Magnetism is a relativistic phenomenon

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Abstract

Many publications ([5], [6], to list a few) and textbooks ([1], [3]) when trying to explain magnetism as a relativistic phenomenon uses the approach with the infinitely long conductor/wire where charges/current move with the velocity v in one reference frame and are fixed in the second one. I had and still have problems with understanding of that way of demonstrating the magnetism as a relativistic effect. I thought. Wouldn't it be possible to explain this phenomenon just by using two point charges? One would be the source of the field and the second one would move in the field created by the first. As simple as that. Let us check then.

This article aims to provide explanations and derivations proving that magnetism can/shall be treated as a relativistic phenomenon and in all those derivations we will use model where only two point charges are involved. One of the charges will create the electric field and the second one will be moving in the field created be the first.

1 Special case

Let's start our derivations from a scenario which is a special case. A case where charges q_1 and q_2 are fixed in frame $S^{'}$, and the same, are moving with the same uniform velocity $\boldsymbol{u}_1 = \boldsymbol{u}_2 = \boldsymbol{v}$ in frame S. $\boldsymbol{v}$ is the velocity confined to the x direction of the relative motion of two coordinate frames S and $S^{'}$. $S^{'}$ (the primed frame) is seen from the unprimed frame S as moving with velocity $v = |\boldsymbol{v}|$ along the x axis. See figure 1.

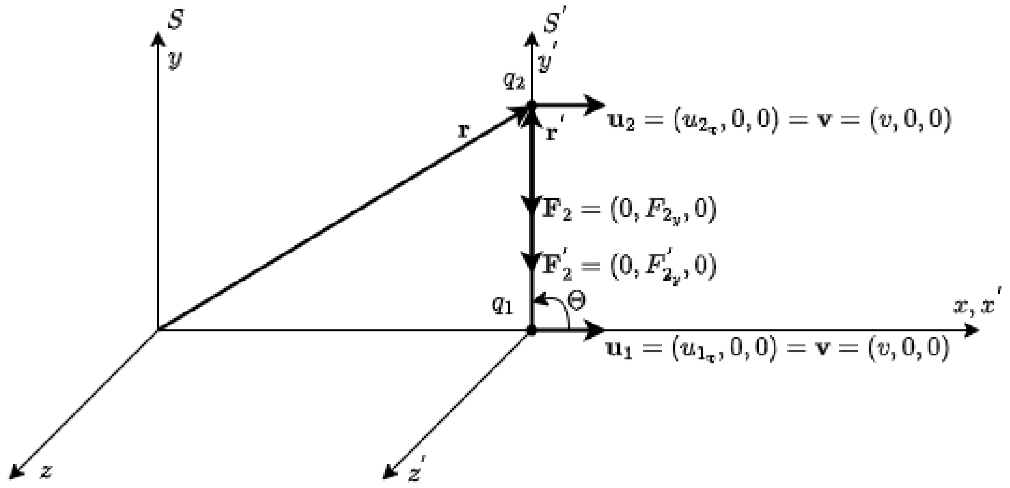

Figure 1: Special case - charges are fixed in $S^{'}$ and are moving with uniform velocity in S.

The force acting on q_2 in $S^{'}$ is thus purely electrical. Here we have classical electrostatic case, the Coulomb's law. Thus we write

$$\boldsymbol{F}_2^{'} = (F_{2_x}^{'}, F_{2_y}^{'}, F_{2_z}^{'}) = (0, \frac{q_1 q_2}{4\pi\epsilon_0}\frac{1}{(y^{'})^2}, 0). \tag{1.1}$$

Let us transform now that force to frame S. We will use following set of transformation functions (please see (3.47))

$$
\begin{aligned}
F_x &= \frac{F_x^{'} + \frac{v}{c^2}\left(\boldsymbol{F}^{'}\boldsymbol{u}^{'}\right)}{1 + \frac{u_x^{'}v}{c^2}} \\
F_y &= \frac{F_y^{'}}{\gamma\left(1 + \frac{u_x^{'}v}{c^2}\right)} \\
F_z &= \frac{F_z^{'}}{\gamma\left(1 + \frac{u_x^{'}v}{c^2}\right)}.
\end{aligned}
\tag{1.2}
$$

Section 3 contains physical supplement where above equations are derived.
Hence

$$
\boldsymbol{F}_2 = (F_{2_x}, F_{2_y}, F_{2_z}) = (0, \frac{F_{2_y}^{'}}{\gamma\left(1 + \frac{u_{2_x}^{'}v}{c^2}\right)}, 0). \tag{1.3}
$$

Noting that $F_{2_y}^{'} = \frac{q_1 q_2}{4\pi\epsilon_0}\frac{1}{(y^{'})^2}$, $u_{2_x}^{'} = 0$ and $y^{'} = y$, we finally get

$$
\boldsymbol{F}_2 = \left(0, \frac{1}{\gamma}\frac{q_1 q_2}{4\pi\epsilon_0}\frac{1}{y^2}, 0\right). \tag{1.4}
$$

$\boldsymbol{F}_2$ is the force acting on q_2 as measured in frame S where both charges q_1 and q_2 are moving with uniform velocity v parallel to the x axis. $\boldsymbol{F}_2$ is smaller by a factor of $1/\gamma = \sqrt{1 - v^2/c^2}$ comparing to $\boldsymbol{F}_2^{'}$ as measured in $S^{'}$.

But we shall be able to evaluate $\boldsymbol{F}_2$ doing our calculus purely in S. In frame S, the electric field which is sensed by q_2 is created by a moving q_1. To find value of that field, we will use following expression

$$
\boldsymbol{E}(\boldsymbol{r}, t) = \frac{q}{4\pi\epsilon_0}\frac{1 - \frac{v^2}{c^2}}{\left(1 - \frac{v^2}{c^2}\sin^2\Theta\right)^{\frac{3}{2}}}\frac{\boldsymbol{w}}{w^3} \tag{1.5}
$$

as found in [4] (please note that Greiner uses Gaussian system of units in his textbook, so I had to augment his formula by $\frac{1}{4\pi\epsilon_0}$ factor). $\boldsymbol{w} = \boldsymbol{r} - \boldsymbol{v}t$ is a vector which points from the actual (not retarded) position of q_1 to the point of observation q_2. Θ is the angle between $\boldsymbol{w}$ and $\boldsymbol{v}$ which in our special case is all the time equal to 90°. We quickly find out, that

$$\boldsymbol{E}_2 = \left(0, \gamma \frac{q_1}{4\pi\epsilon_0} \frac{1}{y^2}, 0\right). \tag{1.6}$$

So if the electric field were the only field to be acting on q_2 as is the case in $S^{'}$ then we have a problem. The value

$$\boldsymbol{F}_2 = q_2 \boldsymbol{E}_2 = \left(0, \gamma \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{y^2}, 0\right) \tag{1.7}$$

does not correspond to (1.4). Not only it doesn't correspond to the expression (1.4), but it does not reflect the measurements coming out of experiments. So we need to augment $\boldsymbol{F}_2$ so that it reflects the experiments and is compliant with Lorentz transformations. We already know that this will be the $\boldsymbol{v} \times \boldsymbol{B}$ term in the Lorentz force, where $\boldsymbol{v}$ is the velocity of the charge and $\boldsymbol{B}$ is the magnetic field in which the charge is moving. We also know that $\boldsymbol{B}$ created by a charge moving with a uniform velocity is related to $\boldsymbol{E}$ by the following equation

$$\boldsymbol{B} = \frac{\boldsymbol{v} \times \boldsymbol{E}}{c^2}. \tag{1.8}$$

Applying formula (1.8) to our case, we get following relation for the augmented $\boldsymbol{F}_2$

$$\boldsymbol{F}_2 = q_2 \left(\boldsymbol{E}_2 + \boldsymbol{u}_2 \times \boldsymbol{B}_2\right) = q_2 \left(\boldsymbol{E}_2 + \boldsymbol{u}_2 \times \frac{\boldsymbol{u}_1 \times \boldsymbol{E}_2}{c^2}\right). \tag{1.9}$$

Further noting that velocities $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are equal, and equal to the velocity $\boldsymbol{v}$ of $S^{'}$ as measured in S, and perpendicular to the $\boldsymbol{E}_2$ and $\boldsymbol{B}_2$ we write

$$\boldsymbol{F}_2 = q_2 \left(0, E_{2_y} \left(1 - \frac{v^2}{c^2}\right), 0\right). \tag{1.10}$$

Finally using the equation (1.6) we get

$$\boldsymbol{F}_2 = \left(0, \frac{1}{\gamma} \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{y^2}, 0\right). \tag{1.11}$$

Okey, now the equation (1.11) matches (1.4). It matches the experiments and it matches the Lorentz transformations. We only had to add to it a term dependent on some additional field $\boldsymbol{B}$, which appears only in frames where q_1 (a charge which is the source of the field) is moving with some velocity $\boldsymbol{v}$. Thus we may/shall treat $\boldsymbol{B}$ as a some "helper" field, which is a function of electric field $\boldsymbol{E}$ and charge velocity $\boldsymbol{v}$, and which disappears in frames where $\boldsymbol{v} = \boldsymbol{0}$.

But this is a special case. A case where charges q_1 and q_2 are at rest in frame $S^{'}$, and are moving with uniform velocity at S. But what if charge q_2 is moving in

$S^{'}$ with some arbitrary velocity $\boldsymbol{u}_2^{'}$ and the origin of the $S^{'}$ is bound to the charge q_1? That will be covered in the next chapter.

2 General case

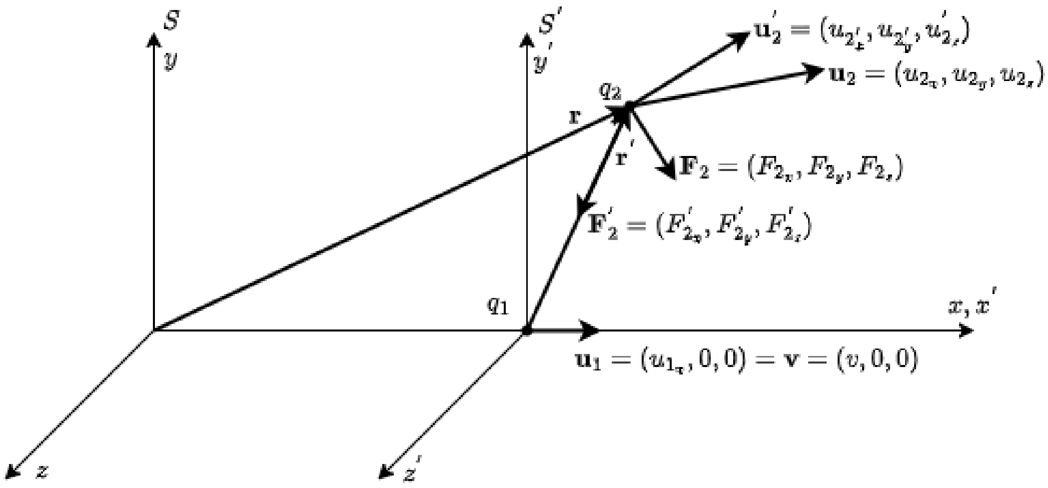

Figure 2: General case - charge q_2 is moving with velocity $\boldsymbol{u}_2^{'}$ in $S^{'}$, charge q_1 is bound to the origin of $S^{'}$.

In general case, the force acting on q_2 in $S^{'}$ is still purely electrical. This is because, we deliberately selected origin of the $S^{'}$ to coincide with the position of charge q_1. Thus charge q_1 is at rest in $S^{'}$ and the same we still consider classical electrostatic case here, the Coulomb's law.

Hence in $S^{'}$ we write

$$\boldsymbol{F}_2^{'} = (F_{2_x}^{'}, F_{2_y}^{'}, F_{2_z}^{'}) = \left(k\frac{x^{'}}{(r^{'})^3}, k\frac{y^{'}}{(r^{'})^3}, k\frac{z^{'}}{(r^{'})^3} \right) \tag{2.1}$$

where $k = \dfrac{q_1 q_2}{4\pi\epsilon_0}$.

Let us transform now that force to frame S. We are yet again using equations (3.47).

$$F_{2_x} = \frac{F'_{2_x} + \frac{v}{c^2}\left(\boldsymbol{F}'_2 \boldsymbol{u}'_2\right)}{1 + \frac{u'_{2_x} v}{c^2}} = \frac{k\frac{x'}{(r')^3} + \frac{v}{c^2}\left(k\frac{x'}{(r')^3}u'_{2_x} + k\frac{y'}{(r')^3}u'_{2_y} + k\frac{z'}{(r')^3}u'_{2_z}\right)}{1 + \frac{u'_{2_x} v}{c^2}} \tag{2.2}$$

$$F_{2_y} = \frac{F'_{2_y}}{\gamma\left(1 + \frac{u'_{2_x} v}{c^2}\right)} = \frac{k\frac{y'}{(r')^3}}{\gamma\left(1 + \frac{u'_{2_x} v}{c^2}\right)} \tag{2.3}$$

$$F_{2_z} = \frac{F'_{2_z}}{\gamma\left(1 + \frac{u'_{2_x} v}{c^2}\right)} = \frac{k\frac{z'}{(r')^3}}{\gamma\left(1 + \frac{u'_{2_x} v}{c^2}\right)} \tag{2.4}$$

To express $\boldsymbol{F}_2 = (F_{2_x}, F_{2_y}, F_{2_z})$ as a function of only unprimed variables, we need to transform all primed variables (x', y', z', r' and u'_{2_x}, u'_{2_y}, u'_{2_z}) to the S frame. Using (3.2) we may write

$$\begin{aligned} \frac{x'}{(r')^3} &= \frac{\gamma(x - vt)}{\left(\gamma^2 (x - vt)^2 + y^2 + z^2\right)^{\frac{3}{2}}} \\ \frac{y'}{(r')^3} &= \frac{y}{\left(\gamma^2 (x - vt)^2 + y^2 + z^2\right)^{\frac{3}{2}}} \\ \frac{z'}{(r')^3} &= \frac{z}{\left(\gamma^2 (x - vt)^2 + y^2 + z^2\right)^{\frac{3}{2}}} \end{aligned} \tag{2.5}$$

Using (3.18) we write

$$1 + \frac{u'_{2_x} v}{c^2} = 1 + \frac{u_{2_x} - v}{1 - \frac{u_{2_x} v}{c^2}}\frac{v}{c^2} = 1 + \frac{u_{2_x} v - v^2}{c^2 - u_{2_x} v} = \frac{c^2 - u_{2_x} v + u_{2_x} v - v^2}{c^2 - u_{2_x} v} =$$

$$\frac{c^2 - v^2}{c^2 - u_{2_x} v} = \frac{1 - \frac{v^2}{c^2}}{1 - \frac{u_{2_x} v}{c^2}} = \frac{1}{\gamma^2}\frac{1}{1 - \frac{u_{2_x} v}{c^2}}. \tag{2.6}$$

So, using above relations force $\boldsymbol{F}_2$ can be expressed as

$$
\begin{aligned}
F_{2_x} &= \gamma k \frac{1}{\left[\gamma^2 (x-vt)^2 + y^2 + z^2\right]^{3/2}} \left(x - vt + \frac{v}{c^2}\left(u_{2_y} y + u_{2_z} z\right)\right) \\
F_{2_y} &= \gamma k \frac{y}{\left[\gamma^2 (x-vt)^2 + y^2 + z^2\right]^{3/2}} \left(1 - \frac{u_{2_x} v}{c^2}\right) \\
F_{2_z} &= \gamma k \frac{z}{\left[\gamma^2 (x-vt)^2 + y^2 + z^2\right]^{3/2}} \left(1 - \frac{u_{2_x} v}{c^2}\right).
\end{aligned}
\tag{2.7}
$$

Equations (2.7) were obtained as a result of Lorentz transformation of force from frame $S^{'}$ to frame S. Now, repeating the same steps as in the special case (chapter (1)), we shall be able to calculate that force doing our calculus purely in S frame (directly from the Maxwell's equations). Let us start then.

And we start from recalling equation

$$
\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \boldsymbol{A}}{\partial t} \tag{2.8}
$$

where ϕ is the scalar, and $\boldsymbol{A}$ vector electric potential respectively. Thus, we shall calculate these potentials first to be able to calculate electric field $\boldsymbol{E}$. And all these potentials shall be derived for a charge moving with uniform velocity in a straight line. Fortunately someone already has done that for us. For example Mr. Feynman in its lecture *"Solutions of Maxwell's Equations with Currents and Charges"* [2]. Let us rewrite them here.

$$
\phi(x,y,z,t) = \frac{q_1}{4\pi\epsilon_0} \frac{1}{\sqrt{1-\dfrac{v^2}{c^2}}} \frac{1}{\left[\left(\dfrac{x-vt}{\sqrt{1-v^2/c^2}}\right)^2 + y^2 + z^2\right]^{1/2}} = \gamma \frac{q_1}{4\pi\epsilon_0} \frac{1}{\left[\gamma^2 (x-vt)^2 + y^2 + z^2\right]^{1/2}} \tag{2.9}
$$

$$
\boldsymbol{A} = \frac{\boldsymbol{v}}{c^2}\phi = \gamma \frac{q_1}{4\pi\epsilon_0} \frac{\dfrac{\boldsymbol{v}}{c^2}}{\left[\gamma^2 (x-vt)^2 + y^2 + z^2\right]^{1/2}} \tag{2.10}
$$

Once we have got potentials, what is left, is to do some math. We need to calculate $\boldsymbol{\nabla}\phi$ and $\dfrac{\partial \boldsymbol{A}}{\partial t}$. Let's begin with $\boldsymbol{\nabla}\phi$.

$$
\begin{aligned}
\frac{\partial\phi}{\partial x} &= -\gamma^3\frac{q_1}{4\pi\epsilon_0}\frac{x-vt}{\left[\gamma^2\left(x-vt\right)^2+y^2+z^2\right]^{3/2}}\\
\frac{\partial\phi}{\partial y} &= -\gamma\frac{q_1}{4\pi\epsilon_0}\frac{y}{\left[\gamma^2\left(x-vt\right)^2+y^2+z^2\right]^{3/2}}\\
\frac{\partial\phi}{\partial z} &= -\gamma\frac{q_1}{4\pi\epsilon_0}\frac{z}{\left[\gamma^2\left(x-vt\right)^2+y^2+z^2\right]^{3/2}}
\end{aligned}
\tag{2.11}
$$

$\dfrac{\partial \boldsymbol{A}}{\partial t}$ gives

$$
\frac{\partial \boldsymbol{A}}{\partial t} = \gamma^3\frac{q_1}{4\pi\epsilon_0}\frac{(x-vt)v\dfrac{\boldsymbol{v}}{c^2}}{\left[\gamma^2\left(x-vt\right)^2+y^2+z^2\right]^{3/2}}
\tag{2.12}
$$

Because $\boldsymbol{v}=(v,0,0)$, the electric field $\boldsymbol{E}_2=-\boldsymbol{\nabla}\phi-\dfrac{\partial \boldsymbol{A}}{\partial t}$ gains following form

$$
E_{2_x} = \gamma^3\frac{q_1}{4\pi\epsilon_0}\frac{(x-vt)}{\left[\gamma^2\left(x-vt\right)^2+y^2+z^2\right]^{3/2}} - \gamma^3\frac{q_1}{4\pi\epsilon_0}\frac{(x-vt)\dfrac{v^2}{c^2}}{\left[\gamma^2\left(x-vt\right)^2+y^2+z^2\right]^{3/2}} =
$$

$$
\gamma^3\frac{q_1}{4\pi\epsilon_0}\frac{(x-vt)(1-\dfrac{v^2}{c^2})}{\left[\gamma^2\left(x-vt\right)^2+y^2+z^2\right]^{3/2}} = \gamma\frac{q_1}{4\pi\epsilon_0}\frac{(x-vt)}{\left[\gamma^2\left(x-vt\right)^2+y^2+z^2\right]^{3/2}}.
$$

$$
\begin{aligned}
E_{2_y} &= \gamma\frac{q_1}{4\pi\epsilon_0}\frac{y}{\left[\gamma^2\left(x-vt\right)^2+y^2+z^2\right]^{3/2}}\\
E_{2_z} &= \gamma\frac{q_1}{4\pi\epsilon_0}\frac{z}{\left[\gamma^2\left(x-vt\right)^2+y^2+z^2\right]^{3/2}}
\end{aligned}
\tag{2.13}
$$

So, yet again we have the same problem as already revealed when discussing the special case (chapter (1)). If the electric field were the only field to be acting on q_2 in frame S, then force $\boldsymbol{F}_2=q_2\boldsymbol{E}_2=(F_{2_x},F_{2_y},F_{2_z})$, where

$$
\begin{aligned}
F_{2_x} &= \gamma k\frac{x-vt}{\left[\gamma^2\left(x-vt\right)^2+y^2+z^2\right]^{3/2}}\\
F_{2_y} &= \gamma k\frac{y}{\left[\gamma^2\left(x-vt\right)^2+y^2+z^2\right]^{3/2}}\\
F_{2_z} &= \gamma k\frac{z}{\left[\gamma^2\left(x-vt\right)^2+y^2+z^2\right]^{3/2}}
\end{aligned}
\tag{2.14}
$$

does not correspond to force (2.7), which is the force derived using Lorentz transformations. So we need to augment $\boldsymbol{F}_2$ so that it matches (2.7). We already did so in chapter (1) and we will repeat the same steps here. We will augment $\boldsymbol{F}_2$ by $\boldsymbol{v} \times \boldsymbol{B}$ term, where $\boldsymbol{v}$ is the velocity of the charge and $\boldsymbol{B}$ is the magnetic field in which the charge is moving. I could also write *"and $\boldsymbol{B}$ is some additional, pseudo field, which we have to introduce, so that our equations describing force acting on a moving charge are valid"*. And I think, I would be right. This is, after all, the thesis I am trying to prove in this article. So, what stopped me from so? Maybe the fact, that the term *"magnetic field"* was bring to that world before we learned about Einstein's relativity. Maybe the fact, that special relativity was actually build on top of the discoveries and problems revealed by the classical electrodynamics. Or maybe just I didn't want to be too cocky. Nevertheless, special relativity applies not only to electromagnetism. It is a wider theory. Thus I will stick with my statement that *"Magnetism is a relativistic phenomenon"*, but I will not be crossing swords over it. Maybe it just does not make sense. Let's come back to calculus.

Augmented $\boldsymbol{F}_2$ takes following form (it is well known Lorentz force)

$$\boldsymbol{F}_2 = q_2\left(\boldsymbol{E}_2 + \boldsymbol{u}_2 \times \boldsymbol{B}_2\right) = q_2\left(\boldsymbol{E}_2 + \boldsymbol{u}_2 \times \frac{\boldsymbol{u}_1 \times \boldsymbol{E}_2}{c^2}\right). \tag{2.15}$$

where $\boldsymbol{u}_1 = (u_{1_x}, u_{1_y}, u_{1_z}) = \boldsymbol{v} = (v, 0, 0)$ is the velocity of the charge q_1 (the source of the field) and $\boldsymbol{u}_2$ is the velocity of the charge q_2.

Thus

$$\begin{aligned} F_{2_x} &= q_2\left(E_{2_x} + u_{2_y}B_{2_z} - u_{2_z}B_{2_y}\right) \\ F_{2_y} &= q_2\left(E_{2_y} + u_{2_z}B_{2_x} - u_{2_x}B_{2_z}\right) \\ F_{2_z} &= q_2\left(E_{2_z} + u_{2_x}B_{2_y} - u_{2_y}B_{2_x}\right). \end{aligned} \tag{2.16}$$

Noting that

$$\begin{aligned} B_{2_x} &= \frac{1}{c^2}\left(u_{1_y}E_{2_z} - u_{1_z}E_{2_y}\right) = 0 \\ B_{2_y} &= \frac{1}{c^2}\left(u_{1_z}E_{2_x} - u_{1_x}E_{2_z}\right) = -\frac{u_{1_x}E_{2_z}}{c^2} = -\frac{vE_{2_z}}{c^2} \\ B_{2_z} &= \frac{1}{c^2}\left(u_{1_x}E_{2_y} - u_{1_y}E_{2_x}\right) = \frac{u_{1_x}E_{2_y}}{c^2} = \frac{vE_{2_y}}{c^2} \end{aligned} \tag{2.17}$$

we may further write

$$
\begin{aligned}
F_{2_x} &= q_2 \left(E_{2_x} + u_{2_y} \frac{v E_{2_y}}{c^2} + u_{2_z} \frac{v E_{2_z}}{c^2} \right) \\
F_{2_y} &= q_2 \left(E_{2_y} - u_{2_x} \frac{v E_{2_y}}{c^2} \right) = q_2 E_{2_y} \left(1 - \frac{u_{2_x} v}{c^2} \right) \\
F_{2_z} &= q_2 \left(E_{2_z} - u_{2_x} \frac{v E_{2_z}}{c^2} \right) = q_2 E_{2_z} \left(1 - \frac{u_{2_x} v}{c^2} \right) .
\end{aligned}
\tag{2.18}
$$

Now, using the formulas for electric field (2.13), we finally get

$$
\begin{aligned}
F_{2_x} &= \gamma k \frac{1}{\left[\gamma^2 (x - vt)^2 + y^2 + z^2\right]^{3/2}} \left(x - vt + \frac{v}{c^2} \left(u_{2_y} y + u_{2_z} z \right) \right) \\
F_{2_y} &= \gamma k \frac{y}{\left[\gamma^2 (x - vt)^2 + y^2 + z^2\right]^{3/2}} \left(1 - \frac{u_{2_x} v}{c^2} \right) \\
F_{2_z} &= \gamma k \frac{z}{\left[\gamma^2 (x - vt)^2 + y^2 + z^2\right]^{3/2}} \left(1 - \frac{u_{2_x} v}{c^2} \right) .
\end{aligned}
\tag{2.19}
$$

Okey, this is it. Now equations (2.19) are identical with (2.7). We only had to introduce some "helper", pseudo field $\boldsymbol{B}$ to make our equations consistent between frames where charge q_1 is at rest, and frames where this charge is moving with some uniform velocity $\boldsymbol{v}$. Thus *magnetism is a relativistic phenomenon* which appears in frames where q_1 (a charge which is the source of the field) is moving with some velocity $\boldsymbol{v}$, and which disappears in frames where this velocity vanishes as well.

3 Cheetsheet

3.1 Transformation of position and time coordinates

We will start this supplement section by providing/recalling the most common form of the transformations where v is the uniform velocity confined to the x direction of the relative motion of two coordinate frames S and $S^{'}$. $S^{'}$ (the primed frame) is seen from the unprimed frame S as moving with velocity v along the x axis.

$$\begin{aligned} x &= \gamma\left(x^{'} + vt^{'}\right) \\ y &= y^{'} \\ z &= z^{'} \\ t &= \gamma\left(t^{'} + \frac{vx^{'}}{c^2}\right) \end{aligned} \tag{3.1}$$	$$\begin{aligned} x^{'} &= \gamma\left(x - vt\right) \\ y^{'} &= y \\ z^{'} &= z \\ t^{'} &= \gamma\left(t - \frac{vx}{c^2}\right) \end{aligned} \tag{3.2}$$

The above formulas are valid in case where relative motion of two reference frames is confined to the x direction. When relative motion of frames is not confined to any direction and is described by an arbitrary velocity vector $\boldsymbol{v}$, then the transformation relations can be expressed in a vectorial form. Let us derive this vectorial form. Please note that only time and the position coordinates parallel to the direction of relative motion undergo transformation. Thus we may write

$$\begin{aligned} \boldsymbol{r}_{\parallel} &= \gamma\left(\boldsymbol{r}^{'}_{\parallel} + \boldsymbol{v}t\right) \\ \boldsymbol{r}_{\perp} &= \boldsymbol{r}^{'}_{\perp} \\ t &= \gamma\left(t^{'} + \frac{\boldsymbol{v}\boldsymbol{r}^{'}_{\parallel}}{c^2}\right) \end{aligned} \tag{3.3}$$

where $\boldsymbol{r}_{\parallel}$ and $\boldsymbol{r}^{'}_{\parallel}$ are parallel, and $\boldsymbol{r}_{\perp}$ and $\boldsymbol{r}^{'}_{\perp}$ are perpendicular to $\boldsymbol{v}$, and

$$\begin{aligned} \boldsymbol{r} &= \boldsymbol{r}_{\parallel} + \boldsymbol{r}_{\perp} \\ \boldsymbol{r}^{'} &= \boldsymbol{r}^{'}_{\parallel} + \boldsymbol{r}^{'}_{\perp}. \end{aligned} \tag{3.4}$$

Further noting that, vector component of $\boldsymbol{r}$ $(\boldsymbol{r}^{'})$ parallel to $\boldsymbol{v}$ can be expressed as

$$\boldsymbol{r}_{\parallel} = \frac{\boldsymbol{r}\boldsymbol{v}}{v^2}\boldsymbol{v} \tag{3.5}$$

$$\boldsymbol{r}^{'}_{\parallel} = \frac{\boldsymbol{r}^{'}\boldsymbol{v}}{v^2}\boldsymbol{v} \tag{3.6}$$

and vector component perpendicular to $\boldsymbol{v}$ as

$$\boldsymbol{r}_{\perp} = \boldsymbol{r} - \boldsymbol{r}_{\parallel} = \boldsymbol{r} - \frac{\boldsymbol{r}\boldsymbol{v}}{v^2}\boldsymbol{v} \tag{3.7}$$

$$\boldsymbol{r}^{'}_{\perp} = \boldsymbol{r}^{'} - \boldsymbol{r}^{'}_{\parallel} = \boldsymbol{r}^{'} - \frac{\boldsymbol{r}^{'}\boldsymbol{v}}{v^2}\boldsymbol{v} \tag{3.8}$$

we may write

$$\begin{aligned}\boldsymbol{r} = \boldsymbol{r}_{\parallel} + \boldsymbol{r}_{\perp} = \gamma\left(\boldsymbol{r}^{'}_{\parallel} + \boldsymbol{v}t\right) + \boldsymbol{r}^{'}_{\perp} = \\ \gamma\left(\frac{\boldsymbol{r}^{'}\boldsymbol{v}}{v^2}\boldsymbol{v} + \boldsymbol{v}t\right) + \boldsymbol{r}^{'} - \frac{\boldsymbol{r}^{'}\boldsymbol{v}}{v^2}\boldsymbol{v} = \boldsymbol{r}^{'} + (\gamma - 1)\frac{\boldsymbol{r}^{'}\boldsymbol{v}}{v^2}\boldsymbol{v} + \gamma\boldsymbol{v}t^{'}\end{aligned} \tag{3.9}$$

$$t = \gamma\left(t^{'} + \frac{\boldsymbol{v}\boldsymbol{r}^{'}_{\parallel}}{c^2}\right) = \gamma\left(t^{'} + \frac{\boldsymbol{v}\left(\boldsymbol{r}^{'} - \boldsymbol{r}^{'}_{\perp}\right)}{c^2}\right) = \gamma\left(t^{'} + \frac{\boldsymbol{v}\boldsymbol{r}^{'}}{c^2}\right). \tag{3.10}$$

Putting all those transformations (forward and inverse) into a table we get

$\begin{aligned}\boldsymbol{r} &= \boldsymbol{r}^{'} + (\gamma - 1)\frac{\boldsymbol{r}^{'}\boldsymbol{v}}{v^2}\boldsymbol{v} + \gamma\boldsymbol{v}t^{'} \\ t &= \gamma\left(t^{'} + \frac{\boldsymbol{v}\boldsymbol{r}^{'}}{c^2}\right)\end{aligned}$ (3.11)	$\begin{aligned}\boldsymbol{r}^{'} &= \boldsymbol{r} + (\gamma - 1)\frac{\boldsymbol{r}\boldsymbol{v}}{v^2}\boldsymbol{v} - \gamma\boldsymbol{v}t \\ t^{'} &= \gamma\left(t - \frac{\boldsymbol{v}\boldsymbol{r}}{c^2}\right)\end{aligned}$ (3.12)

3.2 Transformation of velocities

Yet again, let us start from the case, where v is the uniform velocity confined to the x direction of the relative motion of two coordinate frames S and $S^{'}$. Velocities of objects in those frames are defined as usual. Let us assign them symbols $\boldsymbol{u}$ and $\boldsymbol{u}^{'}$ as letter v is already reserved to the relative velocity of the frames. Thus

$$\boldsymbol{u} = (u_x, u_y, u_z) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = \frac{d\boldsymbol{r}}{dt} \tag{3.13}$$

and

$$\boldsymbol{u}^{'} = \left(u_x^{'}, u_y^{'}, u_z^{'}\right) = \left(\frac{dx^{'}}{dt^{'}}, \frac{dy^{'}}{dt^{'}}, \frac{dz^{'}}{dt^{'}}\right) = \frac{d\boldsymbol{r}^{'}}{dt^{'}}. \tag{3.14}$$

From (3.1) we have

$$dx = \gamma\left(dx^{'} + vdt^{'}\right), dy = dy^{'}, dz = dz^{'}, dt = \gamma\left(dt^{'} + \frac{vdx^{'}}{c^2}\right), \tag{3.15}$$

and from (3.2), analogously

$$dx^{'} = \gamma\left(dx - vdt\right), dy^{'} = dy, dz^{'} = dz, dt^{'} = \gamma\left(dt - \frac{vdx}{c^2}\right). \tag{3.16}$$

Dividing dx, dy, and dz by the dt we get formulas transforming velocity coordinates when moving from $S^{'}$ to S, and dividing $dx^{'}$, $dy^{'}$, and $dz^{'}$ by the $dt^{'}$ we get formulas transforming velocity coordinates when moving from S to $S^{'}$ (please see derivations for vectorial form below). All those transformations are covered in the following table.

$$u_x = \frac{u_x^{'} + v}{1 + \frac{u_x^{'}v}{c^2}} \quad u_y = \frac{u_y^{'}}{\gamma\left(1 + \frac{u_x^{'}v}{c^2}\right)} \quad u_z = \frac{u_z^{'}}{\gamma\left(1 + \frac{u_x^{'}v}{c^2}\right)} \tag{3.17}$$	$$u_x^{'} = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \quad u_y^{'} = \frac{u_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} \quad u_z^{'} = \frac{u_z}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} \tag{3.18}$$

If the relative motion of two reference frames is described by an arbitrary velocity vector $\boldsymbol{v}$, then we shall start from differentiation of (3.11) or (3.12) if we would like to get vectorial form of the transformation equations. I will provide derivations, starting from differentiation of (3.11). Inverse transformation is symmetrical. Only the sign of $\boldsymbol{v}$ changes. Hence

$$d\boldsymbol{r} = d\boldsymbol{r}^{'} + (\gamma - 1)\,\frac{\boldsymbol{v}d\boldsymbol{r}^{'}}{v^2}\boldsymbol{v} + \gamma\boldsymbol{v}dt^{'} \tag{3.19}$$

and

$$dt = \gamma\left(dt^{'} + \frac{\boldsymbol{v}d\boldsymbol{r}^{'}}{c^2}\right). \tag{3.20}$$

Thus

$$\boldsymbol{u} = \frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}^{'} + (\gamma - 1)\,\frac{\boldsymbol{v}d\boldsymbol{r}^{'}}{v^2}\boldsymbol{v} + \gamma\boldsymbol{v}dt^{'}}{\gamma\left(dt^{'} + \frac{\boldsymbol{v}d\boldsymbol{r}^{'}}{c^2}\right)}. \tag{3.21}$$

Dividing both numerator and denominator by the $\gamma dt^{'}$ and noting that $\boldsymbol{u}^{'} = \frac{d\boldsymbol{r}^{'}}{dt^{'}}$ we further get

$$\boldsymbol{u} = \frac{\frac{1}{\gamma}\frac{d\boldsymbol{r}^{'}}{dt'} + \frac{\gamma-1}{\gamma}\frac{\boldsymbol{v}}{v^2}\frac{d\boldsymbol{r}^{'}}{dt'}\boldsymbol{v} + \boldsymbol{v}}{1 + \frac{\boldsymbol{v}}{c^2}\frac{d\boldsymbol{r}^{'}}{dt'}} = \frac{\frac{1}{\gamma}\boldsymbol{u}^{'} + \frac{\gamma-1}{\gamma}\frac{\boldsymbol{v}\boldsymbol{u}^{'}}{v^2}\boldsymbol{v} + \boldsymbol{v}}{1 + \frac{\boldsymbol{v}\boldsymbol{u}^{'}}{c^2}}. \tag{3.22}$$

Putting these results into a table we finally have (as already mentioned, inverse transformation (3.24) is obtained analogously)

$$\boldsymbol{u} = \frac{\frac{1}{\gamma}\boldsymbol{u}^{'} + \frac{\gamma-1}{\gamma}\frac{\boldsymbol{v}\boldsymbol{u}^{'}}{v^2}\boldsymbol{v} + \boldsymbol{v}}{1 + \frac{\boldsymbol{v}\boldsymbol{u}^{'}}{c^2}} \qquad (3.23)$$	$$\boldsymbol{u}^{'} = \frac{\frac{1}{\gamma}\boldsymbol{u} + \frac{\gamma-1}{\gamma}\frac{\boldsymbol{v}\boldsymbol{u}}{v^2}\boldsymbol{v} - \boldsymbol{v}}{1 - \frac{\boldsymbol{v}\boldsymbol{u}}{c^2}}. \qquad (3.24)$$

3.3 Transformation of momentum and energy

Let's consider infinitesimally small change of the position of the object done in the infinitesimally small time interval. In reference frame S, this will be dx, dy, dz and dt. We already know transformations of those quantities when moving from $S^{'}$ to S (see (3.15)). These are accordingly

$$dx = \gamma\left(dx^{'} + vdt^{'}\right), dy = dy^{'}, dz = dz^{'}, c^2dt = \gamma\left(c^2dt^{'} + vdx^{'}\right). \tag{3.25}$$

Please note, that there is one tiny difference in (3.25) when comparing to (3.15). In (3.25) time interval dt has been multiplied by c^2. We can do so as c is of course invariant to a Lorentz transformation. But we also know that mass m and proper time $d\tau$ are invariant as well. Thus we may multiply by m and divide by $d\tau$ all 4 equations in (3.25) gaining

$$\begin{aligned}
m\frac{dx}{d\tau} &= \gamma\left(m\frac{dx^{'}}{d\tau} + mv\frac{dt^{'}}{d\tau}\right) \\
m\frac{dy}{d\tau} &= m\frac{dy^{'}}{d\tau} \\
m\frac{dz}{d\tau} &= m\frac{dz^{'}}{d\tau} \\
mc^2\frac{dt}{d\tau} &= \gamma\left(mc^2\frac{dt^{'}}{d\tau} + mv\frac{dx^{'}}{d\tau}\right).
\end{aligned} \tag{3.26}$$

Now, noting that

$$p_x = m\frac{dx}{d\tau}, p_y = m\frac{dy}{d\tau}, p_z = m\frac{dz}{d\tau}, E = mc^2\frac{dt}{d\tau} \tag{3.27}$$

and

$$p_x^{'} = m\frac{dx^{'}}{d\tau}, p_y^{'} = m\frac{dy^{'}}{d\tau}, p_z^{'} = m\frac{dz^{'}}{d\tau}, E^{'} = mc^2\frac{dt^{'}}{d\tau} \tag{3.28}$$

we may rewrite (3.26) as

$$\begin{aligned}
p_x &= \gamma\left(p_x^{'} + \frac{vE^{'}}{c^2}\right) \\
p_y &= p_y^{'} \\
p_z &= p_z^{'} \\
E &= \gamma\left(E^{'} + vp_x^{'}\right).
\end{aligned} \tag{3.29}$$

Putting (3.29) and inverse transformations into a table we get

$$\begin{aligned} p_x &= \gamma\left(p_x^{'} + \frac{vE^{'}}{c^2}\right) \\ p_y &= p_y^{'} \\ p_z &= p_z^{'} \\ E &= \gamma\left(E^{'} + vp_x^{'}\right) \end{aligned} \qquad (3.30)$$	$$\begin{aligned} p_x^{'} &= \gamma\left(p_x - \frac{vE}{c^2}\right) \\ p_y^{'} &= p_y \\ p_z^{'} &= p_z \\ E^{'} &= \gamma\left(E - vp_x\right). \end{aligned} \qquad (3.31)$$

The above equations are given for the case where v is the uniform velocity confined to the x direction of the relative motion of two coordinate frames S and $S^{'}$. Now we will provide derivations for the case where relative motion of two reference frames is described by an arbitrary velocity vector $\boldsymbol{v}$. Then we shall get the vectorial form of the transformation equations. We will follow the same approach as above, but now we will start from recalling (3.19) and (3.20). Let's repeat them here:

$$d\boldsymbol{r} = d\boldsymbol{r}^{'} + (\gamma - 1)\frac{\boldsymbol{v}d\boldsymbol{r}^{'}}{v^2}\boldsymbol{v} + \gamma\boldsymbol{v}dt^{'}$$

$$c^2dt = \gamma\left(c^2dt^{'} + \boldsymbol{v}d\boldsymbol{r}^{'}\right).$$

Yet again, we multiplied both sides of (3.20) by c^2. Next step is exactly the same as previously. We multiply both equations by mass m and divide them by the proper time $d\tau$. Hence

$$m\frac{d\boldsymbol{r}}{d\tau} = m\frac{d\boldsymbol{r}^{'}}{d\tau} + (\gamma - 1)\frac{\boldsymbol{v}m\frac{d\boldsymbol{r}^{'}}{d\tau}}{v^2}\boldsymbol{v} + \gamma m\boldsymbol{v}\frac{dt^{'}}{d\tau} \qquad (3.32)$$

$$mc^2\frac{dt}{d\tau} = \gamma\left(mc^2\frac{dt^{'}}{d\tau} + m\boldsymbol{v}\frac{d\boldsymbol{r}^{'}}{d\tau}\right). \qquad (3.33)$$

Noting that

$$\boldsymbol{p} = m\frac{d\boldsymbol{r}}{d\tau}, E = mc^2\frac{dt}{d\tau} \qquad (3.34)$$

and

$$\boldsymbol{p}^{'} = m\frac{d\boldsymbol{r}^{'}}{d\tau}, E^{'} = mc^2\frac{dt^{'}}{d\tau} \qquad (3.35)$$

we may rewrite (3.32) as

$$\boldsymbol{p} = \boldsymbol{p}^{'} + (\gamma - 1)\,\frac{\boldsymbol{v}\boldsymbol{p}^{'}}{v^2}\boldsymbol{v} + \frac{\gamma E^{'}}{c^2}\boldsymbol{v} \tag{3.36}$$

and (3.33) as

$$E = \gamma\left(E^{'} + \boldsymbol{v}\boldsymbol{p}^{'}\right). \tag{3.37}$$

Putting (3.36), (3.37) and inverse transformations into a table we finally get

$$\begin{aligned}\boldsymbol{p} &= \boldsymbol{p}^{'} + (\gamma - 1)\,\frac{\boldsymbol{v}\boldsymbol{p}^{'}}{v^2}\boldsymbol{v} + \frac{\gamma E^{'}}{c^2}\boldsymbol{v}\\ E &= \gamma\left(E^{'} + \boldsymbol{v}\boldsymbol{p}^{'}\right)\end{aligned} \tag{3.38}$$	$$\begin{aligned}\boldsymbol{p}^{'} &= \boldsymbol{p} + (\gamma - 1)\,\frac{\boldsymbol{v}\boldsymbol{p}}{v^2}\boldsymbol{v} - \frac{\gamma E}{c^2}\boldsymbol{v}\\ E^{'} &= \gamma\left(E - \boldsymbol{v}\boldsymbol{p}\right).\end{aligned} \tag{3.39}$$

3.4 Transformation of forces

To find transformations for forces, we start from the Newton's second law

$$\boldsymbol{F} = (F_x, F_y, F_z) = \left(\frac{dp_x}{dt}, \frac{dp_y}{dt}, \frac{dp_z}{dt}\right) = \frac{d\boldsymbol{p}}{dt}, \tag{3.40}$$

$$\boldsymbol{F}^{'} = \left(F_x^{'}, F_y^{'}, F_z^{'}\right) = \left(\frac{dp_x^{'}}{dt^{'}}, \frac{dp_y^{'}}{dt^{'}}, \frac{dp_z^{'}}{dt^{'}}\right) = \frac{d\boldsymbol{p}^{'}}{dt^{'}}. \tag{3.41}$$

The only difference, comparing to the classical mechanics is, that now we shall use the relativistic expression for momentum

$$\boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1 - v^2/c^2}} = \frac{m}{\sqrt{1 - v^2/c^2}}\frac{d\boldsymbol{r}}{dt} = m\frac{d\boldsymbol{r}}{d\tau}. \tag{3.42}$$

And we already know from the previous chapter how momentum and energy transform when moving from one inertial frame to another.

Thus using (3.30) we may write

$$dp_x = \gamma\left(dp_x^{'} + \frac{vdE^{'}}{c^2}\right), dp_y = dp_y^{'}, dp_z = dp_z^{'}, dE = \gamma\left(dE^{'} + vdp_x^{'}\right), \tag{3.43}$$

and using (3.31) analogously we get

$$dp_x^{'} = \gamma \left(dp_x - \frac{vdE}{c^2} \right), dp_y^{'} = dp_y, dp_z^{'} = dp_z, dE^{'} = \gamma \left(dE - vdp_x \right). \tag{3.44}$$

Please note, that here we are using the same approach as used in chapter (3.2) when trying to derive transformations for velocities. Thus dividing dp_x, dp_y, and dp_z by the dt we get formulas transforming forces when moving from $S^{'}$ to S, and dividing $dp_x^{'}$, $dp_y^{'}$, and $dp_z^{'}$ by the $dt^{'}$ we get formulas transforming forces when moving from S to $S^{'}$. To complete our derivations, we need to find relations between dt and $dt^{'}$ yet. From (3.15) we have

$$dt = \gamma \left(dt^{'} + \frac{vdx^{'}}{c^2} \right) = \gamma \left(1 + \frac{vu_x^{'}}{c^2} \right) dt^{'} \tag{3.45}$$

and from (3.16) we get

$$dt^{'} = \gamma \left(dt - \frac{vdx}{c^2} \right) = \gamma \left(1 - \frac{vu_x}{c^2} \right) dt. \tag{3.46}$$

Having got all those relations we finally write

$$F_x = \frac{dp_x}{dt} = \frac{\gamma \left(dp_x^{'} + \frac{vdE^{'}}{c^2} \right)}{\gamma \left(1 + \frac{vu_x^{'}}{c^2} \right) dt^{'}} = \frac{\frac{dp_x^{'}}{dt^{'}} + \frac{v}{c^2} \frac{dE^{'}}{dt^{'}}}{1 + \frac{vu_x^{'}}{c^2}} = \frac{F_x^{'} + \frac{v}{c^2} \boldsymbol{F}^{'} \boldsymbol{u}^{'}}{1 + \frac{vu_x^{'}}{c^2}}$$

$$F_y = \frac{dp_y}{dt} = \frac{dp_y^{'}}{\gamma \left(1 + \frac{vu_x^{'}}{c^2} \right) dt^{'}} = \frac{F_y^{'}}{\gamma \left(1 + \frac{vu_x^{'}}{c^2} \right)}$$

$$F_z = \frac{dp_z}{dt} = \frac{dp_z^{'}}{\gamma \left(1 + \frac{vu_x^{'}}{c^2} \right) dt^{'}} = \frac{F_z^{'}}{\gamma \left(1 + \frac{vu_x^{'}}{c^2} \right)}$$

Formulas for $F_x^{'}$, $F_y^{'}$ and $F_z^{'}$ are retrieved in symmetrical way. As usual, let's put them all in the table.

$$F_x = \frac{F_x^{'} + \frac{v}{c^2}\boldsymbol{F}^{'}\boldsymbol{u}^{'}}{1 + \frac{vu_x^{'}}{c^2}} \quad F_y = \frac{F_y^{'}}{\gamma\left(1 + \frac{vu_x^{'}}{c^2}\right)} \quad F_z = \frac{F_z^{'}}{\gamma\left(1 + \frac{vu_x^{'}}{c^2}\right)} \qquad (3.47)$$	$$F_x^{'} = \frac{F_x - \frac{v}{c^2}\boldsymbol{F}\boldsymbol{u}}{1 - \frac{vu_x}{c^2}} \quad F_y^{'} = \frac{F_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} \quad F_z^{'} = \frac{F_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} \qquad (3.48)$$

The last transformations which are still missing are the vectorial forms for the force transformation generalized for arbitrary directions of $\boldsymbol{u}$ and $\boldsymbol{v}$. To get such transforming equations we follow the same approach as above, but now we start from the vectorial form of the transformations for momentum and time. Hence, from (3.38) we get

$$d\boldsymbol{p} = d\boldsymbol{p}^{'} + (\gamma - 1)\frac{\boldsymbol{v}d\boldsymbol{p}^{'}}{v^2}\boldsymbol{v} + \frac{\gamma dE^{'}}{c^2}\boldsymbol{v}, \qquad (3.49)$$

and from (3.39) we have

$$d\boldsymbol{p}^{'} = d\boldsymbol{p} + (\gamma - 1)\frac{\boldsymbol{v}d\boldsymbol{p}}{v^2}\boldsymbol{v} - \frac{\gamma dE}{c^2}\boldsymbol{v}. \qquad (3.50)$$

Time transformations we get from (3.11) and (3.12)

$$dt = \gamma\left(dt^{'} + \frac{\boldsymbol{v}d\boldsymbol{r}^{'}}{c^2}\right) = \gamma\left(1 + \frac{\boldsymbol{v}\boldsymbol{u}^{'}}{c^2}\right)dt^{'}, \qquad (3.51)$$

$$dt^{'} = \gamma\left(dt - \frac{\boldsymbol{v}d\boldsymbol{r}}{c^2}\right) = \gamma\left(1 - \frac{\boldsymbol{v}\boldsymbol{u}}{c^2}\right)dt. \qquad (3.52)$$

Now, dividing $d\boldsymbol{p}$ by the dt we get formulas transforming forces when moving from $S^{'}$ to S, and dividing $d\boldsymbol{p}^{'}$ by the $dt^{'}$ we get formulas transforming forces when moving from S to $S^{'}$. Let us provide derivations for $\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt}$. Derivations for $\boldsymbol{F}^{'} = \frac{d\boldsymbol{p}^{'}}{dt^{'}}$ will be analogous. Hence

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} = \frac{d\boldsymbol{p}' + (\gamma - 1)\,\dfrac{\boldsymbol{v}d\boldsymbol{p}'}{v^2}\boldsymbol{v} + \dfrac{\gamma dE'}{c^2}\boldsymbol{v}}{\gamma\left(1 + \dfrac{\boldsymbol{v}\boldsymbol{u}'}{c^2}\right)dt'}. \tag{3.53}$$

Dividing numerator and denominator by $\gamma dt'$, and noting that $\dfrac{dE'}{dt'} = \boldsymbol{F}'\boldsymbol{u}'$ we write

$$\boldsymbol{F} = \frac{\dfrac{1}{\gamma}\dfrac{d\boldsymbol{p}'}{dt'} + \dfrac{(\gamma-1)}{\gamma}\dfrac{\boldsymbol{v}}{v^2}\dfrac{d\boldsymbol{p}'}{dt'}\boldsymbol{v} + \dfrac{1}{c^2}\dfrac{dE'}{dt'}\boldsymbol{v}}{1 + \dfrac{\boldsymbol{v}\boldsymbol{u}'}{c^2}} = \frac{\dfrac{\boldsymbol{F}'}{\gamma} + \dfrac{(\gamma-1)}{\gamma}\left(\boldsymbol{F}'\boldsymbol{v}\right)\dfrac{\boldsymbol{v}}{v^2} + \dfrac{1}{c^2}\left(\boldsymbol{F}'\boldsymbol{u}'\right)\boldsymbol{v}}{1 + \dfrac{\boldsymbol{v}\boldsymbol{u}'}{c^2}}. \tag{3.54}$$

One last thing to be done is to put those transformations (forward and inverse) as usual into a table.

$$\boldsymbol{F} = \frac{\dfrac{\boldsymbol{F}'}{\gamma} + \dfrac{(\gamma-1)}{\gamma}\left(\boldsymbol{F}'\boldsymbol{v}\right)\dfrac{\boldsymbol{v}}{v^2} + \dfrac{1}{c^2}\left(\boldsymbol{F}'\boldsymbol{u}'\right)\boldsymbol{v}}{1 + \dfrac{\boldsymbol{v}\boldsymbol{u}'}{c^2}} \tag{3.55}$$
$$\boldsymbol{F}' = \frac{\dfrac{\boldsymbol{F}}{\gamma} + \dfrac{(\gamma-1)}{\gamma}\left(\boldsymbol{F}\boldsymbol{v}\right)\dfrac{\boldsymbol{v}}{v^2} - \dfrac{1}{c^2}\left(\boldsymbol{F}\boldsymbol{u}\right)\boldsymbol{v}}{1 - \dfrac{\boldsymbol{v}\boldsymbol{u}}{c^2}} \tag{3.56}$$

4 Acknowledgments

This article was brought to you by letters E and B and by three digits: 3, 5 and 7.

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