MATH 412 - Homework 2

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Problem 2.2.2.

a.) $L(u) = \dfrac{\partial}{\partial x}\left(K_0(x)\dfrac{\partial u}{\partial x}\right) = \dfrac{dK_0}{dx}\dfrac{\partial u}{\partial x} + K_0(x)\dfrac{\partial^2 u}{\partial x^2}$

$$\begin{aligned}
L(c_1u_1 + c_2u_2) &= \frac{dK_0}{dx}\frac{\partial}{\partial x}(c_1u_1 + c_2u_2) + K_0(x)\frac{\partial^2}{\partial x^2}(c_1u_1 + c_2u_2) \\
&= \frac{dK_0}{dx}\left(c_1\frac{\partial u_1}{\partial x} + c_2\frac{\partial u_2}{\partial x}\right) + K_0(x)\left(c_1\frac{\partial^2 u_1}{\partial x^2} + c_2\frac{\partial^2 u_2}{\partial x^2}\right) \\
&= c_1\frac{dK_0}{dx}\frac{\partial u_1}{\partial x} + c_2\frac{dK_0}{dx}\frac{\partial u_2}{\partial x} + c_1K_0(x)\frac{\partial^2 u_1}{\partial x^2} + c_2K_0(x)\frac{\partial^2 u_2}{\partial x^2} \\
&= c_1\frac{dK_0}{dx}\frac{\partial u_1}{\partial x} + c_1K_0(x)\frac{\partial^2 u_1}{\partial x^2} + c_2\frac{dK_0}{dx}\frac{\partial u_2}{\partial x} + c_2K_0(x)\frac{\partial^2 u_2}{\partial x^2} \\
&= c_1L(u_1) + c_2L(u_2)
\end{aligned}$$

b.) $L(u) = \dfrac{\partial}{\partial x}\left(K_0(x,u)\dfrac{\partial u}{\partial x}\right)$

$$\begin{aligned}
L(c_1u_1 + c_2u_2) &= \frac{\partial}{\partial x}\left(K_0(x, c_1u_1 + c_2u_2)\frac{\partial}{\partial x}(c_1u_1 + c_2u_2)\right) \\
&= \frac{\partial}{\partial x}\left(K_0(x, c_1u_1 + c_2u_2)\left(c_1\frac{\partial u_1}{\partial x} + c_2\frac{\partial u_2}{\partial x}\right)\right) \\
&= \frac{\partial}{\partial x}\left(c_1K_0(x, c_1u_1 + c_2u_2)\frac{\partial u_1}{\partial x} + c_2K_0(x, c_1u_1 + c_2u_2)\frac{\partial u_2}{\partial x}\right) \\
&\neq c_1\frac{\partial}{\partial x}\left(K_0(x,u_1)\frac{\partial u_1}{\partial x}\right) + c_2\frac{\partial}{\partial x}\left(K_0(x,u_2)\frac{\partial u_2}{\partial x}\right)
\end{aligned}$$

Problem 2.3.2c.

$\dfrac{d^2\phi}{dx^2} + \lambda\phi = 0, \quad \dfrac{d\phi}{dx}(0) = \dfrac{d\phi}{dx}(L) = 0$

$r^2 + \lambda = 0 \;\Rightarrow\; r = \pm\sqrt{-\lambda}$

If $\lambda > 0$

$$\begin{aligned}
&\phi(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x) \\
&\phi'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x) \\
&\phi'(0) = 0 = c_2\sqrt{\lambda} \;\Rightarrow\; c_2 = 0 \\
&\phi'(L) = 0 = -c_1\sin(\sqrt{\lambda}L) \\
&\qquad\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2,\ n = \pm1, \pm2, \ldots
\end{aligned}$$

If $\lambda = 0$

$$\begin{aligned}
\phi(x) &= c_1 x + c_2 \\
\phi'(x) &= c_1 \\
\phi'(0) &= 0 = c_2 \\
\phi'(L) &= 0 = c_1 \\
&\Rightarrow \lambda = 0
\end{aligned}$$

If $\lambda < 0$

$$\begin{aligned}
&\phi(x) = c_1 \cosh(\sqrt{-\lambda}x) + c_2 \sinh(\sqrt{-\lambda}x) \\
&\phi'(x) = -c_1\sqrt{-\lambda}\sinh(\sqrt{-\lambda}x) + c_2\sqrt{-\lambda}\cosh(\sqrt{-\lambda}x) \\
&\phi'(0) = 0 = c_2 \\
&\phi'(L) = 0 = -c_1\sqrt{-\lambda}\sinh(\sqrt{-\lambda}x) \ \Rightarrow c_1 = 0
\end{aligned}$$

There are no eigenvalues for $\lambda < 0$.

Problem 2.3.2e.

$\dfrac{d^2\phi}{dx^2} + \lambda\phi = 0, \quad \dfrac{d\phi}{dx}(0) = \phi(L) = 0$

$r^2 + \lambda = 0 \ \Rightarrow \ r = \pm\sqrt{-\lambda}$

If $\lambda > 0$

$$\begin{aligned}
&\phi(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x) \\
&\phi'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x) \\
&\phi'(0) = 0 = c_2 \\
&\phi(L) = 0 = c_1\cos(\sqrt{\lambda}L) + c_2\sin(\sqrt{\lambda}L) \\
&\qquad \Rightarrow \sqrt{\lambda}L = (2n-1)\frac{\pi}{2} \ \Rightarrow \ \lambda = \left(\frac{(2n-1)\pi}{2L}\right)^2, \, n = \pm 1, \pm 2, \ldots
\end{aligned}$$

If $\lambda = 0$

$$\begin{aligned}
\phi(x) &= c_1 x + c_2 \\
\phi'(x) &= c_1 \\
\phi'(0) &= 0 = c_1 \\
\phi(L) &= 0 = c_1 L + c_2 \\
&\Rightarrow c_2 = 0
\end{aligned}$$

There are no eigenvalues for $\lambda = 0$.

If $\lambda < 0$

$$\begin{aligned}
&\phi(x) = c_1 \cosh(\sqrt{-\lambda}x) + c_2 \sinh(\sqrt{-\lambda}x) \\
&\phi'(x) = -c_1\sqrt{-\lambda}\sinh(\sqrt{-\lambda}x) + c_2\sqrt{-\lambda}\cosh(\sqrt{-\lambda}x) \\
&\phi'(0) = 0 = c_2 \\
&\phi(L) = 0 = c_1\cosh(\sqrt{-\lambda}L) + c_2\sinh(\sqrt{-\lambda}L) \ \Rightarrow \ c_1 = 0
\end{aligned}$$

There are no eigenvalues for $\lambda < 0$.

Problem 2.3.3a.
$u(x,0) = f(x) = 6\sin\left(\frac{9\pi x}{L}\right), \quad u(0,t) = u(L,t) = 0$

By the principle of superposition,

$$u(x,0) = \sum_{n=1}^{\infty} B_n e^{-(n\pi/L)^2 kt}\sin\left(\frac{n\pi x}{L}\right) = 6\sin\left(\frac{9\pi x}{L}\right)$$

which implies that $B_9 = 6$. Thus,

$$\boxed{u(x,t) = 6\sin\left(\frac{9\pi x}{L}\right)e^{-(9\pi/L)^2 kt}}$$

Problem 2.3.3b.
Starting with $f(x)$,

$$u(x,0) = f(x) = 3\sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L}\right) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$
$$\Rightarrow B_1 = 3, B_3 = -1$$

By the principle of superposition, our solution becomes

$$\boxed{u(x,t) = 3\sin\left(\frac{\pi x}{L}\right)e^{-(n\pi/L)^2 kt} - \sin\left(\frac{3\pi x}{L}\right)e^{-(n\pi/L)^2 kt}}$$

Problem 2.3.5.
Using the fact that $\sin(a)\sin(b) = \frac{1}{2}(\cos(a-b) - \cos(a+b))$,

$$\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) = \frac{1}{2}\left(\cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(n+m)\pi x}{L}\right)\right)$$

We then have the following 2 cases:

<u>Case 1</u>: $n = m \neq 0$

$$\begin{aligned}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,dx &= \int_0^L \frac{1}{2}\left(1 - \cos\left(\frac{2n\pi x}{L}\right)\right)dx \\ &= \frac{1}{2}\left(x\Big|_0^L - \frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\Big|_0^L\right) \\ &= \frac{L}{2}\end{aligned}$$

<u>Case 2</u>: $n \neq m$

$$\begin{aligned}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,dx &= \frac{1}{2}\left(\int_0^L \cos\left(\frac{(n-m)\pi x}{L}\right)dx - \int_0^L \cos\left(\frac{(n+m)\pi x}{L}\right)dx\right) \\ &= \frac{1}{2}\left(\frac{L}{(n-m)\pi}\sin\left(\frac{(n-m)\pi x}{L}\right)\Big|_0^L - \frac{L}{(n+m)\pi}\sin\left(\frac{(n+m)\pi x}{L}\right)\Big|_0^L\right) \\ &= \frac{L}{2(n-m)\pi}(\sin((n-m)\pi) - \sin(0)) - \frac{L}{2(n+m)\pi}(\sin((n+m)\pi) - \sin(0)) \\ &= 0\end{aligned}$$

$$\therefore \boxed{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} L/2 & n = m \\ 0 & n \neq m\end{cases}}$$

Problem 2.4.3.

$\dfrac{d^2\phi}{dx^2} + \lambda\phi = 0, \quad \phi(0) = \phi(2\pi), \quad \dfrac{d\phi}{dx}(0) = \dfrac{d\phi}{dx}(2\pi)$

$r^2 + \lambda = 0 \;\Rightarrow\; r = \pm\sqrt{-\lambda}$

If $\lambda > 0$

$$\begin{aligned}
&\phi(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x) \\
&\phi(0) = c_1 = c_1 \cos(\sqrt{\lambda}2\pi) + c_2 \sin(\sqrt{\lambda}2\pi) \\
&\phi'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x) \\
&\phi'(0) = c_2 = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}2\pi) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}2\pi) \\
&\qquad \Rightarrow \sqrt{\lambda}2\pi = 1 \\
&\qquad \Rightarrow \lambda = n^2,\, n = \pm 1, \pm 2, \ldots
\end{aligned}$$

If $\lambda = 0$

$$\begin{aligned}
&\phi(x) = c_1 x + c_2 \\
&\phi(0) = 0 = c_1 \Rightarrow c_2 = 0
\end{aligned}$$

There are no eigenvalues for $\lambda = 0$ If $\lambda < 0$

$$\begin{aligned}
&\phi(x) = c_1 \cosh(\sqrt{-\lambda}x) + c_2 \sinh(\sqrt{-\lambda}x) \\
&\phi(0) = 0 \;\Rightarrow\; c_1 = c_2 = 0
\end{aligned}$$

There are no eigenvalues for $\lambda < 0$.

Problem 2.4.4.

$\dfrac{d^2\phi}{dx^2} + \lambda\phi = 0, \quad \dfrac{d\phi}{dx}(0) = \dfrac{d\phi}{dx}(L) = 0$

If $\lambda < 0$

$$\begin{aligned}
&\phi(x) = c_1 \cosh(\sqrt{-\lambda}x) + c_2 \sinh(\sqrt{-\lambda}x) \\
&\phi'(x) = -c_1\sqrt{-\lambda}\sinh(\sqrt{-\lambda}x) + c_2\sqrt{-\lambda}\cosh(\sqrt{-\lambda}x) \\
&\phi'(0) \;\Rightarrow\; c_2 = -c_1 \\
&\Rightarrow \sin(\sqrt{-\lambda}L) > 0 \text{ so } c_1 = 0 \Rightarrow \phi(x) = 0
\end{aligned}$$

So there are no negative eigenvalues.

Problem 2.4.6.

a.) For equilibrium, we know that $\partial u/\partial t = 0$. So

$$\frac{d^2u}{dx^2} = 0 \Rightarrow u(x) = c_1 x + c_2$$

From $u(-L) = u(L)$, we have

$$-c_1 L + c_2 = c_1 L + c_2$$

Thus $c_1 = 0$. The second condition $u'(-L) = u'(L)$ yields $c_1 = c_1$. Therefore the equilibrium solution is

$$u(x) = c_2$$

If a system is in equilibrium, the total energy is constant, and the initial temperature is equal to the final temperature, i.e, $u(x,0) = f(x)$ and $u(x) = c_2$. So we have

$$\begin{aligned}\int_{-L}^{L} f(x)\,dx &= \int_{-L}^{L} c_2\,dx \\ &= 2Lc_2 \\ &\Rightarrow c_2 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx\end{aligned}$$

Therefore the equilibrium temperature distribution is

$$u(x) = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

b.) By equation (2.4.38), we know that the solution to the time dependent problem is

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-(n\pi/L)^2 kt} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-(n\pi/L)^2 kt}$$

and by equation (2.4.43),

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

Since $\lim\limits_{t\to\infty} u(x,t) = a_0$, this implies that

$$u(x) = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$