SOLUTIONS FOR PROBLEM SET 7
TIME SERIES II AND PORTFOLIO THEORY

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Abstract. This is document contains solutions to problem Set 7 for MIT online course "Mathematics for Applications in Finance" available at url.

1.

Consider a bivariate random variable:

$$\boldsymbol{X}_t = \begin{bmatrix} X_1 \\ X_2, \end{bmatrix}$$

with mean and covariance:
$(\epsilon_{1,t}, \epsilon_{2,t})^T$ are $i.i.d.N(0_2, \Sigma)$, and

$$E[X] = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, and \qquad Cov[X] = \Sigma = \begin{bmatrix} \Sigma_{1,1} & \Sigma_{1,2} \\ \Sigma_{2,1} & \Sigma_{2,2} \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix},$$

where $\sigma_1 = \sqrt{\Sigma_{1,1}}$, $\sigma_2 = \sqrt{\Sigma_{2,2}}$ and ρ is the correlation between $\boldsymbol{X}_1 and \boldsymbol{X}_2$.

Conduct the Principal Components Analysis (PCA) of $\boldsymbol{X}$:

1.1. **a.** Compute the **eigenvalues** $\Sigma : \lambda_1 \geq \lambda_2 \geq 0$.

Solution 1.
Calculating eigenvalues:

$$\begin{aligned} (\sigma_1^2 - \lambda)(\sigma_2^2 - \lambda) - \rho^2\sigma_1^2\sigma_2^2 = 0 \\ \lambda^2 - \lambda(\sigma_1^2 + \sigma_2^2) + (1 - \rho^2)\sigma_1^2\sigma_2^2 = 0 \end{aligned} \tag{1}$$

Eigenvalues quadratic equation solution :

$$\begin{aligned} \Delta = & \quad (\sigma_1^2 + \sigma_2^2)^2 - 4(1 - \rho^2)\sigma_1^2\sigma_2^2 \\ = & \quad \sigma_1^4 + \sigma_2^4 + 2\sigma_1^2\sigma_2^2 - (4 - 4\rho^2)\sigma_1^2\sigma_2^2 \\ = & \quad \sigma_1^4 + \sigma_2^4 - 2(1 - 2\rho^2)\sigma_1^2\sigma_2^2 \end{aligned} \tag{2}$$

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Computing quadratic equation values :

$$\begin{aligned}\lambda_1 = &\quad \frac{(\sigma_1^2+\sigma_2^2)+\sqrt{\Delta}}{2}\\ \lambda_2 = &\quad \frac{(\sigma_1^2+\sigma_2^2)-\sqrt{\Delta}}{2}\end{aligned} \tag{3}$$

1.2. **b.** Compute the **eigenvectors** γ_1, γ_2:

$$\begin{aligned}\Sigma\gamma_i &= \lambda_i, \quad i=1,2\\ \gamma_i'\gamma_i &= 1, \quad i=1,2\\ \gamma_1'\gamma_2 &= 0\end{aligned}$$

Solution 2.

$$\begin{aligned}\begin{bmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{bmatrix}\begin{bmatrix}\gamma_{i,1}\\ \gamma_{i,2}\end{bmatrix} &= \begin{bmatrix}\lambda_i & 0\\ 0 & \lambda_i\end{bmatrix}\begin{bmatrix}\gamma_{i,1}\\ \gamma_{i,2}\end{bmatrix}\\ \begin{bmatrix}\sigma_1^2-\lambda_i & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2-\lambda_i\end{bmatrix}\begin{bmatrix}\gamma_{i,1}\\ \gamma_{i,2}\end{bmatrix} &= [0]\end{aligned} \tag{4}$$

Computing $\gamma_{i,1} = -\frac{\rho\sigma_1\sigma_2\gamma_{i,2}}{\sigma_1^2-\lambda_i}$*: Substituting to second row:*

$$\begin{aligned}-\rho\sigma_1\sigma_2\frac{\rho\sigma_1\sigma_2\gamma_{i,2}}{\sigma_1^2-\lambda_i}+(\sigma_2^2-\lambda_i)\gamma_{i,2} &= 0\\ -\rho\sigma_1\sigma_2\rho\sigma_1\sigma_2\gamma_{i,2}+(\sigma_1^2-\lambda_i)(\sigma_2^2-\lambda_i)\gamma_{i,2} &= 0\\ -\rho^2\sigma_1^2\sigma_2^2\gamma_{i,2}+(-\sigma_1^2\lambda_i-\sigma_2^2\lambda_i+\sigma_1^2\sigma_2^2+\lambda_i^2)\gamma_{i,2} &= 0\\ (\rho^2\sigma_1^2\sigma_2^2-\sigma_1^2\lambda_i-\sigma_2^2\lambda_i+\sigma_1^2\sigma_2^2+\lambda_i^2)\gamma_{i,2} &= 0\\ \left[\lambda_i^2-(\sigma_1^2+\sigma_2^2)\lambda_i-\rho^2\sigma_1^2\sigma_2^2+\sigma_1^2\sigma_2^2\right]\gamma_{i,2} &= 0\\ \left[\lambda_i^2-(\sigma_1^2+\sigma_2^2)\lambda_i+(1-\rho^2)\sigma_1^2\sigma_2^2\right])\gamma_{i,2} &= 0\end{aligned} \tag{5}$$

From Equation 1 we know that: $\lambda_i^2-(\sigma_1^2+\sigma_2^2)\lambda_i+(1-\rho^2)\sigma_1^2\sigma_2^2 = 0$ *therefore* $\gamma_{i,2} = c, c\in\mathbb{C}$*. Setting* $\gamma_{i,2} = \rho\sigma_1\sigma_2$ *one can compute* $\gamma_{i,1}$*.*

Using second row from Equation 4 we get : $\gamma_{i,1} = \lambda_i - \sigma_2^2$*.*

Finally eigenvectors will be:

$$\begin{bmatrix}\lambda_1-\sigma_2^2\\ \rho\sigma_1\sigma_2\end{bmatrix}, \begin{bmatrix}\lambda_2-\sigma_2^2\\ \rho\sigma_1\sigma_2\end{bmatrix} \tag{6}$$

2.