MATH 407 - Homework 2

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Problem 2.53c.
Let $z = x + iy$, then

$$\begin{aligned}
f(z) &= \frac{1-z}{1+z} \\
&= \frac{1-(x+iy)}{1+(x+iy)} \\
&= \frac{(1-x)+iy}{(1+x)+iy} \cdot \frac{(1+x-iy)}{(1+x)-iy} \\
&= \frac{x^2-1-i(y+yx)+i(y+yx)-y^2-i2y}{(x+1)^2-i(y+yx)+i(y+yx)-y^2} \\
&= \frac{(1-x^2-y^2)-i2y}{(1+x^2)^2+y^2} \\
&= \frac{1-x^2-y^2}{(1+x^2)^2+y^2} - i\frac{2y}{(1+x^2)^2+y^2} \\
\therefore &\boxed{u(x,y) = \frac{1-x^2-y^2}{(1+x^2)^2+y^2}\ ,\ v(x,y) = \frac{-2y}{(1+x^2)^2+y^2}}
\end{aligned}$$

Problem 2.58a.

Proof. Let $z_1 = a+ib, z_2 = c+id$. By the addition property, we have

$$\begin{aligned}
e^{z_1+(-z_2)} &= e^{a+ib+(-c-id)} \\
&= e^{a+ib}e^{-c-id}
\end{aligned}$$

Using the reciprocal property, we know that $e^{-c-id} = 1/e^{c+id}$. Combining these, we obtain

$$e^{a+ib}\left(\frac{1}{e^{c+id}}\right) = \frac{e^{a+ib}}{e^{c+id}}$$

□

Problem 2.61b.
In polar form, $i = e^{\pi i/2}$. We then have

$$\begin{aligned}
e^{4z} &= e^{\pi i/2} \\
e^{4z} &= e^{(\pi/2+2k\pi)i} \\
4z &= \left(\frac{\pi}{2}+2k\pi\right)i \\
\therefore &\boxed{z = \left(\frac{\pi}{8}+\frac{\pi}{2}k\right)i\ ,\ k = \pm1, \pm2, \dots}
\end{aligned}$$

Problem 2.62b.

Proof. We know that

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \ , \ \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

We then have

$$\begin{aligned}
\cos^2(z) - \sin^2(z) &= \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 \\
&= \frac{1}{4}(e^{iz} + e^{-iz})^2 + \frac{1}{4}(e^{iz} - e^{-iz})^2 \\
&= \frac{1}{4}(e^{2iz} + 2 + e^{-2iz} + e^{2iz} - 2 + e^{-2iz}) \\
&= \frac{1}{2}(e^{2iz} + e^{-2iz}) \\
&= \cos(2z)
\end{aligned}$$

□

Problem 2.68d.
Let $z = x + iy$. We then have

$$e^{2z} = e^{2(x+iy)} = e^{2x+i2y} = e^{2x}e^{i2y} = e^{2x}(\cos(2y) + i\sin(2y))$$

and

$$z^2 = (x + iy)^2 = (x^2 - y^2) + i2xy$$

Combining these, we get

$$\begin{aligned}
z^2e^{2z} &= ((x^2 - y^2) + i2xy)(e^{2x}\cos(2y) + ie^{2x}\sin(2y)) \\
&= e^{2x}x^2\cos(2y) + ix^2e^{2x}\sin(2y) - y^2e^{2x}\cos(2y) - iy^2e^{2x}\sin(2y) + i2xye^{2x}\cos(2y) - 2xye^{2x}\sin(2y) \\
&= e^{2x}(x^2\cos(2y) - y^2\cos(2y) - 2xy\sin(2y)) + ie^{2x}(x^2\sin(2y) - y^2\sin(2y) + 2xy\cos(2y)) \\
&= e^{2x}((x^2 - y^2)\cos(2y) - 2xy\sin(2y)) + ie^{2x}(2xy\cos(2y) + (x^2 - y^2)\sin(2y))
\end{aligned}$$

$$\boxed{\begin{aligned}
u(x, y) &= e^{2x}((x^2 - y^2)\cos(2y) - 2xy\sin(2y)) \\
v(x, y) &= e^{2x}(2xy\cos(2y) + (x^2 - y^2)\sin(2y))
\end{aligned}}$$