MATH 407 - Homework 4

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September 27, 2018

Problem 2.102.

First note that $f(z) = \dfrac{z^2+4}{z-2i} = \dfrac{(z-2i)(z+2i)}{z-2i} = z+2i$.

a.) We must show that for any $\epsilon > 0$, we can find $\delta > 0$ such that $|z+2i-L| < \epsilon$ when $0 < |z-i| < \delta$. In this case $L = 3i$. So we have

$$|z+2i-3i| < \epsilon \Rightarrow |z-i| < \epsilon$$

So if we take $\delta = \epsilon$, the definition is fulfilled. Thus $\lim\limits_{z\to i} f(z)$ exists and $\lim\limits_{z\to i} f(z) = 3i$.

b.) No, if we compute this limit as $z \to 0$ on the real and on the imaginary axis, we will see that for the real axis, $\lim_{x\to 2i} f(z) = (4+y)i$. When we compute the limit on the imaginary axis, we see that $\lim_{y\to 2i} = x+4i$. Thus the limit depends on the direction in which we approach $2i$, which implies that $f(z)$ is not continuous at $z = 2i$.

Problem 2.142.

$$\begin{aligned}
\tan(z) = \frac{\sin(z)}{\cos(z)} &= \frac{\sin(x)\cosh(y) + i\cos(x)\sinh(y)}{\cos(x)\cosh(y) - i\sin(x)\sinh(y)} \cdot \frac{\cos(x)\cosh(y) + i\sin(x)\sinh(y)}{\cos(x)\cosh(y) + i\sin(x)\sinh(y)} \\
&= \frac{\sin(x)\cos(x) + i\sinh(y)\cosh(y)}{\cosh^2(y) - \sin^2(x)} \\
&= \frac{\sin(2x) + i\sinh(2y)}{2\cosh^2(y) - 2\sinh^2(x) + 1 - 1} \\
&= \frac{\sin(2x) + i\sinh(2y)}{\cos(2x) + \cosh(2y)} \\
&\therefore \boxed{u(x,y) = \frac{\sin(2x)}{\cos(2x)+\cosh(2y)}\,,\ v(x,y) = \frac{\sinh(2y)}{\cos(2x)+\cosh(2y)}}
\end{aligned}$$

Problem 3.45.

$f(z) = |z|^2 = x^2 + y^2$

$$\frac{\partial u}{\partial x} = 2x\,,\ \frac{\partial u}{\partial y} = 2y\,,\ \frac{\partial v}{\partial x} = 0\,,\ \frac{\partial v}{\partial y} = 0$$

By the Cauchy-Riemann equations, $\partial u/\partial x \neq \partial v/\partial y$ and $\partial v/\partial x \neq -\partial u/\partial y$. Thus $f(z)$ is not differentiable anywhere.

Problem 3.48.

$f(z) = x^2 + iy^3$

$$\frac{\partial u}{\partial x} = 2x\,,\ \frac{\partial u}{\partial y} = 0\,,\ \frac{\partial v}{\partial x} = 0\,,\ \frac{\partial v}{\partial y} = 3y^2$$

By the Cauchy-Riemann equations, $\partial v/\partial x = -\partial u/\partial y$, but $\partial u/\partial x \neq \partial v/\partial y$. Thus $f(z)$ is not analytic.

Problem 3.50.

a.) $u(x,y) = 2x(1-y)$

$$\frac{\partial u}{\partial x} = 2(1-y), \frac{\partial u}{\partial y} = -2x \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} = 0$$

So $u(x,y)$ satisfies Laplace's equation and is therefore harmonic.

b.) By the first Cauchy-Riemann equation,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 2(1-y) = \frac{\partial v}{\partial y} \Rightarrow v(x,y) = \int 2(1-y)\,dy = 2y - y^2 + h(x)$$

And by the second Cauchy-Riemann equation,

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \Rightarrow 2x = h'(x) \Rightarrow h(x) = x^2$$

$$\therefore \boxed{v(x,y) = 2y - y^2 + x^2}$$

c.) We know that $f(z) = u(x,y) + iv(x,y)$. So

$$\begin{aligned} f(z) &= 2x(1-y) + i(2y - y^2 + x^2) \\ &= 2x - 2xy + i2y - iy^2 + ix^2 \\ &= ix^2 - iy^2 - 2xy + 2x + i2y \\ &= i(x^2 - y^2 + i2xy) + 2(x + iy) \\ &= \boxed{iz^2 + 2z} \end{aligned}$$