Notes on WL-WAL Data Analysis

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1 Hikami-Larkin-Nagaoka Theory

The conductance in magnetic field H, when the scattering can be due to 1) normal impurities (τ_ϵ), 2) spin-orbit scattering (τ_{SO}), 3) scattering from localized spins (τ_S), is given by the combination of digamma functions

$$\sigma = \sigma_0 - \frac{e^2}{2\pi^2\hbar}\Big[\psi\Big(\frac{1}{2}+\frac{1}{a\tau}\Big) - \psi\Big(\frac{1}{2}+\frac{1}{a\tau_1}\Big) + \frac{1}{2}\psi\Big(\frac{1}{2}+\frac{1}{a\tau_2}\Big) - \frac{1}{2}\psi\Big(\frac{1}{2}+\frac{1}{a\tau_3}\Big)\Big], \tag{1}$$

where

$$\frac{1}{\tau} = \frac{1}{\tau_{SO}^z} + \frac{2}{\tau_{SO}^x} + \frac{1}{\tau_S^z} + \frac{2}{\tau_S^x} + \frac{1}{\tau_\epsilon} \tag{2}$$

$$\frac{1}{\tau_1} = \frac{1}{\tau_{SO}^z} + \frac{2}{\tau_{SO}^x} + \frac{2}{\tau_S^x} + \frac{1}{\tau_\epsilon} \tag{3}$$

$$\frac{1}{\tau_2} = \frac{2}{\tau_S^z} + \frac{4}{\tau_S^x} + \frac{1}{\tau_\epsilon} \tag{4}$$

$$\frac{1}{\tau_3} = \frac{2}{\tau_S^z} + \frac{4}{\tau_{SO}^x} + \frac{1}{\tau_\epsilon} \tag{5}$$

and

$$a = \frac{4DeH}{\hbar c}, \quad H \text{ is a diffusion constant.} \tag{6}$$

Depending on the relative and absolute magnitude of the parameters

$$a\tau_i = \frac{4DeH\tau_i}{\hbar c} = \frac{B}{B_i}, \quad B_i \equiv \frac{\hbar c}{4De\tau_i} \tag{7}$$

the expression for $\sigma(B)$ may simplify. For example, an often used approximation is when any one scattering mechanism dominates and the condition $1/a\tau \gg 1$ is satisfied, in which case the conductance is given by

$$\sigma(B) = \sigma(0) - \frac{\alpha e^2}{2\pi^2\hbar} f\left(\frac{B_\epsilon}{B}\right), \tag{8}$$

where

$$f(x) = \ln(x) - \psi\left(\frac{1}{2}+x\right). \tag{9}$$

Using the asymptotic series expansion for $\psi(x)$, valid for large x (i.e. when $B \ll B_\epsilon$)

$$\psi(x) = \ln(x) - \frac{1}{x} - \sum_{n=1}^{\infty} \frac{\beta_{2n}}{2nx^{2n}}, \quad \beta_k \text{ (is the } k\text{-th Bernoulli number)} \tag{10}$$

we get the following polynomial approximation of the $f(B_\epsilon/B)$ at $B \ll B_\epsilon$

$$f(B_\epsilon/B) \approx -\frac{1}{24}\left(\frac{B}{B_\epsilon}\right)^2 + \frac{7}{960}\left(\frac{B}{B_\epsilon}\right)^4 - \frac{31}{8064}\left(\frac{B}{B_\epsilon}\right)^6 \tag{11}$$

It must be noted that polynomial approximation fails miserably for $B \geq B_\epsilon$. For the

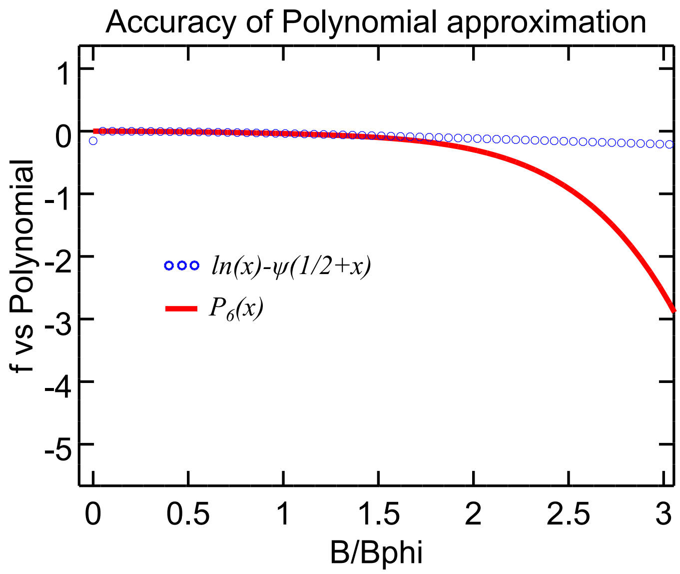

Figure 1: Comparison of $f(B)$ with the polynomial approximation of the 6th degree. At very low fields the truncated Hikami-Larkin-Nagaoka function $\ln(x) - \psi(1/2 + x)$ looks like parabola.

fields $B \gg B_\epsilon$ the conductance increases logarithmically (see also HLN paper, Eq. (19)):

$$\sigma(B) = \sigma(0) - \frac{\alpha e^2}{2\pi^2\hbar} \ln \frac{B}{B_\epsilon} \tag{12}$$

If there is any additional contribution into magnetoconductance, like normal quadratic, then it will most likely dominate the logarithmic term.

In summary, the validity of the truncated expression for the conductance

$$\sigma(B) = \sigma(0) - \frac{\alpha e^2}{2\pi^2\hbar}\left[\ln\left(\frac{B_\epsilon}{B}\right) - \psi\left(\frac{1}{2} + \frac{B_\epsilon}{B}\right)\right] \tag{13}$$

requires that the total scattering rate τ is large enough for the condition $1/a\tau \gg 1$ to be satisfied. From the definition of τ it follows that to apply the truncated expression we need

$$\frac{1}{a\tau_{SO}^{z}} + \frac{2}{a\tau_{SO}^{x}} + \frac{1}{a\tau_{S}^{z}} + \frac{2}{a\tau_{S}^{x}} + \frac{1}{a\tau_{\epsilon}} \gg 1. \tag{14}$$

If the spin-orbit interaction is very strong (like in TIs), this requirement is satisfied due to the large values of the τ_{SO}-related terms. The magnitude of α, according to Hikami-Larkin-Nagaoka, indicates which mechanism is diminant in scattering: $\alpha = 1$ corresponds to the absence of spin-orbit and magnetic scattering; 2) $\alpha = 0$ corresponds to strong magnetic scattering; 3) $\alpha = -1/2$ corresponds to absence of magnetic scattering and strong spin-orbit interactions.

It must be kept in mind that "*In strict two dimensions, the spin-orbit interaction has only the z-component* ($1/\tau_{SO}^{x} = 0$, $1/\tau_{SO}^{z} \neq 0$)*. Then, if* $1/\tau_S = 0$*, the behavior of the system becomes the same as unitary case without logarithmic term*". It means that $\alpha = 0$ may also correspond to strict two-dimensions with strong spin-orbit and without magnetic scattering!

2 Data Fitting Strategies

The most straightforward, although not necessarily the easiest, strategy would be to fit the experimental data

$$\frac{\sigma_0 - \sigma(B)}{G_0/2\pi}, \quad \left(G_0 = \frac{2e^2}{h} - \text{conductance quantum}\right) \tag{15}$$

with the four-parameter function

$$s(B) = \psi\left(\frac{1}{2} + \frac{B_1}{B}\right) - \psi\left(\frac{1}{2} + \frac{B_2}{B}\right) + \frac{1}{2}\psi\left(\frac{1}{2} + \frac{B_3}{B}\right) - \frac{1}{2}\psi\left(\frac{1}{2} + \frac{B_4}{B}\right). \tag{16}$$

The number of independent parameters is actually three (3) since there is a relation between B_1, B_2, B_3, and B_4 such that the condition $s(0) = 0$ is satisfied. Indeed, considering that $\psi(1/2 + x) \approx \ln(x)$ for large x (small fields) we have

$$s(0) = \ln\left(\frac{B_1}{B}\right) - \ln\left(\frac{B_2}{B}\right) + \frac{1}{2}\ln\left(\frac{B_3}{B}\right) - \frac{1}{2}\ln\left(\frac{B_4}{B}\right) = \frac{1}{2}\ln\frac{B_1^2 B_3}{B_2^2 B_4} = 0 \tag{17}$$

so

$$\frac{B_1^2 B_3}{B_2^2 B_4} = 1 \quad \rightarrow \quad B_4 = B_3\left(\frac{B_1}{B_2}\right)^2. \tag{18}$$