GPS signal tracking
Digital PLL/DLL loop filters

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Abstract

While implementing digital PLL and DLL loop filters required in the process of GPS signal tracking I used design and filter coefficients as described in [1]. Table 5.6 "Loop Filter Characteristics" of that reference provides loop filter coefficients, but these are hardcoded values such as $1.1\omega_n^2$ or $2.4\omega_n$. I applied those values in my implementation (`https://github.com/lukasz-wiecaszek/gr-gnss-gnuradio-3.10`), they work fine, but since then I felt discomfort as I did not understand how those values were derived. This article aims to provide explanations and derivations so that it shall be clear what is the origin of the filter coefficients, how loop noise bandwidth was derived and what is the relationship between loop noise bandwidth and filter coefficients.

1 Phase-Locked Loop Model

A phase-locked loop is a control theory system that generates an output signal whose phase is "aligned" (equal or of constant difference) to the phase of the input signal. Keeping input and output phases "aligned" means that the input and output frequencies of the signals must be the same.

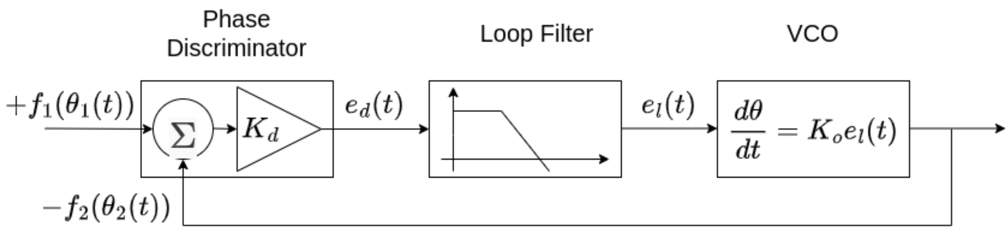

Figure 1: Phased Locked Loop model (time domain)

Figure 1 presents model of a phase-locked loop system of an arbitrary order. It consists of

- a phase discriminator (the terms phase detector and phase comparator are also used) whose output is some function of the phase difference between the input and output signals,
- a loop filter whose objective is to reduce noise present at its input,
- and a voltage controlled oscillator that produces output/reference signal.

Depending whether output of the phase discriminator is a linear or non-linear function of the phase difference, the entire loop is said to be working in linear or non-linear mode respectively (assuming linear loop filter design). For example if the phase discriminator is realized as an analog sinusoidal multiplier, it can be treated as linear when the phase difference is "small" (works in this "linear" slope of sine function).

So, assuming operation in the linear mode the equations can be derived. Let's use Laplace domain. Then we have

$$E_d(s) = K_d\left(\Theta_1(s) - \Theta_2(s)\right) \tag{1}$$

$$E_l(s) = F(s)E_d(s) \tag{2}$$

$$\Theta_2(s) = N(s)E_l(s) = \frac{K_o}{s}E_l(s) \tag{3}$$

Which further gives us

$$\begin{aligned}\Theta_2(s) &= \frac{K_o}{s}F(s)E_d(s)\\ &= \frac{K_o}{s}F(s)\left[K_d\left(\Theta_1(s)-\Theta_2(s)\right)\right]\\ &= \frac{K_dK_o}{s}F(s)\Theta_1(s)-\frac{K_dK_o}{s}F(s)\Theta_2(s)\end{aligned} \tag{4}$$

The closed-loop transfer function for the PLL operating in linear mode is then

$$H(s)=\frac{\Theta_2(s)}{\Theta_1(s)}=\frac{K_dF(s)N(s)}{1+K_dF(s)N(s)}=\frac{K_dK_oF(s)}{s+K_dK_oF(s)} \tag{5}$$

Figure 2 shows diagram of a phase-locked loop system using Laplace domain terminology.

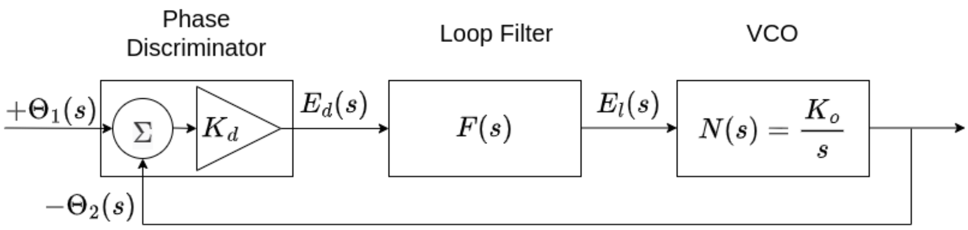

Figure 2: Phased Locked Loop model (Laplace domain)

2 Second-Order PLL

The second-order PLL system contains a first-order loop filter. Let us choose loop filter design such that its transfer function is

$$F(s) = \frac{1+\tau_2 s}{\tau_1 s} \tag{6}$$

Putting that filter transfer function into (5) yields

$$H(s) = \frac{K_d K_o \frac{1+\tau_2 s}{\tau_1 s}}{s + K_d K_o \frac{1+\tau_2 s}{\tau_1 s}} = \frac{K_d K_o \tau_2 s + K_d K_o}{\tau_1 s^2 + K_d K_o \tau_2 s + K_d K_o} \tag{7}$$

which finally gives

$$H(s) = \frac{K_d K_o \frac{\tau_2}{\tau_1} s + K_d K_o \frac{1}{\tau_1}}{s^2 + K_d K_o \frac{\tau_2}{\tau_1} s + K_d K_o \frac{1}{\tau_1}} \tag{8}$$

Introducing natural frequency $\omega_n^2 = \frac{K_d K_o}{\tau_1}$ and the damping ratio $\zeta = \frac{\tau_2 \omega_n}{2}$ the PLL transfer function gains following customary shape

$$H(s) = \frac{2\zeta\omega_n s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{9}$$

We want to find coefficiets of the digial loop filter and express them as the function of natural frequency and damping ratio. Let's start from converting analog version of transfer function (6) into digital one. To do so, bilinear transformation is used.

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \tag{10}$$

Substituting (10) into (6) we get

$$F(z) = \frac{1+\tau_2 \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}{\tau_1 \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{T\left(1+z^{-1}\right) + 2\tau_2\left(1-z^{-1}\right)}{2\tau_1\left(1-z^{-1}\right)} \tag{11}$$

Dividing numerator and denominator by $2\tau_1$ we receive

$$F(z) = \frac{\frac{T}{2}\frac{1}{\tau_1}\left(1+z^{-1}\right) + \frac{\tau_2}{\tau_1}\left(1-z^{-1}\right)}{1-z^{-1}} \tag{12}$$

And finally

$$F(z) = \frac{\left(\frac{T}{2}\frac{1}{\tau_1} + \frac{\tau_2}{\tau_1}\right) + \left(\frac{T}{2}\frac{1}{\tau_1} - \frac{\tau_2}{\tau_1}\right) z^{-1}}{1 - z^{-1}} \tag{13}$$

Now we use formulas for natural frequency $\omega_n^2 = \frac{K_d K_o}{\tau_1}$ and the damping ratio $\zeta = \frac{\tau_2 \omega_n}{2}$. Rearanging them we get

$$\frac{1}{\tau_1} = \omega_n^2 \frac{1}{K_d K_o} \tag{14}$$

and

$$\frac{\tau_2}{\tau_1} = 2\zeta\omega_n \frac{1}{K_d K_o} \tag{15}$$

And substituting them into digital filter transfer function (13) yields

$$F(z) = \frac{\left(\frac{T}{2}\omega_n^2 \frac{1}{K_d K_o} + 2\zeta\omega_n \frac{1}{K_d K_o}\right) + \left(\frac{T}{2}\omega_n^2 \frac{1}{K_d K_o} - 2\zeta\omega_n \frac{1}{K_d K_o}\right) z^{-1}}{1 - z^{-1}} \tag{16}$$

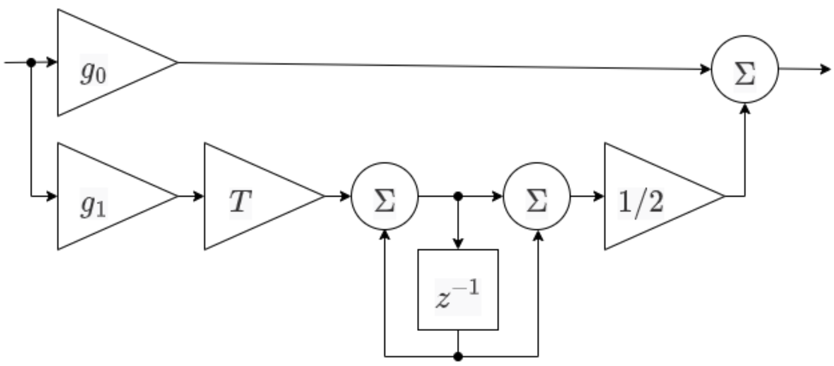

Figure 3: First-order digital loop filter diagram

Digital first-order filter as implemented in GPS tracking loops is presented in figure 3. It is the result of the bilinear transformation applied to analog counterpart present in figure 4.

The z-domain transfer function of the loop filter is

$$F(z) = g_0 + g_1 \frac{T}{2} \frac{1 + z^{-1}}{1 - z^{-1}} = \frac{\left(g_0 + g_1 \frac{T}{2}\right) + \left(-g_0 + g_1 \frac{T}{2}\right) z^{-1}}{1 - z^{-1}} \tag{17}$$

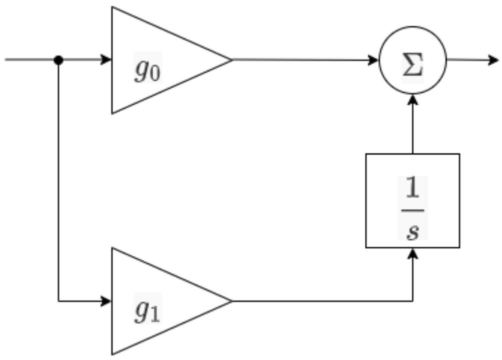

Figure 4: First-order analog loop filter diagram

Comparing (16) and (17) we quickly find out that

$$g_0 = 2\zeta\omega_n \frac{1}{K_d K_o} \tag{18}$$

and

$$g_1 = \omega_n^2 \frac{1}{K_d K_o} \tag{19}$$

Table 5.6 "Loop Filter Characteristics" of [1] for second-order loop gives following values for loop filter coefficients

$$a_2\omega_0 = 1.414\omega_0 \tag{20}$$
$$\omega_0^2 \tag{21}$$

This is it. g_0 and g_1 match the form of the coefficietns in [1]. ω_0 corresponds to loop natural frequency ω_n and $a_2 = 2\zeta$ with $\zeta = \frac{1}{\sqrt{2}}$. $K_d K_o$ shall be put equal to 1. This is what I wanted to achive.

Once we have got g_0 and g_1, let me derive yet coefficiets of the digital loop (IIR) filter as a function of g_0 and g_1. Because z-transfer function can always be represented as ratio of two polynomials in z, we can write for 1st order loop filter

$$F(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}} \tag{22}$$

This gives us following digital loop filter coefficients

$$
\begin{aligned}
b_0 &= +g_0 + g1\frac{T}{2} \\
b_1 &= -g_0 + g1\frac{T}{2} \\
a_1 &= -1
\end{aligned}
\tag{23}
$$

The last thing to find out is the relationship between natural frequency and the closed loop noise bandwidth B_L. According to [1] it is

$$
B_L = \frac{\omega_0 \left(1 + a_2^2\right)}{4a_2} \tag{24}
$$

But how this noise bandwith was derivied. To find out let's start from the definition. The loop noise bandwith (single-sided) is defined as

$$
B_L = \int_0^{\infty} |H(s)|^2 \, df = \frac{1}{2}\int_{-\infty}^{\infty} |H(s)|^2 \, df \tag{25}
$$

In our case, the second-order loop transfer function $H(s)$ is

$$
H(s) = \frac{2\zeta\omega_n s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{26}
$$

Transforming the s domain into frequency domain with $s = j\omega$, gives us

$$
H(j\omega) = \frac{2\zeta\omega_n j\omega + \omega_n^2}{-\omega^2 + 2\zeta\omega_n j\omega + \omega_n^2} \tag{27}
$$

Dividing numerator and denominator by ω_n^2 we reach

$$
H(j\omega) = \frac{2\zeta j\left(\dfrac{\omega}{\omega_n}\right) + 1}{-\left(\dfrac{\omega}{\omega_n}\right)^2 + 2\zeta j\left(\dfrac{\omega}{\omega_n}\right) + 1} \tag{28}
$$

Because the conjugate of a product (quotient) is the product (quotient) of the conjugates of the individual complex numbers we can write

$$
H^*(j\omega) = \frac{-2\zeta j\left(\dfrac{\omega}{\omega_n}\right) + 1}{-\left(\dfrac{\omega}{\omega_n}\right)^2 - 2\zeta j\left(\dfrac{\omega}{\omega_n}\right) + 1} \tag{29}
$$

Which gives us

$$|H(j\omega)|^2 = H(j\omega)\cdot H^*(j\omega) = \frac{1+\left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2} \tag{30}$$

$$|H(j\omega)|^2 = \frac{1+4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}{1+2\left(2\zeta^2-1\right)\left(\frac{\omega}{\omega_n}\right)^2+\left(\frac{\omega}{\omega_n}\right)^4} \tag{31}$$

Hence, what we need to calculate is the following integral

$$\begin{aligned} B_L &= \int_0^\infty |H(s)|^2\, df = \frac{1}{2\pi}\int_0^\infty |H(j\omega)|^2\, d\omega \\ &= \frac{1}{2\pi}\int_0^\infty \frac{1+4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}{1+2\left(2\zeta^2-1\right)\left(\frac{\omega}{\omega_n}\right)^2+\left(\frac{\omega}{\omega_n}\right)^4}\, d\omega \end{aligned} \tag{32}$$

Equation (32) can be written as

$$B_L = \frac{\omega_n}{2\pi}\int_0^\infty \frac{1+4\zeta^2x^2}{1+2\left(2\zeta^2-1\right)x^2+x^4}\, dx = \frac{\omega_n}{2\pi}I_1 + \frac{\omega_n}{2\pi}4\zeta^2 I_2 \tag{33}$$

where

$$I_1 = \int_0^\infty \frac{1}{1+2\left(2\zeta^2-1\right)x^2+x^4}\, dx \tag{34}$$

$$I_2 = \int_0^\infty \frac{x^2}{1+2\left(2\zeta^2-1\right)x^2+x^4}\, dx \tag{35}$$

To solve I_1 and I_2 we will make use of the following relationship. It is taken from [3], equation 3.264-2.

$$\int_0^\infty \frac{x^{\mu-1}}{\left(\beta+x^2\right)\left(\gamma+x^2\right)}\, dx = \frac{\pi}{2}\frac{\gamma^{\frac{\mu}{2}-1}-\beta^{\frac{\mu}{2}-1}}{\beta-\gamma} cosec\frac{\mu\pi}{2} \tag{36}$$

We will have to find β and γ so that

$$\left(\beta+x^2\right)\left(\gamma+x^2\right) = \beta\gamma + \left(\beta+\gamma\right)x^2 + x^4 = 1+2\left(2\zeta^2-1\right)x^2+x^4 \tag{37}$$

To do so, we have to solve following system of equations

$$\begin{cases} \beta\gamma = 1 \\ \beta + \gamma = 2\left(2\zeta^2 - 1\right) \end{cases} \tag{38}$$

Doing so, we get

$$\begin{aligned} \beta &= \left(\zeta + \sqrt{\zeta^2 - 1}\right)^2 \\ \gamma &= \left(\zeta - \sqrt{\zeta^2 - 1}\right)^2 \end{aligned} \tag{39}$$

Thus the solution to integral I_1, where $\mu = 1$ is

$$\begin{aligned} I_1 &= \frac{\pi}{2}\frac{\gamma^{\frac{1}{2}-1} - \beta^{\frac{1}{2}-1}}{\beta - \gamma} cosec\frac{\pi}{2} \\ &= \frac{\pi}{2}\frac{\frac{1}{\left(\zeta - \sqrt{\zeta^2-1}\right)} - \frac{1}{\left(\zeta + \sqrt{\zeta^2-1}\right)}}{\left(\zeta + \sqrt{\zeta^2-1}\right)^2 - \left(\zeta - \sqrt{\zeta^2-1}\right)^2} cosec\frac{\pi}{2} \\ &= \frac{\pi}{2}\frac{2\sqrt{\zeta^2-1}}{4\zeta\sqrt{\zeta^2-1}} cosec\frac{\pi}{2} = \frac{\pi}{2}\frac{1}{2\zeta}1 = \frac{\pi}{4\zeta} \end{aligned} \tag{40}$$

For I_2 we use $\mu = 3$

$$\begin{aligned} I_2 &= \frac{\pi}{2}\frac{\gamma^{\frac{3}{2}-1} - \beta^{\frac{3}{2}-1}}{\beta - \gamma} cosec\frac{3\pi}{2} \\ &= \frac{\pi}{2}\frac{\left(\zeta - \sqrt{\zeta^2-1}\right) - \left(\zeta + \sqrt{\zeta^2-1}\right)}{\left(\zeta + \sqrt{\zeta^2-1}\right)^2 - \left(\zeta - \sqrt{\zeta^2-1}\right)^2} cosec\frac{3\pi}{2} \\ &= \frac{\pi}{2}\frac{-2\sqrt{\zeta^2-1}}{4\zeta\sqrt{\zeta^2-1}} cosec\frac{3\pi}{2} = \frac{\pi}{2}\frac{-1}{2\zeta}(-1) = \frac{\pi}{4\zeta} \end{aligned} \tag{41}$$

And finally noise bandwith B_L is equal

$$B_L = \frac{\omega_n}{2\pi}I_1 + \frac{\omega_n}{2\pi}4\zeta^2 I_2 = \frac{\omega_n}{2\pi}\frac{\pi}{4\zeta} + \frac{\omega_n}{2\pi}4\zeta^2\frac{\pi}{4\zeta} = \omega_n\frac{1+4\zeta^2}{8\zeta} \tag{42}$$

That is exactly what I wanted to achive ;-)

Please note that [1] gives following formula for loop noise bandwidth

$$B_L = \frac{\omega_0 \left(1 + a_2^2\right)}{4a_2} \tag{43}$$

But with the $\omega_0 = \omega_n$ and with the $a_2 = 2\zeta$, this is the same.

3 Third-Order PLL

The third-order PLL system must contain a second-order loop filter. Let us choose loop filter design such that its transfer function is

$$F(s) = \frac{1 + \tau_2 s + \tau_3 s^2}{\tau_1 s^2} \tag{44}$$

Putting that filter transfer function into (5) yields

$$H(s) = \frac{K_d K_o \dfrac{1 + \tau_2 s + \tau_3 s^2}{\tau_1 s^2}}{s + K_d K_o \dfrac{1 + \tau_2 s + \tau_3 s^2}{\tau_1 s^2}} = \frac{K_d K_o \tau_3 s^2 + K_d K_o \tau_2 s + K_d K_o}{\tau_1 s^3 + K_d K_o \tau_3 s^2 + K_d K_o \tau_2 s + K_d K_o} \tag{45}$$

which finally gives

$$H(s) = \frac{K_d K_o \dfrac{\tau_3}{\tau_1} s^2 + K_d K_o \dfrac{\tau_2}{\tau_1} s + K_d K_o \dfrac{1}{\tau_1}}{s^3 + K_d K_o \dfrac{\tau_3}{\tau_1} s^2 + K_d K_o \dfrac{\tau_2}{\tau_1} s + K_d K_o \dfrac{1}{\tau_1}} \tag{46}$$

Introducing natural frequency $\omega_n^3 = \dfrac{K_d K_o}{\tau_1}$, $\zeta = \tau_2 \omega_n$ and $\xi = \tau_3 \omega_n^2$, the third-order PLL loop transfer function gains following shape

$$H(s) = \frac{\xi \omega_n s^2 + \zeta \omega_n^2 s + \omega_n^3}{s^3 + \xi \omega_n s^2 + \zeta \omega_n^2 s + \omega_n^3} \tag{47}$$

As in the case of the first-order loop filter, now we want to find coefficiets of the digial second-order loop filter and express them as function of ω_n, ζ and ξ. Again, let's start from converting analog version of transfer function (44) into digital one. To do so, bilinear transformation (10) is used. Substituting (10) into (44) we get

$$\begin{aligned} F(z) &= \frac{1 + \tau_2 \dfrac{2}{T} \dfrac{1 - z^{-1}}{1 + z^{-1}} + \tau_3 \left(\dfrac{2}{T} \dfrac{1 - z^{-1}}{1 + z^{-1}} \right)^2}{\tau_1 \left(\dfrac{2}{T} \dfrac{1 - z^{-1}}{1 + z^{-1}} \right)^2} \\ &= \frac{T^2 \left(1 + z^{-1}\right)^2 + 2T\tau_2 \left(1 + z^{-1}\right)\left(1 - z^{-1}\right) + 4\tau_3 \left(1 - z^{-1}\right)^2}{4\tau_1 \left(1 - z^{-1}\right)^2} \end{aligned} \tag{48}$$

Dividing numerator and denominator by $4\tau_1$ we receive

$$F(z) = \frac{\frac{T^2}{4}\frac{1}{\tau_1}\left(1+2z^{-1}+z^{-2}\right) + \frac{T}{2}\frac{\tau_2}{\tau_1}\left(1-z^{-2}\right) + \frac{\tau_3}{\tau_1}\left(1-2z^{-1}+z^{-2}\right)}{1-2z^{-1}+z^{-2}} \tag{49}$$

And finally

$$F(z) = \frac{\left(\frac{T}{2}\frac{T}{2}\frac{1}{\tau_1} + \frac{T}{2}\frac{\tau_2}{\tau_1} + \frac{\tau_3}{\tau_1}\right) + \left(2\frac{T}{2}\frac{T}{2}\frac{1}{\tau_1} - 2\frac{\tau_3}{\tau_1}\right)z^{-1} + \left(\frac{T}{2}\frac{T}{2}\frac{1}{\tau_1} - \frac{T}{2}\frac{\tau_2}{\tau_1} + \frac{\tau_3}{\tau_1}\right)z^{-2}}{1-2z^{-1}+z^{-2}} \tag{50}$$

Now we use formulas for natural frequency $\omega_n^3 = \frac{K_d K_o}{\tau_1}$, $\zeta = \tau_2\omega_n$ and $\xi = \tau_3\omega_n^2$. Rearanging them we get

$$\frac{1}{\tau_1} = \omega_n^3 \frac{1}{K_d K_o} \tag{51}$$

$$\frac{\tau_2}{\tau_1} = \zeta\omega_n^2 \frac{1}{K_d K_o} \tag{52}$$

$$\frac{\tau_3}{\tau_1} = \xi\omega_n \frac{1}{K_d K_o} \tag{53}$$

And substituting them into digital filter transfer function (50) yields

$$F(z) = \frac{\left(\frac{T}{2}\frac{T}{2}\frac{\omega_n^3}{K} + \frac{T}{2}\frac{\zeta\omega_n^2}{K} + \frac{\xi\omega_n}{K}\right) + \left(2\frac{T}{2}\frac{T}{2}\frac{\omega_n^3}{K} - 2\frac{\xi\omega_n}{K}\right)z^{-1} + \left(\frac{T}{2}\frac{T}{2}\frac{\omega_n^3}{K} - \frac{T}{2}\frac{\zeta\omega_n^2}{K} + \frac{\xi\omega_n}{K}\right)z^{-2}}{1-2z^{-1}+z^{-2}} \tag{54}$$

where $K = K_d K_o$

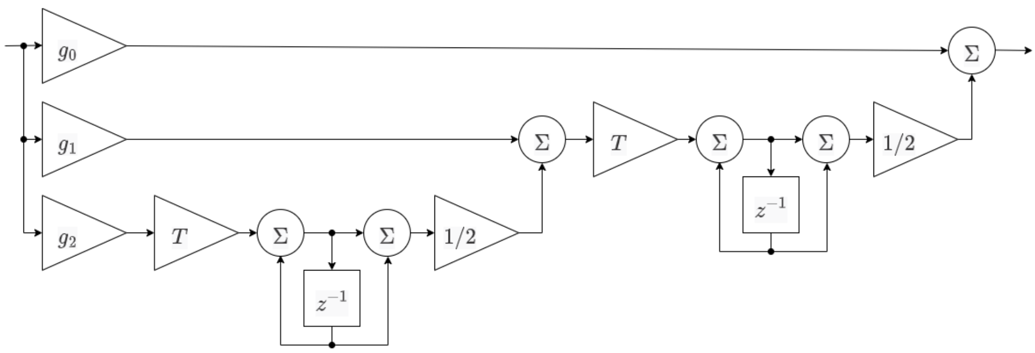

Figure 5: Second-order digital loop filter diagram

Digital second-order filter as implemented in GPS tracking loops is present in figure 5. It is the result of the bilinear transformation applied to analog counterpart present in figure 6.

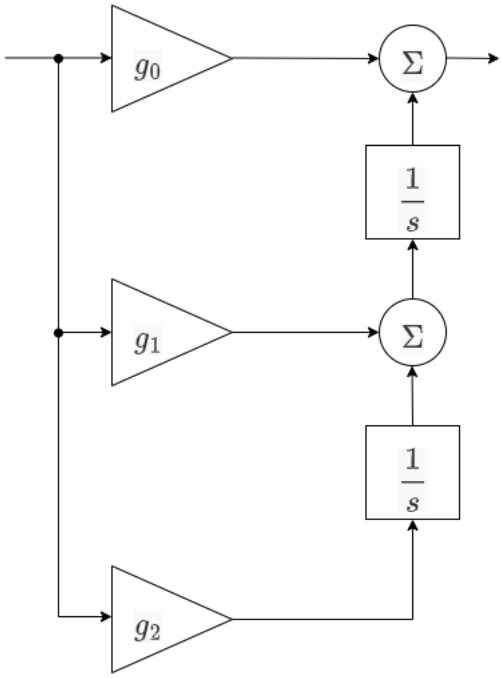

Figure 6: Second-order analog loop filter diagram

The z-domain transfer function of the loop filter is

$$
\begin{aligned}
F(z) &= g_0 + \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}\left(g_1 + g_2\frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}\right) \\
&= g_0 + \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}\frac{\left(g_1+g_2\frac{T}{2}\right)+\left(-g_1+g_2\frac{T}{2}\right)z^{-1}}{1-z^{-1}} \\
&= \frac{\left(g_0+\frac{T}{2}\left(g_1+g_2\frac{T}{2}\right)\right)+\left(-2g_0+2g_2\frac{T}{2}\frac{T}{2}\right)z^{-1}+\left(g_0+\frac{T}{2}\left(-g_1+g_2\frac{T}{2}\right)\right)z^{-2}}{1-2z^{-1}+z^{-2}} \\
&= \frac{\left(g_0+\frac{T}{2}g_1+\frac{T}{2}\frac{T}{2}g_2\right)+\left(-2g_0+2g_2\frac{T}{2}\frac{T}{2}\right)z^{-1}+\left(g_0-\frac{T}{2}g_1+\frac{T}{2}\frac{T}{2}g_2\right)z^{-2}}{1-2z^{-1}+z^{-2}}
\end{aligned}
\tag{55}
$$

Comparing (50) and (55) we quickly find out that

$$
\begin{aligned}
g_0 &= \xi\omega_n \frac{1}{K_d K_o} \\
g_1 &= \zeta\omega_n^2 \frac{1}{K_d K_o} \\
g_2 &= \omega_n^3 \frac{1}{K_d K_o}
\end{aligned} \tag{56}
$$

Table 5.6 "Loop Filter Characteristics" of [1] for third-order loop gives following values for loop filter coefficients

$$b_3\omega_0 = 2.4\omega_0 \tag{57}$$

$$a_3\omega_0^2 = 1.1\omega_0^2 \tag{58}$$

$$\omega_0^3 \tag{59}$$

This is it. g_0, g_1 and g_2 match the form of the coefficietns in [1]. ω_0 corresponds to the loop natural frequency, $a_3 = \zeta$ and $b_3 = \xi$. $K_d K_o$ shall be put equal to 1. This is what I wanted to achive.

Once we have got g_0, g_1 and g_2, let us derive yet coefficients of the digital loop (IIR) filter as a function of g_0, g_1 and g_2. Because z-transfer function can always be represented as ratio of two polynomials in z, we can write for 2nd order loop filter

$$F(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \tag{60}$$

This gives us following digital loop filter coefficients

$$
\begin{aligned}
b_0 &= g_0 + \frac{T}{2}\left(g_1 + g_2\frac{T}{2}\right) \\
b_1 &= -2g_0 + 2g_2\frac{T}{2}\frac{T}{2} \\
b_2 &= g_0 + \frac{T}{2}\left(-g_1 + g_2\frac{T}{2}\right) \\
a_1 &= -2 \\
a_2 &= 1
\end{aligned} \tag{61}
$$

And the last thing to find out is the relationship between natural frequency and the closed loop noise bandwidth B_L. According to [1] for third-order loop it is

$$B_L = \frac{\omega_0\left(a_3 b_3^2 + a_3^2 - b_3\right)}{4\left(a_3 b_3 - 1\right)} \tag{62}$$

But how this noise bandwith was derivied. To find out let's start as previously from the definition (25) of the noise bandwidth.

In our case, the third-order loop transfer function $H(s)$ is

$$H(s) = \frac{\xi\omega_n s^2 + \zeta\omega_n^2 s + \omega_n^3}{s^3 + \xi\omega_n s^2 + \zeta\omega_n^2 s + \omega_n^3} \tag{63}$$

Transforming the s domain into frequency domain with $s = j\omega$, gives us

$$H(j\omega) = \frac{-\xi\omega_n\omega^2 + \zeta\omega_n^2 j\omega + \omega_n^3}{-j\omega^3 - \xi\omega_n\omega^2 + \zeta\omega_n^2 j\omega + \omega_n^3} \tag{64}$$

Dividing numerator and denominator by ω_n^3 we reach

$$H(j\omega) = \frac{-\xi\left(\frac{\omega}{\omega_n}\right)^2 + j\zeta\left(\frac{\omega}{\omega_n}\right) + 1}{-j\left(\frac{\omega}{\omega_n}\right)^3 - \xi\left(\frac{\omega}{\omega_n}\right)^2 + j\zeta\left(\frac{\omega}{\omega_n}\right) + 1} \tag{65}$$

Because the conjugate of a product (quotient) is the product (quotient) of the conjugates of the individual complex numbers we can write

$$H^*(j\omega) = \frac{-\xi\left(\frac{\omega}{\omega_n}\right)^2 - j\zeta\left(\frac{\omega}{\omega_n}\right) + 1}{j\left(\frac{\omega}{\omega_n}\right)^3 - \xi\left(\frac{\omega}{\omega_n}\right)^2 - j\zeta\left(\frac{\omega}{\omega_n}\right) + 1} \tag{66}$$

Which gives us

$$|H(j\omega)|^2 = H(j\omega)\cdot H^*(j\omega) =$$
$$\frac{-\xi\left(\frac{\omega}{\omega_n}\right)^2 + j\zeta\left(\frac{\omega}{\omega_n}\right) + 1}{-j\left(\frac{\omega}{\omega_n}\right)^3 - \xi\left(\frac{\omega}{\omega_n}\right)^2 + j\zeta\left(\frac{\omega}{\omega_n}\right) + 1} \cdot \frac{-\xi\left(\frac{\omega}{\omega_n}\right)^2 - j\zeta\left(\frac{\omega}{\omega_n}\right) + 1}{j\left(\frac{\omega}{\omega_n}\right)^3 - \xi\left(\frac{\omega}{\omega_n}\right)^2 - j\zeta\left(\frac{\omega}{\omega_n}\right) + 1}$$

And further

$$|H(j\omega)|^2 =$$

$$\frac{1 + \left(\zeta^2 - 2\xi\right)\left(\frac{\omega}{\omega_n}\right)^2 + \xi^2\left(\frac{\omega}{\omega_n}\right)^4}{\left[-j\left(\frac{\omega}{\omega_n}\right)^3 - \xi\left(\frac{\omega}{\omega_n}\right)^2 + j\zeta\left(\frac{\omega}{\omega_n}\right) + 1\right] \cdot \left[j\left(\frac{\omega}{\omega_n}\right)^3 - \xi\left(\frac{\omega}{\omega_n}\right)^2 - j\zeta\left(\frac{\omega}{\omega_n}\right) + 1\right]}$$

Hence, what we need to calculate is the following integral

$$B_L = \int_0^\infty |H(s)|^2\, df = \frac{1}{2\pi}\int_0^\infty |H(j\omega)|^2\, d\omega = \frac{1}{4\pi}\int_{-\infty}^\infty |H(j\omega)|^2\, d\omega =$$

$$\frac{1}{4\pi}\int_{-\infty}^\infty \frac{1 + \left(\zeta^2 - 2\xi\right)\left(\frac{\omega}{\omega_n}\right)^2 + \xi^2\left(\frac{\omega}{\omega_n}\right)^4}{\left[-j\left(\frac{\omega}{\omega_n}\right)^3 - \xi\left(\frac{\omega}{\omega_n}\right)^2 + j\zeta\left(\frac{\omega}{\omega_n}\right) + 1\right] \cdot \left[j\left(\frac{\omega}{\omega_n}\right)^3 - \xi\left(\frac{\omega}{\omega_n}\right)^2 - j\zeta\left(\frac{\omega}{\omega_n}\right) + 1\right]}\, d\omega \tag{67}$$

Equation (67) can be written as

$$B_L = \frac{\omega_n}{4\pi}\int_{-\infty}^\infty \frac{1 + \left(\zeta^2 - 2\xi\right)x^2 + \xi^2 x^4}{\left[-jx^3 - \xi x^2 + j\zeta x + 1\right] \cdot \left[jx^3 - \xi x^2 - j\zeta x + 1\right]}\, dx \tag{68}$$

To solve it, we will one more time use [3]. This time we will elaborate equation 3.112-4. It states

$$\int_{-\infty}^\infty \frac{g_3(x)}{h_3(x)h_3(-x)}\, dx = \pi j \frac{-a_2 b_0 + a_0 b_1 - \frac{a_0 a_1 b_2}{a_3}}{a_0\left(a_0 a_3 - a_1 a_2\right)} \tag{69}$$

where

$$g_3 = b_0 x^4 + b_1 x^2 + b_2 \tag{70}$$

and

$$h_3 = a_0 x^3 + a_1 x^2 + a_2 x + a_3 \tag{71}$$

Comparing (68) with (70) and (71) we find out that

$$b_0 = \xi^2, b_1 = \zeta^2 - 2\xi, b_2 = 1$$
$$a_0 = -j, a_1 = -\xi, a_2 = j\zeta, a_3 = 1$$

Thus we can write

$$B_L = \frac{\omega_n}{4\pi}\pi j \frac{-j\zeta\xi^2 - j\left(\zeta^2 - 2\xi\right) - j\xi}{-j\left(-j + j\zeta\xi\right)} \tag{72}$$

What finally evaluates to

$$B_L = \frac{\omega_n}{4}\frac{\zeta\xi^2 + \zeta^2 - \xi}{\zeta\xi - 1} \tag{73}$$

Please note that [1] gives following formula for loop noise bandwidth

$$B_L = \frac{\omega_0\left(a_3 b_3^2 + a_3^2 - b_3\right)}{4\left(a_3 b_3 - 1\right)} \tag{74}$$

But with the $\omega_n = \omega_0$ and with $a_3 = \zeta$ and $b_3 = \xi$ this is exactly the same. That's all. That is what I wanted to achive.

References

[1] Elliot D. Kaplan, Christopher J. Hegarty *Understanding GPS - Principleas and Applications* (Second Edition).

[2] Roland E. Best *Phase-Locked Loops - Design, Simulation and Applications* (Fifth Edition).

[3] I. S. Gradshteyn, I. M. Ryzhik *Table of Integrals, Series, and Products* (Sevents Edition).