MATH 407 – Homework 9

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Problem 6.91.

a.) $|z| < 3$

$$f(z) = \frac{1}{z-3} = \frac{-1}{3-z} = \frac{-1}{3(1-z/3)} = -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n = \sum_{n=0}^{\infty}\frac{-z^n}{3^{n+1}},\ |z| < 3.$$

b.) $|z| > 3$

$$f(z) = \frac{1}{z-3} = \frac{1}{z(1-3/z)} = \sum_{n=0}^{\infty}\frac{3^{n-1}}{z^n},\ |z| > 3.$$

Problem 6.92abc.

a.) $|z| < 1$

$$\begin{aligned}
f(z) = \frac{z}{(z-1)(2-z)} &= \frac{-1}{1-z} + \frac{2}{2-z} \\
&= \frac{-1}{1-z} + \frac{1}{1-z/2} \\
&= -(1 + z + z^2 + z^3 + \dots) + \left(1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots\right) \\
&= -\frac{z}{z} - \frac{3z^2}{4} - \frac{7z^3}{8} - \dots
\end{aligned}$$

b.) $1 < |z| < 2$

$$\begin{aligned}
f(z) &= \frac{1}{z(1-1/z)} + \frac{1}{1-z/2} \\
&= \frac{1}{z}\left(1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \dots\right) + \left(1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots\right) \\
&= \dots + \left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^2 + \frac{1}{z} + 1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots
\end{aligned}$$

c.) $|z| > 2$

$$\begin{aligned}
f(z) &= \frac{1}{z(1-1/z)} + \frac{1}{1-z/2} \\
&= \frac{1}{z}\left(1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \dots\right) - \frac{z}{2}\left(1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \dots\right) \\
&= -\frac{1}{z} - \frac{3}{z^2} - \frac{7}{z^3} - \frac{15}{z^4} - \dots
\end{aligned}$$

Problem 6.92de.

d.) $|z-1| > 1 \Rightarrow \frac{1}{|z-1|} < 1$

$$\begin{aligned}
f(z) &= \frac{1}{z-1} - \frac{2}{z-2} \\
&= \frac{1}{z-1} - \frac{2}{z-1-1} \\
&= \frac{1}{z-1} - \frac{2}{(z-1)\left(1-\frac{1}{z-1}\right)} \\
&= \frac{1}{z-1} - \frac{2}{z-1}\left(1 + \frac{1}{z-1} + \left(\frac{1}{z-1}\right)^2 + \dots\right) \\
&= -\frac{1}{z-1} - \frac{2}{(z-1)^2} - \frac{2}{(z-1)^3} - \dots
\end{aligned}$$

e.) $0 < |z-2| < 1$

$$\begin{aligned}
f(z) &= \frac{1}{z-1} - \frac{2}{z-2} \\
&= \frac{1}{z-2+1} - \frac{2}{z-2} \\
&= \left(1-(z-2)-(z-2)^2 - \dots\right) - \frac{2}{z-2} \\
&= -\frac{2}{z-2} + 1 - (z-2) + (z-2)^2 + (z-2)^3 + \dots
\end{aligned}$$

Problem 6.93.

a.) $0 < |z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{1}{z(z-2)} = \frac{-1}{2z\,(1-z/2)} = \frac{1}{2z}\sum_{n=1}^{\infty}\left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty}\frac{z^{n-1}}{2^{n+1}}$$

b.) $|z| > 2 \Rightarrow \left|\frac{2}{z}\right| < 1$

$$f(z) = \frac{1}{z(z-2)} = \frac{1}{z^2\,(1-2/z)} = \frac{1}{z^2}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n = \sum_{n=1}^{\infty}\frac{2^n}{z^{n+2}}$$

Problem 6.114.

Let $f(z) = z\ln(z)$. We have that

$$\begin{array}{lllll}
f'(z) = 1+\ln(z) & f''(z) = 1/z & f'''(z) = -1/z^2 & f^{(4)}(z) = 2/z^3 & f^{(5)}(z) = -6/z^4 \\
f'(1) = 1 & f''(1) = 1 & f'''(1) = -1 & f^{(4)}(1) = 2 & f^{(5)}(1) = -6
\end{array}$$

By $f(z)$'s Taylor series, we have

$$\begin{aligned}
f(z) &= \frac{(z-1)}{1!} + \frac{(z-1)^2}{2!} + \frac{-(z-1)^3}{3!} + \frac{2(z-1)^4}{4!} + \dots \\
&= (z-1) + \frac{(z-1)^2}{1\cdot 2} - \frac{(z-1)^3}{2\cdot 3} + \frac{(z-1)^4}{3\cdot 4} + \dots
\end{aligned}$$