MATH 407 — Complex Variables

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The Complex Number Plane
Introduction to complex numbers, the complex plane, point sets in the plane, stereographic projection; the extended complex plane, curves and regions. [1]

Functions of a Complex Variable
Functions and limits, differentiability and analyticity, the Cauchy-Riemann conditions, linear fractional transformations, transcendental functions, Riemann surfaces. [2]

Integration in the Complex Plane
Line integrals, Cauchy's theorem, Cauchy formulas, Maximum Modulus Principle. [3]

Sequences and Series
Sequences of complex numbers; functions, infinite series, power series, analytic continuation, Laurent series, Double series, infinite products, improper integrals, the Gamma function. [4]

Residue Calculus
The Residue theorem, evaluation of real integrals, the principle of the argument, meromorphic functions, entire functions. [5]

Contents

1 The Complex Numbers

Question: Does $x^2 + 1 = 0$ have any solutions?

- No: If we look for real solutions
- Yes: If we have a more general notion of numbers

We introduce i so that $i^2 = -1$.

Definition. $\mathbb{C}$ is the set of complex numbers formed as $z = x + iy$, $x, y \in \mathbb{R}$. The *real part*, x, is written as $\mathrm{Re}(z)$, and the *imaginary part*, y, is written as $\mathrm{Im}(z)$.

Note. We can identify $\mathbb{C}$ with points in $\mathbb{R}^2$ by the correspondence $(x + iy) \leftrightarrow (x, y)$.

Definition (Addition/Subtraction)**.** We add/subtract complex numbers by their real and imaginary components respectively.

$$(a + ib) \pm (c + id) = (a \pm c) + i(b \pm d)$$

Definition. The *Modulus* (absolute value) of $z = x + iy$ is the length of the vector (x, y) which is $\sqrt{x^2 + y^2}$, and is denoted as $|z|$.

Definition (Triangle Inequality)**.** For vectors, we have the triangle inequality

$$|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$$

For $\mathbb{C}$, this translates to

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

There is also a useful variant:

Apply to $z = (z - w) + w$, where $z, w \in \mathbb{C}$.

$$\begin{aligned} |z| &\leq |z - w| + |w| \\ |z - w| &\geq |z| - |w| \\ |w| - |z| &\leq |w - z| = |z - w| \\ \Rightarrow |z - w| &\geq \big||z| - |w|\big| \end{aligned}$$

Definition (Multiplication)**.** For multiplying two complex numbers, we expand each binomial and collect the real and imaginary parts respectively.

$$(a + ib)(c + id) = (ac - bd) + i(bc + ad)$$

The usual algebraic rules still apply for complex numbers.

$$\begin{aligned} (z_1 z_2) z_3 &= z_1 (z_2 z_3) && \text{associative} \\ z_1 z_2 &= z_2 z_1 && \text{commutative} \\ z_1(z_2 + z_3) &= z_1 z_2 + z_1 z_3 && \text{distributive} \end{aligned}$$

Example (Proof of commutative)**.**

Proof.

$$(a+ib)(c+id) = (ac-bd) + i(bc+ad)$$
$$(c+id)(a+ib) = (ca-db) + i(cb+da)$$

□

Definition (Complex conjugate)**.** If $z = x + iy$, then its complex conjugate, denoted as $\bar{z}$, is $\bar{z} = x - iy$.

Example. If we compute $z\bar{z}$, we get

$$\begin{aligned}(z+iy)(z-iy) &= x^2 + y^2 + i(yx - xy)\\ &= x^2 + y^2\\ &= |z|^2\end{aligned}$$

i.e $z\bar{z} = |z|^2$.

Some other useful properties:

$$\begin{aligned}\overline{(z+w)} &= \bar{z} + \bar{w}\\ \overline{zw} &= \bar{z}\bar{w}\\ |\bar{z}| &= |z|\end{aligned}$$

Note. $z + \bar{z}/2 = x$, $z - \bar{z}/2i = y$.

Definition (Inverses)**.** If $z \neq 0$, then $1/z$ exists as a complex number. We have

$$z\bar{z} = |z^2|$$

or

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

Definition (Division)**.** It follows from the definition of a complex inverse,

$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{z\bar{w}}{|w|^2}$$

Theorem (Fundamental Theorem of Algebra)**.** Every polynomial can be expressed as a product of linear factors over $\mathbb{C}$. This is equivalent to the statement that every polynomial equation $p(z) = 0$ has a solution.

Proof. Suppose a polynomial $p(z)$ factors as $(z-z_1)(z-z_2)\ldots(z-z_n)$, then we have solutions $z_1, \ldots, z_n$ for $p(z) = 0$.

Suppose all equations $p(z) = 0$ have solutions. We want all $p(z)$'s to factor. Do induction on the degree of $p(z)$.

If $\deg p(z) = 1$, then $p(z) = (z - z_1)$, so it factors. Suppose polynomials of $\deg n - 1$ factor, and consider an n degree polynomial $p(x)$, so $p(z) = 0$ has a solution. So there is z_1 with $p(z_1) = 0$. We can write

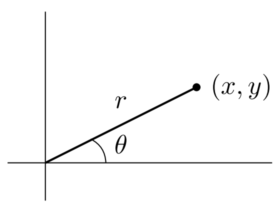
$$p(z) = (z - z_1)q(z)$$

with $\deg q(z) = n - 1$. Then $q(z)$ factors as $(z - z_1)(z - z_2)\dots(z - z_n)$, so $p(z)$ factors. $\square$

Remark. If the Fundamental Theorem is true, then for any complex number w, the equation $z^n - w = 0$ has a solution, i.e all complex numbers have n^{th} roots.

Polar Form

Sometimes, working in a different coordinate system helps simplify things. We introduce the polar coordinate system, where r is the distance from the point to the origin, and θ is the angle from the point to the x-axis.



To convert from Cartesian to polar and back, we use the following formulas

$$x = r\cos\theta \qquad y = r\sin\theta$$

We also adopt the convention that $-\pi \leq \theta \leq \pi$.

Definition (Argument)**.** We define the *argument* of z to be all θ's so that $z = r\cos\theta + ir\sin\theta$. We write this as $\arg z$. There is always one value in $(-\pi, \pi]$ and this the *principal value*, written as $\operatorname{Arg} z$. We then have that

$$\arg z = \{\operatorname{Arg} z + 2\pi n \mid n \in \mathbb{Z}\}$$

Example.

a.) $\operatorname{Arg}(-1) = \pi$, $\arg(-1) = \{(2n+1)\pi \mid n \in \mathbb{Z}\}$.

b.) $\operatorname{Arg}(i) = \pi/2$, $\arg(i) = \{(2n + \frac{1}{2})\pi \mid n \in \mathbb{Z}\}$.

In Calculus, we have the following Taylor Series

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots
\end{aligned}
$$

This suggests

$$
\begin{aligned}
e^{i\theta} &= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \dots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right) \\
&= \cos\theta + i\sin\theta
\end{aligned}
$$

Since any z can be written in polar as $z = r\cos\theta + ir\sin\theta$, we can write this as $z = re^{i\theta}$.