MATH 407 - Homework 3

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Problem 2.75c.
We know that $\ln(z) = \ln|z| + i\arg(z)$

$|z| = |\sqrt{3} - i| = \sqrt{4} = 2$

$\arg(z) = \arg(\sqrt{3} - i) = \dfrac{11\pi}{6} + 2\pi k$

$$\therefore \boxed{\ln(\sqrt{3} - i) = \ln(2) + i\left(\frac{11\pi}{6} + 2\pi k\right),\ \mathrm{Ln}(\sqrt{3} - i) = \ln(2) + i\frac{11\pi}{6}}$$

Problem 2.82a.
Using the fact that for complex numbers z, w, $z^w = e^{w(\ln|z| + i\arg(z))}$,

$$\ln(1 - i) = \ln(\sqrt{2}) + i\left(\frac{7\pi}{4} + 2\pi k\right)$$

Thus

$$\begin{aligned}
(1-i)^{1+i} &= e^{(1+i)(\ln(2) + i(7\pi/4 + 2\pi k))} \\
&= e^{\ln(\sqrt{2}) - 7\pi/4 - 2\pi k} e^{i(\ln(\sqrt{2}) + 7\pi/4 + 2\pi k)} \\
&= e^{\ln(\sqrt{2}) - 7\pi/4 - 2\pi k} \cos\left(\ln(\sqrt{2}) + \frac{7\pi}{4}\right) \\
&= \boxed{e^{\frac{1}{2}\ln(2) - 7\pi/4 - 2\pi k} \cos\left(\frac{1}{2}\ln(2) + \frac{7\pi}{4}\right)}
\end{aligned}$$

Problem 2.95.

$$\begin{aligned}
\lim_{z \to e^{\pi i/3}} \left(z - e^{\pi i/3}\right)\left(\frac{z}{z^3 + 1}\right) &= \lim_{z \to e^{\pi i/3}} \frac{z^2 - ze^{\pi i/3}}{z^3 + 1} \\
&\overset{\text{L.H}}{=} \lim_{z \to e^{\pi i/3}} \frac{2z - e^{\pi i/3}}{3z^2} \\
&= \frac{2e^{\pi i/3} - e^{\pi i/3}}{3\left(e^{\pi i/3}\right)^2} \\
&= \frac{1}{3} e^{-\pi i/3} \\
&= \frac{1}{3}\left(\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right) \\
&= \boxed{\frac{1}{6} - i\frac{\sqrt{3}}{6}}
\end{aligned}$$

Problem 2.97.

Proof.

$$\begin{aligned}
\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h} &= \lim_{h\to 0}\frac{\dfrac{2z_0+2h-1}{3z_0+3h+2}-\dfrac{2z_0-1}{3z_0+2}}{h} \\
&= \lim_{h\to 0}\frac{(3z_0+2)(2z_0+2h-1)-(2z_0-1)(3z_0+3h+2)}{h(3z_0+2)(3z_0+3h+2)} \\
&= \lim_{h\to 0}\frac{6z_0^2+4z_0+6hz_0+4h-3z_0-2-(6z_0^2-3z_0+6hz_0-3h+4z_0-2)}{h(3z_0+2)(3z_0+3h+2)} \\
&= \lim_{h\to 0}\frac{7h}{h(3z_0+2)(3z_0+3h+2)} \\
&= \lim_{h\to 0}\frac{7}{(3z_0+2)(3z_0+3h+2)} \\
&= \frac{7}{(3z_0+2)^2}\ ,\ z_0\neq -2/3
\end{aligned}$$

□

Problem 2.140.

Proof.

$$\begin{aligned}
z &= (1-i)^{\sqrt{2}i} \\
&= \left(\sqrt{2}e^{-\pi i/4-2\pi k i}\right)^{\sqrt{2}i} \\
&= \sqrt{2}^{\sqrt{2}i}e^{\sqrt{2}(\pi/4+2\pi k)} \\
&= e^{\sqrt{2}(\pi/4+2\pi k)}e^{i\sqrt{2}\ln(\sqrt{2})}
\end{aligned}$$

At this point, we know that all of the values of z are fixed since the term $e^{i\sqrt{2}\ln(\sqrt{2})}$ has a fixed angle of $\sqrt{2}\ln(\sqrt{2})$ radians. When we vary k we're varying the left factor. When we vary the magnitude and keep the direction constant, we generate points on a line through the origin. To find the line, we use the fact that $\theta=\tan^{-1}(y/x)$, so the line happens to be

$$y=\tan(\sqrt{2}\ln(\sqrt{2}))x$$

□