MATH 412 – Homework 10

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Problem 9.2.2.

Solving

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q(x,t)$$
$$u(0,t) = u(L,t) = 0 \quad u(x,0) = g(x)$$

Let

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t)\phi_n$$

Plugging into the PDE yields

$$\sum_{n=0}^{\infty}\frac{da_n(t)}{dt}c\rho\phi_n = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q(x,t)$$
$$\frac{da_n(t)}{dt} = \frac{\int_0^L \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right)\phi_n\,dx + \int_0^L Q(x,t)\phi_n\,dx}{\int_0^L (\phi_n)^2\,c\rho\,dx}$$

We can use Green's formula on the first integral in the numerator:

$$\int_0^L \frac{\partial}{\partial x}\left(K_0\frac{\partial \phi_n}{\partial x}\right) - \phi_n\frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right)\,dx = u\frac{d\phi_n}{dx} - \phi_n\left.\frac{\partial u}{\partial x}\right|_0^L$$
$$-\lambda_n\int_0^L uc\rho\phi_n\,dx - \int_0^L \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right)\phi_n\,dx = 0$$
$$\int_0^L \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right)\phi_n\,dx = -\lambda_n\int_0^L uc\rho\phi_n\,dx$$

We then have

$$\begin{aligned}\frac{da_n(t)}{dt} &= -\lambda_n\frac{\int_0^L uc\rho\phi_n\,dx}{\int_0^L c\rho(\phi_n)^2\,dx} + \frac{\int_0^L Q(x,t)c\rho\phi_n\,dx}{\int_0^L c\rho(\phi_n)^2\,dx}\\ &= -\lambda_n a_n(t) + q_n(t)\end{aligned}$$

This is just a linear first order ODE. By using an integrating factor, we have

$$a_n(t) = a_n(0)e^{-\lambda_n t} + e^{-\lambda_n t}\int_0^t q_n(\bar{t})e^{\lambda\bar{t}}\,d\bar{t}$$

where

$$a_n(0) = \frac{\int_0^L c\rho g(x)\phi_n\,dx}{\int_0^L c\rho(\phi_n)^2\,dx}$$

Thus

$$u(x,t) = \sum_{n=0}^{\infty} \left(\frac{\int_0^L c\rho g(x)\phi_n\,dx}{\int_0^L c\rho(\phi_n)^2\,dx} e^{-\lambda_n t} + e^{-\lambda_n t} \int_0^t \frac{\int_0^L Q(x,t)c\rho\phi_n\,dx}{\int_0^L c\rho(\phi_n)^2\,dx} e^{\lambda_n \bar{t}}\,d\bar{t} \right) \phi_n$$

By changing the order of summation and integrating, we have

$$u(x,t) = \int_0^L g(\bar{x}) \sum_{n=0}^{\infty} \left(\frac{\phi_n(\bar{x})\phi_n(x)e^{-\lambda_n t}c\rho}{\int_0^L c\rho(\phi_n)^2\,dx} \right) d\bar{x} + \int_0^t \int_0^L Q(\bar{x},\bar{t}) \sum_{n=1}^{\infty} \left(\frac{\phi_n(\bar{x})\phi_n(x)e^{-\lambda_n(t-\bar{t})}c\rho}{\int_0^L c\rho(\phi_n)^2\,dx} \right) d\bar{x}\,d\bar{t}$$

If we let our Green function to be

$$G(x,t;\bar{x},\bar{t}) = \sum_{n=1}^{\infty} \left(\frac{\phi_n(\bar{x})\phi_n(x)e^{-\lambda_n(t-\bar{t})}c\rho}{\int_0^L c\rho(\phi_n)^2\,dx} \right)$$

we obtain

$$\boxed{u(x,t) = \int_0^L g(\bar{x})G(x,t;\bar{x},0)\,d\bar{x} + \int_0^t \int_0^L Q(\bar{x},\bar{t})G(x,t;\bar{x},\bar{t})\,d\bar{x}\,d\bar{t}}$$

Problem 9.3.5.

$\dfrac{d^2u}{dx^2} = f(x) \quad u(0) = \dfrac{du}{dt}(L) = 0$

a.) Integrating twice, we have

$$\begin{aligned} u(x) - u(0) &= \int_0^x \int_L^{x_0} f(\bar{x})\,d\bar{x}\,dx_0 \\ u(x) &= \int_0^x \int_L^{x_0} f(\bar{x})\,d\bar{x}\,dx_0 \end{aligned}$$

Integrating by parts,

$$\begin{aligned} u(x) &= x_0 \int_L^x f(\bar{x})\,d\bar{x}\Big|_L^x - \int_0^x x_0 f(x_0)\,dx_0 \\ &= \boxed{x\int_L^x f(\bar{x})\,d\bar{x} - \int_0^x x_0 f(x_0)\,dx_0} \end{aligned}$$

b.) Consider a basis of homogeneous solutions $u_1 = x, u_2 = 1$. The general solution is then

$$\begin{aligned} u &= u_1v_1 + u_2v_2 \\ &= x\int_0^x f(x_0)\,dx_0 - \int_0^x x_0 f(x_0)\,dx_0 + c_1x + c_2 \\ u(0) &= 0 = c_2 \\ \frac{du}{dx}(L) &= 0 = \int_0^L f(x_0)\,dx_0 + L(f(x_0))\Big|_0^L - \int_0^L x_0 f(x_0)\,dx_0 + c_1 \\ &\Rightarrow c_1 = -\int_0^L f(x_0)\,dx_0 \end{aligned}$$

Thus

$$\begin{aligned} u(x) &= x\int_0^x f(x_0)\,dx_0 - \int_0^x x_0 f(x_0)\,dx_0 - x\int_0^L f(x_0)\,dx_0 \\ &= \boxed{\int_0^x (x - x_0)f(x_0)\,dx_0 - x\int_0^L x_0 f(x_0)\,dx_0} \end{aligned}$$

c.)

$$u(x) = -\int_x^L xf(x_0)\,dx_0 - \int_0^L x_0 f(x_0)\,dx_0$$

$$\boxed{\begin{aligned} u(x) &= \int_0^L f(x_0)G(x,x_0)\,dx_0 \\ G(x,x_0) &= \begin{cases} -x & x < x_0 \\ -x_0 & x_0 > x \end{cases} \end{aligned}}$$

d.) Our eigenfunction is

$$\phi_n = \sin(\sqrt{\lambda_n}x)\,, \quad \lambda_n = \left(\frac{(2n+1)\pi}{2L}\right)^2$$

Let

$$u(x) = \sum_{n=0}^{\infty} a_n \sin(\lambda_n x)$$

and plug into the ODE to obtain

$$-\sum_{n=0}^{\infty} a_n\lambda_n \sin(\sqrt{\lambda_n}x) = f(x)$$

By orthogonality, we have

$$a_n = -\frac{\int_0^L f(x)\sin(\sqrt{\lambda_n}x)\,dx}{\lambda_n \int_0^L \left(\sin(\sqrt{\lambda_n}x)\right)^2\,dx}$$

Thus

$$u(x) = \sum_{n=0}^{\infty} -\frac{\int_0^L f(x_0)\sin(\sqrt{\lambda_n}x_0)\sin(\lambda_n x)\,dx_0}{\lambda_n \int_0^L \left(\sin(\sqrt{\lambda_n}x)\right)^2\,dx}$$

By changing the order of summation and integrating, we have

$$u(x) = \int_0^L f(x_0)\sum_{n=0}^{\infty} \frac{\sin(\sqrt{\lambda_n}x_0)\sin(\lambda_n x)\,dx_0}{-\lambda_n \int_0^L \left(\sin(\sqrt{\lambda_n}x)\right)^2\,dx}$$

Thus

$$\boxed{\begin{aligned} u(x) &= \int_0^L f(x_0)G(x,x_0)\,dx_0 \\ G(x,x_0) &= \sum_{n=0}^{\infty} \frac{\sin(\sqrt{\lambda_n}x_0)\sin(\lambda_n x)}{-\lambda_n \int_0^L \left(\sin(\sqrt{\lambda_n}x)\right)^2\,dx} \end{aligned}}$$

Problem 9.3.6.

$\dfrac{d^2G}{dx^2} = \delta(x-x_0) \quad G(0,x_0) = \dfrac{dG}{dx}(L,x_0) = 0$

a.) If $x \neq x_0$, we have

$$\frac{d^2G}{dx^2} = 0$$

Thus the solution is the following

$$G(x,x_0) = \begin{cases} c_1x + c_2 & x < x_0 \\ d_1x + d_2 & x > x_0 \end{cases}$$

Applying boundary conditions,

$$G(0,x_0) = 0 = c_2$$
$$\frac{dG}{dx}(L,x_0) = 0 = d_1$$

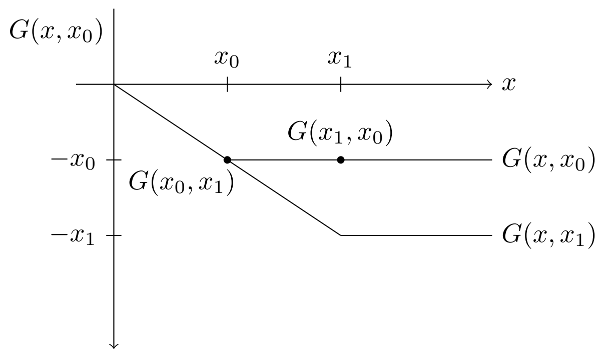
If G is continuous at $x = x_0$, $c_1 x_0 = d_2$. Integrating the ODE, we have

$$\frac{dG}{dx}\bigg|_{x_0^-}^{x_0^+} = \int_{x_0^-}^{x_0^+} \delta(x-x_0)\,dx$$
$$\frac{dG}{dx}(x_0^+) - \frac{dG}{dx}(x_0^-) = 1$$
$$0 - c_1 = 1$$
$$c_1 = -1,\ d_2 = -x_0$$

Thus

$$\boxed{G(x,x_0) = \begin{cases} -x & x < x_0 \\ -x_0 & x > x_0 \end{cases}}$$

b.) The graph of $G(x,x_0) = G(x_0,x)$



c.) The Green function is the same as $G(x,x_0)$ solved for in part (a).

Problem 9.3.11.

$\dfrac{d^2G}{dx^2} + G = \delta(x-x_0) \quad G(0,x_0) = G(L,x_0) = 0$

a.) If $x \neq x_0$, then we have

$$\frac{d^2G}{dx^2} + G = 0$$

which has the following general solution

$$G(x,x_0) = \begin{cases} A\cos(x) + B\sin(x) & x < x_0 \\ C\cos(x) + D\sin(x) & x > x_0 \end{cases}$$

Applying boundary conditions,

$$G(0,x_0) = 0 = A$$
$$G(L,x_0) = 0 = C\cos(L) + D\sin(L)$$
$$C = -D\tan(L)$$

If G is continuous at $x = x_0$, we have

$$\begin{aligned} B\sin(x_0) &= -D\tan(L)\cos(x_0) + D\sin(x_0) \\ B &= -D\tan(L)\cot(x_0) + D \\ B &= D(1 - \tan(L)\cot(x_0)) \end{aligned}$$

Integrating the ODE, we have

$$\begin{aligned} \left.\frac{dG}{dx}\right|_{x_0^-}^{x_0^+} + \int_{x_0^-}^{x_0^+} G\,dx &= \int_{x_0^-}^{x_0^+} \delta(x - x_0)\,dx \\ \frac{dG}{dx}(x_0^+) - \frac{dG}{dx}(x_0^-) &= 1 \\ D\tan(L)\sin(x_0) + D\cos(x_0) - D(1 - \tan(L)\cot(x_0))\cos(x_0) &= 1 \\ D\tan(L) &= \sin(x_0) \\ D &= \frac{\sin(x_0)}{\tan(L)} \end{aligned}$$

Thus

$$\boxed{G(x, x_0) = \begin{cases} \dfrac{\sin(x_0)}{\tan(L)}(1 - \tan(L)\cot(x_0))\sin(x) & x < x_0 \\ -\sin(x_0)\cos(x) + \dfrac{\sin(x_0)}{\tan(L)}\sin(x) & x > x_0 \end{cases}}$$

We needed to assume $L \neq n\pi$ because $\tan(x) = 0$ for $x = n\pi, n \in \mathbb{Z}$.

b.) From $G(x, x_0)$,

$$G(x_0, x) = \begin{cases} \dfrac{\sin(x)}{\tan(L)}(1 - \tan(L)\cot(x))\sin(x_0) & x > x_0 \\ -\sin(x)\cos(x_0) + \dfrac{\sin(x)}{\tan(L)}\sin(x_0) & x < x_0 \end{cases}$$

For $x > x_0$ we need the following to be true

$$\begin{aligned} -\sin(x_0)\cos(x) + \frac{\sin(x_0)}{\tan(L)}\sin(x) &= \frac{\sin(x)}{\tan(L)}(1 - \tan(L)\cot(x))\sin(x_0) \\ &= \left(\frac{\sin(x)}{\tan(L)} - \sin(x)\frac{\cos(x)}{\sin(x)}\right)\sin(x_0) \\ &= \frac{\sin(x)}{\tan(L)}\sin(x_0) - \cos(x)\sin(x_0) \end{aligned}$$

For $x < x_0$ we need the following to be true

$$\begin{aligned} -\sin(x)\cos(x_0) + \frac{\sin(x)}{\tan(L)}\sin(x_0) &= \frac{\sin(x_0)}{\tan(L)}(1 - \tan(L)\cot(x_0))\sin(x) \\ &= \left(\frac{\sin(x_0)}{\tan(L)} - \sin(x_0)\frac{\cos(x_0)}{\sin(x_0)}\right)\sin(x) \\ &= \frac{\sin(x_0)}{\tan(L)}\sin(x) - \cos(x_0)\sin(x) \end{aligned}$$

Thus $G(x, x_0) = G(x_0, x)$.

Problem 9.3.21.

Solving

$$\frac{dG}{dx} = \delta(x - x_0) \quad G(0, x_0) = 0$$

By integrating the ODE, we have

$$\begin{aligned} G(x, x_0)\Big|_{x_0^-}^{x_0^+} &= \int_{x_0^-}^{x_0^+} \delta(x - x_0)\, dx \\ &= 1 \end{aligned}$$

Since there is a jump at $x = x_0$ we have

$$\boxed{G(x, x_0) = \begin{cases} 0 & x < x_0 \\ 1 & x > x_0 \end{cases}}$$

The result of the delta function $\delta(x - x_0)$ is symmetric, i.e. $\delta(x - x_0) = \delta(x_0 - x)$. Taking the boundary condition $G(x_0, 0) = 0$ for $G(x_0, x)$ and doing the same steps as above results in $G(x, x_0) \neq G(x_0, 0)$. So $G(x, x_0)$ is not symmetric.