MATH 412 – Bonus Homework

Lukas Zamora

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Problem 1.

From the characteristic,

$$\frac{dx}{dt} = 0$$

Thus $u = c$ and

$$\frac{dt}{1} = \frac{dx}{x} \quad \Rightarrow \quad \log(x) + \tilde{c} \quad \begin{aligned} &\Rightarrow x = \tilde{c}e^{t} \\ &\Rightarrow \tilde{c} = xe^{-t} \end{aligned}$$

Hence our solution is of the form $u = f(xe^{-t})$.

For $x < 0$:

$$4 = u(x,0) = f(x)$$

For $3 < x < 4$:

$$u(x,0) = 1 = f(x)$$

For $0 < x < 3$:

$$4 - x = u(x,0) = f(x)$$

For $x > 4$:

$$12 = u(x,0) = f(x)$$

Thus

$$u(x,t) = \begin{cases} 4 & x < 0 \\ 4 - xe^{-t} & 0 < x < 3 \\ 1 & 3 < x < 4 \\ 12 & x > 4 \end{cases}$$

Problem 3.

Consider de'Alembert's formula,

$$u(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\, d\bar{x}$$

Here, $c = 3$ and $g(x) = 0$ from the initial condition. We then have

$$\begin{aligned} u(x,t) &= \frac{1}{2}(f(x-3t) + f(x+3t)) \\ &= \frac{1}{2}\left(e^{2(x-3t)} + e^{2(x+3t)}\right) \\ &= \frac{1}{2}e^{2x}\left(e^{-6t} + e^{6t}\right) \\ &= e^{2x}\cosh(6t) \end{aligned}$$

Checking our solution,

$$u_{tt} = 36e^{2x}\cosh(6t) = 4e^{2x}\cosh(6t) = 9u_{xx}$$

Thus

$$u(x,t) = e^{2x}\cosh(6t)$$