A Coinductive Confluence Proof for Infinitary Lambda-Calculus

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Presentation plan

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Presentation plan

1. Coinduction.

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- 1. Coinduction.
- 2. Infinitary lambda-calculus.

Coinductive definitions

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$$\mathbb{T} ::= V \parallel A(\mathbb{T}) \parallel B(\mathbb{T}, \mathbb{T})$$

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The set \mathbb{T} consists of all *finite and infinite* terms built up from variables and the constructors A and B.

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The set of all possibly infinite labelled trees with labels specified by the grammar.

Guarded corecursion

For $t \in \mathbb{T}$, $x \in V$, $\mathrm{subst}_x^t : \mathbb{T} \to \mathbb{T}$.

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Each (co)recursive call of $\operatorname{subst}_{x}^{t}$ occurs *directly* inside a constructor for \mathbb{T} .

Coinductive definitions of relations

$$\frac{t \to t'}{\overline{A(t)} \to A(t')} (2)$$

$$\frac{s \to s' \quad t \to t'}{\overline{B(s,t)} \to B(s',t')} (3) \qquad \frac{t \to t'}{\overline{A(t)} \to B(t',t')} (4)$$

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The relation \to is the greatest fixpoint νF of a function $F: \mathcal{P}(\mathbb{T} \times \mathbb{T}) \to \mathcal{P}(\mathbb{T} \times \mathbb{T})$ defined as follows.

$$F(R) = \{\langle t_1, t_2 \rangle \mid (t_1 \equiv t_2 \equiv x) \lor \\ \exists t, t' (t_1 \equiv A(t) \land t_2 \equiv A(t') \land R(t, t')) \lor \\ \exists s, t, s', t' (t_1 \equiv B(s, t) \land t_2 \equiv B(s', t') \land \\ R(s, s') \land R(t, t')) \lor \\ \exists t, t' (t_1 \equiv A(t) \land t_2 \equiv B(t', t') \land R(t, t')) \}$$

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The relation \rightarrow is the greatest fixpoint νF of a function $F: \mathcal{P}(\mathbb{T} \times \mathbb{T}) \to \mathcal{P}(\mathbb{T} \times \mathbb{T})$ defined as follows.

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F is monotone, i.e., $F(R) \subseteq F(S)$ for $R \subseteq S$.



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 $t \equiv A(t') \rightarrow A(t') \equiv t$ by rule (2).

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Usual coinduction principle

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$$\nu F = \bigcup \{X \in \mathcal{P}(A) \mid X \subseteq F(X)\}.$$

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This yields the following proof principles

$$\frac{F(X) \subseteq X}{\mu F \subseteq X} \text{ (IND)} \quad \frac{X \subseteq F(X)}{X \subseteq \nu F} \text{ (COIND)}$$

where $X \in \mathcal{P}(A)$.

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- $\blacktriangleright \ \nu^{\alpha} {\it F} = \bigcap_{\beta < \alpha} \nu^{\beta} {\it F} \ {\it if} \ \alpha \ {\it is a limit ordinal}.$

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There exists an ordinal ζ such that $\nu^{\zeta}F = \nu F$.

(In all definitions in the paper we actually have $\zeta=\omega$)

Coinductive definitions of relations

Where $\rightarrow^{\alpha} = \nu^{\alpha} F$.

$$\frac{t \to^{\alpha} t'}{A(t) \to^{\alpha+1} A(t')} (2)$$

$$\frac{s \to^{\alpha} s' \quad t \to^{\alpha} t'}{B(s,t) \to^{\alpha+1} B(s',t')} (3) \qquad \frac{t \to^{\alpha} t'}{A(t) \to^{\alpha+1} B(t',t')} (4)$$

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• $\alpha = 0$: $\forall x \in A(\varphi(x) \to x \in A)$ holds trivially.

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- ▶ $\alpha = 0$: $\forall x \in A(\varphi(x) \to x \in A)$ holds trivially.
- ▶ α a limit ordinal: $\forall x \in A(\varphi(x) \to x \in \bigcap_{\beta < \alpha} \nu^{\beta} F)$ is equivalent to the inductive hypothesis $\forall \beta < \alpha \forall x \in A(\varphi(x) \to x \in \nu^{\beta} F)$.

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- \blacktriangleright So it remains to show the inductive step for α a successor ordinal.

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We show: for all $t \in \mathbb{T}$, $t \to t$.

If $t \equiv x$ then this follows by rule (1).

If $t \equiv A(t')$ then $t' \to t'$ by the coinductive hypothesis, so $t \equiv A(t') \to A(t') \equiv t$ by rule (2).

If $t \equiv B(t_1, t_2)$ then $t_1 \to t_1$ and $t_2 \to t_2$ by the coinductive hypothesis, so $t \to t$ by rule (3).

Sample coinductive proof

We show: for all $t \in \mathbb{T}$, $t \to^{\alpha} t$ (just the inductive step for $\alpha + 1$).

If $t \equiv x$ then this follows by rule (1).

If $t \equiv A(t')$ then $t' \to {}^{\alpha} t'$ by the coinductive hypothesis, so $t \equiv A(t') \to {}^{\alpha+1} A(t') \equiv t$ by rule (2).

If $t \equiv B(t_1, t_2)$ then $t_1 \to^{\alpha} t_1$ and $t_2 \to^{\alpha} t_2$ by the coinductive hypothesis, so $t \to^{\alpha+1} t$ by rule (3).

Existentials

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How to show statements of the following form?

$$\forall x, y, z \in A(\varphi(x, y, z) \to \exists a \in A(\langle x, a \rangle \in \nu F \land \langle y, a \rangle \in \nu F))$$

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Instead show

$$\forall x, y, z \in A(\varphi(x, y, z) \to (\langle x, f(x, y, z) \rangle \in \nu F \land \langle y, f(x, y, z) \rangle \in \nu F))$$

for some function $f: A^3 \to A$.

Sample coinductive proof with existentials

We show: for all $s, t, t' \in \mathbb{T}$, if $s \to t$ and $s \to t'$ then there exists $s' \in \mathbb{T}$ with $t \to s'$ and $t' \to s'$.

Sample coinductive proof with existentials

We show: for all $s, t, t' \in \mathbb{T}$, if $s \to t$ and $s \to t'$ then there exists $s' \in \mathbb{T}$ with $t \to s'$ and $t' \to s'$.

It suffices to find a function $f : \mathbb{T} \times \mathbb{T} \to \mathbb{T}$ such that:

 (\star) if $s \to t$ and $s \to t'$ then $t \to f(t, t')$ and $t' \to f(t, t')$.

Sample coinductive proof with existentials

The rules for \rightarrow suggest a definition of f:

```
\begin{array}{rcl} f(x,x) & = & x \\ f(A(t),A(t')) & = & A(f(t,t')) \\ f(A(t),B(t',t')) & = & B(f(t,t'),f(t,t')) \\ f(B(t,t),A(t')) & = & B(f(t,t'),f(t,t')) \\ f(B(t,t),B(t',t')) & = & B(f(t,t'),f(t,t')) \\ f(B(t_1,t_2),B(t'_1,t'_2)) & = & B(f(t_1,t'_1),f(t_2,t'_2)) \\ f(t,t') & = & \text{some arbitrary term} \\ & & \text{if none of the above matches} \end{array}
```

The definition is guarded, so f is well-defined.

Sample coinductive proof with existentials

We show one case of a coinductive proof of:

$$(\star)$$
 if $s \to t$ and $s \to t'$ then $t \to f(t, t')$ and $t' \to f(t, t')$.

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If
$$s\equiv A(s_1)$$
, $t\equiv A(t_1)$ and $t'\equiv A(t_1')$ with $s_1\to t_1$ and $s_1\to t_1'$,

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If $s \equiv A(s_1)$, $t \equiv A(t_1)$ and $t' \equiv A(t_1')$ with $s_1 \to t_1$ and $s_1 \to t_1'$, then by the coinductive hypothesis $t_1 \to f(s_1, t_1, t_1')$ and $t_1' \to f(s_1, t_1, t_1')$.

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Sample coinductive proof with existentials – shorter formulation

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We show one case of a coinductive proof of: if $s \to t$ and $s \to t'$ then there exists $s' \in \mathbb{T}$ with $t \to s'$ and $t' \to s'$.

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If $s \equiv A(s_1)$, $t \equiv A(t_1)$ and $t' \equiv A(t'_1)$ with $s_1 \to t_1$ and $s_1 \to t'_1$, then by the coinductive hypothesis we obtain s'_1 such that $t_1 \to s'_1$ and $t'_1 \to s'_1$.

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The definition of $f: \mathbb{T} \times \mathbb{T} \to \mathbb{T}$ is left implicit and follows straightforwardly from the given proof.

Definitions

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Definition (Terms)

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Definition (Infinitary β -reduction)

$$\frac{s \to_{\beta}^* x}{\overline{s \to_{\beta}^{\infty} x}}$$

$$\frac{s \to_{\beta}^* t_1 t_2 \quad t_1 \to_{\beta}^{\infty} t_1' \quad t_2 \to_{\beta}^{\infty} t_2'}{s \to_{\beta}^{\infty} t_1' t_2'}$$

$$\frac{s \to_{\beta}^* \lambda x.r \quad r \to_{\beta}^{\infty} r'}{s \to_{\beta}^{\infty} \lambda x.r'}$$

Equivalence with strongly convergent reductions

Theorem (Endrullis, Polonsky. TYPES 2011)

 $s \to_{\beta}^{\infty} t$ iff there exists a strongly convergent β -reduction from s to t

Infinitary lambda-calculus Definitions

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t is root-stable if either

- ▶ $t \equiv x$ with $x \not\equiv \bot$, or
- $ightharpoonup t \equiv \lambda x.t'$, or
- ▶ $t \equiv t_1 t_2$ and there does not exist s such that $t_1 \to_{\beta}^{\infty} \lambda x.s.$

Definitions

Definition (Root-stable)

t is root-stable if either

- ▶ $t \equiv x$ with $x \not\equiv \bot$, or
- $ightharpoonup t \equiv \lambda x.t'$, or
- $t \equiv t_1 t_2$ and there does not exist s such that $t_1 \to_{\beta}^{\infty} \lambda x.s.$

Definition (Root-active)

t is $root\text{-}active}$ if there does not exist a root-stable s such that $t\to_\beta^\infty s$

Definitions

Definition (Equivalence of root-active subterms)

$$\frac{t, s \text{ are root-active}}{t \sim s}$$

$$\frac{t \sim s}{\overline{\lambda}x.t \sim \lambda x.s}$$

$$\frac{t_1 \sim s_1 \quad t_2 \sim s_2}{t_1t_2 \sim s_1s_2}$$

Main result

Theorem (Confluence of infinitary β -reduction up to equivalence of root-active subterms)

If $t \sim t'$, $t \to_{\beta}^{\infty} s$ and $t' \to_{\beta}^{\infty} s'$, then there exist r, r' such that $s \to_{\beta}^{\infty} r$, $s' \to_{\beta}^{\infty} r'$ and $r \sim r'$.

Main result

Theorem (Confluence of infinitary β -reduction up to equivalence of root-active subterms)

If $t \sim t'$, $t \to_{\beta}^{\infty} s$ and $t' \to_{\beta}^{\infty} s'$, then there exist r, r' such that $s \to_{\beta}^{\infty} r$, $s' \to_{\beta}^{\infty} r'$ and $r \sim r'$.

This was already obtained by Kennaway, Klop, Sleep and de Vries in 1995, but we give a different coinductive proof.

Böhm reduction - definitions

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Definition (⊥-rules)

 $t \rightarrow \bot$ if t is root-active and $t \not\equiv \bot$

Böhm reduction - definitions

Definition (⊥-rules)

$$t \rightarrow \bot$$
 if t is root-active and $t \not\equiv \bot$

Definition (Infinitary Böhm reduction)

$$\frac{s \to_{\beta \perp}^* x}{s \to_{\beta \perp}^\infty x}$$

$$\frac{s \to_{\beta \perp}^* t_1 t_2 \quad t_1 \to_{\beta \perp}^\infty t_1' \quad t_2 \to_{\beta \perp}^\infty t_2'}{s \to_{\beta \perp}^\infty t_1' t_2'}$$

$$\frac{s \to_{\beta \perp}^* \lambda x. r \quad r \to_{\beta \perp}^\infty r'}{s \to_{\beta \perp}^\infty \lambda x. r'}$$

Böhm reduction

Theorem (Confluence of infinitary Böhm reduction)

If $t \to_{\beta\perp}^{\infty} t_1$ and $t \to_{\beta\perp}^{\infty} t_2$ then there exists t_3 such that $t_1 \to_{\beta\perp}^{\infty} t_3$ and $t_2 \to_{\beta\perp}^{\infty} t_3$.

Böhm reduction

Theorem (Confluence of infinitary Böhm reduction)

If $t \to_{\beta\perp}^{\infty} t_1$ and $t \to_{\beta\perp}^{\infty} t_2$ then there exists t_3 such that $t_1 \to_{\beta\perp}^{\infty} t_3$ and $t_2 \to_{\beta\perp}^{\infty} t_3$.

This was also obtained by Kennaway et al., but our proof uses coinduction.

Confluence proof

To prove confluence of infinitary β -reduction up to equivalence of root-active subterms, we introduce ϵ -calculus (similar to the ϵ -calculus in Kennaway et al.).

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Definition (ϵ -contraction)

$$\epsilon^n(\lambda x.s)t \to_{\epsilon} \epsilon(s[t/x])$$

Confluence proof

Definition (Parallel ϵ -reduction)

$$\frac{s \to_1 s'}{\overline{\lambda x.s} \to_1 \lambda x.s'}$$

$$\frac{s \to_1 s' \quad t \to_1 t'}{st \to_1 s't'} \qquad \frac{t \to_1 t'}{\overline{\epsilon(t)} \to_1 \epsilon(t')}$$

$$\frac{t_1[t_2/x] \to_1 t'}{\overline{\epsilon^n(\lambda x.t_1)t_2 \to_1 \epsilon(t')}}$$

Confluence proof

Definition (Infinitary ϵ -reduction)

$$\frac{s \to_{\epsilon}^{\infty} s'}{\overline{\lambda x. s \to_{\epsilon}^{\infty} \lambda x. s'}}$$

$$\frac{s \to_{\epsilon}^{\infty} s'}{st \to_{\epsilon}^{\infty} s' t'}$$

$$\frac{s \to_{\epsilon}^{\infty} s' t \to_{\epsilon}^{\infty} t'}{s \to_{\epsilon}^{\infty} s' t'}$$

$$\frac{s \to_{\epsilon}^{\infty} s' t}{s \to_{\epsilon}^{\infty} s' t'}$$

$$\frac{s \to_{\epsilon}^{*} s(t) t \to_{\epsilon}^{\infty} t'}{s \to_{\epsilon}^{\infty} s(t')}$$

Confluence proof

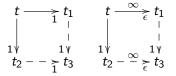
Definition (Erasure)

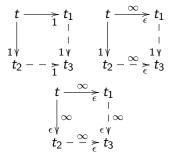
$$\frac{s' \text{ is root-active}}{\epsilon^{n}(x) \succ x}$$

$$\frac{s \succ s'}{\epsilon^{n}(\lambda x.s) \succ \lambda x.s'}$$

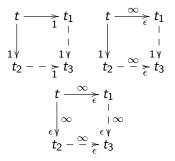
$$\frac{s \succ s' \qquad t \succ t'}{\epsilon^{n}(st) \succ s't'}$$







Confluence proof



 $t_1 \sim t_2$ iff there exists s with $s \succ t_1$ and $s \succ t_2$

