MATH 323 - Homework 6

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October 23, 2018

Problem 4.5.6.

a.) Suppose $\mathbf{x}_1, \mathbf{x}_2$ are solutions, then $\mathbf{A}\mathbf{x}_1 = \mathbf{0}$ and $\mathbf{A}\mathbf{x}_2 = \mathbf{0}$. Then for any scalars $c_1, c_2 \in \mathbb{R}$, $\mathbf{A}(c_1\mathbf{x}_1 + c_2\mathbf{x}_2) = c_1\mathbf{A}\mathbf{x}_1 + c_2\mathbf{A}\mathbf{x}_2 = c_1(\mathbf{0}) + c_2(\mathbf{0}) = \mathbf{0}$. Thus $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ is also a solution. Hence $\mathcal{W}$ is a subspace of $\mathbb{R}^4$.

b.) Converting $\mathbf{A}$ into row echelon form,

$$\mathbf{A} = \begin{bmatrix} -1 & -2 & -4 & 8 \\ 3 & 4 & 6 & -14 \\ -2 & -1 & 1 & 7 \\ 1 & 0 & -2 & 0 \\ 2 & 3 & 5 & -12 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 4 & 12 & -14 \\ 0 & -1 & -3 & 7 \\ 0 & -2 & -6 & 8 \\ 0 & 3 & 9 & -12 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We have 3 nonzero rows. Let $x_1 = 2\alpha, x_2 = -3\alpha, x_3 = \alpha, x_4 = 0$. Then our basis for $\mathcal{W}$ is

$$\left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \end{pmatrix} \right\}$$

c.) From the basis, $\dim(\mathcal{W}) = 1$, and from the reduced row echelon form of $\mathbf{A}$, $\operatorname{rank}(\mathbf{A}) = 3$. Thus $\dim(\mathcal{W}) + \operatorname{rank}(\mathbf{A}) = 1 + 3 = 4$.

Problem 4.5.7.

First note that $f^{(k)}$ is a $(n-k)^{th}$ polynomial and it cannot be expressed as a linear combination of the previous degree polynomials, i.e, $f^{(k)} \notin \operatorname{span}\left(\{f^{(k+1)}, f^{(k+2)}, \ldots, f^{(n)}\}\right)$.

Suppose $S = \{f, f', f'', \ldots, f^{(n)}\}$ is linearly dependent, i.e, $a_0 f + a_1 f' + a_2 f'' + \cdots + a_n f^{(n)} = 0$ for some $a_0, \ldots, a_n \in \mathbb{R}$ (not all 0). Suppose a_k is the minimum coefficient. Then

$$\begin{aligned}
& a_0 f + a_1 f' + a_2 f'' + \cdots + a_n f^{(n)} = 0 \\
\Rightarrow\; & 0f + 0f' + \cdots + 0f^{(k-1)} + a_k f^{(k)} + a_{k+1} f^{(k+1)} + \cdots + a_n f^{(n)} = 0 \\
\Rightarrow\; & a_k f^{(k)} = (-a_{k+1}) f^{(k+1)} + (-a_{k+2}) f^{(k+2)} + \cdots + (-a_n) f^{(n)} \\
\Rightarrow\; & f^{(k)} = \left(-\frac{a_{k+1}}{a_k}\right) f^{(k+1)} + \left(-\frac{a_{k+2}}{a_k}\right) f^{(k+2)} + \cdots + \left(-\frac{a_n}{a_k}\right) f^{(n)} \\
\Rightarrow\; & f^{(k)} \in \operatorname{span}\left(\left\{f^{(k+1)}, f^{(k+2)}, \ldots, f^{(n)}\right\}\right)
\end{aligned}$$

which is a contradiction. Thus S is linearly independent.

We know that in a vector space $\mathcal{V}$ with dimension n, $|S|$ is always $\leq n$. If $|S| = n$, then S is a basis for $\mathcal{V}$. So we have $\dim(\mathcal{P}_n) = n+1$ and $|S| = n+1$. Since $|S| = \dim(\mathcal{P}_n)$, S is a basis for $\mathcal{P}_n$.

Problem 4.5.8.

a.) Let $\mathbf{A}$ be a 2×2 matrix. Then the charateristic polynomial of $\mathbf{A}$ is

$$\lambda^2 - \operatorname{tr}(\mathbf{A})\lambda + \det(\mathbf{A}) = 0$$

Since every matrix satisfies its characteristic polynomial, we have

$$\mathbf{A}^2 - \operatorname{tr}(\mathbf{A})\mathbf{A} + \det(\mathbf{A})\mathbf{I}_2 = \mathbf{0}_2$$

Hence there is a nonzero polynomial of degree 2. This actually has degree 4 if we take the coefficients of $\mathbf{A}^3$ and $\mathbf{A}^4$ to both be zero.

b.) By part (a), we know that the characteristic polynomial of $\mathbf{B}$ will be of degree n, which will be satisfied by $\mathbf{B}$. Hence there exists a nonzero polynomial $\mathbf{p} \in \mathcal{P}_n \subseteq \mathcal{P}_{n^2}$ such that $\mathbf{p}(\mathbf{B}) = \mathbf{0}_n$.

Problem 4.5.13.

Proof. By contrapositive method. If S spans $\mathcal{V}$, then S is a basis by Theorem 4.12. We also have that If S is linearly independent, then S is a basis by Theorem 4.12. □

Problem 4.5.14.

Proof. Suppose S spans $\mathcal{V}$, then S is a basis for $\mathcal{V}$ by Theorem 4.12. Similarly, suppose S is linearly independent and also $|S| = \dim(\mathcal{V})$. Then by Theorem 4.12, S is a basis for $\mathcal{V}$, and thus S spans $\mathcal{V}$. □