MATH 407 - Homework 1

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Problem 1.53(g).

$$\begin{aligned}
\frac{(2+i)(3-2i)(1+2i)}{(1-i)^2} &= \frac{(6-4i)+3i+2(1+2i)}{(1-i)(1-i)} \\
&= \frac{(8-i)(1+2i)}{(1-i-i-1)} \\
&= \frac{8+16i-i+2}{-2i} \\
&= \frac{10+15i}{-2i} \\
&= \frac{(10+15i)}{(-2i)} \cdot \frac{2i}{2i} \\
&= \frac{-30+20i}{4} \\
&= \boxed{-\frac{15}{2}+5i}
\end{aligned}$$

Problem 1.54(e).

$$\begin{aligned}
\left|\frac{z_1+z_2+1}{z_1-z_2+i}\right| &= \left|\frac{(1-i)+(-2+4i)+1}{(1-i)-(-2+4i)+i}\right| \\
&= \left|\frac{3i}{3-4i}\right| \\
&= \left|\frac{3i(3+4i)}{(3-4i)(3+4i)}\right| \\
&= \left|\frac{-12+9i}{25}\right| \\
&= \left|-\frac{12}{25}+\frac{9}{25}i\right| \\
&= \sqrt{\left(\frac{12}{25}\right)^2+\left(\frac{9}{25}\right)^2} \\
&= \boxed{\frac{3}{5}}
\end{aligned}$$

Problem 1.73(b).

Starting with the definition of z,

$$z^2 = (x+iy)^2 = (x^2-y^2)+2xyi$$

This implies that $\mathrm{Re}(z^2) = x^2-y^2$. We then have the region $x^2-y^2 > 1$ which is a region that resembles a hyperbola.

Problem 1.81(f).

Converting from Cartesian to polar coordinates, we need to find r, θ such that $x = r\cos\theta, y = r\sin\theta$.

$$r = |-2\sqrt{3} - 2i| = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = \sqrt{16} = 4$$

To find theta, we use the fact that $\tan\theta = (y/x)$.

$$\tan\theta = \frac{y}{x}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-2}{-2\sqrt{3}}\right) = \frac{\pi}{6}$$

However $z = -2\sqrt{3} - 2i$ is located in the 3rd quadrant, so $\theta = \frac{\pi}{6} + \pi = \frac{7\pi}{6}$. Thus $z = \boxed{4e^{7\pi i/6}}$.

Problem 1.95(b).

We are trying to solve the equation $z^5 = w$ where $w = -4 + 4i$. If $w = \rho e^{i\varphi}$ and $z = re^{i\theta}$, then by De Moivre's formula we have

$$r^5 e^{i5\theta} = \rho e^{i\varphi}$$

where $\rho = \sqrt{(-4)^2 + 4^2} = \sqrt{32}$ and $\varphi = \tan^{-1}(1) = \frac{\pi}{4} - \pi = -\frac{3\pi}{4} + k$ for some $k \in \mathbb{Z}$. We then have

$$r^5 e^{i5\theta} = \sqrt{32} e^{i(-3\pi/4 + 2\pi k)}$$

Equating coefficients we are left with $r^5 = \sqrt{32}$ and $5\theta = -3\pi/4 + 2\pi k$. So $r = \sqrt{2}$ and

$$\theta = -\frac{3}{20} + \frac{2\pi k}{5} = \frac{-3\pi + 8\pi k}{20}$$

We observe that the distinct values of k are when $k = -2, -1, 0, 1, 2$. So our 5 distinct roots of the complex number $w = -4 + 4i$ are

$$w_1 = \sqrt{2}e^{\pi i/4}$$
$$w_2 = \sqrt{2}e^{13\pi i/20}$$
$$w_3 = \sqrt{2}e^{-19\pi i/20}$$
$$w_4 = \sqrt{2}e^{-11\pi i/20}$$
$$w_5 = \sqrt{2}e^{-3\pi i/20}$$