MATH 323 - Homework 7

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Problem 5.

a.)

$$A = \begin{bmatrix} 3 & 4 & -1 & 2 \\ 6 & 4 & -10 & 13 \\ 3 & 2 & -5 & 7 \\ 6 & 7 & -4 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $\text{span}(S) = \left\{\alpha(x^3 - 3x) + \beta(x^2 + 2x) + \gamma \,|\, \alpha, \beta, \gamma \in \mathbb{R}\right\}$. But every vector in $\mathcal{P}_3$ cannot be expressed as $\alpha x^3 + \beta x^2 - (3\alpha + 2\beta)x + \gamma$. Thus $\text{span}(S) \neq \mathcal{P}_3$.

b.) From part (a), the row reduced form of A yields 3 nonzero rows, thus a basis B for $\text{span}(S)$ is

$$B = \left\{x^3 - 3x, x^2 + 2x, 1\right\}$$

and $\dim(\text{span}(S)) = |B| = 3$.

Problem 10.

Proof. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a finite subset of a vector space $\mathcal{V}$ and $\mathbf{v} \in \text{span}(S)$, with $\mathbf{v} \neq \text{span}(S)$. Since $\mathbf{v} \in T$, it can be written as $\mathbf{v} = 1 \cdot \mathbf{v}$. Since $\mathbf{v} \in \text{span}(S)$, $\mathbf{v}$ can also be written as $\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{v}_i$, $a_i \in \mathbb{R}$. So there are two different ways to write $\mathbf{v}$. $\square$

Problem 19.

$$\begin{aligned} \mathbf{p} &= ax^4 + bx^3 + (3a - 2b)x^2 + cx + (a - b + 3c) \\ &= ax^4 + bx^3 + 3ax^2 - 2bx^2 + cx + a - b + 3c \\ &= a(x^4 + 3x^2 + 1) + b(x^3 - 2x^2 - 1) + c(x + 3) \end{aligned}$$

So $\text{span}(S) = \{x^4 + 3x^2 + 1, x^3 - 2x^2 - 1, x + 3\}$. Let A be a matrix of the coefficients in $\text{span}(S)$,

$$A = \begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

Reducing A to its reduced row echelon form, we obtain a basis B for $\text{span}(S)$,

$$A = \begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

So $B = \{x^4 + 3x^2 + 1, x^3 - 2x^2 - 1, x + 3\}$.

Problem 20b.

$$B = \left[\begin{array}{rrr|r} 5 & -9 & 6 & -16 \\ -1 & 3 & -1 & 5 \\ 3 & -3 & 4 & -6 \\ 1 & 2 & 1 & -3 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & -2 & 1 & -3 \\ -1 & 3 & -1 & 5 \\ 3 & -3 & 4 & -6 \\ 5 & -9 & 6 & -16 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{rrr|r} 1 & -2 & 1 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & 1 & 3 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $[\mathbf{v}]_B = [4, 2, -3]$.

Problem 22.

a.) $\mathbf{P}$ from B to C.

$$\left[\begin{array}{rrrr|rrrr} 5 & 6 & 4 & 3 & 10 & 4 & 15 & 18 \\ 5 & -2 & 7 & 4 & 5 & -3 & 10 & 9 \\ 4 & 5 & -1 & 6 & 4 & 7 & 8 & 10 \\ 3 & 0 & 3 & 2 & 3 & -1 & 6 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right]$$

Thus $\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

b.) $\mathbf{Q}$ from C to D.

$$\left[\begin{array}{rrrr|rrrr} 3 & 2 & 3 & 2 & 5 & 6 & 4 & 8 \\ -1 & 6 & -1 & 1 & 5 & -2 & 7 & 4 \\ 2 & 1 & 3 & -2 & 4 & 5 & -1 & 6 \\ -1 & 2 & 1 & 1 & 3 & 0 & 3 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

Thus $\mathbf{Q} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

c.) $\mathbf{R}$ from B to $D = \mathbf{QP}$.

$$\left[\begin{array}{rrrr|rrrr} 3 & 2 & 3 & 2 & 10 & 4 & 15 & 18 \\ -1 & 6 & -1 & 1 & 5 & -3 & 10 & 9 \\ 2 & 1 & 3 & -2 & 4 & 7 & 8 & 10 \\ -1 & 2 & 1 & 1 & 3 & -1 & 6 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 & 1 \end{array}\right]$$

So $\mathbf{R} = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & -1 & 1 & 1 \end{bmatrix}$.

$$\mathbf{QP} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & -1 & 1 & 1 \end{bmatrix} = \mathbf{R}$$