MATH 412 – Homework 7

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October 26, 2018

Problem 12.4.1.

$\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = \dfrac{\partial u}{\partial t}(x,0) = 0, \quad u(0,t) = h(t)$

The solution is of the form

$$u(x,t) = F(x-ct) + G(x+ct)$$

For $x > 0$, the initial condition yields

$$F(x) = G(x) = 0 \quad x > 0$$

And for $t > 0$, the boundary condition yields

$$h(t) = F(-ct) + G(ct) \quad t > 0$$

So if $x > ct$, F, G are both positive thus $F = G = 0 \Rightarrow u(x,t) = 0$. If $x < ct$, then F is negative, so we have

$$\begin{aligned} u(x,t) &= F(x-ct) + G(x+ct) \\ &= h\left(t - \frac{x}{c}\right) - G(t-x) + G(ct+x) \end{aligned}$$

Since $x < ct$, both $ct - x > 0$ and $ct + x > 0$. Thus

$$\boxed{u(x,t) = \begin{cases} 0 & x > ct \\ h\left(t - \frac{x}{c}\right) & x < ct \end{cases}}$$

Problem 12.4.2.

$\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = \cos(x), \quad \dfrac{\partial u}{\partial t}(x,0) = 0, \quad u(0,t) = e^{-t}$

The solution is of the form

$$u(x,t) = F(x-ct) + G(x+ct)$$

For $x < 0$, the initial condition yields

$$\cos(x) = F(x) + G(x)$$

implying that $F(x) = G(x) = \frac{1}{2}\cos(x)$. For $x < -ct$, we have

$$\begin{aligned} u(x,t) &= \frac{1}{2}\cos(x-ct) + \frac{1}{2}\cos(x+ct) \\ &= \frac{1}{2}(2\cos(x-ct)\cos(x+ct)) \\ &= \cos(x)\cos(ct) \end{aligned}$$

If $x + ct > 0$, then both F, G are negative, thus

$$u(x,t) = e^{-(t+x/c)} + \frac{1}{2}\cos(x-ct) - \frac{1}{2}\cos(-x-ct)$$

Therefore

$$\boxed{u(x,t) = \begin{cases} \cos(x)\cos(ct) & x + ct < 0 \\ e^{-(t+x/c)} + \sin(x)\sin(ct) & x + ct > 0 \end{cases}}$$

Problem 12.5.1.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad \begin{cases} u(x,0) = f(x), & 0 < x < L \\ \dfrac{\partial u}{\partial t}(x,0) = g(x), & 0 < x < L \\ u(0,t) = u(L,t) = 0 \end{cases}$$

a.) Using separation of variables, let $u(x,t) = \phi(x)h(t)$. We then have

$$\frac{h''}{c^2 h} = \frac{\phi''}{\phi} = -\lambda$$

Solving for $\phi(x)$,

$$\phi'' = -\lambda\phi \qquad \phi(0) = \phi(L) = 0$$
$$\phi(x) = \sin\left(\frac{n\pi x}{L}\right), \quad \lambda = \left(\frac{n\pi}{L}\right)^2$$

Solving for $h(t)$,

$$\begin{aligned} h'' &= -\lambda c^2 h \\ &= -c^2 \frac{n^2\pi^2}{L^2} h \\ \Rightarrow h(t) &= A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \left(\frac{n\pi ct}{L}\right) \end{aligned}$$

We then have

$$u(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left(A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \left(\frac{n\pi ct}{L}\right)\right)$$

and by superposition,

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \left(\frac{n\pi ct}{L}\right)\right)}$$

where

$$\boxed{\begin{aligned} A_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx, \\ B_n &= \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \end{aligned}}$$

b.) If $g(x) = 0$, then $B_n = 0$, yielding

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$$

where $f(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/L)$, $0 < x < L$. But this series does not $f(x)$ outside of $0 < x < L$. Instead we use the periodic extension of $f(x)$:

$$\bar{f}(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$$

where $\bar{f}(x)$ denotes the periodic extension of $f(x)$. Using the identity $\sin\theta\cos\phi = \frac{1}{2}\sin(\theta+\phi) + \frac{1}{2}\sin(\theta-\phi)$, it follows that

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty}\left[A_n \sin\frac{n\pi}{L}(x+ct) + A_n \sin\frac{n\pi}{L}(x-ct)\right]$$

or

$$\boxed{u(x,t) = \frac{1}{2}(\bar{f}(x+ct) + \bar{f}(x-ct))}$$

c.) If $f(x) = 0$, then $A_n = 0$, yielding

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\left(\frac{n\pi ct}{L}\right)$$

following the same logic from part (b), we conclude that

$$\boxed{u(x,t) = \frac{1}{2}(\bar{f}(x+ct) + \bar{f}(x-ct))}$$