MATH 412 - Homework 4

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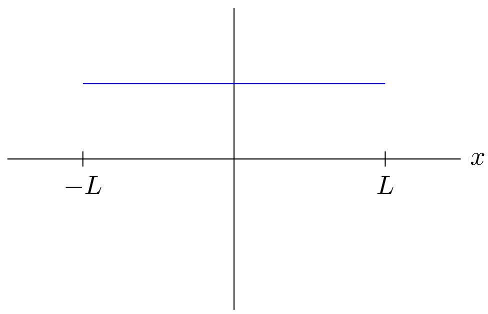
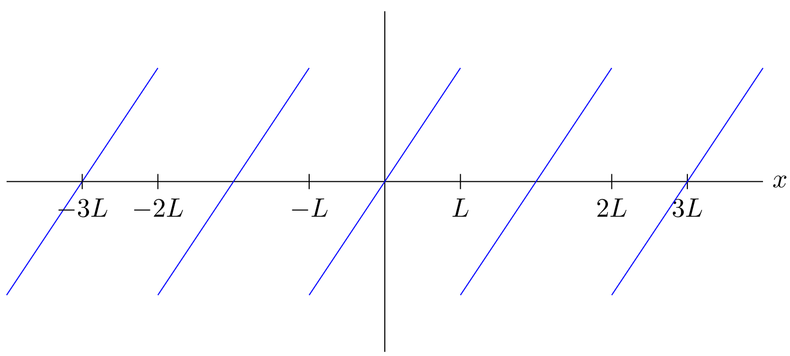
Problem 3.2.1a.
$f(x) = 1$



The graph of the function $f(x)$ and its Fourier series are the same.

Problem 3.2.1c.
$f(x) = 1 + x$



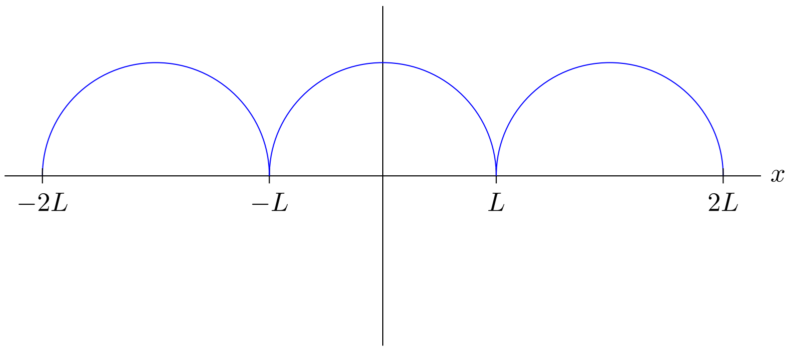
The Fourier series of $f(x)$ crosses the x-axis at odd values of L.

Problem 3.2.4.
If $f(x)$ is piecewise smooth then the Fourier series of $f(x)$ converges to

$$\frac{f(L) + f(-L)}{2}$$

on the endpoints.

Problem 3.3.4.
$f(x) = \sin\left(\frac{\pi x}{L}\right)$



Problem 3.4.1.

a.) Using an elementary integral property,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

this holds for any $c \in [a,b]$. So we have

$$\begin{aligned}
\int_a^b u\frac{dv}{dx}\,dx &= \int_a^{c^-} u\frac{dv}{dx}\,dx + \int_{c^+}^b u\frac{dv}{dx}\,dx \\
&= uv\Big|_a^{c^-} - \int_a^{c^-} v\frac{du}{dx}\,dx + uv\Big|_{c^+}^b - \int_{c^+}^b v\frac{du}{dx}\,dx \\
&= \boxed{uv\Big|_a^b - uv\Big|_{c^-}^{c^+} + \int_a^b v\frac{du}{dx}\,dx}
\end{aligned}$$

b.) If u and v are continuous at $x = c$, then $u(c^-) = u(c^+)$ and $v(c^-) = v(c^+)$, thus

$$\int_a^b u\frac{dv}{dx}\,dx = uv\Big|_a^b - \int_a^b v\frac{du}{dx}\,dx$$

Problem 3.4.3.

a.) Using integration by parts and assuming $\sin(n\pi x/L)$ is continuous everywhere,

$$\begin{aligned}
b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \\
&= \frac{2}{L}\int_0^{x_0^-} \frac{df}{dx}\sin\left(\frac{n\pi x}{L}\right)\,dx + \frac{2}{L}\int_{x_0^+}^L \frac{df}{dx}\sin\left(\frac{n\pi x}{L}\right)\,dx \\
&= \frac{2}{L}\left[f(x)\sin\left(\frac{n\pi x}{L}\right)\Big|_0^{x_0^-} - \frac{n\pi}{L}\int_0^{x_0^-} f(x)\cos\left(\frac{\pi x}{L}\right)\,dx + f(x)\sin\left(\frac{n\pi x}{L}\right)\Big|_{x_0^+}^L \right. \\
&\qquad \left. -\frac{n\pi}{L}\int_{x_0^+}^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx\right] \\
&= \frac{2}{L}\left[f(x)\sin\left(\frac{n\pi x}{L}\right)\Big|_0^{x_0^-} + f(x)\sin\left(\frac{n\pi x}{L}\right)\Big|_{x_0^+}^L - \frac{n\pi}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx\right] \\
&= \frac{2}{L}\left[f(x_0^-)\sin\left(\frac{n\pi x_0^-}{L}\right) - f(x_0^+)\sin\left(\frac{n\pi x_0^+}{L}\right) - \frac{n\pi}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx\right] \\
&= \boxed{\frac{2}{L}(\alpha - \beta)\sin\left(\frac{n\pi x_0}{L}\right) - \frac{n\pi}{L}a_n}
\end{aligned}$$

b.) Using integration by parts and assuming $\cos(n\pi x/L)$ is continuous everywhere,

$$\begin{aligned}
a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \\
&= \frac{2}{L}\int_0^{x_0^-} \frac{df}{dx}\cos\left(\frac{n\pi x}{L}\right)\,dx + \frac{2}{L}\int_{x_0^+}^L \frac{df}{dx}\cos\left(\frac{n\pi x}{L}\right)\,dx \\
&= \frac{2}{L}\left[f(x)\cos\left(\frac{n\pi x}{L}\right)\Big|_0^{x_0^-} + \frac{n\pi}{L}\int_0^{x_0^-} f(x)\sin\left(\frac{\pi x}{L}\right)\,dx + f(x)\cos\left(\frac{n\pi x}{L}\right)\Big|_{x_0^+}^L\right. \\
&\qquad \left. + \frac{n\pi}{L}\int_{x_0^+}^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx\right] \\
&= \frac{2}{L}\left[f(x)\cos\left(\frac{n\pi x}{L}\right)\Big|_0^{x_0^-} + f(x)\cos\left(\frac{n\pi x}{L}\right)\Big|_{x_0^+}^L + \frac{n\pi}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx\right] \\
&= \frac{2}{L}\left[f(x_0^-)\cos\left(\frac{n\pi x_0^-}{L}\right) - f(0) + (-1)^n f(L) + f(x_0^+)\cos\left(\frac{n\pi x_0^+}{L}\right) + \frac{n\pi}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx\right] \\
&= \boxed{\frac{2}{L}\left((\alpha-\beta)\cos\left(\frac{n\pi x_0}{L}\right) - f(0) + (-1)^n f(L) + \frac{n\pi}{L}b_n\right)}
\end{aligned}$$

Problem 3.4.6.

Start with the Fourier series of e^x,

$$e^x \sim A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

The function e^x is continuous, so its Fourier series is also continuous and this can be differentiated term by term:

$$\frac{d}{dx}e^x \sim -\sum_{n=1}^{\infty} \frac{n\pi}{L} A_n \sin\left(\frac{n\pi x}{L}\right) \tag{1}$$

But this Fourier series is not continuous, since $f(0) = 1 \neq 0 \neq e^L = f(L)$. So by equation 3.4.13 in the book,

$$\frac{d}{dx}e^x \sim \frac{e^L - 1}{L} + \sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\left(-\frac{n\pi}{L}\right)A_n + \frac{2}{L}((-1)^n e^L - 1)\right)\cos\left(\frac{n\pi x}{L}\right) \tag{2}$$

Equating (1) and (2),

$$\boxed{A_0 = \frac{e^L - 1}{L}} \qquad\qquad \begin{aligned} A_n &= -\frac{n^2\pi^2}{L^2}A_n + \frac{2(-1)^n e^L - 1}{n^2\pi^2 + L^2} \\ \Rightarrow &\boxed{A_n = \frac{2L(-1)^n e^L - 1}{n^2\pi^2 + L^2}} \end{aligned}$$