Doppler effect in GPS
How theory corresponds to the measurements

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Abstract

One of the most fabulous aspects of GPS (or more generally GNSS) is that it combines many branches of science in one technology. We have got physics, math, astronomy, control theory and probably some other domains packed here in one place. When learning all those aspects of GPS and then trying to implement them using the GNU Radio framework, I discovered that Doppler shift for the received GPS signal plays an important part in the overall decoding/demodulation process. This article doesn't bring anything new to what we know about Doppler shifts for electromagnetic waves, but summarizes several Doppler related aspects and packs them in one place. Finally it shows in a quantitative way how the theory of the Doppler effect corresponds to the measurements which need to be done to properly demodulate a received GSP signal.

1 Theory

The Doppler effect for EM (electromagnetic) waves differs from that we know for mechanical waves. First of all, it contains in its formula only relative speed between transmitter and receiver. No medium is involved, thus there are no relative speeds to that medium as present in the mechanical case. The second difference is related with the direction of the relative motion of transmitter and receiver. In the case of mechanical waves, Doppler effect occurs only in the direction of "Line Of Sight". It is sometimes called longitudinal or radial Doppler effect. The medium relative speeds must be casted to the LOS direction (the direction of line joining transmitter and receiver). In the case of EM waves, besides longitudinal Doppler effect, we observe also transverse Doppler effect. In this transverse case Doppler shift is also observed for cases where transmitter and receiver move at some angle to the LOS, even though the distance between them is not changing.

1.1 Longitudinal Doppler effect

Longitudinal Doppler effect can be derived using special theory of relativity and Lorentz transformations. That's yet another amazing thing about GPS. Einstein's special and general theory of relativity are here. Without corrections implying from those theories, the system would not be so accurate as it is. But, let's come back to the derivations. Let's assume that in the frame moving relative to us at speed v_l (v_l is this longitudinal component of velocity) we have two events: $(0,0)$ and $(T',0)$. Both of them happen at the same point (origin of the frame) and are spaced in time by interval T'. Interval T' is the period of the wave transmitted in that moving (prime) frame and is related with wave frequency by $f' = 1/T'$. To see what will be the frequency observed/received in our rest frame, we have to use Lorentz transformations and transform those two events from moving (prime) frame to our rest (non-prime) frame. As it usually happens, we assume that both frames overlap/coincide at time $t = 0$. That way the first event is exactly the same in both frames and is equal $(0,0)$. Now using the Lorentz transformations,

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right) \tag{1}$$

$$x = \gamma\left(x' + vt'\right) \tag{2}$$

we can derive the second event in our rest frame. This is $(\gamma T', \gamma v_l T')$. Now, if we calculate the time needed for a light to travel the distance from $\gamma v_l T'$ to the origin of our rest frame, we will be able to tell when we see the second event at the origin. This is $\gamma T' + (\gamma v_l T')/c$ and this is exactly the period of the received wave measured in our rest frame. So we've got

$$T = \gamma T' + \frac{\gamma v_l T'}{c} = \gamma T' \left(1 + \frac{v_l}{c}\right) = \gamma T' \sqrt{1 + \frac{v_l}{c}} \sqrt{1 + \frac{v_l}{c}} \tag{3}$$

Because

$$\gamma = \frac{1}{\sqrt{1 - \frac{v_l^2}{c^2}}} = \frac{1}{\sqrt{1 + \frac{v_l}{c}} \sqrt{1 - \frac{v_l}{c}}} \tag{4}$$

We finally get

$$T = T' \sqrt{\frac{1 + \frac{v_l}{c}}{1 - \frac{v_l}{c}}} = T' \sqrt{\frac{c + v_l}{c - v_l}} \tag{5}$$

The above is the formula for longitudinal Doppler effect expressed as a function of wave periods. If we want to express it as a function of frequencies, we note that $f = 1/T$, $f' = 1/T'$ and we get

$$f = f' \sqrt{\frac{c - v_l}{c + v_l}} \tag{6}$$

where f is the frequency observed/measured in our rest frame and f' is the frequency transmitted in a frame moving relative to us at speed v_l.

1.2 Transverse Doppler effect

There are many flavours of transverse Doppler effect, depending on relative motion of transmitter and receiver, but in all cases observed/received frequency depends on the square of the transverse component of velocity (v_t^2). For example when the transmitter is in circular motion around the stationary receiver (that is the GPS case), the transverse Doppler frequency is calculated as follows

$$f = f' \sqrt{1 - \frac{v_t^2}{c^2}} \tag{7}$$

1.3 GPS case

1.3.1 Longitudinal Doppler shift

Longitudinal component of satellite velocity varies between -800 m/s till +800 m/s. The sign is introduced to distinguish between cases where the satellite is moving towards the user/receiver or is moving away (recedes) from the user/receiver. When the satellite is at its closest position relative to the user the longitudinal component of satellite velocity is 0. So applying

$$f = f' \sqrt{\frac{c - v_l}{c + v_l}} \tag{8}$$

and using GPS L1 frequency (1575420000 Hz) as the transmitted frequency we get f(-800 m/s) = 1575424204 Hz and f(+800 m/s) = 1575415796 Hz. Thus the longitudinal Doppler shift varies in the range from Δf = +4204 Hz till Δf = -4204 Hz. When the satellite is at its closest position relative to the user, the longitudinal Doppler shift is 0.

1.3.2 Transverse Doppler shift

Satellite speed in the Earth Centered Earth Fixed (ECEF) reference frame varies between 2600 m/s and 3100 m/s. Because eccentricity of a GPS orbit is around e = 0.005 I believe we can treat GPS orbits as circular in this context and assume that the whole speed is concentrated in transverse component of velocity. So applying

$$f = f' \sqrt{1 - \frac{v_t^2}{c^2}} \tag{9}$$

and using GPS L1 frequency (1575420000 Hz) as the transmitted frequency we get f(2600 m/s) = 1575419999.94075 Hz and f(3100 m/s) = 1575419999.91577 Hz. Thus the transverse Doppler shift is less then 0.1 Hz for GPS satellites and what's more important changes with the range of 0.02 Hz when frequency changes from 2600 Hz till 3100 Hz. Those are the reasons that the transverse Doppler shift can be omitted/ignored in the further analysis of Doppler related shifts.

2 Measurements

To be able to receive and then demodulate and decode GPS signal we need to find the frequency at which that signal is available at the receiver. Usually this process has two stages. In the first stage (called acquisition) we roughly estimate the receiving frequency. Once we know the estimation we have to find exact frequency at which GPS signal is available. The exact frequency is needed in the demodulation process where we generate carrier replica which shall be identical in frequency and phase with the incoming signal. But we already know that this received frequency is not constant. As satellite moves the received frequency changes. So it is not enough to find exact received frequency but we must also track this frequency and correct for any detected changes. This stage is called tracking. And the frequency tracking is usually done by means of Phase (and/or Frequency) Locked Loops.

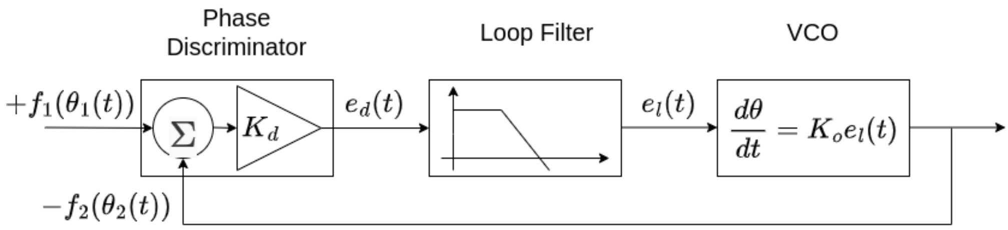

Figure 1: Linearized Phased Locked Loop model

Linearized Phased Locked Loop model is presented on figure 1. The Phase Discriminator block is used to find the phase error between incoming signal and local replica from Voltage Controlled Oscillator (VCO). The output of the discriminator (which is some function of the phase error) is then filtered by the Loop Filter and fed to the VCO. VCO adjusts the frequency of the local replica so that the phase error is minimized. In that way the local replica shall have the same frequency and phase as the incoming signal. This is needed to properly wipe off the carrier (IF) of the incoming signal where we use multiplication of the incoming signal and locally generated frequency replica. And this locally generated frequency replica is just reflecting changes due to satellite movement and thus Doppler shift. If we log that frequency we will have one source of information about Doppler shift. That will come from the PLL (carrier) tracking loop. We may also know the satellite position and satellite velocity from navigation messages. User position can also be known. Either because we already found a fix or from other sources. Now casting satellite velocity $\boldsymbol{v}$ to the line of sight from the satellite to the user we can get longitudinal component of velocity v_l which directly can be used to calculate longitudinal Doppler shift. That will be the second source of Doppler shift.

$$\Delta f = f - f' = f'\sqrt{\frac{c - v_l}{c + v_l}} - f' = f'\left(\sqrt{\frac{c - v_l}{c + v_l}} - 1\right) \tag{10}$$

Where

$$v_l = \boldsymbol{v}\boldsymbol{e_l} \tag{11}$$

and $\boldsymbol{e_l}$ is a unit vector pointing along the line of sight from the user to the satellite.

Those two frequency shifts, the one from PLL (carrier) tracking loop and one from applying theoretical equations shall be equal. So let's check it.

3 GNURadio

To do a verification I will use GNURadio framework and several gr-gnss blocks (`https://github.com/lukasz-wiecaszek/gr-gnss-gnuradio-3.10`). The most important in this context are:

- Acquisition And Tracking
- Doppler Shift

3.1 Acquisition And Tracking

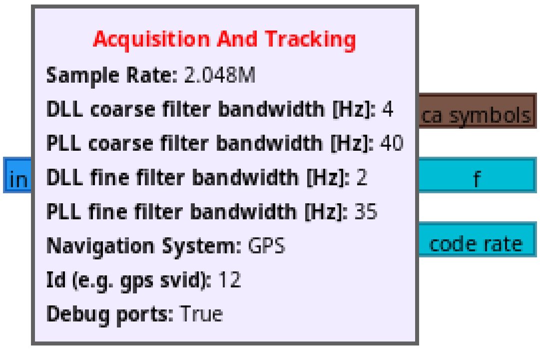

Figure 2: Acquisition And Tracking gr-gnss block

Acquisition and Tracking block does two things. As the name of the block suggests it is signal acquisition followed by signal tracking. Acquisition stage is responsible for fast detection and coarse estimation of two factors in the incoming signal. These are

- C/A (CDMA) code phase
- Doppler shift

Tracking stage takes output from the acquisition (rough estimation of code phase and Doppler shift) and then refines and keeps tracking of those two parameters as times flow. Tracking of the Doppler shift is performed via PLL described in section (2) Measurements. When "Debug ports" are activated ("Debug ports" parameter set to True) then on the "f" port we will get exact VCO frequency used to wipe off the carrier (IF) of the incoming signal. All we have to do now is to log that frequency data samples.

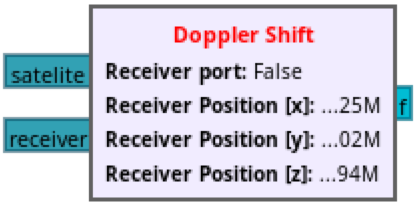

Figure 3: Doppler Shift gr-gnss block

3.2 Doppler Shift

Doppler Shift block takes samples describing satellite position in ECEF reference frame. When constructed it is given user/receiver position (also in ECEF frame). Having got user position and satellite positions it calculates satellite velocity. Longitudinal velocity component is then retrieved and using following formula

$$\Delta f = f' \left(\sqrt{\frac{c - v_l}{c + v_l}} - 1 \right) \tag{12}$$

Doppler shift is calculated. This Doppler shift is made available as the output of the block. So yet again, all we have to do is to log that Doppler shift samples.

4 Verification

4.1 Satellite approaches the receiver

4.1.1 Frequency shift from Acquisition And Tracking block

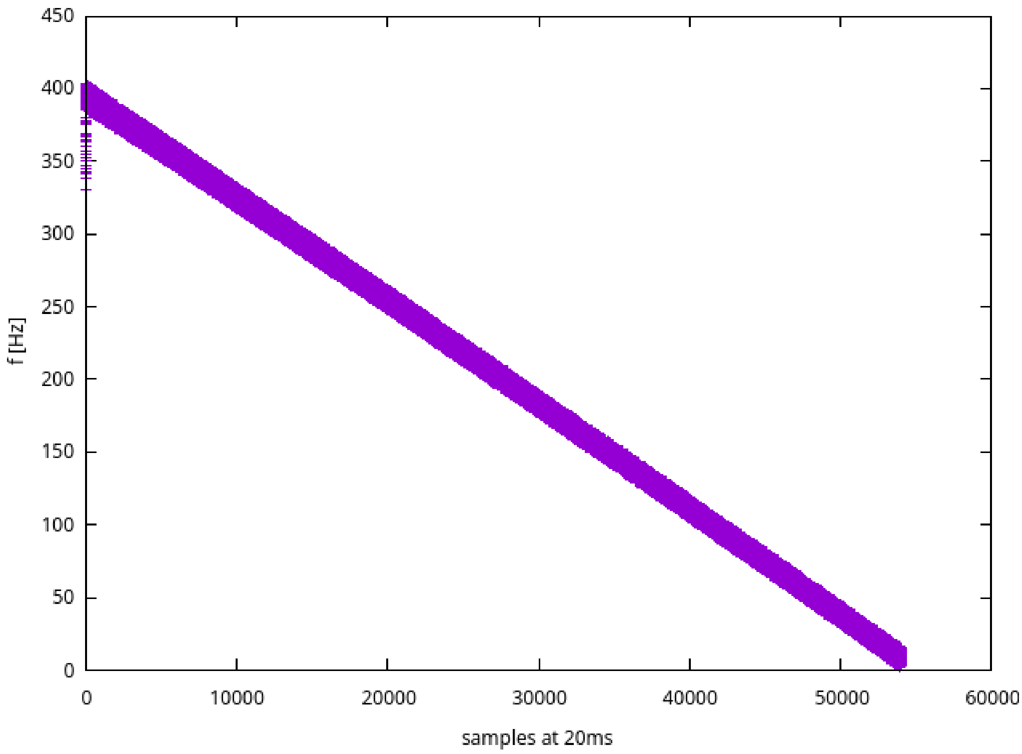

Figure 4: Frequency shift from Acquisition And Tracking block (satellite approaches the receiver)

Figure (4) shows case where satellite approaches the receiver. During 50000 samples (1000 seconds) the measured Doppler shift changes by 350 Hz. Additionally at the very beginning of the graph we see how tracking algorithm refines the frequency estimated in the acquisition state. This period when tracking refines the frequency is also called pull-in state.

4.1.2 Frequency shift from Doppler Shift block

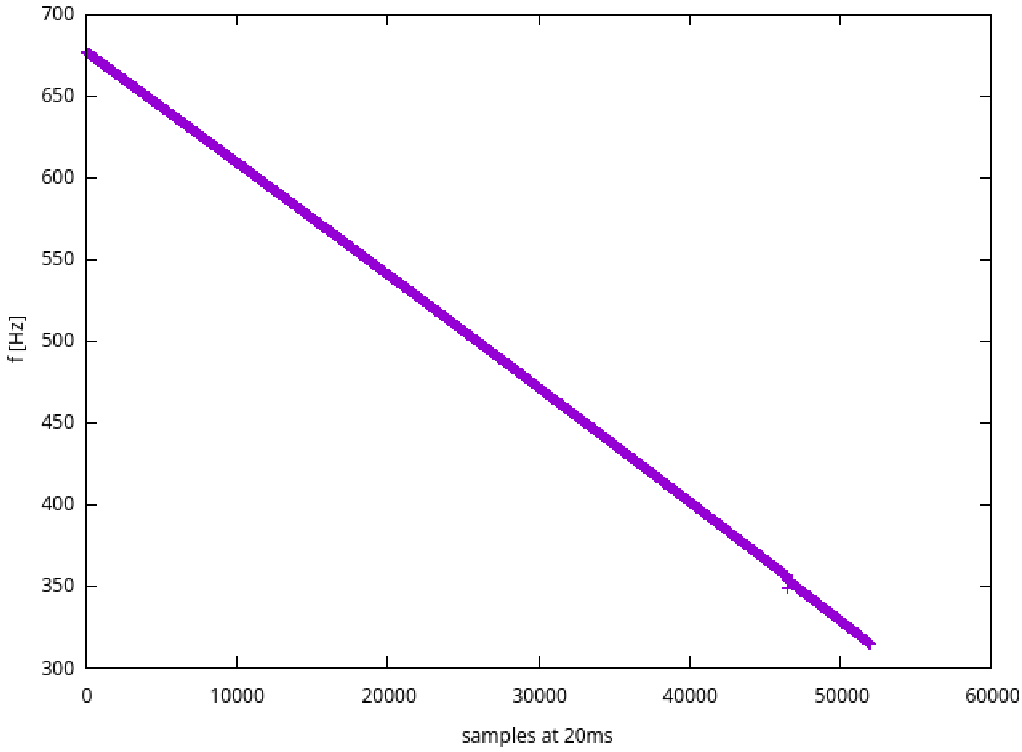

Figure 5: Frequency shift from Doppler Shift block (satellite approaches the receiver)

During the same time (50000 samples), the evaluated Doppler shift also changes by 350 Hz (see figure (5)). That's very good. We can say that theory matches the measurements. Let's us also note, that positive values of frequency on the graph mean that satellite approaches the receiver.

4.2 Satellite passes over the receiver

4.2.1 Frequency shift from Acquisition And Tracking block

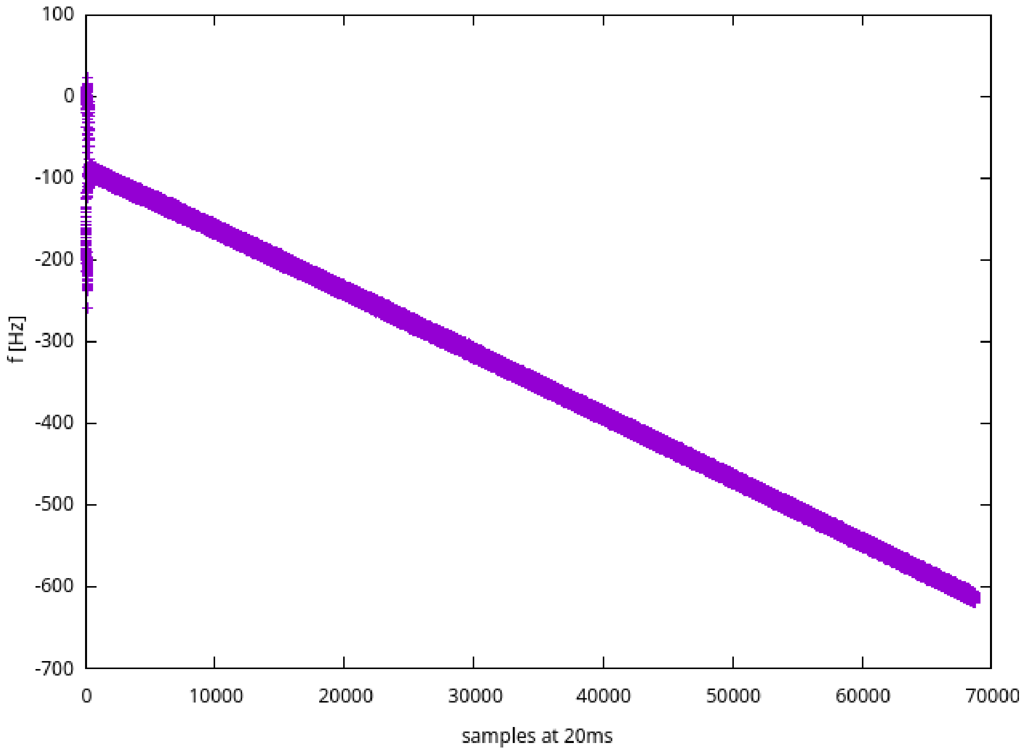

Figure 6: Frequency shift from Acquisition And Tracking block (satellite passes over the receiver)

Figure (6) shows case where satellite passes over the receiver (its longitudinal velocity changes sign relative to the receiver). During 65000 samples (1300 seconds) the measured Doppler shift changes by 480 Hz. The very beginning of the graph reveals the same feature as present in the figure (4) (the frequency refinement in the pull-in state).

4.2.2 Frequency shift from Doppler Shift block

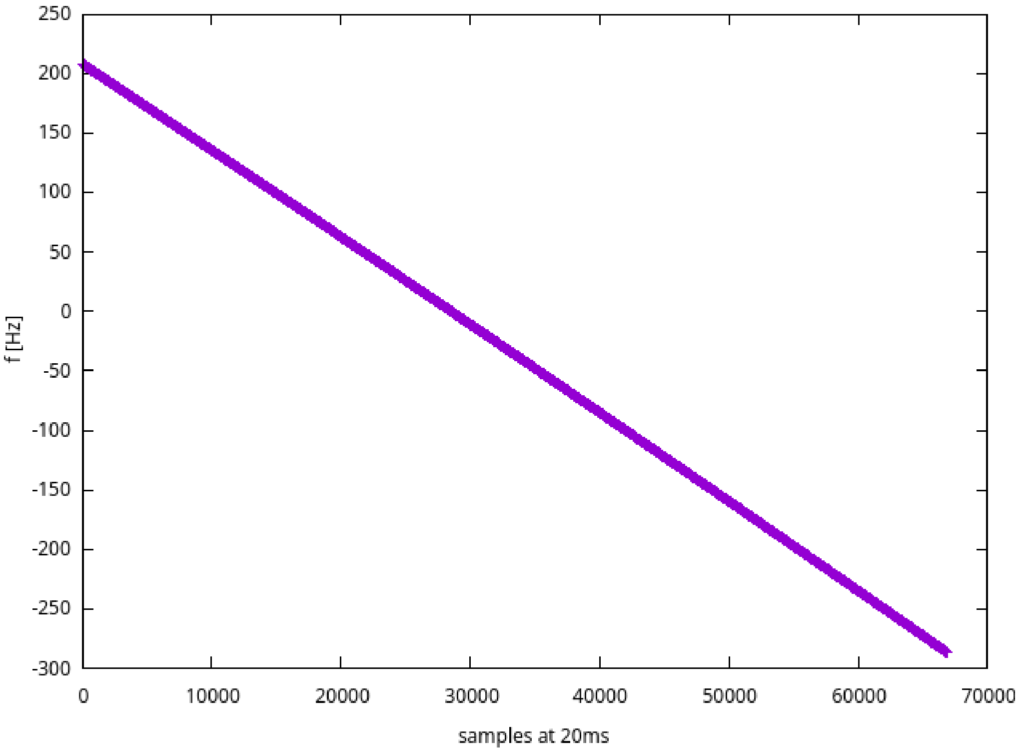

Figure 7: Frequency shift from Doppler Shift block (satellite passes over the receiver)

During the same time (65000 samples), the evaluated Doppler shift also changes by 480 Hz (see figure (7)). That's yet again very good. Yet again we can say that theory matches the measurements. Let's us also note, that this time satellite's Doppler shift crosses the 0 frequency, which means that satellite crosses the point where it is closest to the receiver.

4.3 Satellite recedes from the receiver

4.3.1 Frequency shift from Acquisition And Tracking block

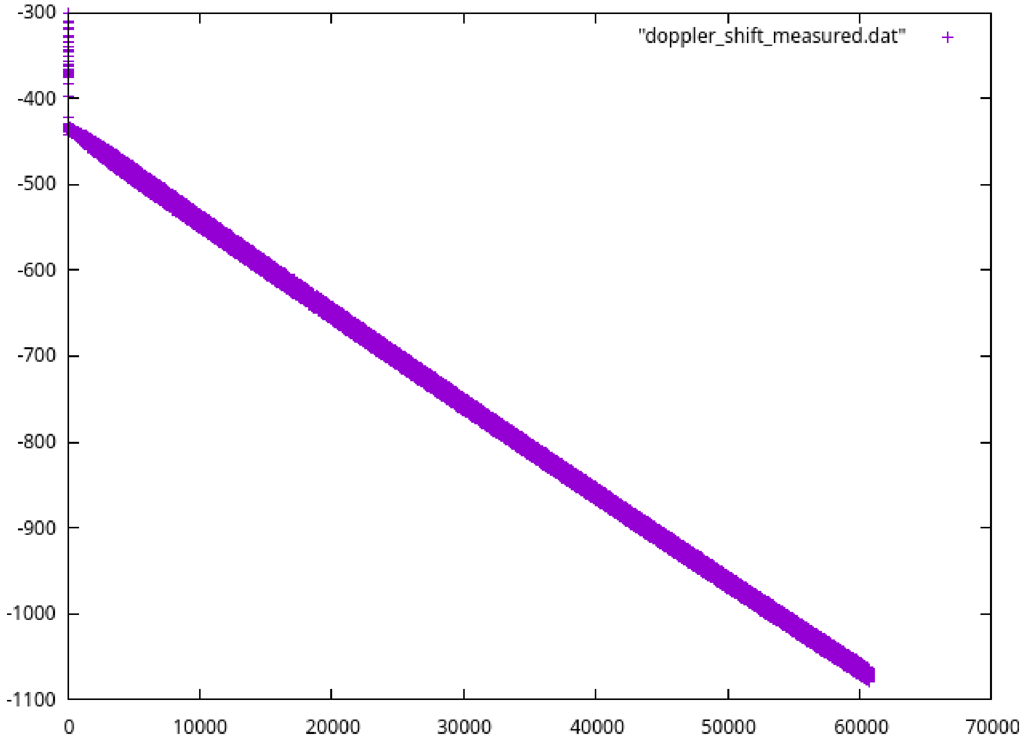

Figure 8: Frequency shift from Acquisition And Tracking block (satellite recedes from the receiver)

The last verification is done for case where satellite recedes from the receiver (figure 8). During 60000 samples (1200 seconds) the measured Doppler shift changes by 460 Hz. And yet again we see frequency refinement in the pull-in state.

4.3.2 Frequency shift from Doppler Shift block

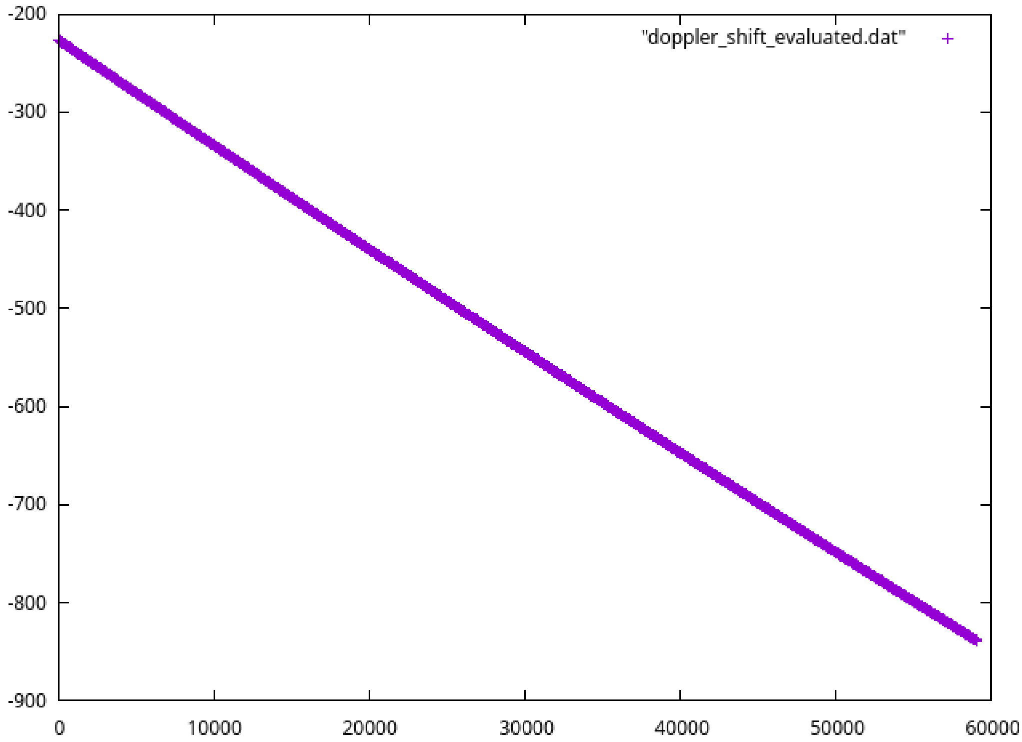

Figure 9: Frequency shift from Doppler Shift block (satellite recedes from the receiver)

During the same time (60000 samples), the evaluated Doppler shift also changes by 640 Hz (see figure (9)). That is our last comparision and yet again this is very good result. Yet again we can say that theory matches the measurements.

4.4 Summary

When I started writing this article I was using standard RTL-SDR V3 USB dongle as my SDR receiver (see `https://www.rtl-sdr.com/`). And that taught me a lot. Especially how important the oscillator is. All the graphs for measured Doppler shift were actually rubbish in the context of this article. Please see (10) and (11) as an example. The reason for such anomaly at the beginning of the graph is the sensitivity to the temperature changes of the original TCXO. The RTL-SDR dongle tends to get quite hot and when it is getting hot at the beginning, the TCXO changes its frequency significantly.

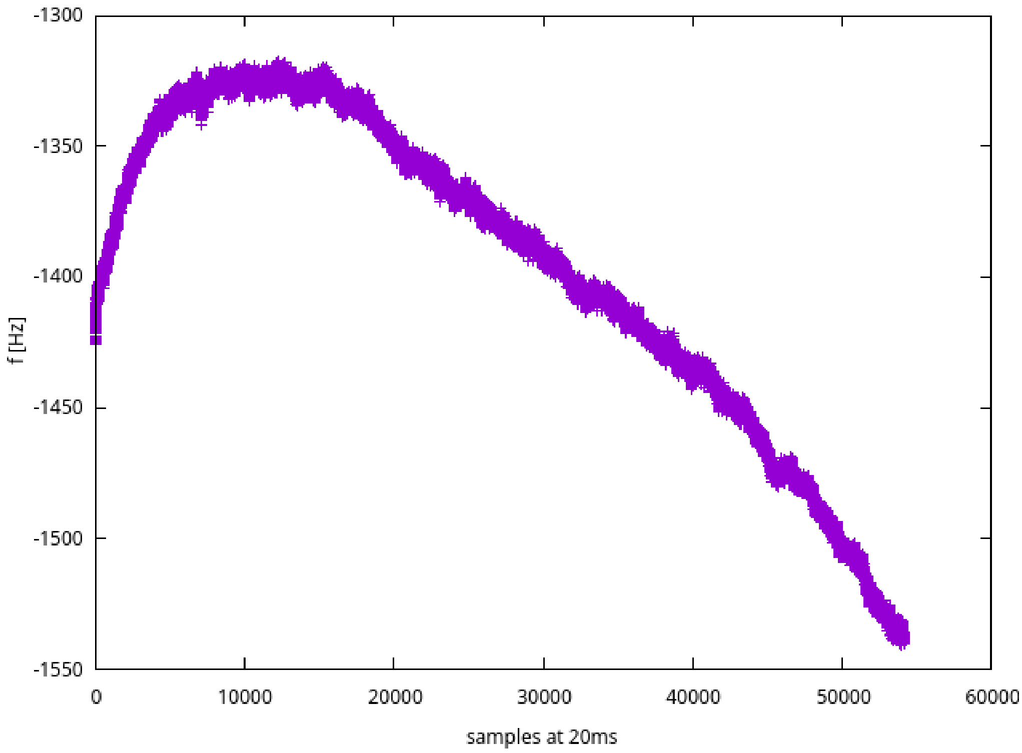

Figure 10: Frequency shift from Acquisition And Tracking block (original TCXO oscillator)

To overcome this problem I had to replace the original built-in TCXO oscilator by something more stable. I selected Leo Bodnar's "Mini Precision GPS Reference Clock". And that was good choice as shown in the previous chapter. At the end I present part of my setup which I have been using to capture all the samples needed for this article.

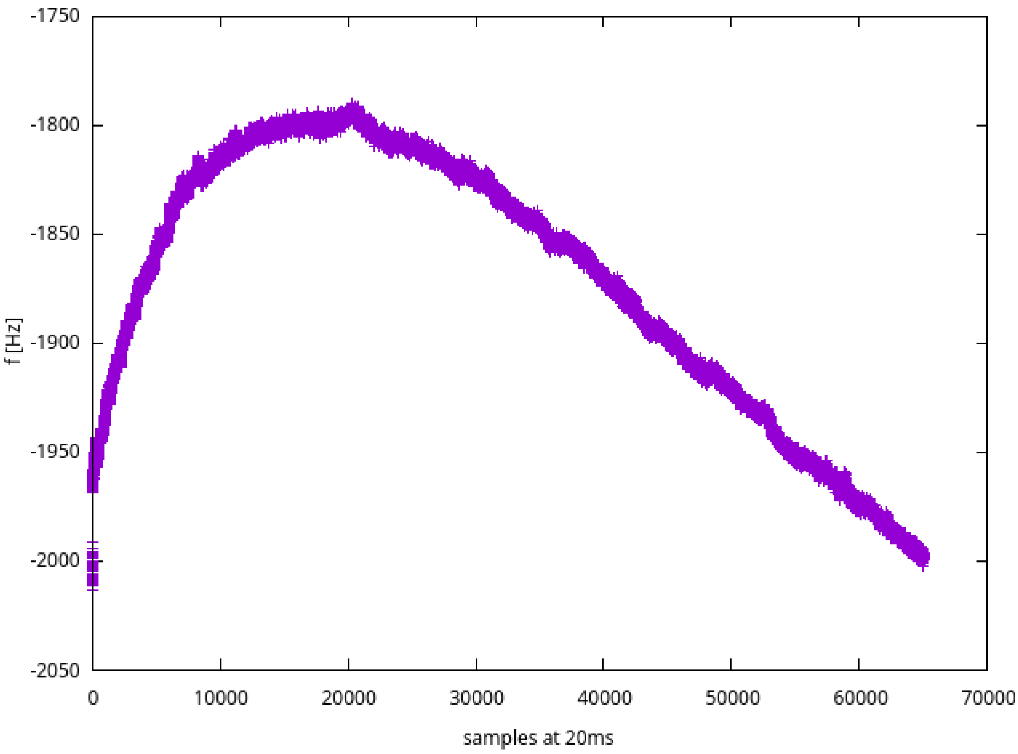

Figure 11: Frequency shift from Acquisition And Tracking block (original TCXO oscilator)

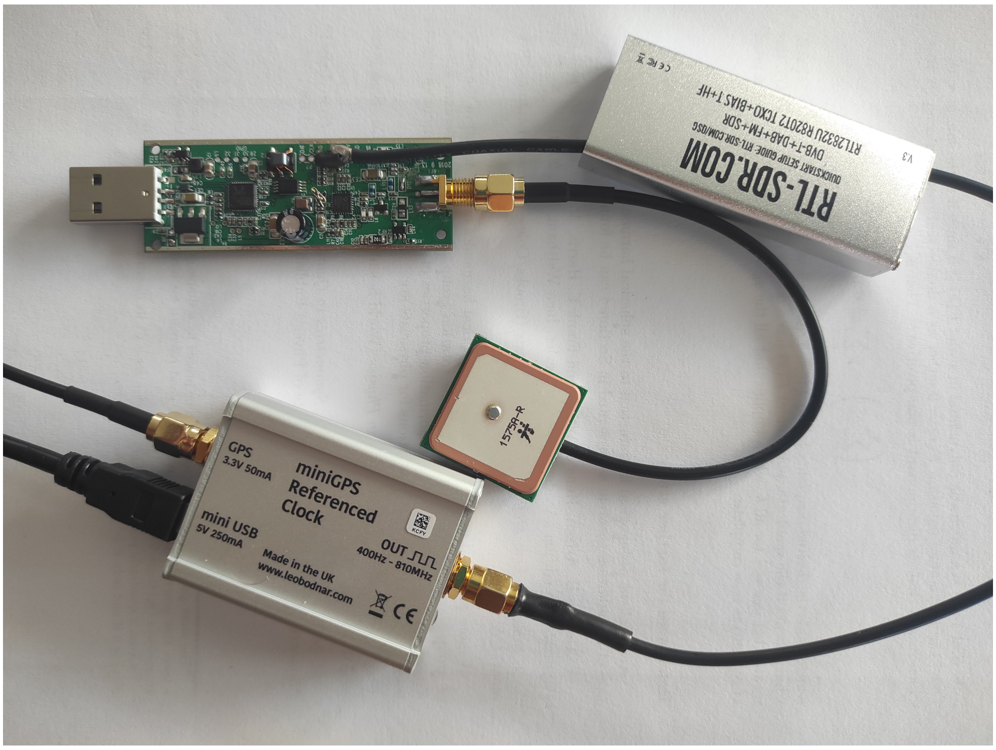

Figure 12: GPSDO as oscilator for RTL-SDR