac{\mu}{r_R} + \frac{v_R^2}{2} - \frac{3\mu}{2a_S}\right) - \frac{2\mu e_S}{c^2 a_S}\left(\frac{\cos E_S(\tau_R)}{1 - e_S\cos E_S(\tau_R)}\right). \tag{5.21}$$

At some time τ_0 the clocks at satellites and the ground stations were synchronized. To find out how far out of the synchronization the clocks are after some time $\delta\tau$ we have to integrate equation (5.21)

$$\tau_S = \tau_R + \frac{1}{c^2}\left(\frac{\mu}{r_R} + \frac{v_R^2}{2} - \frac{3\mu}{2a_S}\right)\tau_R - \frac{2\mu e_S}{c^2 a_S}\int \frac{\cos E_S(\tau_R)}{1 - e_S\cos E_S(\tau_R)}\, d\tau_R + C. \tag{5.22}$$

Using Kepler's equation (4.43),

$$E_S(\tau_R) - e_S \sin E_S(\tau_R) = \sqrt{\frac{\mu}{a_S^3}}\left(\tau_R - \tau_{R_0}\right) \tag{5.23}$$

or more precisely integral (4.42) we may write

$$d\tau_R = \sqrt{\frac{a_S^3}{\mu}}\,(1 - e_S \cos E_S(\tau_R))\, dE_S. \tag{5.24}$$

Substituting (5.24) into (5.22) we get

$$\tau_S = \tau_R + \frac{1}{c^2}\left(\frac{\mu}{r_R} + \frac{v_R^2}{2} - \frac{3\mu}{2a_S}\right)\tau_R - \frac{2\sqrt{\mu a_S}e_S}{c^2}\int \cos E_S(\tau_R)\, dE_S + C. \tag{5.25}$$

Finally, integrating (5.25), and introducing constant $F = \dfrac{2\sqrt{\mu}}{c^2}$ we get

$$\tau_S = \tau_R + \frac{1}{c^2}\left(\frac{\mu}{r_R} + \frac{v_R^2}{2} - \frac{3\mu}{2a_S}\right)\tau_R - F\sqrt{a_S}e_S \sin E_S(\tau_R) + C. \tag{5.26}$$

If, by τ_{S_e} we mark the time of emission as measured by satellite clock, by τ_{R_e} we mark the time of emission as measured by receiver (ground station) clock and by τ_{R_r} we mark the reception time as measured by ground station clock, then we may write the following formulas for time of emission, uncorrected and corrected pseudoranges.

$$\tau_{S_e} = \tau_{R_e} + \frac{1}{c^2}\left(\frac{\mu}{r_R} + \frac{v_R^2}{2} - \frac{3\mu}{2a_S}\right)\tau_{R_e} - F\sqrt{a_S}e_S \sin E_S(\tau_{R_e}) + C \tag{5.27}$$

uncorrected range	$\rho_u = c\,(\tau_{R_r} - \tau_{S_e})$
corrected range	$\rho_c = c\,(\tau_{R_r} - \tau_{R_e})$

Thus

$$\rho_c = \rho_u + c\tau_{S_e} - c\tau_{R_e} = \rho_u + c\Delta t_r \tag{5.28}$$

where Δt_r is the term we are looking for and is called "relativistic correction term". It is defined as the difference between emission time as measured by satellite clock and emission time as measured by receiver clock $\tau_{S_e} - c\tau_{R_e}$. The term $\dfrac{1}{c^2}\left(\dfrac{\mu}{r_R} + \dfrac{v_R^2}{2} - \dfrac{3\mu}{2a_S}\right)\tau_{R_e}$ is constant and according to [2] it "causes GPS clock to run fast compared with a station clock. Prior to launch the satellite clock is purposely set low to 10.22999999545 MHz or $4.45 \cdot 10^{-10}$ low in frequency relative to a nominal 10.23 MHz. This in effect removes this constant term and leaves only the periodic relativistic term". Thus

$$\boxed{\Delta t_r = F\sqrt{a_S}e_S \sin E_S(\tau_{R_e})} \tag{5.29}$$

which is the equation we were looking for. Finally we are done for formula 1.

5.3 Relativistic correction term - formula 2

The GPS specification [1] provides also an alternative but equivalent formula for the relativistic correction term. This is

$$\Delta t_r = -\frac{2\boldsymbol{Rv}}{c^2}. \tag{5.30}$$

Let us now provide derivations to get this "alternative" form. We will start from bringing out one more time equation (1.5). This time we will differentiate it with respect to time

$$\frac{dr}{dt} = ae\frac{dE}{dt}\sin E. \tag{5.31}$$

Differentiating Kepler's equation (4.43) with respect to time gives

$$\frac{dE}{dt}(1 - ecosE) = \sqrt{\frac{\mu}{a^3}}. \tag{5.32}$$

When equation (5.32) is solved for $\frac{dE}{dt}$ and substituted into (5.31) we get

$$\frac{dr}{dt} = \sqrt{\frac{\mu}{a^3}}\frac{1}{1 - ecosE}ae\sin E. \tag{5.33}$$

Using one more time equation (1.5) to eliminate the quantity $1 - ecosE$ we get

$$\frac{dr}{dt} = \sqrt{\frac{\mu}{a^3}}\frac{a}{r}ae\sin E. \tag{5.34}$$

After some rearrangement

$$r\frac{dr}{dt} = \sqrt{\mu a}e\sin E. \tag{5.35}$$

Now using equation (3.21) and multiplying both sides by $\frac{2}{c^2}$ we get

$$\boxed{\frac{2\boldsymbol{Rv}}{c^2} = \frac{2\mu}{c^2}\sqrt{a}e \sin E = \Delta t_r} \tag{5.36}$$

And we are done for formula 2.

6 GNU Radio

In this chapter I will present in practice how the relativistic correction term changes the location of position fixes. To do so I will use GNU Radio framework and several gr-gnss blocks from my project (`https://github.com/lukasz-wiecaszek/gr-gnss-gnuradio-3.10`). The data (IQ samples) were captured at the roof of CTTC (Centre Tecnològic de Telecomunicacions de Catalunya), Barcelona (approximated coordinates 41.27484422 N, 1.987756 E). Knowing that location we will be able to see how much error is introduced when we do not apply the relativistic correction term and where the position fixes are, when relativistic correction term is used.

6.1 Position fixes without relativistic correction term

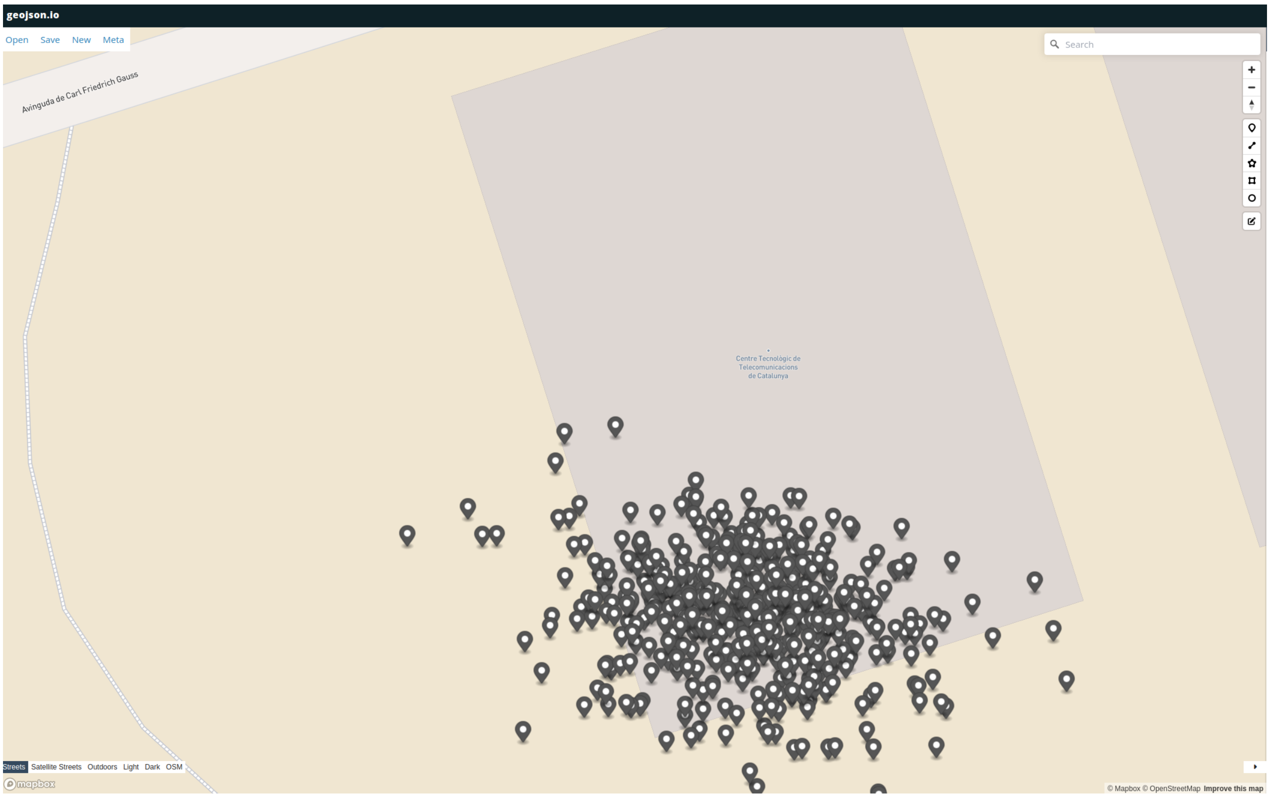

Figure 4: Position fixes without relativistic correction term

Figure 4 presents position fixes calculated when relativistic correction term was not applied.

6.2 Position fixes with relativistic correction term

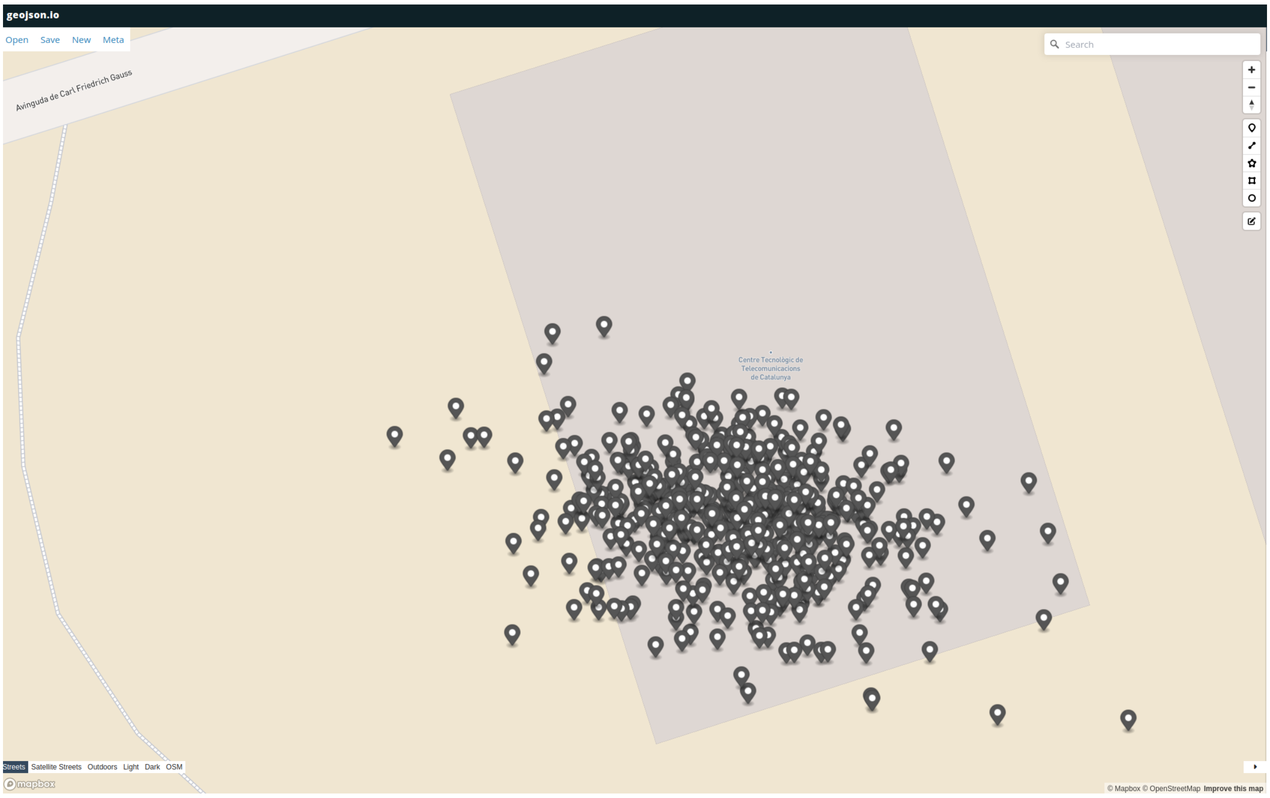

Figure 5: Position fixes with relativistic correction term

Figure 5 presents position fixes calculated when relativistic correction term was applied.

6.3 Summary

Unfortunatelly there is still big variance between position fixes in either case (with and without relativistic correction term). On the other hand, I think it is visible that the averaged position differs in both cases. The difference between mean value of the position fixes with and without relativistic correction term is about 5 meters. Hence such correction cannot be neglated and relativistic correction term must be applied in the process of calculation of GNSS receiver position.

7 Acknowledgments

This article was brought to you by letters G, P and S but more importantly also by three digits 3, 5 and 7. Thank you for that sponsorship.

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