MATH 323 - Homework 3

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Problem 1.3.14.

Proof. Suppose that $||\operatorname{proj}_{\mathbf{x}} \mathbf{y}|| = \mathbf{x} \cdot \mathbf{y}$. Since $\cos\theta = \frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||\,||\mathbf{y}||}$, we have

$$\begin{aligned} ||\operatorname{proj}_{\mathbf{x}} \mathbf{y}|| &= ||\mathbf{x}||\,||\mathbf{y}|| \cos\theta \Leftrightarrow \\ ||(\mathbf{x}\cdot\mathbf{y})\mathbf{x}|| &= ||\mathbf{x}||\,||\mathbf{y}|| \cos\theta \Leftrightarrow \\ ||\mathbf{x}||^2 \cdot \mathbf{y} &= ||\mathbf{x}||\,||\mathbf{y}|| \cos\theta \Leftrightarrow \\ ||\mathbf{x}|| &= \cos\theta \end{aligned}$$

which is a contradiction. Thus $\mathbf{x}$ is not a unit vector. $\square$

Problem 1.4.3.

We have the fact that $A = S + V$, where $S = \frac{1}{2}(A + A^T)$ and $V = \frac{1}{2}(A - A^T)$.

(a) $A = \begin{bmatrix} 3 & -1 & 4 \\ 0 & 2 & 5 \\ 1 & -3 & 2 \end{bmatrix}$

$$\begin{aligned} S = \frac{1}{2}(A + A^T) &= \frac{1}{2}\left(\begin{bmatrix} 3 & -1 & 4 \\ 0 & 2 & 5 \\ 1 & -3 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 1 \\ -1 & 2 & -3 \\ 4 & 5 & 2 \end{bmatrix}\right) \\ &= \frac{1}{2}\begin{bmatrix} 6 & -1 & 5 \\ -1 & 4 & 2 \\ 5 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -1/2 & 5/2 \\ -1/2 & 2 & 1 \\ 5/2 & 1 & 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} V = \frac{1}{2}(A - A^T) &= \frac{1}{2}\left(\begin{bmatrix} 3 & -1 & 4 \\ 0 & 2 & 5 \\ 1 & -3 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 1 \\ -1 & 2 & -3 \\ 4 & 5 & 2 \end{bmatrix}\right) \\ &= \frac{1}{2}\begin{bmatrix} 0 & -1 & 3 \\ 1 & 3 & 6 \\ -3 & -8 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1/2 & 3/2 \\ 1/2 & 3/2 & 3 \\ -3/2 & -4 & 0 \end{bmatrix} \end{aligned}$$

$$\therefore \boxed{A = \begin{bmatrix} 3 & -1/2 & 5/2 \\ -1/2 & 2 & 1 \\ 5/2 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -1/2 & 3/2 \\ 1/2 & 3/2 & 3 \\ -3/2 & -4 & 0 \end{bmatrix}}$$

(b) $A = \begin{bmatrix} 1 & 0 & -4 \\ 3 & 3 & -1 \\ 4 & -1 & 0 \end{bmatrix}$

$$\begin{aligned} S = \frac{1}{2}(A + A^T) &= \frac{1}{2}\left(\begin{bmatrix} 1 & 0 & -4 \\ 3 & 3 & -1 \\ 4 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 4 \\ 0 & 3 & -1 \\ -4 & -1 & 0 \end{bmatrix}\right) \\ &= \frac{1}{2}\begin{bmatrix} 2 & 3 & 0 \\ 3 & 6 & -2 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3/2 & 0 \\ 3/2 & 3 & -1 \\ 0 & -1 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} V = \frac{1}{2}(A - A^T) &= \left(\begin{bmatrix} 1 & 0 & -4 \\ 3 & 3 & -1 \\ 4 & -1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 4 \\ 0 & 3 & -1 \\ -4 & -1 & 0 \end{bmatrix}\right) \\ &= \frac{1}{2}\begin{bmatrix} 0 & -3 & -8 \\ 3 & 0 & 0 \\ 8 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -3/2 & -4 \\ 3/2 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$\therefore \boxed{A = \begin{bmatrix} 1 & 3/2 & 0 \\ 3/2 & 3 & -1 \\ 0 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -3/2 & -4 \\ 3/2 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix}}$$

(c) $A = \begin{bmatrix} 2 & 3 & 4 & -1 \\ -3 & 5 & -1 & 2 \\ -4 & 1 & -2 & 0 \\ 1 & -2 & 0 & 5 \end{bmatrix}$

$$\begin{aligned} S = \frac{1}{2}(A + A^T) &= \frac{1}{2}\left(\begin{bmatrix} 2 & 3 & 4 & -1 \\ -3 & 5 & -1 & 2 \\ -4 & 1 & -2 & 0 \\ 1 & -2 & 0 & 5 \end{bmatrix} + \begin{bmatrix} 2 & -3 & -4 & 1 \\ 3 & 5 & 1 & -2 \\ 4 & -1 & -2 & 0 \\ -1 & 2 & 0 & 5 \end{bmatrix}\right) \\ &= \frac{1}{2}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} V = \frac{1}{2}(A - A^T) &= \left(\begin{bmatrix} 2 & 3 & 4 & -1 \\ -3 & 5 & -1 & 2 \\ -4 & 1 & -2 & 0 \\ 1 & -2 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 2 & -3 & -4 & 1 \\ 3 & 5 & 1 & -2 \\ 4 & -1 & -2 & 0 \\ -1 & 2 & 0 & 5 \end{bmatrix}\right) \\ &= \frac{1}{2}\begin{bmatrix} 0 & 6 & 2 & -2 \\ -6 & 0 & -2 & 4 \\ -8 & 2 & 0 & 0 \\ 2 & -4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 4 & -1 \\ -3 & 0 & -1 & 2 \\ -4 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$\therefore \boxed{A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 3 & 4 & -1 \\ -3 & 0 & -1 & 2 \\ -4 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{bmatrix}}$$

(d) $A = \begin{bmatrix} -3 & 3 & 5 & -4 \\ 11 & 4 & 5 & -1 \\ -9 & 1 & 5 & -14 \\ 2 & -11 & -2 & -5 \end{bmatrix}$

$$\begin{aligned} S = \frac{1}{2}(A + A^T) &= \frac{1}{2}\left(\begin{bmatrix} -3 & 3 & 5 & -4 \\ 11 & 4 & 5 & -1 \\ -9 & 1 & 5 & -14 \\ 2 & -11 & -2 & -5 \end{bmatrix} + \begin{bmatrix} -3 & 11 & -9 & 2 \\ 3 & 4 & 1 & -11 \\ 5 & 5 & 5 & -2 \\ -4 & -1 & -14 & -5 \end{bmatrix} \right) \\ &= \frac{1}{2}\begin{bmatrix} -6 & 14 & -4 & -2 \\ 14 & 8 & 6 & -12 \\ -4 & 6 & 10 & -16 \\ -2 & -12 & -16 & -10 \end{bmatrix} = \begin{bmatrix} -3 & 7 & -2 & -1 \\ 7 & 4 & 3 & -6 \\ -2 & 3 & 5 & -8 \\ -1 & -6 & -8 & -5 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} V = \frac{1}{2}(A - A^T) &= \frac{1}{2}\left(\begin{bmatrix} -3 & 3 & 5 & -4 \\ 11 & 4 & 5 & -1 \\ -9 & 1 & 5 & -14 \\ 2 & -11 & -2 & -5 \end{bmatrix} - \begin{bmatrix} -3 & 11 & -9 & 2 \\ 3 & 4 & 1 & -11 \\ 5 & 5 & 5 & -2 \\ -4 & -1 & -14 & -5 \end{bmatrix} \right) \\ &= \frac{1}{2}\begin{bmatrix} 0 & -8 & 14 & -6 \\ 8 & 0 & 4 & 10 \\ -14 & -4 & 0 & -12 \\ 6 & -10 & 12 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -4 & 7 & -3 \\ 4 & 0 & 2 & 5 \\ -7 & -2 & 0 & -6 \\ 3 & -5 & 6 & 0 \end{bmatrix} \end{aligned}$$

$$\therefore \boxed{A = \begin{bmatrix} -3 & 7 & -2 & -1 \\ 7 & 4 & 3 & -6 \\ -2 & 3 & 5 & -8 \\ -1 & -6 & -8 & -5 \end{bmatrix} + \begin{bmatrix} 0 & -4 & 7 & -3 \\ 4 & 0 & 2 & 5 \\ -7 & -2 & 0 & -6 \\ 3 & -5 & 6 & 0 \end{bmatrix}}$$

Problem 1.5.2b.

$$\begin{aligned} GH &= \begin{bmatrix} 5 & 1 & 0 \\ 0 & -2 & -1 \\ 1 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 6 & 3 & 1 \\ 1 & -15 & -5 \\ -2 & -1 & 10 \end{bmatrix} = \begin{bmatrix} 31 & 0 & 0 \\ 0 & 31 & 0 \\ 0 & 0 & 31 \end{bmatrix} \\ HG &= \begin{bmatrix} 6 & 3 & 1 \\ 1 & -15 & -5 \\ -2 & -1 & 10 \end{bmatrix} \cdot \begin{bmatrix} 5 & 1 & 0 \\ 0 & -2 & -1 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 31 & 0 & 0 \\ 0 & 31 & 0 \\ 0 & 0 & 31 \end{bmatrix} \end{aligned}$$

Since $GH = HG$, the matrices G and H commute.

Problem 1.5.2e.

FQ will have size 4×4 while QF will have size 2×2. Thus the matrices F and Q do not commute.

Problem 1.5.23.

(a) *Proof.* Let $A \in \mathbf{R}^{m \times n}, B \in \mathbf{R}^{n \times p}$. So $AB \in \mathbf{R}^{m \times p}$. However, if $m = n = p$, we can conclude that $BA \in \mathbf{R}^{p \times m}$. Thus AB is commutative. □

(b) *Proof.* Let $A \in \mathbf{R}^{m \times m}, B \in \mathbf{R}^{n \times n}$. From (a), we know that AB is commutative, i.e. $AB = BA$. We then have

$$\begin{aligned} (A+B)^2 &= A^2 + AB + BA + B^2 \\ &= A^2 + 2AB + B^2 \end{aligned}$$

□

Problem Ch. 1 Review 19.

Proof. By Mathematical Induction.

Base Case: ($k = 2$). Suppose A and B are upper triangular $n \times n$ matrices, and let $C = AB$. Then $a_{ij} = b_{ij} = 0$ for $i > j$. Hence for $i > j$,

$$c_{ij} = \sum_{m=1}^{n} a_{im} b_{mj} = \sum_{m=1}^{i-1} 0 \cdot b_{mj} + a_{ii} b_{ij} + \sum_{m=i+1}^{n} a_{im} \cdot 0 = a_{ii}(0) = 0$$

Thus C is upper triangular.

Inductive Step: Let $A_1, A_2, \ldots, A_{k+1}$ be upper triangular matrices. Then the product $C = A_1 A_2 \ldots A_k$ is upper triangular by the inductive hypothesis, and so the product $A_1 A_2 \ldots A_{k+1} = C A_{k+1}$ is upper triangular by the base step.

□