# MATH 323 - Homework 5

Lukas Zamora

October 16, 2018

**Problem 4.1.6.**

*Proof.* Let $I$ be the $n \times n$ identity matrix. Both $I$ and $-I$ are invertible, but $I + (-I) = \mathbf{0}$, which is not invertible. Since the zero vector is not invertible, the set of $n \times n$ invertible matrices is not a vector space. $\square$

**Problem 4.2.5.**

Setting $a = b = c = 0$, we see that $\mathbf{0} \in \mathcal{V}$. Thus $\mathcal{V}$ is nonempty.

Let $\mathbf{x_1} = \begin{pmatrix} 2a_1 - 3b_1 \\ a_1 - 5c_1 \\ a_1 \\ 4c_1 - b_1 \\ c_1 \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} 2a_2 - 3b_2 \\ a_2 - 5c_2 \\ a_2 \\ 4c_2 - b_2 \\ c_2 \end{pmatrix} \in \mathbb{R}^5$. Then

$$\mathbf{x_1} + \mathbf{x_2} = \begin{pmatrix} 2a_1 - 3b_1 \\ a_1 - 5c_1 \\ a_1 \\ 4c_1 - b_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} 2a_2 - 3b_2 \\ a_2 - 5c_2 \\ a_2 \\ 4c_2 - b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2a_1 - 3b_1 + 2a_2 - 3b_2 \\ a_1 - 5c_1 + a_2 - 5c_2 \\ a_1 + a_2 \\ 4c_1 - b_1 + 4c_2 - b_2 \\ c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 2(a_1 + a_2) - 3(b_1 + b_2) \\ (a_1 + a_2) - (5c_1 + c_2) \\ a_1 + a_2 \\ 4(c_1 + c_2) - (b_1 + b_2) \\ c_1 + c_2 \end{pmatrix} \in \mathbb{R}^5,$$

since $a_1 + a_2, b_1 + b_2, c_1 + c_2 \in \mathbb{R}$.

Let $\alpha \in \mathbb{R}$, then

$$\alpha \mathbf{x_1} = \alpha \begin{pmatrix} 2a_1 - 3b_1 \\ a_1 - 5c_1 \\ a_1 \\ 4c_1 - b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} \alpha(2a_1 - 3b_1) \\ \alpha(a_1 - 5c_1) \\ \alpha(a_1) \\ \alpha(4c_1 - b_1) \\ \alpha(c_1) \end{pmatrix} \in \mathbb{R}^5$$

Since $\mathbf{x}$ is closed under vector addition and scalar multiplication, $\mathbf{x}$ is a subspace of $\mathbb{R}^5$.

**Problem 4.2.18.**

*Proof.* Since $\mathcal{W}_1$ and $\mathcal{W}_2$ are subspaces, then $0 \in \mathcal{W}_1$ and $0 \in \mathcal{W}_2$. Thus $0 \in \mathcal{W}_1 \cap \mathcal{W}_2$.

Suppose that $\mathbf{x}, \mathbf{y} \in \mathcal{W}_1 \cap \mathcal{W}_2$. Then $\mathbf{x}$ is in both $\mathcal{W}_1$ and $\mathcal{W}_2$, and $\mathbf{y}$ is in both $\mathcal{W}_1$ and $\mathcal{W}_2$. Since $\mathcal{W}_1$ is a subspace of $\mathcal{V}$, then $\mathbf{x}+\mathbf{y} \in \mathcal{W}_1$. Similarly, since $\mathcal{W}_2$ is a subspace of $\mathcal{V}$, then $\mathbf{x}+\mathbf{y} \in \mathcal{W}_2$. Thus $\mathbf{x}+\mathbf{y} \in \mathcal{W}_1 \cap \mathcal{W}_2$.

Let $\mathbf{x} \in \mathcal{W}_1 \cap \mathcal{W}_2$, and $\alpha \in \mathbb{R}$. Since $\mathbf{x}$ is in both $\mathcal{W}_1$ and $\mathcal{W}_2$, and $\mathcal{W}_1$ and $\mathcal{W}_2$ are subspaces of $\mathcal{V}$, then both $\mathcal{W}_1$ and $\mathcal{W}_2$ are closed under scalar multiplication. Thus $\alpha\mathbf{x} \in \mathcal{W}_1$ and $\alpha\mathbf{x} \in \mathcal{W}_2$, which implies that $\alpha\mathbf{x} \in \mathcal{W}_1 \cap \mathcal{W}_2$.

Therefore $\mathcal{W}_1 \cap \mathcal{W}_2$ is a subspace of $\mathcal{V}$. $\square$

**Problem 4.3.6.**

*Proof.* Let $A = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 7 & -3 \\ 4 & 8 & -7 \end{pmatrix}$. Using Gaussian elimination,

$$\begin{pmatrix} 1 & 3 & -1 \\ 2 & 7 & -3 \\ 4 & 8 & -7 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \\ 0 & -4 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & -7 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since $\operatorname{rank}(A) = 3$, $S$ spans $\mathbb{R}^3$. $\square$

**Problem 4.3.16.**

*Proof.* $\operatorname{Span}(S_1) \subseteq \operatorname{span}(S_2)$, since $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = (-a_1)(-\mathbf{v}_1) + \cdots + (-a_n)(-\mathbf{v}_n)$, and $\operatorname{span}(S_2) \subseteq \operatorname{span}(S_1)$, since $a_1(-\mathbf{v}_1) + \cdots + a_n(-\mathbf{v}_n) = (-a_1)\mathbf{v}_1 + \cdots + (-a_n)\mathbf{v}_n$. Thus $\operatorname{span}(S_1) = \operatorname{span}(S_2)$. $\square$