MATH 323 - Homework 9

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Problem 6.1.10.

Proof. Let $\mathcal{W}$ be the subspace spanned by $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$. By expanding the basis, we have $\mathcal{W} = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$. Since $\mathbf{v} \in \mathbb{R}^n$, by Theorem 6.3, we have

$$\mathbf{v} \cdot \mathbf{v} = (\mathbf{v} \cdot \mathbf{u}_1)(\mathbf{v} \cdot \mathbf{u}_1) + (\mathbf{v} \cdot \mathbf{u}_2)(\mathbf{v} \cdot \mathbf{u}_2) + \cdots + (\mathbf{v} \cdot \mathbf{u}_k)(\mathbf{v} \cdot \mathbf{u}_k) + (\mathbf{v} \cdot \mathbf{u}_{k+1})(\mathbf{v} \cdot \mathbf{u}_{k+1}) + \cdots + (\mathbf{v} \cdot \mathbf{u}_n)(\mathbf{v} \cdot \mathbf{u}_n)$$

or

$$\begin{aligned}
||\mathbf{v}||^2 &= (\mathbf{v} \cdot \mathbf{u}_1)^2 + (\mathbf{v} \cdot \mathbf{u}_2)^2 + \cdots + (\mathbf{v} \cdot \mathbf{u}_k)^2 + (\mathbf{v} \cdot \mathbf{u}_{k+1})^2 + \cdots + (\mathbf{v} \cdot \mathbf{u}_n)^2 \\
&= \left[(\mathbf{v} \cdot \mathbf{u}_1)^2 + (\mathbf{v} \cdot \mathbf{u}_2)^2 + \cdots + (\mathbf{v} \cdot \mathbf{u}_k)^2\right] + \left[(\mathbf{v} \cdot \mathbf{u}_{k+1})^2 + \cdots + (\mathbf{v} \cdot \mathbf{u}_n)^2\right] \\
&\geq (\mathbf{v} \cdot \mathbf{u}_1)^2 + \cdots + (\mathbf{v} \cdot \mathbf{u}_n)^2
\end{aligned}$$

□

Problem 6.1.12.

Proof. Suppose $\mathbf{A}$ is symmetric, i.e. $\mathbf{A} = \mathbf{A}^T$. Since $\mathbf{A}^2 = \mathbf{I}_n$, we have $\mathbf{A} = \mathbf{A}^{-1}$. Then $\mathbf{A}^T = (\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1} = \mathbf{A}^{-1}$. The fact that $\mathbf{A}^T = \mathbf{A}^{-1}$ imples that $\mathbf{A}$ is orthogonal.
Suppose $\mathbf{A}$ is orthogonal, i.e $\mathbf{A}^T = \mathbf{A}^{-1}$. Since $\mathbf{A} = \mathbf{A}^{-1}$, we have $\mathbf{A}^T = \mathbf{A}$, thus $\mathbf{A}$ is symmetric. □

Problem 6.2.12.

Proof. Consider the orthogonal basis of $\mathcal{W}$ is $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$. Then consider the orthogonal subset of $\mathcal{W}_1$ and $\mathcal{W}_2$ such that $\mathcal{W}_1 \subseteq \mathcal{W}_2$, where $\mathcal{W}_1 = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$ and $\mathcal{W}_2 = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{k+1}, \dots, \mathbf{u}_m\}$. We then have $\mathcal{W}_1^\perp = \text{span}\{\mathbf{u}_{k+1}, \mathbf{u}_{k+2}, \dots, \mathbf{u}_n\}$ and similarly, $\mathcal{W}_2^\perp = \text{span}\{\mathbf{u}_{m+1}, \mathbf{u}_{m+2}, \dots, \mathbf{u}_n\}$. Thus $\mathcal{W}_2^\perp \subseteq \mathcal{W}_1^\perp$. □

Problem 6.2.13.

a.) Consider the orthogonal basis of $\mathcal{W}$ as $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$. Let $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$ for some $a_1, \dots, a_n \in \mathbb{R}$. Thus

$$\begin{aligned}
\text{proj}_{\mathcal{W}}\, \mathbf{v} &= (\mathbf{v} \cdot \mathbf{v}_1)\mathbf{v}_1 + (\mathbf{v} \cdot \mathbf{v}_2)\mathbf{v}_2 + \cdots + (\mathbf{v} \cdot \mathbf{v}_k)\mathbf{v}_k \\
&= ((a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n) \cdot \mathbf{v}_1)\mathbf{v}_1 + ((a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n) \cdot \mathbf{v}_2)\mathbf{v}_2 + \\
&\quad \cdots + ((a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n) \cdot \mathbf{v}_k)\mathbf{v}_k \\
&= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots a_k\mathbf{v}_k \\
&= \mathbf{v}
\end{aligned}$$

Thus if $\mathbf{v} \in \mathcal{W}$, then $\text{proj}_{\mathcal{W}}\, \mathbf{v} = \mathbf{v}$.

b.) Suppose $\mathbf{v} \in \mathcal{W}^{\perp}$ and $\mathbf{v} = a_{k+1}\mathbf{v}_{k+1} + a_{k+2}\mathbf{v}_{k+2} \cdots + a_n\mathbf{v}_n$, for some $a_1, \ldots, a_n \in \mathbb{R}$. Then

$$\begin{aligned}
\operatorname{proj}_{\mathcal{W}} \mathbf{v} &= (\mathbf{v} \cdot \mathbf{v}_1)\mathbf{v}_1 + (\mathbf{v} \cdot \mathbf{v}_2)\mathbf{v}_2 + \cdots + (\mathbf{v} \cdot \mathbf{v}_k)\mathbf{v}_k \\
&= ((a_{k+1}\mathbf{v}_{k+1} + a_{k+2}\mathbf{v}_{k+2} + \cdots + a_n\mathbf{v}_n) \cdot \mathbf{v}_1)\mathbf{v}_1 + \\
&\quad ((a_{k+1}\mathbf{v}_{k+1} + a_{k+2}\mathbf{v}_{k+2} + \cdots + a_n\mathbf{v}_n) \cdot \mathbf{v}_2)\mathbf{v}_2 + \ldots \\
&\quad ((a_{k+1}\mathbf{v}_{k+1} + a_{k+2}\mathbf{v}_{k+2} + \cdots + a_n\mathbf{v}_n) \cdot \mathbf{v}_k)\mathbf{v}_k \\
&= (a_{k+1}\mathbf{v}_{k+1} \cdot \mathbf{v}_1)\mathbf{v}_1 + (a_{k+2}\mathbf{v}_{k+2} \cdot \mathbf{v}_1)\mathbf{v}_1 + \cdots + (a_n\mathbf{v}_n \cdot \mathbf{v}_1)\mathbf{v}_1 + \ldots \\
&\quad + (a_{k+1}\mathbf{v}_{k+1} \cdot \mathbf{v}_k)\mathbf{v}_k + (a_{k+2}\mathbf{v}_{k+2} \cdot \mathbf{v}_k)\mathbf{v}_k + \cdots + (a_n\mathbf{v}_n \cdot \mathbf{v}_k)\mathbf{v}_k \\
&= 0
\end{aligned}$$

Thus if $\mathbf{v} \in \mathcal{W}^{\perp}$, then $\operatorname{proj}_{\mathcal{W}} \mathbf{v} = 0$.

Problem 6.2.21.

Proof. Suppose $T(\mathbf{v}) = T(\mathbf{w})$. We need to show that $\mathbf{v} = \mathbf{w}$. Since T is linear, $T(\mathbf{v} - \mathbf{w}) = \mathbf{0}$. Also since $\mathbf{v}, \mathbf{w} \in (\ker L)^{\perp}$ and, by definition, $T(\mathbf{v} - \mathbf{w}) = L(\mathbf{v} - \mathbf{w})$, then $\mathbf{v} - \mathbf{w} \in \ker L$. So $\mathbf{v} - \mathbf{w} \in \ker L \cap (\ker L)^{\perp}$. Using Theorem 6.10, we know that $\ker L \cap (\ker L)^{\perp} = \{\mathbf{0}\}$, thus $\mathbf{v} - \mathbf{w} = \mathbf{0} \Rightarrow \mathbf{v} = \mathbf{w}$. Hence T is one-to-one. □