MATH 412 - Homework 5

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Problem 4.4.3.

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}$$

a.) In this equation, $\frac{\partial u}{\partial t}$ represents the velocity of $u(x,t)$. For a dampened vibration, the acceleration is opposite to the velocity, thus $\beta > 0$.

b.) Let $u(x,t) = \phi(x)h(t)$. We then have

$$\rho_0 \phi(x) h''(t) = T_0 \phi''(x) h(t) - \beta h'(t) = -\lambda$$

or

$$\frac{\rho_0}{T_0}\frac{h''}{h} + \frac{\beta}{T_0}\frac{\phi''}{\phi} = -\lambda$$

Solving for $\phi(x)$:

$$\phi'' = -\lambda\phi \qquad \phi(0) = \phi(L) = 0$$
$$\phi = \sin\left(\frac{n\pi x}{L}\right), \quad \lambda = \left(\frac{n\pi}{L}\right)^2$$

Solving for $h(t)$:

Our characteristic equation looks like

$$r^2 + \frac{\beta}{\rho_0} r + \frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2$$
$$r = \frac{-\beta}{\rho_0} \pm \sqrt{\left(\frac{\beta}{\rho_0}\right)^2 - 4\frac{T_0}{\rho_0}\frac{n^2\pi^2}{L^2}}$$

Let $\psi = \left(\frac{\beta}{\rho_0}\right)^2 - 4\frac{T_0}{\rho_0}\frac{n^2\pi^2}{L^2} = \frac{1}{\rho_0^2}\left(\beta^2 - \frac{4T_0\rho_0 n^2\pi^2}{L^2}\right)$.

Since $\beta^2 < \frac{4T_0\rho_0 n^2\pi^2}{L^2}$, implies that $\psi < 0$, we have two complex solutions:

$$r_{1,2} = \frac{\frac{-\beta}{\rho_0} \pm \sqrt{\psi}}{2} = \frac{-\beta}{2\rho_0} \pm i\frac{\sqrt{-\psi}}{2}$$