SOLUTIONS FOR PROBLEM SET 5

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Abstract. This is document contains solutions to problem set 5 for MIT online course "Mathematics for Applications in Finance" available at url.

1. Sample Estimators of Diffusion Process Volatility and Drift

1.1. **b.** Derive the distribution of $\hat{\mu}_n$; give specific formulas for the expectation and variance of $\hat{\mu}_n$.

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} Y_i \tag{1}$$

where: $Y_i \sim \boldsymbol{N}\left(\mu, \sigma^2\right)$.

$$Y_i \sim \boldsymbol{N}(\mu, \sigma^2) \Rightarrow \frac{Y_i}{n} \sim \boldsymbol{N}(\frac{\mu}{n}, \frac{\sigma^2}{n^2}) \tag{2}$$

$$\boldsymbol{M}_{\frac{Y_i}{n}}(t) = \exp\left(\frac{\mu t}{n} + \frac{\sigma^2 t^2}{2n^2}\right) \tag{3}$$

where: $\boldsymbol{M}_X(t)$ is moment generating function of r.v X.

Solution 1. *From properties of moment generating function we know that MGF of sum of r.v. is equal to product of individual MGF's:*

$$\mathbf{M}_{\sum_{i=1}^{n} X_i}(t) = \prod_{i=1}^{n} \mathbf{M}_{X_i}(t) \tag{4}$$

Therefore:

$$\mathbf{M}_{\hat{\mu}}(t) = \prod_{i=1}^{n} \mathbf{M}_{\frac{Y_i}{n}}(t) = \exp\left(\frac{n\mu t}{n} + \frac{n\sigma^2 t^2}{2n^2}\right) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2n}\right) \tag{5}$$

From this follows that $\hat{\mu} \sim \mathbf{N}\left(\mu, \frac{\sigma^2}{n}\right)$ *and:*

$$\begin{aligned} \mathbb{E}\left(\hat{\mu}\right) &= \mu \\ \mathbb{V}\left(\hat{\mu}\right) &= \frac{\sigma^2}{n}\square \end{aligned} \tag{6}$$

Date: January 28, 2016.

1.2. **c.** Derive the distribution of $\hat{\sigma}^2$.

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (Y_i - \hat{\mu})^2 \tag{7}$$

Solution 2. *Let's derive distribution of* $\frac{(Y_i-\hat{\mu})^2}{n}$ *starting by finding distribution of* $\frac{(Y_i-\hat{\mu})}{\sqrt{n}}$*:*

From (4) *we know that:*

$$\frac{(Y_i - \hat{\mu})}{\sqrt{n}} \sim \mathbf{N}() \tag{8}$$

However individual sums $(Y_i - \hat{\mu})$ *are not independent r.v.'s therefore we need to split them to sum of independent ones to be able to to find distribution of sum of residuals* $(Y_i - \hat{\mu})^2$*.*

$$\begin{aligned} (Y_i - \hat{\mu}) &= ((Y_i - \mu) - (\hat{\mu} - \mu)) \\ &= (Y_i - \mu) - \sum_{i=1}^{n} \frac{(Y_i - \mu)}{n} = \frac{n-1}{n}(Y_i - \mu) - \sum_{i \neq j}^{n} \frac{(Y_j - \mu)}{n} \end{aligned} \tag{9}$$

$$\begin{aligned} (Y_i - \hat{\mu})^2 &= \left(\frac{n-1}{n}(Y_i - \mu) - \sum_{i \neq j}^{n} \frac{(Y_j - \mu)}{n} \right)^2 \\ &= \left(\frac{(n-1)^2}{n^2}(Y_i - \mu)^2 + \sum_{j \neq i}^{n} \frac{(Y_j - \mu)^2}{n^2} \right. \\ &\qquad \left. - 2 \sum_{j \neq i}^{n} \frac{(n-1)(Y_i - \mu)(Y_j - \mu)}{n^2} + 2 \sum_{k \neq j \neq i}^{n} \frac{(Y_k - \mu)(Y_j - \mu)}{n^2} \right) \end{aligned} \tag{10}$$

$$\begin{aligned} \sum_{i=1}^{n} (Y_i - \hat{\mu})^2 &= \sum_{i=1}^{n} \frac{(n-1)^2}{n^2}(Y_i - \mu)^2 + (n-1) \sum_{j \neq i}^{n} \frac{(Y_i - \mu)^2}{n^2} \\ &\qquad - 2 \sum_{i}^{n} \sum_{j \neq i}^{n} \frac{(n-1)(Y_i - \mu)(Y_j - \mu)}{n^2} + (n-2) \sum_{i}^{n} \sum_{j \neq i}^{n} \frac{(Y_i - \mu)(Y_j - \mu)}{n^2} \\ &= \frac{(n-1)}{n} \sum_{i=1}^{n} (Y_i - \mu)^2 - \sum_{i}^{n} \sum_{j \neq i}^{n} \frac{(Y_i - \mu)(Y_j - \mu)}{n} \end{aligned} \tag{11}$$

From (11) *one can easily derive expected value of* $\hat{\sigma}^2$*:*

$$
\begin{aligned}
\mathbb{E}(\hat{\sigma^2}) &= \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n}(Y_i-\hat{\mu})^2\right) \\
&= \mathbb{E}\left(\frac{(n-1)}{n^2}\sum_{i=1}^{n}(Y_i-\mu)^2 - \sum_{i}^{n}\sum_{j\neq i}^{n}\frac{(Y_i-\mu)(Y_j-\mu)}{n^2}\right) \\
&= \frac{(n-1)}{n^2}\sigma^2 - \sum_{i}^{n}\sum_{j\neq i}^{n}\mathbb{E}\left(\frac{(Y_i-\mu)(Y_j-\mu)}{n^2}\right) = \frac{(n-1)}{n^2}\sigma^2\square
\end{aligned} \tag{12}
$$

From the expected value of $\mathbb{E}(\hat{\sigma}^2)$ we can see that it is not equal to σ therefore $\hat{\sigma}^2$ is an biased estimator.

According to many publications [1, p.341][3, p.92][4, p.72], $\frac{n\hat{\sigma}^2}{\sigma} \sim \chi^2_{n-1}$*. Proof is derived in Appendix C. This gives:*

$$
\frac{n\hat{\sigma}^2}{\sigma} \sim \chi^2_{n-1}; \tag{13}
$$

The variance of χ^2_{n-1} is $2(n-1)$ this gives variance:

$$
\begin{aligned}
Var\left(\frac{n\hat{\sigma}^2}{\sigma}\right) &= 2(n-1) \\
Var\left(\hat{\sigma}^2\right) = \sigma_{\hat{\sigma}^2} &= \sigma^4\frac{2(n-1)}{n^2}\square
\end{aligned} \tag{14}
$$

1.3. **e.** A sequence of estimators $\hat{\Theta}_n$ for parameter Θ, is weakly consistent if

$$
\lim_{n\to\infty} Pr(|\hat{\Theta}_n - \Theta|) = 0 \tag{15}
$$

For each of $\hat{\mu}_n$ and $\hat{\sigma}^2$, determine whether the sequence of estimators is weakly consistent.

The limit should be written as this:

$$
\lim_{n\to\infty} Pr(|\hat{\Theta}_n - \Theta| \geq \epsilon) = 0 \tag{16}
$$

Solution 3. *Solution for $\hat{\mu}$.*

The probability can be rewritten as:

$$
\begin{aligned}
Pr(|\hat{\mu}_n - \mu| \geq \epsilon) &= Pr[(\hat{\mu}_n \leq \mu - \epsilon) \quad \cap \quad (\hat{\mu}_n \geq \mu + \epsilon)], \bigvee \epsilon > 0 \\
&= F(\mu - \epsilon) + (1 - F(\mu + \epsilon))
\end{aligned} \tag{17}
$$

where: $F(x)$ is CDF of X.

From 1.1 we know that $\hat{\mu} \sim (N)\left(\mu, \frac{\sigma^2}{n}\right)$ and

$$
\begin{aligned}
Pr(|\hat{\Theta}_n - \Theta| \geq \epsilon) &= \frac{1}{\sqrt{2\pi\frac{\sigma^2}{n}}}\int_{-\infty}^{\mu-\epsilon}\exp\left(\frac{-(x-\mu)^2}{\frac{\sigma^2}{n}}\right)dx + \frac{1}{\sqrt{2\pi\frac{\sigma^2}{n}}}\int_{\mu+\epsilon}^{\infty}\exp\left(\frac{-(x-\mu)^2}{\frac{\sigma^2}{n}}\right)dx \\
&= \frac{2}{\sqrt{2\pi\frac{\sigma^2}{n}}}\int_{-\infty}^{-\epsilon}\exp\left(\frac{-z^2}{\frac{\sigma^2}{n}}\right)dz = 1 - \frac{1}{\sqrt{2\pi\frac{\sigma^2}{n}}}\int_{-\epsilon}^{\epsilon}\exp\left(\frac{-z^2}{\frac{\sigma^2}{n}}\right)dz \\
&= 1 - 1 = 0\square
\end{aligned} \tag{18}
$$

where: $z = x - \mu$.

The solution of last integral is derived in B.

Solution 4. *Solution for* $\hat{\sigma}^2$.

From 1.2 we know that $\frac{n\hat{\sigma}_n^2}{\sigma^2} \sim \chi^2_{n-1}$.

From Central Limit Theorem (CLT) (52) we know that any sequence of n i.i.d r.v's with $\mathbb{E}(X_i) = \mu$ *and* $Var(X_i) = \sigma$ *converges to normal distribution as n approaches infinity.*

Therefore :

$$\begin{aligned} &\frac{\hat{\sigma}^2}{\sigma^2} \sim \frac{1}{n}\chi^2_{n-1} \sim \frac{1}{n}\frac{n-1}{n-1}\chi^2_{n-1} \sim \frac{n-1}{n}\mathbf{N}\left(\mu_{\chi^2_1}, \frac{\sigma^2_{\chi^2_1}}{n-1}\right) \sim \frac{n-1}{n}\mathbf{N}\left(1, \frac{2}{n-1}\right) \\ &\hat{\sigma}^2 \sim \mathbf{N}\left(\frac{n-1}{n}\sigma^2, \frac{2(n-1)}{n^2}\sigma^4\right) \sim \mathbf{N}\left(\mu_{\hat{\sigma}^2}, \sigma^2_{\hat{\sigma}^2}\right) \end{aligned} \tag{19}$$

We can rewrite probability inequality for consistency as :

$$Pr(|\hat{\sigma}_n^2 - \sigma^2| \geq \epsilon) = Pr(\sigma_n^2 - \epsilon \geq \hat{\sigma}^2 \geq \sigma^2 + \epsilon) = 0, \bigvee \epsilon \geq 0, n \to \infty \tag{20}$$

Let's solve for value of the probability

$$Pr(\sigma^2 - \epsilon \geq \hat{\sigma}_n^2 \geq \sigma^2 + \epsilon) = 1 - \int_{\sigma^2-\epsilon}^{\sigma^2+\epsilon} \frac{1}{\sqrt{2\pi\sigma^2_{\hat{\sigma}^2}}} \exp\left(\frac{-(x-\mu_{\hat{\sigma}^2})^2}{2\sigma^2_{\hat{\sigma}^2}}\right) dx \tag{21}$$

Using solution from B and knowing that $\lim_{n\to\infty} \sigma^2_{\hat{\sigma}^2} = \lim_{n\to\infty} \frac{n-1}{n^2} 2\sigma^4 = \lim_{n\to\infty} \frac{1}{n} 2\sigma^4$ *and* $\lim_{n\to\infty} \frac{n-1}{n}\sigma^2 = \sigma^2$ *we can write integral:*

$$\int_{\sigma^2-\epsilon}^{\sigma^2+\epsilon} \frac{1}{\sqrt{2\pi\sigma^2_{\hat{\sigma}^2}}} \exp\left(\frac{-(x-\mu_{\hat{\sigma}^2})^2}{2\sigma^2_{\hat{\sigma}^2}}\right) dx = \int_{-\epsilon}^{\epsilon} \frac{1}{\sqrt{2\pi\sigma^2_{\hat{\sigma}^2}}} \exp\left(\frac{-x^2}{2\sigma^2_{\hat{\sigma}^2}}\right) dx = 1 \tag{22}$$

Therefore:

$$Pr(|\hat{\sigma}_n^2 - \sigma^2| \geq \epsilon) = 1 - 1 = 0\Box \tag{23}$$

In summary both $\hat{\mu}$ and $\hat{\sigma}^2$ are weakly consistent.

2.

2.1. **a.** Derive the MLE for μ and its distribution for a fixed set of sampling increments $\{h_i\} : \sum_{i=1}^n h_i = T$.

Solution 5. *Finding estimate for* μ.

The Likelihood function for MLE for μ*:*

$$\begin{aligned} \mathcal{L}\left(\mu, \sigma^2 | Y_1, Y_2, \ldots, Y_n\right) &= Pr(Y_1, Y_2, \ldots, Y_n | \mu, \sigma^2) \\ &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2 h_i}} \exp\left(\frac{-(Y_i - \mu h_i)^2}{2\sigma^2 h_i}\right) \end{aligned} \tag{24}$$

And the log-likelihood function:

$$l\left(\mu,\sigma^2|Y_1,Y_2,\ldots,Y_n\right)=\sum_{i=1}^{n}\ln\left(\frac{1}{\sqrt{2\pi\sigma^2 h_i}}\right)+\sum_{i=1}^{n}\left(\frac{-(Y_i-\mu h_i)^2}{2\sigma^2 h_i}\right) \tag{25}$$

To find maximum of log-likelihood function we take derivative with respect to μ*:*

$$\frac{\partial}{\partial\mu}l\left(\mu,\sigma|Y_1,Y_2,\ldots,Y_n\right)=\sum_{i=1}^{n}\frac{2h_i(Y_i-\mu h_i)}{2\sigma^2 h_i}=\sum_{i=1}^{n}\frac{(Y_i-\mu h_i)}{\sigma^2} \tag{26}$$

Log-likelihood function will be maximized when $\frac{\partial}{\partial\mu}l\left(\mu,\sigma|Y_1,Y_2,\ldots,Y_n\right)=0$*:*

$$\sum_{i=1}^{n}\frac{(Y_i-\mu h_i)}{\sigma^2}=0\Longrightarrow\sum_{i=1}^{n}(Y_i-\mu h_i)=0\Longrightarrow\sum_{i=1}^{n}Y_i=\mu\sum_{i=1}h_i \tag{27}$$

$$\hat{\mu}=\frac{1}{T}\sum_{i=1}^{n}Y_i \tag{28}$$

Solution 6. *Finding distribution for* $\hat{\mu}$*.*
Since $Y_i\sim\mathbf{N}\left(\mu h_i,\sigma^2 h_i\right)$ *the Moment Generating Function of* Y_i *will be:*

$$\mathbf{M}_{Y_i}(t)=\exp\left(\mu h_i t+\frac{\sigma^2 h_i t^2}{2}\right); \tag{29}$$

The moment generating function of sum of r.v's is equal to product of their MGF's :

$$\begin{aligned}\mathbf{M}_Z(t)&=\prod_{i=1}^{n}\exp\left(\mu h_i t+\frac{\sigma^2 h_i t^2}{2}\right)=\exp\left(\mu t\sum_{i=1}^{n}h_i+\frac{\sigma^2 t^2\sum_{i=1}^{n}h_i}{2}\right)\\&=\exp\left(\mu Tt+\frac{\sigma^2 Tt^2}{2}\right)\end{aligned} \tag{30}$$

where $Z=\sum_{i=1}^{n}Y_i$*.*
The $\mathbf{M}_Z(t)$ *is a MGF of Normal distribution therefore:*

$$\hat{\mu}=\frac{1}{T}Z\sim\mathbf{N}\left(\mu,\frac{\sigma^2}{T}\right)\square \tag{31}$$

2.2. **b.** Derive the MLE for σ^2 and its distribution for a fixed set of sampling increments $\{h_i\}:\sum_{i=1}^{n}h_i=T$.

Solution 7. *Finding MLE for* σ^2*.*
Log-likelihood function:

$$l\left(\mu,\sigma^2|Y_1,Y_2,\ldots,Y_n\right)=\sum_{i=1}^{n}\ln\left(\frac{1}{\sqrt{2\pi\sigma^2 h_i}}\right)+\sum_{i=1}^{n}\left(\frac{-(Y_i-\mu h_i)^2}{2\sigma^2 h_i}\right) \tag{32}$$

To find maximum of log-likelihood function from (25) *we take derivative with respect to* σ^2*:*

$$\frac{\partial}{\partial(\sigma^2)} l\left(\mu, \sigma^2 | Y_1, Y_2, \ldots, Y_n\right) = \sum_{i=1}^{n} \frac{-2\pi h_i \sqrt{2\pi\sigma^2 h_i}}{2(2\pi\sigma^2 h_i)^{\frac{3}{2}}} + \sum_{i=1}^{n} \frac{(Y_i - \hat{\mu} h_i)^2}{2\sigma^4 h_i} \tag{33}$$
$$= \sum_{i=1}^{n} \frac{-1}{2\sigma^2} + \sum_{i=1}^{n} \frac{(Y_i - \hat{\mu} h_i)^2}{2\sigma^4 h_i}$$

And set it to zero:

$$\sum_{i=1}^{n} \frac{-1}{2\sigma^2} + \sum_{i=1}^{n} \frac{(Y_i - \hat{\mu} h_i)^2}{2\sigma^4 h_i} = 0$$
$$\sum_{i=1}^{n} \frac{1}{2\sigma^2} = \sum_{i=1}^{n} \frac{(Y_i - \hat{\mu} h_i)^2}{2\sigma^4 h_i} \tag{34}$$
$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} \frac{(Y_i - \hat{\mu} h_i)^2}{h_i}$$

Solution 8. *Finding Distribution for $\hat{\sigma}^2$.*
Similairly to previous chapter the $\hat{\sigma}^2 \sim \chi^2_{n-1}$. The proof is given in D.

2.3. **c.** If limited to sampling $n+1$ price points for $\{X_t\}$, (including X_0 and X_T) prove that

- For estimating σ^2, sampling, the ML estimators vary with the increment spacing, but the variance of these estimators are all equal, regardless of the increment spacing.
- For estimating μ, all ML estimators are the same and have the same variance, regardless of the increment spacing.

Solution 9. *Finding answer for σ^2 estimator.*
We can see from derivation of ML estimate of that for $n+1$ in 2.2 spaced over period T we have to decide first how to assign weight g_0 to r.v Y_0 and modify weights h_i for subsequent samples. One scheme could be:

$$g_i = \begin{cases} \frac{h_1}{2} & for \quad i = 0 \\ \frac{h_i + h_{i+1}}{2} & for \quad i = 1 \ldots (n-1) \\ \frac{h_n}{2} & for \quad i = n \end{cases} \tag{35}$$

Using above we can re-write formula for $\hat{\sigma}^2$:

$$\hat{\sigma}^2 = \frac{1}{n+1} \left[\frac{(Y_0 - \hat{\mu}\frac{h_1}{2})^2}{\frac{h_1}{2}} + \sum_{i=1}^{n-1} \frac{(Y_i - \hat{\mu}\frac{h_i + h_{i+1}}{2})^2}{\frac{h_i + h_{i+1}}{2}} + \frac{(Y_n - \hat{\mu}\frac{h_n}{2})^2}{\frac{h_1}{2}} \right] \tag{36}$$
$$= \frac{1}{n+1} \sum_{i=0}^{n} \frac{(Y_i - \hat{\mu} g_i)^2}{g_i}$$

where : $\sum_{i=0}^{n} g_i = T$.
From derivation in D69 we can see that:

$$(n+1)\hat{\sigma}^2 = \sum_{i=0}^{n} \frac{(Y_i - \mu g_i)^2}{g_i} - T(\hat{\mu} - \mu)^2$$
$$= \sum_{i=0}^{n} \frac{Y_i^2}{g_i} - 2\mu \sum_{i=0}^{n} Y_i + T\mu^2 - T(\hat{\mu} - \mu)^2 \tag{37}$$
$$= \sum_{i=0}^{n} \frac{Y_i^2}{g_i} - T\hat{\mu}^2$$

Let's chose different set of sampling points $k_1 \dots k_{n-1}$. We can see that the estimate formula will be different for each choice of intervals:

$$\hat{\sigma}_g^2 = \sum_{i=0}^{n} \frac{(Y_i - \mu g_i)^2}{g_i} \neq \hat{\sigma}_k^2 = \sum_{i=0}^{n} \frac{(Z_i - \mu k_i)^2}{k_i} \tag{38}$$

However both for the different choice of sampling intervals $\frac{(n+1)\hat{\sigma}_g^2}{\sigma^2} \sim \frac{(n+1)\hat{\sigma}_k^2}{\sigma^2} \sim \chi_n^2$, and variance of both estimators will be the same. □

Solution 10. *Finding answer for μ estimator.*

For $\hat{\mu}$ = estimator the ML formula for estimator doesn't depend on choice of sampling intervals. For any choice of intervals the ML estimate will result in the same formula, and having the same distribution. □

3. ARCH(1) Model Properties

3.1. **a.** Prove that: $\boldsymbol{E}[\epsilon_t^2] = \alpha_0/(1-\alpha_1)$. Where Z_t is i.i.d. and $\boldsymbol{E}[Z_t] = 0$ and $\boldsymbol{E}[Z_t^2] = 1$.

Solution 11. *Using $\mathbf{E}[\epsilon_t^2] = Z_t\sigma_t$ and $\sigma_t^2 = \alpha_0 + \alpha_1\epsilon_{t-1}^2$:*

$$\begin{aligned}
\mathbf{E}[\epsilon_t^2] &= \mathbf{E}\left[Z_t^2\sigma_t^2\right] \\
&= \mathbf{E}\left[Z_t^2(\alpha_0 + \alpha_1\epsilon_{t-1}^2)\right] \\
&= \mathbf{E}\left[Z_t^2\right]\mathbf{E}\left[\alpha_0 + \alpha_1\epsilon_{t-1}^2\right] \\
&= 1 \cdot \left\{\alpha_0 + \alpha_1\mathbf{E}\left[\alpha_1\epsilon_{t-1}^2\right]\right\} \\
&= \alpha_0 + \alpha_1\mathbf{E}\left[\epsilon_t^2\right] \\
&= \frac{\alpha_0}{1-\alpha_1}\square
\end{aligned} \tag{39}$$

3.2. **b.** Prove that: $\boldsymbol{E}[\epsilon_t^3] = 0$. Suppose that $\boldsymbol{E}[Z_t^3] = 0$.

Solution 12.

$$\begin{aligned}
\mathbf{E}[\epsilon_t^3] &= \mathbf{E}\left[Z_t^3\sigma_t^3\right] \\
&= \mathbf{E}\left[Z_t^3\right]\mathbf{E}\left[\sigma_t^3\right] \\
&= 0 \cdot \mathbf{E}\left[\sigma_t^3\right] \\
&= 0\square.
\end{aligned} \tag{40}$$

because $\mathbf{E}\left[Z_t^3\right] = 0$.

3.3. **c.** Prove that: $\boldsymbol{E}[\epsilon_t^4] = \frac{\kappa\alpha_0^2(1+\alpha_1)}{(1-\alpha_1)(1-\kappa\alpha_1)}$. Suppose that $\boldsymbol{E}[Z_t^4] = \kappa$.

Solution 13.

$$
\begin{aligned}
\mathbf{E}[\epsilon_t^4] &= \mathbf{E}\left[Z_t^4\sigma_t^4\right] \\
&= \mathbf{E}\left[Z_t^4(\alpha_0 + \alpha_1\epsilon_{t-1}^2)^2\right] \\
&= \mathbf{E}\left[Z_t^4\right]\mathbf{E}\left[(\alpha_0 + \alpha_1\epsilon_{t-1}^2)^2)\right] \\
&= \kappa \cdot \left\{\mathbf{E}\left[(\alpha_0^2 + 2\alpha_0\alpha_1\epsilon_{t-1}^2 + \alpha_1^2\epsilon_{t-1}^4)\right]\right\} \\
&= \kappa \cdot \left\{\alpha_0^2 + 2\alpha_0\alpha_1\mathbf{E}\left[\epsilon_{t-1}^2\right] + \alpha_1^2\mathbf{E}\left[\epsilon_{t-1}^4\right]\right\} \\
&= \kappa \cdot \left\{\alpha_0^2 + 2\alpha_0\alpha_1\frac{\alpha_0}{1-\alpha_1} + \alpha_1^2\mathbf{E}\left[\epsilon_{t-1}^4\right]\right\} \\
&= \kappa \cdot \left\{\frac{\alpha_0^2(1-\alpha_1)}{1-\alpha_1} + \frac{2\alpha_0^2\alpha_1}{1-\alpha_1} + \alpha_1^2\mathbf{E}\left[\epsilon_{t-1}^4\right]\right\} \\
&= \kappa \cdot \left\{\frac{\alpha_0^2 - \alpha_0^2\alpha_1 + 2\alpha_0^2\alpha_1}{1-\alpha_1} + \alpha_1^2\mathbf{E}\left[\epsilon_{t-1}^4\right]\right\} \\
&= \kappa\frac{\alpha_0^2 - \alpha_0^2\alpha_1 + 2\alpha_0^2\alpha_1}{1-\alpha_1} + \kappa\alpha_1^2\mathbf{E}\left[\epsilon_{t-1}^4\right] \\
&= \kappa\frac{\alpha_0^2 - \alpha_0^2\alpha_1 + 2\alpha_0^2\alpha_1}{1-\alpha_1} + \kappa\alpha_1^2\mathbf{E}\left[\epsilon_t^4\right] \\
&= \frac{\kappa\alpha_0^2(1+\alpha_1)}{(1-\alpha_1)(1-\kappa\alpha_1^2)}\square.
\end{aligned} \tag{41}
$$

3.4. **d.** What constraints of α_0, α_1 must be made in 3.3, to maintain 4-th order stationarity (bounded).

Solution 14. *For 4-th order stationarity $\alpha_1 \neq 1$ and $|\alpha_1| \neq \kappa\square$.*

3.5. **e.** The kurtosis of ϵ_t is:

$$\kappa_\epsilon = \frac{\boldsymbol{E}[\epsilon_t^4]}{\boldsymbol{E}[\epsilon_t^2]}$$

(The fourth moment is normalized to be scale-free). If the distribution Z_t is Gaussian/Normal (i.e., the scaled, conditional error distribution of ϵ_t), does the unconditional distribution of ϵ_t, have a higher kurtosis than that of the Gaussian distribution, (i.e., heavier tails)?

Solution 15.

$$
\begin{aligned}
\kappa_\epsilon &= \frac{\mathbf{E}[\epsilon_t^4]}{\left\{\mathbf{E}[\epsilon_t^2]\right\}^2} \\
&= \frac{\kappa\alpha_0^2(1+\alpha_1)}{(1-\alpha_1)(1-\kappa\alpha_1^2)}\left(\frac{\alpha_0}{1-\alpha_1}\right)^{-2} \\
&= \frac{\kappa(1-\alpha_1^2)}{(1-\kappa\alpha_1^2)}
\end{aligned} \tag{42}
$$

Setting $\kappa = 3$ for Normal distribution of Z_t:

$$\begin{aligned}\kappa_\epsilon &= \frac{3(1-\alpha_1^2)}{(1-3\alpha_1^2)} \\ &= 3 \cdot \frac{(1-\alpha_1^2)}{(1-3\alpha_1^2)}\end{aligned} \tag{43}$$

for κ_ϵ to be higher than that of Normal ($\kappa_\epsilon = 3$) the factor

$$\frac{(1-\alpha_1^2)}{(1-3\alpha_1^2)}$$

must be greater than 1:

$$\begin{aligned}\frac{(1-\alpha_1^2)}{(1-3\alpha_1^2)} &\geq 1 \\ -\alpha_1^2 &\geq -3\alpha_1^2 \\ \alpha_1^2 &\leq 3\alpha_1^2\end{aligned} \tag{44}$$

As we are comparing squares therefore above is always true. It means that distribution of κ_ϵ has kurtosis higher than normal even if Z_t is normally distributed.$\square$

4. Annual Sample Variances of Logarithmic Returns

4.1. **a.** Under the Gaussian model given n, μ, σ^2 prove that the distribution of $\hat{\sigma}^2$ is:

$$\hat{\sigma}^2 \sim \frac{\sigma^2}{n-1} \times \chi^2_{n-1}$$

Solution 16. *We have proven in C that $\frac{n\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-1}$ for ML estimator:*

$$\hat{\sigma}^2 \sim \frac{\sigma^2}{n} \times \chi^2_{n-1}$$

The proof will be the same for unbiased estimator and we can write:

$$\hat{\sigma}^2 \sim \frac{\sigma^2}{n-1} \times \chi^2_{n-1}$$

From above it follows that:

$$\hat{\sigma}^2 \sim \frac{\sigma^2}{n-1} \times \chi^2_{n-1}\square \tag{45}$$

4.2. **b.**

- Using data for 2008, compute the two-sided 95 % confidence interval for σ^2, based on daily log returns.
- Express the interval in terms of the annualized volatitiliy ($\sqrt{253}\sigma$). Does the sample annual volatility for any other year fall in the confidence interval for 2008?

Solution 17. *For 2008 the daily variance is* $6.677100e-04$. *The quantiles for 95% confidence and 252 d.f. are* $q_{0.025} = 209.9227$ *and* $q_{0.975} = 297.8637$. *Using formula:*

$$Pr\left(\hat{\sigma}^2\frac{n-1}{q_{0.025}} \leq \sigma^2 \leq \hat{\sigma}^2\frac{n-1}{q_{0.975}}\right) = 0.975 \tag{46}$$

we can calculate intervals to be:

$$5.6490e-4 \leq \sigma^2 \leq 8.0155e-4\square \tag{47}$$

Solution 18. *The intervals in terms of annualized volatility are :*

$$0.3780 \leq \sqrt{253}\sigma \leq 0.4503\square \tag{48}$$

None of the other year's volatilities fall into confidence interval of 2008 volatility.

4.3. **c.**

- Compute the test statistic $S = S_0$ for testing the daily return variance for 2008 is equal to the daily return variance for 2007.
- Given the value of the test statistic S_0, determine the $\alpha - level$ at which the null hypothesis is on the boundary of beeing just accepted/rejected. (This level is called the $P - value$ of the test statistic. Reporting a test statistic's $P - value$ provides evidence concernig for/against the test null hypothesis which can be provided without having to specify $\alpha - level$.
- Repeat the previous two questions for testing the equality of the return variance for 2008 so that for 2006. (Note: the degrees of freedom for 2006 are the same as those for 2007 so the same F distribution is applicable).

Solution 19. *The test statistic* S_0 *:*

$$S = \frac{\hat{\sigma}^2_{2008}}{\hat{\sigma}^2_{2007}} = 6.5552 \tag{49}$$

We have to reject null hypothesis that $\frac{\hat{\sigma}^2_{2008}}{\hat{\sigma}^2_{2007}} = 1$.

Solution 20. *For the test statistic the value above the* $\alpha - value$ *is 0 down to machine precision of R-package.*

Solution 21. *Because*

$$S = \frac{\hat{\sigma}^2_{2008}}{\hat{\sigma}^2_{2006}} = 16.7709 \tag{50}$$

We have to reject H_0 *hypothesis and also the* $\alpha - value$ *will be zero.*

References

[1] Rohatgi, V.K., Ehsanes Saleh, A. K. MD.,*An Introduction to Probability and Statistics*, John Wiley & Sons, Inc., 2001

[2] Ahn, S., Fessler, G.,*Standard Errors of Mean, Variance, and Standard Deviation Estimators*, The University of Michigan, 2003

[3] Lehmann, E.L., Casella, G.,*Theory of Point Estimation*, Springer-Verlag New York, Inc., Second Edition, 1998

[4] Forbes, C.,Evans, M., Hastings, N., Peacock, B.,*Statistical Distributions*, John Wiley & Sons, Inc., Hoboken, New Jersey., Fourth Edition, 2010

Appendix A. Derivation of $\hat{\sigma}^2$ distribution

To derive PDF of $\hat{\sigma^2}$ we will assume that the second term in the sum:

$$\sum_{i}^{n}\sum_{j\neq i}^{n}\mathbb{E}\left(\frac{(Y_i-\mu)(Y_j-\mu)}{n^2}\right)=\boldsymbol{O}\left(\frac{1}{n}\right) \tag{51}$$

becomes Gaussian as n increases and it's distribution converges to Normal for larger n.
From Central Limit Theorem (CLT) we know that any sequence of i.i.d r.v's with $\mathbb{E}(X_i)=\mu$ and $Var(X_i)=\sigma$ than as N approaches infinity :

$$\sqrt{n}\left(\left(\frac{1}{n}\sum_{i=1}^{n}nX_i)\right)-\mu\right)\xrightarrow{d}\boldsymbol{N}(0,\sigma^2)\Longrightarrow\left(\left(\frac{1}{n}\sum_{i=1}^{n}nX_i)\right)-\mu\right)\xrightarrow{d}\boldsymbol{N}(0,\frac{\sigma^2}{n}) \tag{52}$$

As Y_i and Y_j are i.i.d for $j\neq i$ therefore $\mathbb{E}((Y_i-\mu)(Y_j-\mu))=0$ and $Var((Y_i-\mu)(Y_j-\mu))=\sigma^4\Rightarrow$:

$$\sum_{i}^{n}\sum_{j\neq i}^{n}\frac{(Y_i-\mu)(Y_j-\mu)}{n^2}=\frac{2}{n^2}\sum_{k=1}^{\frac{n(n-1)}{2}}Z_k=\frac{n(n-1)}{n^2}\frac{1}{\frac{n(n-1)}{2}}\sum_{k=1}^{\frac{n(n-1)}{2}}Z_k \tag{53}$$

where $Z_k=(Y_i-\mu)(Y_j-\mu)$ for $j\neq i$;

The distribution of sum of covariance terms Z_k should converge to Normal distribution:

$$\begin{aligned}\frac{1}{\frac{n(n-1)}{2}}\sum_{k=1}^{\frac{n(n-1)}{2}}Z_k&\sim\boldsymbol{N}\left(0,\frac{2}{n(n-1)}\sigma^4\right)\\ \frac{(n-1)}{n}\frac{1}{\frac{n(n-1)}{2}}\sum_{k=1}^{\frac{n(n-1)}{2}}Z_k&\sim\boldsymbol{N}\left(0,\frac{(n-1)^2}{n^2}\frac{2}{n(n-1)}\sigma^4\right)\sim\boldsymbol{N}\left(0,\frac{2(n-1)}{n^3}\sigma^4\right)\end{aligned} \tag{54}$$

One can see that the $\boldsymbol{O}\left(\frac{1}{n}\right)$ is decreasing with n increasing and can be omitted.

Let's now look at the distribution of $(Y_i-\mu)^2$. For simplicity let's look at case $\left(\frac{Y_i-\mu}{\sigma}\right)^2$ as $\left(\frac{Y_i-\mu}{\sigma}\right)=X\sim\boldsymbol{N}(0,1)$.

The CDF of X^2, $\boldsymbol{F}(X^2)$ is:

$$\boldsymbol{F}(X^2)=\boldsymbol{Pr}(X^2\leq t)=\boldsymbol{Pr}(-\sqrt{t}\leq X\leq\sqrt{t})\quad=\boldsymbol{F}(\sqrt{t})-\boldsymbol{F}(-\sqrt{t}) \tag{55}$$

And the PDF $\boldsymbol{f}(X^2)$:

$$\begin{aligned}\boldsymbol{f}(X^2)&=\frac{d}{dt}\left(\boldsymbol{F}(\sqrt{t})-\boldsymbol{F}(-\sqrt{t})\right)=\frac{d}{dt}\left(\int_{-\infty}^{\sqrt{t}}\boldsymbol{f}(x)dx-\int_{-\infty}^{-\sqrt{t}}\boldsymbol{f}(x)dx\right)\\&=\frac{1}{2\sqrt{t}}\boldsymbol{f}(\sqrt{t})+\frac{1}{2\sqrt{t}}\boldsymbol{f}(-\sqrt{t})=\frac{1}{\sqrt{t}}\boldsymbol{f}(\sqrt{t})=\frac{1}{\sqrt{t}}\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-t}{2}\right)=\frac{1}{\sqrt{t}}\frac{1}{\sqrt{2}\Gamma\left(\frac{1}{2}\right)}\exp\left(\frac{-t}{2}\right)\end{aligned} \tag{56}$$

This is distribution of χ^2 distribution of order 1. This would suggest that $\frac{n\hat{\sigma}^2}{\sigma}\sim\chi^2_n$.

Appendix B. Derivation of solution of Gaussian integral

We will try to derive value for this integral:

$$I = \frac{1}{\sqrt{2\pi\frac{\sigma^2}{n}}}\int_{-\epsilon}^{\epsilon}\exp\left(\frac{-z^2}{\frac{2\sigma^2}{n}}\right)dz \tag{57}$$

We can try to find value for I^2 :

$$\begin{aligned} I^2 &= \frac{1}{\sqrt{2\pi\frac{\sigma^2}{n}}}\int_{-\epsilon}^{\epsilon}\exp\left(\frac{-y^2}{\frac{2\sigma^2}{n}}\right)dy \times \frac{1}{\sqrt{2\pi\frac{\sigma^2}{n}}}\int_{-\epsilon}^{\epsilon}\exp\left(\frac{-z^2}{\frac{2\sigma^2}{n}}\right)dz \\ &= \frac{1}{2\pi\frac{\sigma^2}{n}}\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}\exp\left(\frac{-y^2-z^2}{\frac{2\sigma^2}{n}}\right)dydz \end{aligned} \tag{58}$$

Substituting $y = r\sin(\alpha), z = r\cos(\alpha)$, $dxdy = -rdrd\alpha$

$$\begin{aligned} I^2 &= -\frac{1}{2\pi\frac{\sigma^2}{n}}\iint_{D_\epsilon} r\exp\left(\frac{-r^2}{\frac{2\sigma^2}{n}}\right)drd\alpha \geq -\frac{1}{2\pi\frac{\sigma^2}{n}}\int_0^{2\pi}\int_0^{\epsilon} r\exp\left(\frac{-r^2}{\frac{2\sigma^2}{n}}\right)drd\alpha \\ I^2 &\geq -\frac{1}{2\pi\frac{\sigma^2}{n}}\int_0^{2\pi}d\alpha\int_0^{\epsilon} r\exp\left(\frac{-r^2}{\frac{2\sigma^2}{n}}\right)dr = -\frac{2\pi}{2\pi\frac{\sigma^2}{n}}\int_0^{\epsilon} r\exp\left(\frac{-r^2}{\frac{2\sigma^2}{n}}\right)dr \\ I^2 &\geq -\frac{1}{\frac{\sigma^2}{n}}\frac{\sigma^2}{n}\left|\exp\left(\frac{-r^2}{\frac{2\sigma^2}{n}}\right)\right|_0^{\epsilon} = -\exp\left(\frac{-\epsilon^2}{\frac{2\sigma^2}{n}}\right) + \exp(0) = 1\square \end{aligned} \tag{59}$$

Because for any $\epsilon > 0$ $\lim_{n\to\infty}\exp\left(\frac{-\epsilon^2}{\frac{2\sigma^2}{n}}\right) = 0$. Therefore $I^2 = 1$.

Appendix C. Derivation of distribution of $\hat{\sigma}^2$

Using steps in [1, p. 339-341] let's start by showing that $\hat{\mu}$ and $\hat{\sigma}^2$ are independent.

$$\begin{aligned} \hat{\mu} &= \frac{1}{n}\sum_{i=1}^{n} Y_i \\ \hat{\sigma}^2 &= \frac{1}{n}\sum_{i=1}^{n}(Y_i - \hat{\mu})^2 \end{aligned} \tag{60}$$

We compute MGF of $\hat{\mu}$ and $Y_1 - \hat{\mu}, Y_2 - \hat{\mu}, \ldots, Y_n - \hat{\mu}$ as follows:

$$
\begin{aligned}
\boldsymbol{M}(t,t_1,t_2,\ldots,t_n) &= \boldsymbol{E}\{\exp\left[t\hat{\mu} + t_1(Y_1-\hat{\mu}) + t_2(Y_2-\hat{\mu}) + \cdots + t_n(Y_n-\hat{\mu})\right]\} \\
&= \boldsymbol{E}\left\{\exp\left[\sum_{i=1}^{n} t_i Y_i - \left(\sum_{i=1}^{n} t_i - t\right)\hat{\mu}\right]\right\} \\
&= \boldsymbol{E}\left\{\exp\left[\sum_{i=1}^{n} Y_i\left(t_i - \frac{t_1+t_2+\ldots+t_n-t}{n}\right)\right]\right\} \\
&= \boldsymbol{E}\left\{\exp\left[\sum_{i=1}^{n} Y_i\left(\frac{nt_i - n\bar{t} + t}{n}\right)\right]\right\} \qquad \left(where:\ \bar{t} = \frac{1}{n}\sum_{i=1}^{n} t_i\right) \\
&= \prod_{i=1}^{n} \boldsymbol{E}\left\{\exp\left[Y_i\left(\frac{t+n(t_i-\bar{t})}{n}\right)\right]\right\} \\
&= \prod_{i=1}^{n} \exp\left\{\mu\frac{1}{n}\left[t+n(t_i-\bar{t})\right] + \frac{\sigma^2}{2}\frac{1}{n^2}\left[t+n(t_i-\bar{t})\right]^2\right\} \\
&= \exp\left\{\frac{\mu}{n}\left[nt + n\sum_{i=1}^{n}(t_i-\bar{t})\right] + \frac{\sigma^2}{2n^2}\sum_{i=1}^{n}\left[t+n(t_i-\bar{t})\right]^2\right\} \\
&= \exp(\mu t)\exp\left\{\frac{\sigma^2}{2n^2}\left[nt^2 + n^2\sum_{i=1}^{n}(t_i-\bar{t})^2\right]^2\right\} \\
&= \exp\left(\mu t + \frac{\sigma^2 t^2}{2n}\right)\exp\left[\frac{\sigma^2}{2}\sum_{i=1}^{n}(t_i-\bar{t})^2\right] \\
&= \boldsymbol{M}_{\hat{\mu}}(t)\boldsymbol{M}_{Y_1-\hat{\mu},Y_2-\hat{\mu},\ldots,Y_n-\hat{\mu}}(t_1,t_2,\ldots,t_n) \\
&= \boldsymbol{M}(t,0,0,\ldots,0)\boldsymbol{M}(0,t_1,t_2,\ldots,t_n)
\end{aligned} \tag{61}
$$

Therefore $\hat{\mu}$ and $\hat{\sigma}^2$ are independent.
Now we will try to prove that $\frac{n\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-1}$.
Since :

$$
\sum_{i=1}^{n}\frac{(Y_i-\mu)^2}{\sigma^2} \sim \chi^2_n, \qquad \frac{(\hat{\mu}-\mu)^2}{\sigma^2} \sim \chi^2_1 \tag{62}
$$

Lets derive expression for $n\hat{\sigma}^2$:

$$
\begin{aligned}
n\hat{\sigma}^2 &= \sum_{i=1}^{n}(Y_i - \hat{\mu})^2 \\
&= \sum_{i=1}^{n}\left[(Y_i - \mu) - (\hat{\mu} - \mu)\right]^2 \\
&= \sum_{i=1}^{n}\left[(Y_i - \mu)^2 - 2(Y_i - \mu)(\hat{\mu} - \mu) + (\hat{\mu} - \mu)^2\right] \\
&= \sum_{i=1}^{n}(Y_i - \mu)^2 - 2(\hat{\mu} - \mu)\sum_{i=1}^{n}(Y_i - \mu) + n(\hat{\mu} - \mu)^2 \\
&= \sum_{i=1}^{n}(Y_i - \mu)^2 - 2(\hat{\mu} - \mu)(\sum_{i=1}^{n}Y_i - n\mu) + n(\hat{\mu} - \mu)^2 \\
&= \sum_{i=1}^{n}(Y_i - \mu)^2 - n(\hat{\mu} - \mu)^2
\end{aligned} \tag{63}
$$

From above we can see that:

$$
\begin{aligned}
\boldsymbol{E}\left\{\exp\left[\frac{\sum_{i=1}^{n}(Y_i - \mu)^2}{\sigma^2}\right]\right\} &= \boldsymbol{E}\left\{\exp\left[\frac{n(\hat{\mu} - \mu)^2}{\sigma^2} + \frac{n\hat{\sigma}^2}{\sigma^2}\right]\right\} \\
&= \boldsymbol{E}\left\{\exp\left[\frac{n(\hat{\mu} - \mu)^2}{\sigma^2}\right]\right\}\boldsymbol{E}\left[\exp\left(\frac{n\hat{\sigma}^2}{\sigma^2}\right)\right] \\
(1-2t)^{\frac{-n}{2}} &= (1-2t)^{\frac{-1}{2}}\boldsymbol{E}\left[\exp\left(\frac{n\hat{\sigma}^2}{\sigma^2}\right)\right] \\
\boldsymbol{E}\left[\exp\left(\frac{n\hat{\sigma}^2}{\sigma^2}\right)\right] &= (1-2t)^{\frac{n-1}{2}}
\end{aligned} \tag{64}
$$

From uniqueness of MGF it follows that :

$$
\boldsymbol{E}\left[\exp\left(\frac{n\hat{\sigma}^2}{\sigma^2}\right)\right] \sim \chi^2_{n-1}\square \tag{65}
$$

Appendix D. Derivation of distribution of $\hat{\sigma}^2$

Using steps similar to these in C let's start by showing that $\hat{\mu}$ and $\hat{\sigma}^2$ are independent.

$$
\begin{aligned}
\hat{\mu} &= \frac{1}{T}\sum_{i=1}^{n}Y_i \\
\hat{\sigma}^2 &= \frac{1}{n}\sum_{i=1}^{n}\frac{\left(Y_i - \hat{\mu}h_i\right)^2}{h_i}
\end{aligned} \tag{66}
$$

We compute MGF of $\hat{\mu}$ and $Y_1 - \hat{\mu}h_1, Y_2 - \hat{\mu}h_2, \ldots, Y_n - \hat{\mu}h_n$ as follows:

$$
\begin{aligned}
\boldsymbol{M}(t,t_1,t_2,\ldots,t_n) &= \boldsymbol{E}\{\exp\left[t\hat{\mu}+t_1(Y_1-\hat{\mu}h_1)+t_2(Y_2-\hat{\mu}h_2)+\cdots+t_n(Y_n-\hat{\mu}h_n)\right]\} \\
&= \boldsymbol{E}\left\{\exp\left[\sum_{i=1}^{n} t_iY_i-\left(\sum_{i=1}^{n} t_ih_i-t\right)\hat{\mu}\right]\right\} \\
&= \boldsymbol{E}\left\{\exp\left[\sum_{i=1}^{n} Y_i\left(t_i-\frac{t_1h_1+t_2h_2+\ldots+t_nh_n-t}{T}\right)\right]\right\} \\
&= \boldsymbol{E}\left\{\exp\left[\sum_{i=1}^{n} Y_i\left(\frac{Tt_i-T\bar{t}+t}{T}\right)\right]\right\} \qquad \left(\mathit{where}:\bar{t}=\frac{1}{T}\sum_{i=1}^{n}t_ih_i\right) \\
&= \prod_{i=1}^{n}\boldsymbol{E}\left\{\exp\left[Y_i\left(\frac{t+T(t_i-\bar{t})}{T}\right)\right]\right\} \\
&= \prod_{i=1}^{n}\exp\left\{\mu h_i\frac{1}{T}\left[t+T(t_i-\bar{t})\right]+\frac{\sigma^2h_i}{2}\frac{1}{T^2}\left[t+T(t_i-\bar{t})\right]^2\right\} \\
&= \exp\left\{\frac{\mu}{T}\left[Tt+T\sum_{i=1}^{n}h_i(t_i-\bar{t})\right]+\frac{\sigma^2}{2T^2}\sum_{i=1}^{n}h_i\left[t+T(t_i-\bar{t})\right]^2\right\} \\
&= \exp\left(\mu t\right)\exp\left\{\frac{\sigma^2}{2T^2}\left[Tt^2+T^2\sum_{i=1}^{n}h_i(t_i-\bar{t})^2\right]^2\right\} \\
&= \exp\left(\mu t+\frac{\sigma^2t^2}{2T}\right)\exp\left[\frac{\sigma^2}{2}\sum_{i=1}^{n}h_i(t_i-\bar{t})^2\right] \\
&= \boldsymbol{M}_{\hat{\mu}}(t)\boldsymbol{M}_{Y_1-\hat{\mu}h_1,Y_2-\hat{\mu}h_2,\ldots,Y_n-\hat{\mu}h_n}(t_1,t_2,\ldots,t_n) \\
&= \boldsymbol{M}(t,0,0,\ldots,0)\boldsymbol{M}(0,t_1,t_2,\ldots,t_n)
\end{aligned} \tag{67}
$$

Therefore $\hat{\mu}$ and $\hat{\sigma}^2$ are independent.
Now we will try to prove that $\frac{n\hat{\sigma}^2}{\sigma^2}\sim\chi^2_{n-1}$.
Since :

$$
\sum_{i=1}^{n}\frac{(Y_i-\mu h_i)^2}{\sigma^2}\sim\chi^2_n, \qquad \frac{(\hat{\mu}-\mu)^2}{\sigma^2}\sim\chi^2_1 \tag{68}
$$

Lets derive expression for $n\hat{\sigma}^2$:

$$
\begin{aligned}
n\hat{\sigma}^2 &= \sum_{i=1}^{n} \frac{(Y_i - \hat{\mu} h_i)^2}{h_i} \\
&= \sum_{i=1}^{n} \frac{\left[(Y_i - \mu h_i) - h_i(\hat{\mu} - \mu)\right]^2}{h_i} \\
&= \sum_{i=1}^{n} \frac{\left[(Y_i - \mu h_i)^2 - 2h_i(Y_i - \mu h_i)(\hat{\mu} - \mu) + h_i^2(\hat{\mu} - \mu)^2\right]}{h_i} \\
&= \sum_{i=1}^{n} \frac{(Y_i - \mu h_i)^2}{h_i} - 2(\hat{\mu} - \mu)\sum_{i=1}^{n}(Y_i - \mu h_i) + \sum_{i=1}^{n} h_i(\hat{\mu} - \mu)^2 \\
&= \sum_{i=1}^{n} \frac{(Y_i - \mu h_i)^2}{h_i} - 2(\hat{\mu} - \mu)\left(T\hat{\mu} - T\mu\right) + T(\hat{\mu} - \mu)^2 \\
&= \sum_{i=1}^{n} \frac{(Y_i - \mu h_i)^2}{h_i} - T(\hat{\mu} - \mu)^2
\end{aligned} \tag{69}
$$

From above we can see that:

$$
\begin{aligned}
\boldsymbol{E}\left\{\exp\left[\sum_{i=1}^{n} \frac{(Y_i - \mu h_i)^2}{\sigma^2 h_i}\right]\right\} &= \boldsymbol{E}\left\{\exp\left[\frac{T(\hat{\mu} - \mu)^2}{\sigma^2} + \frac{n\hat{\sigma}^2}{\sigma^2}\right]\right\} \\
&= \boldsymbol{E}\left\{\exp\left[\frac{T(\hat{\mu} - \mu)^2}{\sigma^2}\right]\right\} \boldsymbol{E}\left[\exp\left(\frac{n\hat{\sigma}^2}{\sigma^2}\right)\right] \\
(1-2t)^{\frac{-n}{2}} &= (1-2t)^{\frac{-1}{2}} \boldsymbol{E}\left[\exp\left(\frac{n\hat{\sigma}^2}{\sigma^2}\right)\right] \\
\boldsymbol{E}\left[\exp\left(\frac{n\hat{\sigma}^2}{\sigma^2}\right)\right] &= (1-2t)^{\frac{n-1}{2}}
\end{aligned} \tag{70}
$$

From uniqueness of MGF it follows that :

$$
\boldsymbol{E}\left[\exp\left(\frac{n\hat{\sigma}^2}{\sigma^2}\right)\right] \sim \chi^2_{n-1} \square \tag{71}
$$