# MATH 407 – Homework 7

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October 25, 2018

**Problem 5.31.**

The singularity $z_0 = \pi/2$ is inside $C$, so using Cauchy's Integral Formula, the integral becomes

$$\oint_{|z|=5} \frac{\sin(3z)}{z+\pi/2}\,dz = 2\pi i \sin\left(-\frac{3\pi}{2}\right) = \boxed{2\pi i}$$

**Problem 5.33.**

a.) Notice that

$$f(z) = \frac{\cos(\pi z)}{z^2-1} = \frac{\cos(\pi z)}{(z+1)(z-1)}$$

Both singularities occur inside the rectangle $C$, so using Cauchy's Integral Formula, the integral becomes

$$\begin{aligned}
\frac{1}{2\pi i}\oint_C \frac{\cos(\pi z)}{z^2-1}\,dz &= \frac{1}{2\pi i}\oint_C \frac{\cos(\pi z)/(z-1)}{z+1}\,dz + \frac{1}{2\pi i}\oint_C \frac{\cos(\pi z)/(z+1)}{z-1}\,dz \\
&= \left.\frac{\cos(\pi z)}{z-1}\right|_{z=-1} + \left.\frac{\cos(\pi z)}{z+1}\right|_{z=1} \\
&= \frac{\cos(-\pi)}{-2} + \frac{\cos(\pi)}{2} \\
&= \frac{1}{2} - \frac{1}{2} = \boxed{0}
\end{aligned}$$

b.) The only singularity that occurs inside $C$ is $z_0 = 1$. So, using Cauchy's Integral Formula, the integral becomes

$$\frac{1}{2\pi i}\oint_C \frac{\cos(\pi z)/(z-1)}{z+1}\,dz = \left.\frac{\cos(\pi z)}{z-1}\right|_{z=-1} = \frac{\cos(-\pi)}{-2} = \boxed{-\frac{1}{2}}$$

**Problem 5.34.**

First notice that

$$f(z) = \frac{e^{zt}}{z^2+1} = \frac{e^{zt}}{(z+i)(z-i)}$$

Both singularities occur inside $C$, so by Cauchy's Integral Formula, the integral becomes

$$\begin{aligned}
\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2+1} &= \frac{1}{2\pi i}\oint_C \frac{e^{zt}/(z+i)}{z-i}\,dz + \frac{1}{2\pi i}\oint_C \frac{e^{zt}/(z-i)}{z+i}\,dz \\
&= \left.\frac{e^{zt}}{z+i}\right|_{z=i} + \left.\frac{e^{zt}}{z-i}\right|_{z=-i} \\
&= \frac{e^{it}}{2i} - \frac{e^{-it}}{2i} = \boxed{\sin(t)}
\end{aligned}$$

**Problem 5.39.**

First notice that

$$f(z) = \frac{e^{zt}}{(z^2+1)^2} = \frac{e^{zt}}{(z+i)^2(z-i^2)}$$

Both singularities occur inside $C$, so using Cauchy's Integral Formula, the integral becomes

$$\begin{aligned}
\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{(z^2+1)^2}\,dz &= \frac{1}{2\pi i}\oint_C \frac{e^{zt}/(z+i)^2}{(z-i)^2}\,dz + \frac{1}{2\pi i}\oint_C \frac{e^{zt}/(z-i)^2}{(z+i)^2}\,dz \\
&= \frac{d}{dz}\left(\frac{e^{zt}}{(z+i)^2}\right)\Bigg|_{z=i} + \frac{d}{dz}\left(\frac{e^{zt}}{(z-i)^2}\right)\Bigg|_{z=-i} \\
&= \frac{e^{it}(2it-2)}{-8i} + \frac{e^{-it}(-2it-2)}{8i} \\
&= \frac{e^{it}-e^{-it}}{4i} - \frac{te^{it}+te^{-it}}{4} \\
&= \boxed{\frac{\sin(t)-t\cos(t)}{2}}
\end{aligned}$$

**Problem 5.39.**

First note that $|f(z)| = |z^2-3z+2| = \left|\left(z-\frac{3}{2}\right)^2 - \frac{1}{4}\right|$. From $|z| \le 1$, we have

$$\begin{aligned}
-1 &\le z \le 1 \\
-\frac{5}{2} &\le z - \frac{3}{2} \le -\frac{1}{2} \\
\frac{25}{4} &\ge \left(z-\frac{3}{2}\right)^2 \ge \frac{1}{4} \\
6 &\ge \left(z-\frac{3}{2}\right)^2 - \frac{1}{4} \ge 0 \\
6 &\ge z^2 - 3z + 2 \ge 0
\end{aligned}$$

Thus the maximum of $|f(z)|$ in $|z| \le 1$ is 6.