MATH 323 - Homework 4

Lukas Zamora

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Problem 3.4.3f.

First we need to find the eigenvalues of A:

$$\begin{vmatrix} 3-\lambda & 4 & 12 \\ 4 & -12-\lambda & 3 \\ 12 & 3 & -4-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} -12-\lambda & 3 \\ 3 & -4-\lambda \end{vmatrix} - 4\begin{vmatrix} 4 & 4 \\ 12 & -4-\lambda \end{vmatrix} + 12\begin{vmatrix} 4 & -12-\lambda \\ 12 & 3 \end{vmatrix} = 0$$

$$= (3-\lambda)(\lambda^2+16\lambda+39) - 4(-4\lambda-52) + 12(12\lambda+156) = 0$$

$$= -\lambda^3 - 13\lambda^2 + 169\lambda + 2197 = 0$$

$$= -(\lambda+13)^2(\lambda-13) = 0$$

The roots are $\lambda = -13$ with multiplicity 2, and $\lambda = 13$ with multiplicity 1.

Eigenvector for $\lambda = 13$: $A - 13I = 0$

$$\left[\begin{array}{ccc|c} 3-13 & 4 & 12 & 0 \\ 4 & -12-13 & 3 & 0 \\ 12 & 3 & 4-13 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -10 & 4 & 12 & 0 \\ 4 & -25 & 3 & 0 \\ 12 & 3 & -9 & 0 \end{array}\right] \xRightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

So $\vec{x} = \vec{0}$, thus

$$E_{13} = \left\{ x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \middle| \, x_1, x_2, x_3 \in \mathbf{R} \right\}$$

Eigenvector for $\lambda = -13$: $A + 13I = 0$

$$\left[\begin{array}{ccc|c} 3+13 & 4 & 12 & 0 \\ 4 & -12+13 & 3 & 0 \\ 12 & 3 & 4+13 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 16 & 4 & 12 & 0 \\ 4 & 1 & 3 & 0 \\ 12 & 3 & 17 & 0 \end{array}\right] \xRightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 1/4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So $x_1 = -1/4x_2$, $x_3 = 0$, $x_2 =$ free. Thus

$$E_{-13} = \left\{ x_2 \begin{pmatrix} -1/4 \\ 1 \\ 0 \end{pmatrix} \middle| \, x_2 \in \mathbf{R} \right\}$$

Problem 3.4.4e.

$$A = \begin{bmatrix} -3 & 3 & -1 \\ 2 & 2 & 4 \\ 6 & 3 & 4 \end{bmatrix}$$

$$\begin{vmatrix} -3-\lambda & 3 & -1 \\ 2 & 2-\lambda & 4 \\ 6 & 3 & 4-\lambda \end{vmatrix} = -3-\lambda\begin{vmatrix} 2-\lambda & 4 \\ 3 & 4-\lambda \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ 6 & 4-\lambda \end{vmatrix} - \begin{vmatrix} 2 & 2-\lambda \\ 6 & 3 \end{vmatrix} = 0$$

$$= (-3-\lambda)(\lambda^2-6\lambda+8-12) - 3(8-2\lambda-24) - (6-12-6\lambda) = 0$$

$$= -3\lambda^2 - \lambda^3 + 18\lambda + 6\lambda^2 + 12 + 4\lambda - 24 + 6\lambda + 72 - 6 + 12 + 6\lambda = 0$$

$$= -\lambda^3 + 3\lambda^2 + 22\lambda + 66 = 0$$

The roots to this equation are $\lambda = 7.2728,\ 2.134 + 2.1239i,\ 2.134 - 2.1239i$. Since A only has 1 real eigenvalue/eigenvector, it is not diagonalizable, since the number of eigenvectors must equal $\dim(A)$ for A to be diagonalizable.

Problem 3.4.6.

Proof. Since A and B are similar, there exists an invertible matrix $M \in \mathbf{R}^{n\times n}$ such that $B = MAM^{-1}$. We then have

$$\begin{aligned} p_B(x) &= \det(B - xI) \\ &= \det(MAM^{-1} - xI) \\ &= \det(MAM^{-1} - xMM^{-1}) \\ &= \det(M^{-1}(A - xI)M) \\ &= \det(M^{-1}M)\det(A - xI) \\ &= \det(A - xI) \\ &= p_A(x) \end{aligned}$$

Therefore $p_A(x) = p_B(x)$. □

Problem 3.4.11.

Proof. First note that $p_A(\frac{1}{x}) = \det(\frac{1}{x}I - A) = \det(I - Ax)$. We then have

$$\begin{aligned} p_{A^{-1}}(x) &= \det(Ix - A^{-1}) \\ &= \det(AA^{-1}x - A^{-1}) \\ &= \det(A^{-1}(Ax - I)) \\ &= \det(A^{-1})\det(Ax - I) \\ &= \det(A^{-1})(-x)^n\det(I - Ax) \\ &= \det(A^{-1})(-x)^n p_A\left(\frac{1}{x}\right) \end{aligned}$$

□

Problem 3.4.17.

Proof. Assume A is singular. Then A has a nontrivial nullspace, so we can choose x such that $Ax = 0 = 0x$, from which we see that x is an eigenvalue for $\lambda = 0$. The converse follows by a symmetric argument. □