SOLUTIONS FOR PROBLEM SET 6
TIME SERIES II AND PORTFOLIO THEORY

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Abstract. This is document contains solutions to problem Set 6 for MIT online course "Mathematics for Applications in Finance" available at url.

1.

Suppose $\boldsymbol{X}_t = \begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix}$ follows $VAR(1)$ model where

$$X_{1,t} = 0.3 + 0.8 \cdot X_{1,t-1} + \epsilon_{1,t}$$
$$X_{2,t} = 0.2 + 0.6 \cdot X_{1,t-1} + 0.4 \cdot X_{2,t-1} + \epsilon_{2,t}$$

where $(\epsilon_{1,t}, \epsilon_{2,t})^T$ are $i.i.d.N(0_2, \Sigma)$, and

$$\Sigma = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}.$$

1.1. **a.** Compute $\mu = E[\boldsymbol{X}_t]$.

Solution 1.

$$\begin{aligned}
E(\mathbf{X}_t) &= \begin{bmatrix} E(X_{1,t}) \\ E(X_{2,t}) \end{bmatrix} \\
&= \begin{bmatrix} E(0.3 + 0.8 \cdot X_{1,t-1} + \epsilon_{1,t}) \\ E(0.2 + 0.6 \cdot X_{1,t-1} + 0.4 \cdot X_{2,t-1} + \epsilon_{2,t}) \end{bmatrix} \\
&= \begin{bmatrix} 0.3 & +0.8 \cdot E(X_{1,t-1}) & & +E(\epsilon_{1,t}) \\ 0.2 & +0.6 \cdot E(X_{1,t-1}) & +0.4 \cdot E(X_{2,t-1}) & +E(\epsilon_{2,t}) \end{bmatrix} \\
\begin{bmatrix} E(X_{1,t}) \\ E(X_{2,t}) \end{bmatrix} &= \begin{bmatrix} 0.3 & +0.8 \cdot E(X_{1,t}) & & +0 \\ 0.2 & +0.6 \cdot E(X_{1,t}) & +0.4 \cdot E(X_{2,t}) & +0 \end{bmatrix} \qquad (1) \\
&= \begin{bmatrix} \frac{0.3}{0.2} & & \\ 0.2 & +0.6 \cdot \frac{0.3}{0.2} & +0.4 \cdot E(X_{2,t}) \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{2} & \\ 1.1 & +0.4 \cdot E(X_{2,t}) \end{bmatrix} \\
&= \begin{bmatrix} {}^{3}/_{2} \\ {}^{11}/_{6} \end{bmatrix} \square
\end{aligned}$$

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1.2. **b.** Compute $\Gamma_0 = Cov[\boldsymbol{X}_t]$.

Solution 2.

$$
\begin{aligned}
Cov(\mathbf{X}_t) &= E\left\{[\mathbf{X}_t - E(\mathbf{X}_t)][\mathbf{X}_t - E(\mathbf{X}_t)]^T\right\} \\
&= \begin{bmatrix} Var(X_{1,t}) & Cov(X_{1,t}, X_{2,t}) \\ Cov(X_{2,t}, X_{1,t}) & Var(X_{2,t}) \end{bmatrix} \\
&= \begin{bmatrix} Var(X_{1,t}) & Cov(X_{1,t}, X_{2,t}) \\ Cov(X_{1,t}, X_{2,t}) & Var(X_{2,t}) \end{bmatrix}
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
Var(X_{1,t}) &= E\{[X_{1,t} - E(X_{1,t})][X_{1,t} - E(X_{1,t})]\} \\
&= E(X_{1,t}^2) - [E(X_{1,t})]^2 \\
&= E\left(0.09 + 0.48 \cdot X_{1,t-1} + 0.6 \cdot \epsilon_{1,t} + 0.64 \cdot X_{1,t-1}^2 + 1.6 \cdot X_{1,t-1}\epsilon_{1,t} + \epsilon_{1,t}^2\right) - [E(X_{1,t})]^2 \\
&= 0.09 + 0.48 \cdot E(X_{1,t-1}) + 0.6 \cdot E(\epsilon_{1,t}) + 0.64 \cdot E(X_{1,t-1}^2) \\
&\quad + 1.6 \cdot E(X_{1,t-1}) \cdot E(\epsilon_{1,t}) + E(\epsilon_{1,t}^2) - [E(X_{1,t})]^2 \\
&= 0.09 + 0.48 \cdot E(X_{1,t}) + 0.6 \cdot 0 + 0.64 \cdot \left\{Var(X_{1,t-1}) + [E(X_{1,t-1})]^2\right\} \\
&\quad + 1.6 \cdot E(X_{1,t-1}) \cdot 0 + E(\epsilon_{1,t}^2) - [E(X_{1,t})]^2 \\
&= 0.09 + 0.48 \cdot E(X_{1,t}) + 0.64 \cdot Var(X_{1,t}) + 0.64 \cdot [E(X_{1,t-1})]^2 \\
&\quad + E(\epsilon_{1,t}^2) - [E(X_{1,t})]^2 \\
&= 0.09 + 0.48 \cdot E(X_{1,t}) + 0.64 \cdot Var(X_{1,t}) \\
&\quad + E(\epsilon_{1,t}^2) - 0.36 \cdot [E(X_{1,t})]^2 \\
&= \frac{1}{0.36}\left\{0.09 + 0.48 \cdot E(X_{1,t}) + E(\epsilon_{1,t}^2) - 0.36 \cdot [E(X_{1,t})]^2\right\} \\
&= \frac{1}{0.36}\left[0.09 + 0.48 \cdot \frac{3}{2} + 3 - 0.36 \cdot \left(\frac{3}{2}\right)^2\right] \\
&= \frac{100}{12}
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
Var(X'_{1,t}) &= E\left\{[X'_{1,t}][X'_{1,t}]\right\} \\
&= E({X'}_{1,t}^2) \\
&= E\left(0.64 \cdot {X'}_{1,t-1}^2 + 1.6 \cdot X'_{1,t-1}\epsilon_{1,t} + \epsilon_{1,t}^2\right) \\
&= \frac{1}{0.36}E(\epsilon_{1,t}^2) \\
&= \frac{1}{0.36} \cdot 3 \\
&= \frac{100}{12}
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
Cov(X_{1,t}, X_{2,t}) &= E\left\{[X_{1,t} - E(X_{1,t})][X_{2,t} - E(X_{2,t})]\right\} \\
&= E(X_{1,t}X_{2,t}) - E(X_{1,t})E(X_{2,t}) \\
&= E(0.06 + 0.18 \cdot X_{1,t-1} + 0.12 \cdot X_{2,t-1} + 0.3 \cdot \epsilon_{2,t} \\
&\quad + 0.16 \cdot X_{1,t-1} + 0.48 \cdot X_{1,t-1}^2 + 0.32 \cdot X_{1,t-1}X_{2,t-1} + 0.8 \cdot X_{1,t-1}\epsilon_{1,t} \\
&\quad + 0.2 \cdot \epsilon_{1,t} + 0.6 \cdot X_{1,t-1} \cdot \epsilon_{1,t} + 0.4 \cdot X_{2,t-1}\epsilon_{1,t} + \epsilon_{1,t}\epsilon_{2,t}) - E(X_{1,t})E(X_{2,t}) \\
&= 0.06 + 0.34 \cdot E(X_{1,t-1}) + 0.12 \cdot E(X_{2,t-1}) \\
&\quad + 0.48 \cdot E(X_{1,t-1}^2) + 0.32 \cdot E(X_{1,t-1}X_{2,t-1}) \\
&\quad + E(\epsilon_{1,t}\epsilon_{2,t}) - E(X_{1,t})E(X_{2,t}) \\
&= 0.06 + 0.34 \cdot E(X_{1,t-1}) + 0.12 \cdot E(X_{2,t-1}) \\
&\quad + 0.48 \cdot Var(X_{1,t-1}) + 0.48 \cdot [E(X_{1,t-1})]^2 \\
&\quad + 0.32 \cdot Cov(X_{1,t-1}, X_{2,t-1}) + 0.32 \cdot E(X_{1,t})E(X_{2,t}) \\
&\quad + E(\epsilon_{1,t}\epsilon_{2,t}) - E(X_{1,t})E(X_{2,t}) \\
&= \frac{1}{0.68}\{0.06 + 0.34 \cdot E(X_{1,t-1}) + 0.12 \cdot E(X_{2,t-1}) \\
&\qquad + 0.48 \cdot Var(X_{1,t-1}) + 0.48 \cdot [E(X_{1,t-1})]^2 \\
&\qquad + E(\epsilon_{1,t}\epsilon_{2,t}) - 0.68 \cdot E(X_{1,t})E(X_{2,t})\} \\
&= \frac{1}{0.68}\left\{0.06 + 0.34 \cdot \frac{3}{2} + 0.12 \cdot \frac{11}{6} + 0.48 \cdot \frac{100}{12} + 0.48 \cdot \left(\frac{3}{2}\right)^2 - 1 - 0.68 \cdot \frac{3}{2}\frac{11}{6}\right\} \\
&= \frac{75}{17}
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
Cov(X'_{1,t}, X'_{2,t}) &= E\left\{[X_{1,t}][X_{2,t} - E(X_{2,t})]\right\} \\
&= E(X'_{1,t}X'_{2,t}) \\
&= E(0.48 \cdot {X'}^2_{1,t-1} + 0.32 \cdot X'_{1,t-1}X'_{2,t-1} + 0.8 \cdot X'_{1,t-1}\epsilon_{1,t} \\
&\quad + 0.6 \cdot X'_{1,t-1} \cdot \epsilon_{1,t} + 0.4 \cdot X'_{2,t-1}\epsilon_{1,t} + \epsilon_{1,t}\epsilon_{2,t}) \\
&= 0.48 \cdot E({X'}^2_{1,t-1}) + 0.32 \cdot E(X'_{1,t-1}X'_{2,t-1}) \\
&\quad + E(\epsilon_{1,t}\epsilon_{2,t}) \\
&= 0.48 \cdot Var(X'_{1,t-1}) + 0.32 \cdot Cov(X'_{1,t-1}, X'_{2,t-1}) \\
&\quad + E(\epsilon_{1,t}\epsilon_{2,t}) \\
&= \frac{1}{0.68}\{0.48 \cdot Var(X'_{1,t-1}) \\
&\quad + E(\epsilon_{1,t}\epsilon_{2,t})\} \\
&= \frac{1}{0.68}\left(0.48 \cdot \frac{100}{12} - 1\right) \\
&= \frac{75}{17}
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
Var(X_{2,t}) &= E\left\{[X_{2,t} - E(X_{2,t})][X_{2,t} - E(X_{2,t})]\right\} \\
&= E(X_{2,t}^2) - [E(X_{2,t})]^2 \\
&= E(0.04 + 0.24 \cdot X_{1,t-1} + 0.16 \cdot X_{2,t-1} + 0.4 \cdot \epsilon_{2,t} \\
&\qquad + 0.36 \cdot X_{1,t-1}^2 + 0.48 \cdot X_{1,t-1}X_{2,t-1} + 1.2 \cdot X_{1,t-1}\epsilon_{2,t} \\
&\qquad + 0.16 \cdot X_{2,t-1}^2 + 0.8 \cdot X_{2,t-1}\epsilon_{2,t} + \epsilon_{2,t}^2) - [E(X_{2,t})]^2 \\
&= 0.04 + 0.24 \cdot E(X_{1,t-1}) + 0.16 \cdot E(X_{2,t-1}) + 0.4 \cdot E(\epsilon_{2,t}) \\
&\qquad + 0.36 \cdot E(X_{1,t-1}^2) + 0.48 \cdot E(X_{1,t-1}X_{2,t-1}) + 1.2 \cdot E(X_{1,t-1}\epsilon_{2,t}) \\
&\qquad + 0.16 \cdot E(X_{2,t-1}^2) + 0.8 \cdot E(X_{2,t-1} \cdot \epsilon_{2,t}) + E(\epsilon_{2,t}^2) - [E(X_{2,t})]^2 \\
&= 0.04 + 0.24 \cdot E(X_{1,t-1}) + 0.16 \cdot E(X_{2,t-1}) + 0.4 \cdot 0 \\
&\qquad + 0.36 \cdot E(X_{1,t-1}^2) + 0.48 \cdot [Cov(X_{1,t-1}, X_{2,t-1}) + E(X_{1,t-1}) \cdot E(X_{2,t-1})] \\
&\qquad\qquad + 1.2 \cdot E(X_{1,t-1}) \cdot E(\epsilon_{2,t}) \\
&\qquad + 0.16 \cdot E(X_{2,t-1}^2) + 0.8 \cdot E(X_{2,t-1}) \cdot E(\epsilon_{2,t}) + E(\epsilon_{2,t}^2) - [E(X_{2,t})]^2 \\
&= 0.04 + 0.24 \cdot E(X_{1,t-1}) + 0.16 \cdot E(X_{2,t-1}) \\
&\qquad + 0.36 \cdot E(X_{1,t-1}^2) + 0.48 \cdot Cov(X_{1,t-1}, X_{2,t-1}) + 0.48 \cdot E(X_{1,t-1}) \cdot E(X_{2,t-1}) \\
&\qquad\qquad + 1.2 \cdot E(X_{1,t-1}) \cdot 0 \\
&\qquad + 0.16 \cdot E(X_{2,t-1}^2) + 0.8 \cdot E(X_{2,t-1}) \cdot 0 + E(\epsilon_{2,t}^2) - [E(X_{2,t})]^2 \\
&= 0.04 + 0.24 \cdot E(X_{1,t-1}) + 0.16 \cdot E(X_{2,t-1}) \\
&\qquad + 0.36 \cdot \left\{Var(X_{1,t-1}) + [E(X_{1,t})]^2\right\} + 0.48 \cdot Cov(X_{1,t-1}, X_{2,t-1}) \\
&\qquad\qquad + 0.48 \cdot E(X_{1,t-1}) \cdot E(X_{2,t-1}) \\
&\qquad + 0.16 \cdot \left\{Var(X_{2,t-1}) + [E(X_{2,t})]^2\right\} + E(\epsilon_{2,t}^2) - [E(X_{2,t})]^2 \\
&= 0.04 + 0.24 \cdot E(X_{1,t-1}) + 0.16 \cdot E(X_{2,t-1}) \\
&\qquad + 0.36 \cdot Var(X_{1,t-1}) + 0.36 \cdot [E(X_{1,t})]^2 + 0.48 \cdot Cov(X_{1,t-1}, X_{2,t-1}) \\
&\qquad\qquad + 0.48 \cdot E(X_{1,t-1}) \cdot E(X_{2,t-1}) \\
&\qquad + 0.16 \cdot Var(X_{2,t-1}) + E(\epsilon_{2,t}^2) - 0.84 \cdot [E(X_{2,t})]^2 \\
&= \frac{1}{0.84} \cdot \{0.04 + 0.24 \cdot E(X_{1,t-1}) + 0.16 \cdot E(X_{2,t-1}) \\
&\qquad\qquad + 0.36 \cdot Var(X_{1,t-1}) + 0.36 \cdot [E(X_{1,t})]^2 + 0.48 \cdot Cov(X_{1,t-1}, X_{2,t-1}) \\
&\qquad\qquad + 0.48 \cdot E(X_{1,t-1}) \cdot E(X_{2,t-1}) \\
&\qquad\qquad + E(\epsilon_{2,t}^2) - 0.84 \cdot [E(X_{2,t})]^2\} \\
&= \frac{1}{0.84} \cdot \Bigg\{0.04 + 0.24 \cdot \frac{3}{2} + 0.16 \cdot \frac{11}{6} + 0.36 \cdot \frac{10}{12} + 0.36 \cdot \left(\frac{3}{2}\right)^2 + 0.48 \cdot \frac{75}{17} \\
&\qquad\qquad +0.48 \cdot \frac{3}{2} \cdot \frac{11}{3} + 3 - 0.84 \cdot \left(\frac{11}{6}\right)^2\Bigg\} \\
&= \frac{1150}{119}
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
Var(X'_{2,t}) &= E\left\{[X_{2,t} - E(X_{2,t})][X_{2,t} - E(X_{2,t})]\right\} \\
&= E({X'}^2_{2,t}) \\
&= E(0.36 \cdot {X'}^2_{1,t-1} + 0.48 \cdot X'_{1,t-1}X'_{2,t-1} + 1.2 \cdot X'_{1,t-1}\epsilon_{2,t} \\
&\quad + 0.16 \cdot {X'}^2_{2,t-1} + 0.8 \cdot X'_{2,t-1}\epsilon_{2,t} + \epsilon^2_{2,t}) \\
&= 0.36 \cdot E({X'}^2_{1,t-1}) + 0.48 \cdot Cov(X'_{1,t-1}, X'_{2,t-1}) + 1.2 \cdot E(X'_{1,t-1}\epsilon_{2,t}) \\
&\quad + 0.16 \cdot Var(X'_{2,t-1}) + 0.8 \cdot E(X'_{2,t-1} \cdot \epsilon_{2,t}) + E(\epsilon^2_{2,t}) \\
&= 0.36 \cdot E({X'}^2_{1,t-1}) + 0.48 \cdot Cov(X'_{1,t}, X'_{2,t}) \\
&\quad + 0.16 \cdot Var(X'_{2,t}) + E(\epsilon^2_{2,t}) \\
&= \frac{1}{0.84} \cdot [0.36 \cdot Var(X'_{1,t-1}) + 0.48 \cdot Cov(X'_{1,t-1}, X'_{2,t-1}) + E(\epsilon^2_{2,t})] \\
&= \frac{1}{0.84} \cdot \left[0.36 \cdot \frac{100}{12} + 0.48 \cdot \frac{75}{17} + 3\right] \\
&= \frac{1150}{119}
\end{aligned} \tag{8}
$$

1.3. **c.** Compute $\Gamma_1 = Cov[\boldsymbol{X}_t, \boldsymbol{X}_{t-1}]$.

Solution 3.

$$
\begin{aligned}
Cov(\mathbf{X}_t, \mathbf{X}_{t-1}) &= E\left\{[\mathbf{X}_t - E(\mathbf{X}_t)][\mathbf{X}_{t-1} - E(\mathbf{X}_{t-1})]^T\right\} \\
&= \begin{bmatrix} Cov(X_{1,t}, X_{1,t-1}) & Cov(X_{1,t}, X_{2,t-1}) \\ Cov(X_{2,t}, X_{1,t-1}) & Cov(X_{2,t}, X_{2,t-1}) \end{bmatrix}
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
Cov(X_{1,t}, X_{1,t-1}) &= E\left\{[X_{1,t} - E(X_{1,t})][X_{1,t-1} - E(X_{1,t-1})]\right\} \\
&= E(X_{1,t}X_{1,t-1}) - E(X_{1,t})E(X_{1,t-1}) \\
&= E(X_{1,t}X_{1,t-1}) - [E(X_{1,t})]^2 \\
&= E\left(0.3 \cdot X_{1,t-1} + 0.8 \cdot X^2_{1,t-1} + X_{1,t-1}\epsilon_{1,t}\right) - [E(X_{1,t})]^2 \\
&= 0.3 \cdot E(X_{1,t-1}) + 0.8 \cdot E(X^2_{1,t-1}) + E(X_{1,t-1}\epsilon_{1,t}) - [E(X_{1,t})]^2 \\
&= 0.3 \cdot E(X_{1,t-1}) + 0.8 \cdot Var(X_{1,t-1}) + 0.8 \cdot [E(X_{1,t-1})]^2 + E(X_{1,t-1}) \cdot E(\epsilon_{1,t}) - [E(X_{1,t})]^2 \\
&= 0.3 \cdot E(X_{1,t}) + 0.8 \cdot Var(X_{1,t}) + 0.8 \cdot [E(X_{1,t})]^2 - [E(X_{1,t})]^2 \\
&= 0.3 \cdot E(X_{1,t}) + 0.8 \cdot Var(X_{1,t}) - 0.2 \cdot [E(X_{1,t})]^2 \\
&= 0.3 \cdot \frac{3}{2} + 0.8 \cdot \frac{10}{12} - 0.2 \cdot \left(\frac{3}{2}\right)^2 \\
&= \frac{20}{3}
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
Cov(X'_{1,t}, X'_{1,t-1}) &= E\left\{[X_{1,t} - E(X_{1,t})][X_{1,t-1} - E(X_{1,t-1})]\right\} \\
&= E(X'_{1,t}X'_{1,t-1}) \\
&= E\left(0.8 \cdot {X'}^2_{1,t-1} + X'_{1,t-1}\epsilon_{1,t}\right) \\
&= 0.8 \cdot E({X'}^2_{1,t-1}) \\
&= 0.8 \cdot \frac{100}{12} \\
&= \frac{20}{3}
\end{aligned}
\tag{11}
$$

(12)

$$
\begin{aligned}
Cov(X_{1,t}, X_{2,t-1}) &= E\left\{[X_{1,t} - E(X_{1,t})][X_{2,t-1} - E(X_{2,t-1})]\right\} \\
&= E(X_{1,t}X_{2,t-1}) \\
&= E[(0.54 + 0.64 \cdot X_{1,t-2} + 0.8 \cdot \epsilon_{1,t-1} + \epsilon_{1,t})(0.2 + 0.6 \cdot X_{1,t-2} + 0.4 \cdot X_{2,t-2} + \epsilon_{2,t-1})] \\
&\qquad - E(X_{1,t})E(X_{2,t-1}) \\
&= E(0.102 + 0.324 \cdot X_{1,t-2} + 0.216 \cdot X_{2,t-2} + 0.54 \cdot \epsilon_{2,t-1} \\
&\qquad + 0.128 \cdot X_{1,t-2} + 0.384 \cdot X_{1,t-2}^2 + 0.256 \cdot X_{1,t-2}X_{2,t-2} + 0.64 \cdot X_{1,t-2}\epsilon_{2,t-1} \\
&\qquad + 0.16 \cdot \epsilon_{1,t-1} + 0.48 \cdot X_{1,t-2}\epsilon_{1,t-1} + 0.32 \cdot X_{2,t-2}\epsilon_{1,t-1} + 0.8 \cdot \epsilon_{1,t-1}\epsilon_{2,t-1} \\
&\qquad + 0.2 \cdot \epsilon_{1,t} + 0.6 \cdot X_{1,t-2}\epsilon_{1,t} + 0.4 \cdot X_{2,t-2}\epsilon_{1,t} + \epsilon_{1,t}\epsilon_{2,t-1} \\
&\qquad - E(X_{1,t})E(X_{2,t-1}) \\
&= 0.102 + 0.324 \cdot E(X_{1,t-2}) + 0.216 \cdot E(X_{2,t-2}) + 0.54 \cdot E(\epsilon_{2,t-1}) \\
&\qquad + 0.128 \cdot E(X_{1,t-2}) + 0.384 \cdot E(X_{1,t-2}^2) + 0.256 \cdot E(X_{1,t-2}X_{2,t-2}) \\
&\qquad + 0.64 \cdot E(X_{1,t-2})E(\epsilon_{2,t-1}) \\
&\qquad + 0.16 \cdot E(\epsilon_{1,t-1}) + 0.48 \cdot E(X_{1,t-2})(\epsilon_{1,t-1}) \\
&\qquad + 0.32 \cdot E(X_{2,t-2})E(\epsilon_{1,t-1}) + 0.8 \cdot (\epsilon_{1,t-1}\epsilon_{2,t-1}) \\
&\qquad + 0.2 \cdot E(\epsilon_{1,t}) + 0.6 \cdot E(X_{1,t-2})E(\epsilon_{1,t}) \\
&\qquad + 0.4 \cdot E(X_{2,t-2})E(\epsilon_{1,t}) + E(\epsilon_{1,t})E(\epsilon_{2,t-1}) \\
&\qquad - E(X_{1,t})E(X_{2,t}) \\
&= 0.102 + 0.324 \cdot E(X_{1,t}) + 0.216 \cdot E(X_{2,t}) \\
&\qquad + 0.128 \cdot E(X_{1,t}) + 0.384 \cdot E(X_{1,t}^2) + 0.256 \cdot E(X_{1,t}X_{2,t}) \\
&\qquad + 0.8 \cdot (\epsilon_{1,t}\epsilon_{2,t}) \\
&\qquad - E(X_{1,t})E(X_{2,t-1}) \\
&= 0.102 + 0.452 \cdot E(X_{1,t}) + 0.216 \cdot E(X_{2,t}) \\
&\qquad + 0.384 \cdot Var(X_{1,t}) + 0.384 \cdot [E(X_{1,t})]^2 \\
&\qquad + 0.256 \cdot Cov(X_{1,t}, X_{2,t}) + 0.256 \cdot E(X_{1,t})E(X_{2,t}) \\
&\qquad + 0.8 \cdot (\epsilon_{1,t}\epsilon_{2,t}) \\
&\qquad - E(X_{1,t})E(X_{2,t}) \\
&= 0.102 + 0.452 \cdot E(X_{1,t}) + 0.216 \cdot E(X_{2,t}) \\
&\qquad + 0.384 \cdot Var(X_{1,t}) + 0.384 \cdot [E(X_{1,t})]^2 \\
&\qquad + 0.256 \cdot Cov(X_{1,t}, X_{2,t}) - 0.744 \cdot E(X_{1,t})E(X_{2,t}) \\
&\qquad + 0.8 \cdot (\epsilon_{1,t}\epsilon_{2,t}) \\
&= 0.102 + 0.452 \cdot \frac{3}{2} + 0.216 \cdot \frac{11}{6} + 0.384 \cdot \frac{100}{12} + 0.384 \cdot \left(\frac{3}{2}\right)^2 \\
&\qquad + 0.256 \cdot \frac{75}{17} - 0.744 \cdot \frac{3}{2}\frac{11}{6} + 0.8 \cdot -1 \\
&= \frac{2107}{598}
\end{aligned}
$$

$$
\begin{aligned}
Cov(X'_{1,t}, X'_{2,t-1}) &= E\left\{[X_{1,t} - E(X_{1,t})][X_{2,t-1} - E(X_{2,t-1})]\right\} \\
&= E(X_{1,t}X_{2,t-1}) - E(X_{1,t})E(X_{2,t-1}) \\
&= E[(0.64 \cdot X'_{1,t-2} + 0.8 \cdot \epsilon_{1,t-1} + \epsilon_{1,t})(0.6 \cdot X'_{1,t-2} + 0.4 \cdot X'_{2,t-2} + \epsilon_{2,t-1})] \\
&= E(0.384 \cdot {X'}^2_{1,t-2} + 0.256 \cdot X'_{1,t-2}X'_{2,t-2} + 0.64 \cdot X_{1,t-2}\epsilon_{2,t-1} \\
&\qquad + 0.48 \cdot X'_{1,t-2}\epsilon_{1,t-1} + 0.32 \cdot X'_{2,t-2}\epsilon_{1,t-1} + 0.8 \cdot \epsilon_{1,t-1}\epsilon_{2,t-1} \\
&\qquad + 0.6 \cdot X'_{1,t-2}\epsilon_{1,t} + 0.4 \cdot X'_{2,t-2}\epsilon_{1,t} + \epsilon_{1,t}\epsilon_{2,t-1}) \\
&= 0.384 \cdot E({X'}^2_{1,t-2}) + 0.256 \cdot E(X'_{1,t-2}X'_{2,t-2}) + 0.64 \cdot E(X'_{1,t-2})E(\epsilon_{2,t-1}) \\
&\qquad + 0.48 \cdot E(X'_{1,t-2})(\epsilon_{1,t-1}) + 0.32 \cdot E(X'_{2,t-2})E(\epsilon_{1,t-1}) + 0.8 \cdot (\epsilon_{1,t-1}\epsilon_{2,t-1}) \\
&\qquad + 0.6 \cdot E(X'_{1,t-2})E(\epsilon_{1,t}) + 0.4 \cdot E(X'_{2,t-2})E(\epsilon_{1,t}) + E(\epsilon_{1,t})E(\epsilon_{2,t-1}) \\
&= 0.384 \cdot E{X'}^2_{1,t}) + 0.256 \cdot E(X'_{1,t}X'_{2,t}) \\
&\qquad + 0.8 \cdot (\epsilon_{1,t}\epsilon_{2,t}) \\
&= 0.384 \cdot Var(X'_{1,t}) + 0.256 \cdot Cov(X'_{1,t}, X_{2,t}) + 0.8 \cdot (\epsilon_{1,t}\epsilon_{2,t}) \\
&= 0.384 \cdot \frac{100}{12} + 0.256 \cdot \frac{75}{17} + 0.8 \cdot -1 \\
&= \frac{2107}{598}
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
Cov(X_{2,t}, X_{2,t-1}) &= E\left\{[X_{2,t} - E(X_{2,t})][X_{2,t-1} - E(X_{2,t-1})]\right\} \\
&= E(X_{2,t}X_{2,t-1}) - E(X_{2,t})E(X_{2,t-1}) \\
&= E(X_{2,t}X_{2,t-1}) - [E(X_{2,t})]^2 \\
&= E\left(0.2 \cdot X_{2,t-1} + 0.6 \cdot X_{1,t-1}X_{2,t-1} + 0.4 \cdot X^2_{2,t-1} + X_{2,t-1}\epsilon_{2,t}\right) - [E(X_{2,t})]^2 \\
&= 0.2 \cdot E(X_{2,t-1}) + 0.6 \cdot E(X_{1,t-1}X_{2,t-1}) \\
&\qquad + 0.4 \cdot E(X^2_{2,t-1}) + E(X_{2,t-1})E(\epsilon_{2,t}) - [E(X_{2,t})]^2 \\
&= 0.2 \cdot E(X_{2,t}) + 0.6 \cdot Cov(X_{1,t-1}, X_{2,t-1}) + 0.6 \cdot E(X_{1,t-1})E(X_{2,t-1}) \\
&\qquad + 0.4 \cdot Var(X_{2,t-1}) + 0.4 \cdot [E(X_{2,t-1})]^2 - [E(X_{2,t})]^2 \\
&= 0.2 \cdot E(X_{2,t}) + 0.6 \cdot Cov(X_{1,t}, X_{2,t}) + 0.6 \cdot E(X_{1,t})E(X_{2,t}) \\
&\qquad + 0.4 \cdot Var(X_{2,t}) - 0.6 \cdot [E(X_{2,t})]^2 \\
&= 0.2 \cdot \frac{11}{6} + 0.6 \cdot \frac{100}{12} + 0.6 \cdot \frac{3}{2}\frac{11}{6} + 0.4 \cdot \frac{1150}{119} - 0.6 \cdot \left(\frac{11}{6}\right)^2 \\
&= \frac{1055}{119}
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
Cov(X'_{2,t}, X'_{2,t-1}) &= E\left\{[X_{2,t} - E(X_{2,t})][X_{2,t-1} - E(X_{2,t-1})]\right\} \\
&= E(X_{2,t}X_{2,t-1}) - E(X_{2,t})E(X_{2,t-1}) \\
&= E(X'_{2,t}X'_{2,t-1}) \\
&= E\left(0.6 \cdot X'_{1,t-1}X'_{2,t-1} + 0.4 \cdot {X'}^2_{2,t-1} + X'_{2,t-1}\epsilon_{2,t}\right) \\
&= 0.6 \cdot E(X_{1,t-1}X_{2,t-1}) + 0.4 \cdot E({X'}^2_{2,t-1}) + E(X'_{2,t-1})E(\epsilon_{2,t}) \\
&= 0.6 \cdot Cov(X'_{1,t-1}, X'_{2,t-1}) + 0.4 \cdot Var(X'_{2,t-1}) \\
&= 0.6 \cdot Cov(X'_{1,t}, X'_{2,t}) + 0.4 \cdot Var(X'_{2,t}) \\
&= 0.6 \cdot \frac{100}{12} + 0.4 \cdot \frac{1150}{119} \\
&= \frac{1055}{119}
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
Cov(X_{2,t}, X_{1,t-1}) &= E\left\{[X_{2,t} - E(X_{2,t})][X_{1,t-1} - E(X_{1,t-1})]\right\} \\
&= E(X_{2,t}X_{1,t-1}) - E(X_{2,t})E(X_{1,t-1}) \\
&= E(0.2 \cdot X_{1,t-1} + 0.6 \cdot X^2_{1,t-1} + 0.4 \cdot X_{1,t-1}X_{2,t-1} + X_{1,t-1}\epsilon_{2,t-1}) \\
&\quad - E(X_{2,t})E(X_{1,t-1}) \\
&= 0.2 \cdot E(X_{1,t-1}) + 0.6 \cdot E(X^2_{1,t-1}) \\
&\quad + 0.4 \cdot E(X_{1,t-1}X_{2,t-1}) + E(X_{1,t-1}\epsilon_{2,t-1}) - E(X_{2,t})E(X_{1,t}) \\
&= 0.2 \cdot E(X_{1,t}) + 0.6 \cdot Var(X_{1,t}) + 0.6 \cdot [E(X_{1,t})]^2 \\
&\quad + 0.4 \cdot Cov(X_{1,t}, X_{2,t}) + 0.4 \cdot E(X_{1,t})E(X_{2,t}) - E(X_{2,t})E(X_{1,t}) \\
&= 0.2 \cdot E(X_{1,t}) + 0.6 \cdot Var(X_{1,t}) + 0.6 \cdot [E(X_{1,t})]^2 \\
&\quad + 0.4 \cdot Cov(X_{1,t}, X_{2,t}) - 0.6 \cdot E(X_{1,t})E(X_{2,t}) \\
&= 0.2 \cdot \frac{3}{2} + 0.6 \cdot \frac{2}{5} + 0.6 \cdot \left[\frac{3}{2}\right]^2 \\
&\quad + 0.4 \cdot \frac{35}{17} - 0.6 \cdot \frac{3}{2}\frac{11}{6}
\end{aligned} \tag{16}
$$

1.4. **d.** Derive formula for computing $\Gamma_h = Cov[\boldsymbol{X}_t, \boldsymbol{X}_{t-h}]$.

Solution 4.

$$
\begin{aligned}
Cov(\mathbf{X}_t, \mathbf{X}_{t-h}) &= E\left\{[\mathbf{X}_t - E(\mathbf{X}_t)][\mathbf{X}_{t-1} - E(\mathbf{X}_{t-h})]^T\right\} \\
&= \begin{bmatrix} Cov(X_{1,t}, X_{1,t-h}) & Cov(X_{1,t}, X_{2,t-h}) \\ Cov(X_{2,t}, X_{1,t-h}) & Cov(X_{2,t}, X_{2,t-h}) \end{bmatrix} \\
&= \begin{bmatrix} \Gamma_{1,1}(h) & \Gamma_{1,2}(h) \\ \Gamma_{2.1}(h) & \Gamma_{2,2}(h) \end{bmatrix}
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
Cov(X_{1,t}, X_{1,t-h}) &= E\left\{[X_{1,t} - E(X_{1,t})][X_{1,t-h} - E(X_{1,t-h})]\right\} \\
&= E(X_{1,t}X_{1,t-h}) - E(X_{1,t})E(X_{1,t-h}) \\
&= E(X'_{1,t}X'_{1,t-h}) \\
&= E\left(0.8 \cdot X'_{1,t-1}X'_{1,t-h} + X'_{1,t-h}\epsilon_{1,t}\right) \\
&= 0.8 \cdot E(X'_{1,t-1}X'_{1,t-h}) \\
&= 0.8 \cdot Cov(X'_{1,t-1}, X'_{1,t-h}) \\
&= 0.8 \cdot Cov(X'_{1,t}, X'_{1,t-(h-1)}) \\
&= 0.8 \cdot Cov(X_{1,t}, X_{1,t-(h-1)}) \\
&= 0.8 \cdot \Gamma_{1,1}(h-1)) \\
&= 0.8^{h-1} \cdot \Gamma_{1,1}(0))
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
Cov(X_{1,t}, X_{2,t-h}) &= E\left\{[X_{1,t} - E(X_{1,t})][X_{2,t-h} - E(X_{2,t-h})]\right\} \\
&= E(X_{1,t}X_{2,t-h}) - E(X_{1,t})E(X_{2,t-h}) \\
&= E(X'_{1,t}X'_{2,t-h}) \\
&= E\left(0.8 \cdot X'_{1,t-1}X'_{2,t-h} + X'_{2,t-h}\epsilon_{1,t}\right) \\
&= 0.8 \cdot E(X'_{1,t-1}X'_{2,t-h}) \\
&= 0.8 \cdot E[X'_{1,t-1}(0.6 \cdot X'_{1,t-(h+1)} + 0.4 \cdot X'_{2,t-(h+1)})] \\
&= 0.48 \cdot E(X'_{1,t-1}X'_{1,t-(h+1)}) + 0.32 \cdot E(X'_{1,t-1}X'_{2,t-(h+1)}) \\
&= 0.48 \cdot Cov(X'_{1,t}, X'_{1,t-h}) + 0.32 \cdot Cov(X'_{1,t}, X'_{2,t-h}) \\
&= 0.48 \cdot \Gamma_{1,1}(h) + 0.32 \cdot Cov(X'_{1,t}, X'_{2,t-h}) \\
&= \frac{0.48}{0.68} \cdot \Gamma_{1,1}(h) \\
&= \frac{0.48 \cdot 0.8}{0.68} \cdot \Gamma_{1,1}(h-1)) \\
&= \frac{0.48}{0.68} \cdot 0.8^{h-1} \cdot \Gamma_{1,1}(0))
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
Cov(X_{2,t}, X_{1,t-h}) &= E\left\{[X_{2,t} - E(X_{2,t})][X_{1,t-h} - E(X_{1,t-h})]\right\} \\
&= E(X_{2,t}X_{1,t-h}) - E(X_{2,t})E(X_{1,t-h}) \\
&= E(X'_{2,t}X'_{1,t-h}) \\
&= E\left(0.6 \cdot X'_{1,t-1}X'_{1,t-h} + 0.4 \cdot X'_{2,t-1}X'_{1,t-h} + X'_{1,t-h}\epsilon_{2,t}\right) \\
&= 0.8 \cdot E(X'_{1,t-1}X'_{1,t-h}) + 0.4 \cdot E(X'_{2,t-1}X'_{1,t-h}) \\
&= 0.8 \cdot Cov(X'_{1,t}X'_{1,t-(h-1)}) + 0.4 \cdot Cov(X'_{2,t}X'_{1,t-(h-1)}) \\
&= 0.8 \cdot \Gamma_{1,1}(h-1) + 0.4 \cdot \Gamma_{2,1}(h-1) \\
&= 0.8^{h-1} \cdot \Gamma_{1,1}(0) + 0.4 \cdot \Gamma_{2,1}(h-1)
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
Cov(X_{2,t}, X_{2,t-h}) &= E\left\{[X_{2,t} - E(X_{2,t})][X_{2,t-h} - E(X_{2,t-h})]\right\} \\
&= E(X_{2,t}X_{2,t-h}) - E(X_{2,t})E(X_{2,t-h}) \\
&= E(X'_{2,t}X'_{2,t-h}) \\
&= E\left(0.6 \cdot X'_{1,t-1}X'_{2,t-h} + 0.4 \cdot X'_{2,t-1}X'_{2,t-h} + X'_{2,t-h}\epsilon_{2,t}\right) \\
&= 0.8 \cdot E(X'_{1,t-1}X'_{2,t-h}) + 0.4 \cdot E(X'_{2,t-1}X'_{2,t-h}) \\
&= 0.8 \cdot Cov(X'_{1,t}X'_{2,t-(h-1)}) + 0.4 \cdot Cov(X'_{2,t}X'_{2,t-(h-1)}) \\
&= 0.8 \cdot \Gamma_{1,2}(h-1) + 0.4 \cdot \Gamma_{2,2}(h-1) \\
&= 0.8 \cdot \frac{0.48 \cdot 0.8}{0.68} \cdot \Gamma_{1,1}(h-2) + 0.4 \cdot \Gamma_{2,2}(h-1) \\
&= \frac{0.48 \cdot 0.8}{0.68} \cdot \Gamma_{1,1}(h-1) + 0.4 \cdot \Gamma_{2,2}(h-1)
\end{aligned} \tag{21}
$$

We can rewrite formula for $\Gamma_h = Cov[\mathbf{X}_t, \mathbf{X}_{t-h}]$ *:*

$$
vec(\Gamma_h) = \begin{bmatrix} \Gamma_{1,1}(h) \\ \Gamma_{1,2}(h) \\ \Gamma_{2,1}(h) \\ \Gamma_{2,2}(h) \end{bmatrix} = A \cdot \begin{bmatrix} \Gamma_{1,1}(h-1) \\ \Gamma_{1,2}(h-1) \\ \Gamma_{2,1}(h-1) \\ \Gamma_{2,2}(h-1) \end{bmatrix} \tag{22}
$$

where:

$$
A = \begin{bmatrix} 0.8 & 0 & 0 & 0 \\ 0.8 \cdot \frac{0.48}{0.68} & 0 & 0 & 0 \\ 0.8 & 0 & 0.4 & 0 \\ 0.8 \cdot \frac{0.48}{0.68} & 0 & 0 & 0.4 \end{bmatrix} \tag{23}
$$

To find generic formula we have to do eigenvalues decomposition of matrix A:

$$
A = VDV'. \tag{24}
$$

Because A is triangular matrix D will be just diagonal of A:

$$
D = diag(A) = \begin{bmatrix} 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0.4 \end{bmatrix} \tag{25}
$$

The generic formula would then become:

$$
vec(\Gamma_h) = VD^hV'vec(\Gamma_{h-1}). \tag{26}
$$

2.

For $\{\epsilon_t\}$ $i.i.d.$ $WN(0, \sigma^2)$, define process $\{w_t\}$ and $\{v_t\}$ as follows

$$
\begin{aligned}
w_t &= 5(1 - 0.5L)^{-1}\epsilon_t \\
v_t &= 4(1 - 0.4L)^{-1}\epsilon_t
\end{aligned}
$$

Define $\{x_t\} : x_t = w_t - v_t$.

2.1. **a.** Solve for coefficients θ_i in the infinite order moving average process for $\{x_t\}$:

$$x_t = \epsilon_t + \sum_{i=1}^{\infty} \theta_i \epsilon_{t-i}$$

Solution 5. *From formula for geometric series sum:*

$$\sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}$$

We can find coefficients for processes w_t and v_t:

$$\begin{aligned} w_t &= 5(1-0.5L)^{-1}\epsilon_t = 5\sum_{i=0}^{\infty}(0.5L)^i \epsilon_t \\ v_t &= 4(1-0.4L)^{-1}\epsilon_t = 4\sum_{i=0}^{\infty}(0.4L)^i \epsilon_t \end{aligned} \tag{27}$$

The coefficients θ_i can be derived from sum of the two infinite series:

$$\begin{aligned} x_t &= 5\epsilon_t + 5\sum_{i=1}^{\infty}(0.5L)^i \epsilon_t - 4\epsilon_t - 4\sum_{i=1}^{\infty}(0.4L)^i \epsilon_t \\ &= \epsilon_t + \sum_{i=1}^{\infty}\left[5(0.5)^i - 4(0.4)^i\right] L^i \epsilon_t \end{aligned} \tag{28}$$

Therefore $\theta_i = 5(0.5)^i - 4(0.4)^i$. $\square$

2.2. **b.** Prove that x_t is an $AR(2)$ process.

Solution 6.

$$\begin{aligned} x_t = w_t - v_t &= \frac{5}{1-0.5L}\epsilon_t - \frac{4}{1-0.4L}\epsilon_t \\ &= \frac{5(1-0.4L) - 4(1-0.5L)}{(1-0.5L)(1-0.4L)}\epsilon_t \\ &= \frac{1}{(1-0.5L)(1-0.4L)}\epsilon_t \end{aligned} \tag{29}$$

$$\begin{aligned} (1-0.5L)(1-0.4L)x_t &= \epsilon_t \\ (1-0.9L+0.2L^2)x_t &= \epsilon_t \\ x_t &= 0.9x_{t-1} - 0.2x_{t-2} + \epsilon_t \quad \square. \end{aligned}$$

2.3. **c.** Solve for ϕ_1 and ϕ_2 in the representation

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \epsilon_t$$

Solution 7. *From previous subsection $\phi_1 = 0.9$ and $\phi_2 = -0.2$.* $\square$

2.4. **d.** Prove that any stationary $AR(2)$ process can be expressed as the difference of two (possibly infinite order) moving average processes on the same innovation process $\{\epsilon_t\}$.

Solution 8. *Any $AR(2)$ process $\{x_t\}$ can be represented as:*

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \epsilon_t \tag{30}$$

The process can be rewritten using lag operator L:

$$\begin{aligned} x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2} &= \epsilon_t \\ (1 - \phi_1 L - \phi_2 L^2) x_t &= \epsilon_t \\ (1 - \lambda_1 L)(1 - \lambda_2 L) x_t &= \epsilon_t \end{aligned} \tag{31}$$

where:

$$\lambda_i = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2\phi_2}$$
$$\lambda_2 = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2\phi_2}$$

are roots of characteristic equation $1 - \phi_1 z - \phi_2 z^2 = 0$.

Process $\{x_t\}$ will be invertible if $|\lambda_1| < 1$ and $|\lambda_2| < 1$ and can be written as:

$$\begin{aligned} x_t &= (1 - \lambda_1 L)^{-1} (1 - \lambda_2 L)^{-1} \epsilon_t \\ &= \left[\frac{A}{1 - \lambda_1 L} + \frac{B}{1 - \lambda_2 L} \right] \epsilon_t \\ &= \left[\frac{\lambda_1}{\lambda_1 - \lambda_2} \frac{1}{1 - \lambda_1 L} - \frac{\lambda_2}{\lambda_1 - \lambda_2} \frac{1}{1 - \lambda_2 L} \right] \epsilon_t \\ &= \frac{\lambda_1}{\lambda_1 - \lambda_2} \frac{1}{1 - \lambda_1 L} \epsilon_t - \frac{\lambda_2}{\lambda_1 - \lambda_2} \frac{1}{1 - \lambda_2 L} \epsilon_t \\ &= \frac{\lambda_1}{\lambda_1 - \lambda_2} \sum_{i=0}^{\infty} (\lambda_1 L)^i \epsilon_t - \frac{\lambda_2}{\lambda_1 - \lambda_2} \sum_{i=0}^{\infty} (\lambda_2 L)^i \epsilon_t \\ &= \frac{\lambda_1}{\lambda_1 - \lambda_2} \sum_{i=0}^{\infty} \lambda_1^i \epsilon_{t-i} - \frac{\lambda_2}{\lambda_1 - \lambda_2} \sum_{i=0}^{\infty} \lambda_2^i \epsilon_{t-i} \end{aligned} \tag{32}$$

where A and B were found by solving set of equations:

$$\begin{aligned} A + B &= 1 \\ A\lambda_2 + B\lambda_1 &= 0 \end{aligned} \tag{32}$$

From above one can see that the $AR(2)$ process can be expressed as a difference of two infinite moving average processes. □