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Chris Walshaw,

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A MULTILEVEL APPROACH TO THE TRAVELLING SALESMAN PROBLEM

CHRIS WALSHAW

Computing and Mathematical Sciences, University of Greenwich, Old Royal Naval College, Greenwich, London, SE10 9LS, United Kingdom, c.walshaw@gre.ac.uk

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We motivate, derive, and implement a multilevel approach to the travelling salesman problem. The resulting algorithm progressively coarsens the problem, initialises a tour, and then employs either the Lin-Kernighan (LK) or the Chained Lin-Kernighan (CLK) algorithm to refine the solution on each of the coarsened problems in reverse order. In experiments on a well-established test suite of 80 problem instances we found multilevel configurations that either improved the tour quality by over 25% as compared to the standard CLK algorithm using the same amount of execution time, or that achieved approximately the same tour quality over seven times more rapidly. Moreover, the multilevel variants seem to optimise far better the more clustered instances with which the LK and CLK algorithms have the most difficulties.

1. INTRODUCTION

In this paper we address the Travelling Salesman Problem (TSP), which can be simply stated as follows: given a collection of "cities," find the shortest tour that visits all of them and returns to the starting point. Typically the cities are given coordinates in the plane and then the tour length is measured by the sum of Euclidean distances between each pair on the tour. However, in the more general form, the problem description simply requires a metric that specifies the distance between every pair of cities.

In particular, here we consider the problem of finding low cost tours in reasonable time rather than solving the problem to optimality. We also focus on the Euclidean version of distance and, by default therefore, the symmetric TSP. In other words, if $d(c_1, c_2)$ is the Euclidean distance between cities c_1 and c_2 then $d(c_1, c_2) = d(c_2, c_1)$, and the tour can be executed in either direction for the same cost. However in §5.1 we discuss how our approach might easily be extended to a more general distance metric.

The TSP, a combinatorial optimisation problem, has been shown to be NP-hard (Garey and Johnson 1979) but has a number of features which make it stand out amongst such problems. Firstly, and perhaps because of the fact that the problem is so intuitive and easy to state, it has almost certainly been more widely studied than any other NP-hard combinatorial optimisation problem. For example Johnson and McGeoch (1997) survey a wide range of approaches that run the gamut from local search through simulated annealing, tabu search, and genetic algorithms to neural nets. Remarkably, and despite all this interest, the local search algorithm proposed by Lin and Kernighan (1973) still remains at the heart of the most successful approaches. In fact, Johnson and McGeoch describe the Lin-Kernighan (LK) algorithm as the world champion heuristic for the TSP from 1973 to 1989. Further, this was only conclusively superseded by chained or iterated versions of LK (see §3.3 for clarification), originally proposed by Martin et al. (1991, 1992).

Even recently, in spite of all the work on exotic and complex combinatorial optimisation techniques, Johnson and McGeoch (1997) conclude that an iterated Lin-Kernighan (ILK) scheme provides the highest quality tours for a reasonable cost. This conclusion was backed up by Applegate et al. (2000), who also illustrate the scalability of the algorithm by applying it to random examples containing up to 25,000,000 cities. In fact, it is usually possible to improve on the quality of (suboptimal) chained/iterated LK tours, for example by sophisticated tour merging techniques similar to genetic algorithm crossovers (see, e.g., Applegate et al. 1999), but Johnson and McGeoch suggest that "the ILK variant..., is the most cost effective way to improve on Lin-Kernighan, at least until one reaches stratospheric running times."

Another unusual feature of the TSP is that, for problems that have not yet been solved to optimality (typically with 10,000 or more cities), an extremely good lower bound can be found for the optimal tour length. This bound, known as the Held-Karp Lower Bound (HKLB), was developed in 1970 by Held and Karp (1970, 1971) and usually comes extremely close to known optimal tour lengths (often within 1%—see the results in §4.1). Thus, to measure the quality of an algorithm for a given set of problem instances (especially if some or all of them do not have known optimal tours), we can simply calculate the average percentage excess of tours produced by the algorithm over the HKLB for each instance.

To illustrate this for the instances of the TSP tested in this paper, LK produces tours about 3.86% in excess of the HKLB on average, whilst the chained LK algorithm brings

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Area of review: Optimization.

this down to about a 1.50% excess although on average requires nearly 40 times as long to achieve this. Again, this would appear to be another unusual feature of the TSP, that what are basically local search algorithms can get so close to optimality. (Recall that the HKLB is a *lower* bound, so the results will be even closer to optimality than this.) For example, no such heuristic (and no such lower bound) is known to exist for the graph partitioning problem.

1.1. Overview and Notation

In this paper we describe the motivation, implementation, and testing of a multilevel approach to finding high quality TSP tours. Below we discuss the merits and features of the multilevel paradigm, based on previous multilevel algorithms for the graph partitioning and graph drawing problems, which led us to investigate a similar approach to the TSP. In §3 we give the details of the resulting procedure that was devised and outline the chained LK algorithm, which is used as a basic building block of the scheme. In §4 we test the algorithm on a large suite of problems and attempt to analyse its behaviour. Finally, in §5 we summarise the paper and present some suggestions for further work.

Although we do not require a great deal of notation for this paper, it is worth remarking that we sometimes use graph-based terminology and refer to cities as vertices and intercity distances as edge lengths. We sometimes refer to tours as cycles, and we also use the terms objective function and cost function to denote the tour length, the quantity we are trying to minimise.

2. MULTILEVEL OPTIMISATION

Before describing the strategy we shall attempt to motivate it by describing the process of ideas that led us to conclude that a multilevel strategy might be beneficial for the TSP. Although such process-driven research does not often form a part of the literature, we feel that in this case it is instructive. Broader surveys of the multilevel paradigm and the combinatorial optimisation problems to which it has been applied can be found in Walshaw (2001c, 2002).

2.1. Background

Our interest in the TSP, and in fact behind our approach to the problem, arises from our work in the field of graph partitioning (Walshaw and Cross 2000) and subsequently graph drawing (Walshaw 2001a). Typically a P-way graph partitioning algorithm aims to divide a graph into P disjoint subdomains of equal size and minimise the number of cut edges, another NP-hard combinatorial optimisation problem (Garey et al. 1976). In recent years it has been recognised that an effective way of both accelerating graph partitioning algorithms and, more importantly, giving them a "global" perspective, is to use multilevel techniques. The idea is to match pairs of vertices to form *clusters*, use the clusters to define a new graph, and recursively iterate this procedure until the graph size falls below some threshold. The coarsest graph is then partitioned (often with a crude algorithm), and the partition is successively refined on all the graphs starting with the coarsest and ending with the original. This sequence of contraction followed by repeated refinement loops is known as multilevel partitioning and was successfully developed as a strategy for overcoming the localised nature of the Kernighan-Lin (KL) (1970) and other partition optimisation algorithms. The multilevel partitioning paradigm was first proposed by Barnard and Simon (1994) as a method of speeding up spectral bisection and improved by both Hendrickson and Leland (1995) and Bui and Jones (1993), who generalised it to encompass local refinement algorithms. Several enhancements for carrying out the matching of vertices have been devised by Karypis and Kumar (1998). The multilevel partitioning strategy is widely used and forms the basis of at least four public domain partitioning packages: CHACO, Hendrickson and Leland (1995), JOSTLE, Walshaw and Cross (2000), METIS, Karypis and Kumar (1998), and SCOTCH, Pellegrini and Roman (1996).

In another recent development, multilevel strategies have also been applied to the graph drawing problem (and in particular force-directed placement). Given a graph with no coordinate information, the aim of a graph drawing algorithm is to infer a "nice" layout of the vertices based on the adjacency structure (e.g., a layout with relatively few edge crossings and where edges all have approximately the same length). Force-directed placement (FDP) algorithms achieve this by assuming repulsive forces between every pair of vertices and by modelling the edges as springs, which attempt to maintain their natural length (i.e., neither stretched nor compressed). Strictly speaking, this is not a combinatorial optimisation problem but does share many of the features.

Until recently most FDP algorithms were at least $O(N^3)$ in complexity and were unable to deal with large graphs (for example, in a comprehensive survey of graph drawing from 1998, one FDP algorithm was singled out as exceptional in that it could handle graphs with over 1,000 vertices; see Di Battista et al. 1998). However, Harel and Koren (2001) have used multiscale ideas in combination with an FDP algorithm and are able to easily handle graphs of 6,000 vertices (although their algorithm still contains an $O(N^2)$ component). Meanwhile, Walshaw (2001a) has independently applied multilevel techniques to the same problem (although using a different FDP algorithm) and presents examples with up to 225,000 vertices.

Most recently a multilevel graph colouring approach was tested (Walshaw 2001b). The graph colouring problem is to colour the vertices of a graph so that adjacent vertices have different colours and a minimum number of colours are used. It is known as one of the most challenging NP-hard combinatorial optimisation problems but has a wide range of applications including scheduling and timetabling. Although the multilevel approach was not universally successful for colouring, it did manage to enhance

the performance of the two local search heuristics under investigation, tabu search and the iterated greedy algorithm, for graphs with low edge density (less than about 30% density), a subgroup into which many real-life scheduling applications fall, e.g., Leighton (1979). This augments existing evidence that, although the multilevel framework cannot be considered as a panacea for combinatorial optimisation problems, it can provide a useful addition to the combinatorial optimisation toolkit.

2.2. The Generic Multilevel Paradigm

The important questions about these approaches are: Why do multilevel approaches appear to work, and is there an abstraction of the paradigm that can be applied to other combinatorial optimisation problems (such as the TSP)?

Considered from the point of view of the multilevel procedure, a series of increasingly smaller and coarser versions of the original problem are being constructed. It is hoped that each problem P_l retains the important features of its parent P_{l-1} but the (usually) randomised and irregular nature of the coarsening precludes any rigorous analysis of this process.

On the other hand, viewing the multilevel process from the point of view of the optimisation problem and, in particular, the objective function is considerably more enlightening. For a given problem instance the *solution space*, $\mathscr{X}$, is the set of all possible solutions for that instance. The *objective function* or *cost function*, $f : \mathscr{X} \to \mathbb{R}$, assigns a cost to each solution in $\mathscr{X}$ (e.g., in the case of the TSP $f(x)$ expresses the length of the tour x). Typically, the aim of the problem is to find a state $x \in \mathscr{X}$ at a minimum (or maximum) of the objective function. Iterative refinement algorithms usually attempt to do this by moving stepwise through the solution space (which is hence also known as a *search space*) but often can become trapped in local minima of f.

Suppose then for the partitioning problem that two vertices $u, v \in G_{l-1}$ are matched and coalesced into a single vertex $v' \in G_l$ (where G_l denotes the graph at coarsening level G). When a partition refinement algorithm is subsequently used on G_l and whenever v' is assigned to a subdomain, both u and v are also both being assigned to that subdomain. In this way the partitioning of G_l is being restricted to consider only those configurations in the solution space in which u and v lie in the *same* subdomain. Because many vertex pairs are generally coalesced from all parts of G_{l-1} to form G_l, this set of restrictions is in some way a sampling of the solution space and hence the surface of the objective function.

This is an important point (see also Walshaw 2000). Previously authors have made a case for multilevel partitioning on the basis that the coarsening successively *approximates* the problem. In fact, it is somewhat better than this; the coarsening *samples* the solution space by placing restrictions on which states the refinement algorithm can visit. Furthermore, this methodology is not confined to multilevel partitioning but can be applied to other combinatorial optimisation problems.

We can then hypothesise that, if the coarsening manages to sample the solution space so as to gradually *smooth* the objective function, the multilevel representation of the problem combined with an iterative refinement algorithm should work well as an optimisation *meta-heuristic*. In other words, by coarsening and smoothing the problem, the multilevel component allows the refinement algorithm to find regions of the solution space where the objective function has a low average value (e.g., broad valleys). This does rely on a certain amount of "continuity" in the objective function, but it is not unusual for these sort of problems that changing one or two components of the solution tends not to change the cost very much.

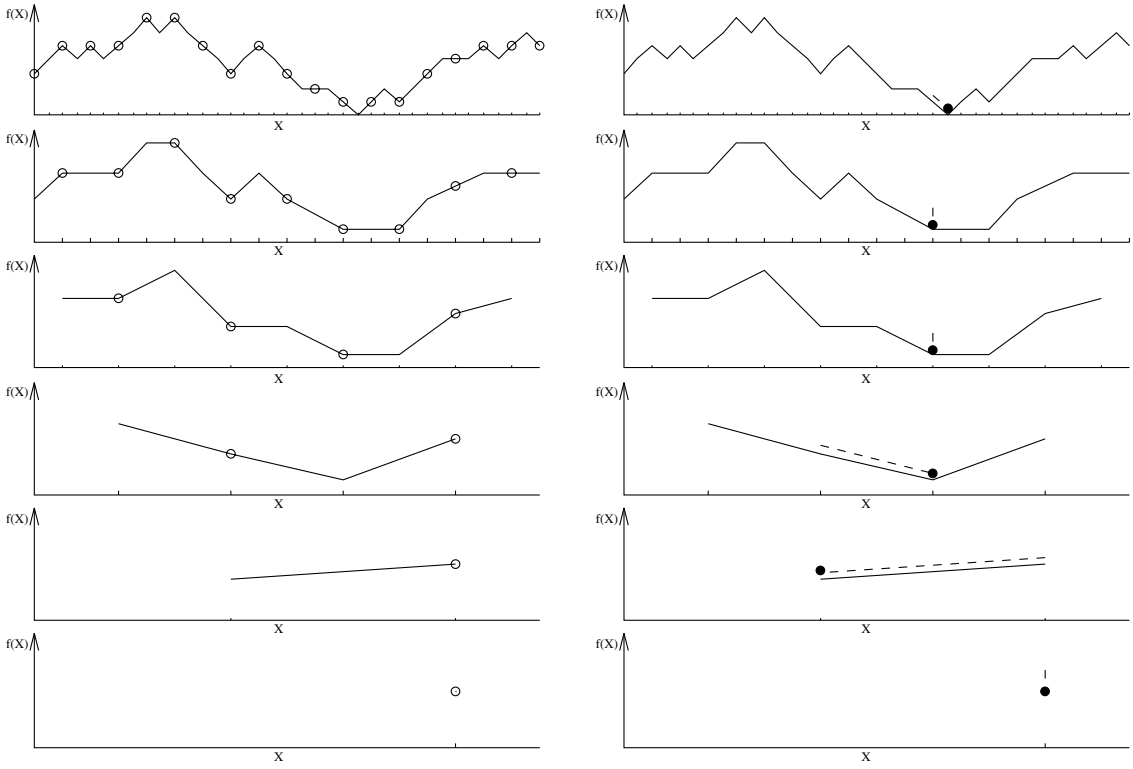
Figure 1 shows an example of how this might work for a search space $\mathscr{X}$ and objective function $f(\mathscr{X})$ which we aim to minimise. On the left-hand side the objective function is gradually sampled and smoothed (the sampled points are circled and all intermediate values removed to give the next coarsest representation). The initial solution for the final coarsened space (shown as a black dot in the bottom right-hand figure) is then trivial (because there is only one possible state), although the resulting configuration is not an optimal solution to the overall problem. However, this state is used as an initial configuration for the next level up, and a *steepest descent* refinement policy finds the nearest local minimum. (Steepest descent refinement will move to a neighbouring configuration only if the value of the objective function is lower there.) Recursing this process keeps the best found solution (indicated by the black dot) in the same region of the solution space. Finally, this gives a good initial configuration for the original problem, and (in this case) the optimal solution can be found. Note, however, that it is possible to pick a different set of sampling points for this example for which the steepest descent policy will fail to find the global minimum, but this only indicates, as might be expected, that the multilevel procedure is somewhat sensitive to the coarsening strategy.

Of course, this motivational example might be considered trivial or unrealistic; in particular, an objective function cannot normally be pictured in two dimensions. However, consider other meta-heuristics such as repeated random starts combined with steepest descent local search, or even simulated annealing, applied to this same objective function; without lucky initial guesses, either might require many iterations to solve this simple problem.

It should be stressed that this hypothesis is nothing more than speculation, and we cannot prove that this process underlies the multilevel paradigm. However, experimental evidence, here and elsewhere (see, e.g., Walshaw 2001c), suggests that the multilevel approach does indeed enhance local search strategies, and we suspect that the sampling/smoothing of the objective function contributes to this.

To summarise the paradigm, multilevel optimisation combines a coarsening strategy together with a refinement

Figure 1. The multilevel scheme in terms of a simple objective function.



algorithm (employed at each level in reverse order) to provide an optimisation meta-heuristic. Figure 2 contains a schematic of this process in pseudocode. Here P_l refers to the coarsened problem after l coarsening steps, $C_l\{P_l\}$ is a solution of this problem and $C_l^0\{P_l\}$ denotes the initial solution.

2.3. Algorithmic Requirements

Assuming that the above analysis does contain some elements of truth, how can we implement a multilevel strategy to test it on a given combinatorial optimisation problem?

First of all, let us assume that we know of a refinement algorithm for the problem, which refines in the sense it can reuse an existing solution and (attempt to) improve it. Typically the refinement algorithm will be a local search strategy that can explore only small regions of the solution space neighbouring to the current solution; however, there is no reason (other than execution time) why it should not be a more sophisticated scheme such as simulated annealing or a genetic algorithm. The refinement algorithm must also be able to cope with any additional restrictions placed on it by using a coarsened problem (e.g., in graph partitioning, the coarser graphs are always weighted, whether the original is or is not). If such a refinement algorithm does not exist (e.g., if the only known heuristics for the problem are based on construction rather than refinement), it is not clear that the multilevel paradigm can be applied.

Figure 2. The multilevel optimisation algorithm.

```
multilevel refinement(input problem instance P_0 , output solution C_0{P_0} )
begin
        l := 0
        while (coarsening)
                P_{l+1} = coarsen( P_l )
                l := l+1
        end
        C_l{P_l} = initialise( P_l )
        while (l > 0)
                l := l-1
                C_l^0{P_l} = extend( C_{l+1}{P_{l+1}}, P_l )
                C_l{P_l} = refine( C_l^0{P_l}, P_l )
        end
end
```

To implement a multilevel algorithm, given a problem and a refinement strategy for it, we then require three additional basic components: a coarsening algorithm, an initialisation algorithm, and an extension algorithm, which takes the solution on one problem and extends it to the parent problem. It is difficult to talk in general terms about these requirements, but the existing examples suggest that the extension is a simple and obvious reversal of the coarsening step that preserves the same cost, e.g., a pair of parent vertices are assigned to the same solution state as their child. The initialisation is also generally a simple canonical mapping, e.g., for partitioning assign P vertices to P subdomains. By canonical we mean that a (nonunique) solution is "obvious" and that the refinement algorithm cannot possibly improve on the initial solution at the coarsest level (because there are no degrees of freedom).

This leaves only the coarsening algorithm, which is then perhaps the key component of a multilevel optimisation

implementation. From existing examples two principles seem to hold:

(C1) Any solution in any of the coarsened spaces should induce a legitimate solution on the original space. Thus, at any stage after initialisation, the current solution could simply be extended through all the problem levels to achieve a solution of the original problem. Furthermore, both solutions (in the coarse space and the original space) should have the same cost with respect to the objective function. This requirement ensures that the coarsening is sampling the solution space (rather than approximating and/or distorting it).

(C2) The number of levels need not be determined a priori, but coarsening should cease when any further coarsening would render the initialisation degenerate.

This still does not tell us *how* to coarsen a given problem. So far, most solutions for the partitioning problem have employed a gradual and fairly uniform reduction, and it has been shown that it is usually more profitable for the coarsening to respect the objective function in some sense (see, e.g., the heavy edge matching strategy in Karypis and Kumar 1998). In this respect it seems likely that the most difficult aspect of finding an effective multilevel algorithm for a given problem and given refinement scheme is the (problem-specific) task of devising the coarsening strategy.

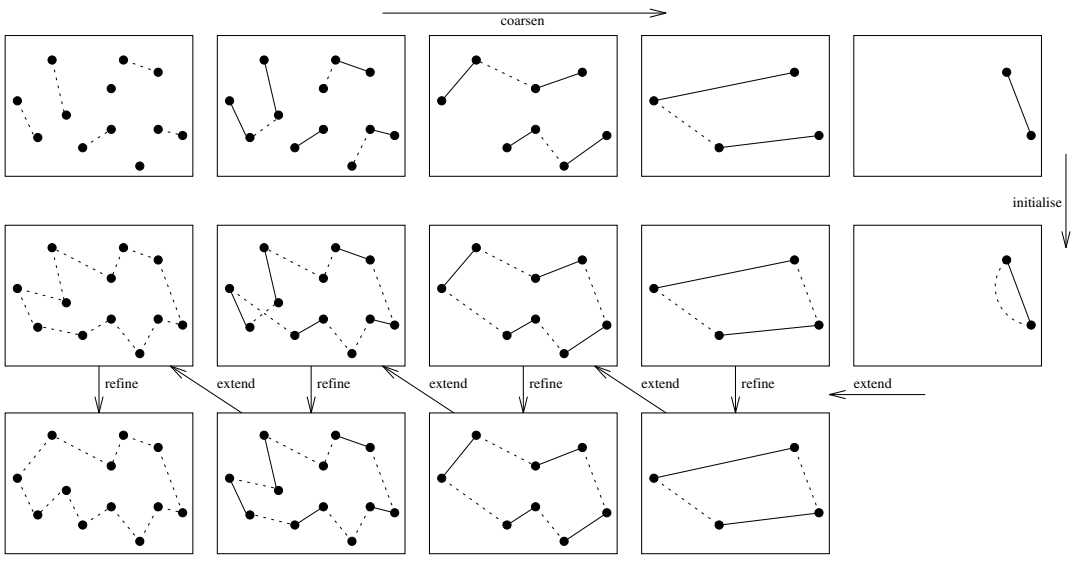
3. A MULTILEVEL ALGORITHM FOR THE TRAVELLING SALESMAN PROBLEM

Having motivated the approach, the next question is: How can the multilevel paradigm be applied to the TSP? Clearly the LK or CLK/ILK algorithms will make a good refinement method, although in principle any iterative refinement procedure including the well-known 2-opt (Croes 1958) and 3-opt (Lin 1965), algorithms could be used. However, with no graph as such, how can the problem be coarsened?

In fact, it seems that the crucial point in devising a coarsening algorithm is the above requirement (C1)—that the solution to each coarsened problem must contain a solution of the original problem (even if it is a poor solution). One way of achieving this is for the coarsening to successively fix edges into the tour. For example, given a TSP instance P of size N, if we fix an edge between cities c_a and c_b, then we create a smaller problem P' of size $N-1$ (because there are $N-1$ edges to be found) where we insist that the final tour of P' must somewhere contain the fixed edge (c_a, c_b). Having found a tour T' for P', we can then return to P and look for better tours using T' as the initial tour. This process is once again equivalent to restricting the solution space (to all tours that contain the edge (c_a, c_b)), and in fact by fixing many distinct edges in one coarsening step we are again sampling the solution space.

Figure 3 shows an example of this. The top row demonstrates the coarsening process where dotted lines represent matchings of vertices (and hence new fixed edges) that are being made in the current coarsening step whilst solid lines represent fixed edges that have been created in previous coarsening steps. Notice, in particular, that from the third step onward, chains of fixed edges are reduced down to a single edge with a vertex at either end, and any vertices internal to such a chain are removed. The coarsening terminates when the problem is reduced to one fixed edge and two vertices, and at this point the tour is initialised. The initialisation is trivial and merely consists of completing the cycle by adding an edge between the two remaining vertices. At this point we could just expand all the fixed edges and get a legitimate tour (and in this sense the coarsening takes the place of an initial tour construction algorithm). However, now the procedure commences the extend/refine loop (as shown in the second and third rows of Figure 3). Again solid lines represent fixed edges whilst dotted lines represent free edges, which may be changed by

Figure 3. An example of a multilevel TSP algorithm at work.



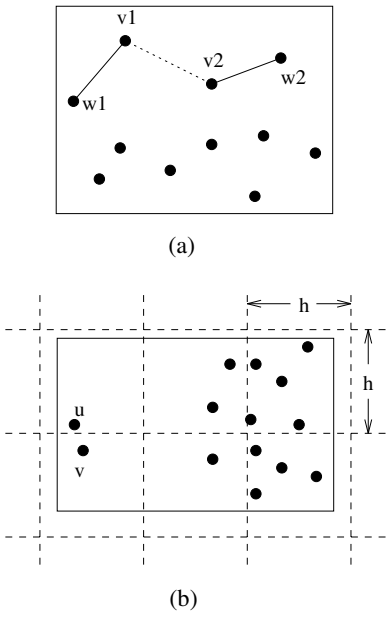
the refinement. The extension itself is straightforward; we simply expand all fixed edges created in the corresponding coarsening step and add the free edges to give an initial tour for the refinement process. The refinement algorithm then attempts to improve on the tour (without changing any of the fixed edges), although notice that for the first two refinement steps in the figure no improvement is possible. The final tour is shown at the bottom left of the figure; note, in particular, that fixing any edge during coarsening does not force it to be in the final tour, since for the final refinement step all edges are free to be changed. However, fixing an edge early on in the coarsening does give it fewer possibilities for being flipped.

3.1. Matching and Coarsening

3.1.1. Data Structures. To implement the above process, although we are matching vertices and then fixing edges between matched pairs, it is more convenient within the code to regard this as the matching of fixed edges. For example, having matched vertex v_1 with vertex v_2 in Figure 4(a) we do not want vertices w_1 and w_2 (the vertices already fixed to v_1 and v_2) to match with any other vertices. Although there is no intrinsic reason why they should not, we feel that this might coarsen the problem too rapidly (by building long multiedge fragments in a single step) and thus miss out on the benefits of the multilevel strategy. This policy also avoids the need to check that a given matching will not create a subcycle (a tour through a subset of the vertices) by matching a series of fixed edges together in a loop.

Our data structure for handling the coarsening thus consists of edge objects. Initially each edge is of zero length and has the same vertex at either end; however, after the first coarsening most edges will have different vertices at either end. Once a given vertex v_1 at one end of edge (v_1, w_1) is matched with another vertex v_2 from edge (v_2, w_2), then all involved vertices v_1, v_2, w_1, w_2 (although some of these may be the same) are prevented from matching again during the same coarsening step.

Figure 4. Matching examples.



3.1.2. Matching. The aim during the matching process should be to fix those edges that are most likely to appear in a high quality tour, thus allowing the refinement to concentrate on the others. For example, consider Figure 4(b); it is difficult to imagine an optimal tour that does not include the edge (u, v), and so ideally the matching should fix it early on in the process. Indeed, if by some good fortune the matching selected *only* optimal edges, then the optimal tour would be found by the end of the matching process and the refinement would have no possible improvements. However, in the absence of any other information about the optimal tour, we have chosen to match vertices with their nearest neighbours.

The computation of matchings of this type is known as the minimum-weight perfect matching problem and polynomial algorithms that can compute optimal matchings are available, e.g., Cook and Rohe (1999). However, they are too expensive for our application (e.g., three minutes for a 100,000-vertex problem), and so we use an algorithm that is fast (e.g., around four seconds for 100,000 vertices) but that finds a matching that has only approximately minimal weight.

We have implemented this strategy with a simple data structure similar to that used previously by Bonomi and Lutton (1984) for excluding long distance edges from random perturbations in a simulated annealing TSP implementation and by Fruchterman and Reingold (1991) for graph drawing. Specifically, for each coarsening level, we choose a maximum matching distance h and allow vertices to be matched only with neighbours closer than this. To accomplish this, we find the smallest rectangle containing all the vertices and overlay it with a square grid of spacing h, e.g., Figure 4(b). For each grid cell, we place all vertices lying within that cell into a linked list in random order. While there are unmatched vertices, we then randomly pick a cell containing unmatched vertices and find the first unmatched vertex, v, in the linked list for that cell. Next, we locate the two closest neighbouring vertices, w_1 and w_2, within distance h of v by visiting all vertices in the cell containing v together with all the vertices in the eight adjacent cells. We then match v with the closest unmatched vertex from w_1 and w_2 (provided of course that v is not already fixed to the chosen vertex). If there is no vertex within distance h of v, or if both w_1 and w_2 are already matched, then v is matched with itself and prevented from matching with any other vertex in that coarsening step.

It might appear that such a process (which allows a vertex to match only with its two nearest neighbours) would be destined to terminate prematurely because the two nearest neighbours of any given vertex, v, might already

be contained within the interior of a tour-fragment of fixed edges. However, recall from above that any such vertices not at either end of a tour fragment are removed from the problem representation, thus giving v a chance to match with vertices which are not its nearest neighbours in the original problem representation.

3.1.3. The Choice of h. Virtually the only variable parameter within this matching process is h, the grid spacing. In fact, it is easier to work with the average number of vertices per grid cell, n, and to calculate h on this basis. If A is the area of the smallest rectangle containing all the vertices, then for the average number of vertices in each cell to be n, the area of each grid cell should be An/N giving $h = \sqrt{An/N}$.

Notice that N here refers to the number of vertices in the current coarsening step. Because this value decreases at every step, the grid spacing gets larger and larger until eventually, when $N < n$, the entire problem is contained in one cell. This prevents the coarsening from coming to a premature end as would happen if h were fixed and there were vertices further apart than h. It also allows matchings to take place at increasingly longer range as the coarsening proceeds.

We have experimented (using the test suite described in §4.1) with n set to 5, 10, 15 and 20. In fact, there was very little difference between any of them, although as n increases, the matching becomes slower (as the code must check for the closest vertices in nine cells). On the other hand, as n decreases the coarsening rate becomes slower (because matching takes place later on in the sparser areas of the problem), so the code must refine more levels. In the end, we chose to use $n = 10$, and this is the value that applies in all the experiments in §4.

Finally, note that if n is too small one can imagine pathological cases where no vertices are within distance h of each other. However, there must be at least one cell with n vertices in it, and it is easy to see that provided $n \geqslant 4$, even if a cell contains four vertices, one at each corner of the cell and with two pairs of fixed edges between them, then at least one match is possible, coarsening can take place, and on the next level h will increase. This is probably true for smaller n, but proof is complicated by the fixed edges (vertices at either end of a fixed edge cannot be matched with each other).

3.2. Initialisation

As suggested in §2.3, principle (C2), the coarsening ceases when further coarsening would cause a degenerate problem, in this case when there remain only two vertices with a fixed edge between them. This is guaranteed to occur because each coarsening level will match at least one pair of vertices, and so the problem size will shrink. Initialisation is then trivial and consists of adding an edge between the two vertices to complete the tour (the other edge of the tour being the fixed one).

3.3. Refinement: the (Chained) Lin-Kernighan Algorithm

We use the chained Lin-Kernighan algorithm for the refinement step of our multilevel procedure. As stated in the introduction, the chained or iterated Lin-Kernighan algorithm is the most successful local search technique for iteratively optimising a TSP tour. It is usually combined with a tour construction heuristic that builds a legitimate initial tour. One such construction technique is Bentley's (1990, 1992) greedy algorithm, which proceeds by sorting all the inter-city distances by length and repeatedly adding in the shortest edge which is not already in the tour and which will not create a subcycle (a tour with fewer than N edges). In this way it progressively builds a series of tour fragments and in many ways resembles the matching and coarsening process described above (although the coarsening tends, at least initially, to create fragments with a uniform number of edges, whereas the greedy algorithm has no such restriction).

Once a tour is constructed, optimisation can take place by "flipping" edges. For example, if the tour contains the edges (v_1, w_1) and (w_2, v_2) in that order, then these two edges can always be flipped to create (v_1, w_2) and (w_1, v_2). This sort of step forms the basis of the 2-opt algorithm due to Croes (1958), which is a steepest descent approach, repeatedly flipping pairs of edges if they improve the tour quality until it reaches a local minimum of the objective function and no more such flips exist. In a similar vein, the 3-opt algorithm of Lin (1965), exchanges three edges at a time. The Lin-Kernighan (LK) algorithm (1973), also referred to as variable-opt, however incorporates a limited amount of hill-climbing by searching for a sequence of exchanges, some of which may individually increase the tour length but combine to form a shorter tour. A vast amount has been written about the LK algorithm, including much on its efficient implementation, together with some additional ideas to improve its quality, and we shall not repeat it here. For an excellent overview of techniques see the survey of Johnson and McGeoch (1997), and for more details of the implementation used here see Applegate et al. (1999) and Applegate (2000).

The basic LK algorithm employs a good deal of randomisation, and for many years the accepted method of finding the shortest tours was simply to run it repeatedly with different random seed values and pick the best (a technique that also had the advantage that it could be executed in parallel on more than one machine at once). An important contribution to the field by Martin et al. (1991, 1992) came with the observation that, instead of restarting the procedure from scratch every time, it was more efficient to perturb the final tour of one LK search and use this as the starting point for the next. In their original approach, Martin et al. referred to their algorithm as chained local optimisation and used it as a form of accelerated simulated annealing. Thus they would perturb or "kick" a tour and use LK to find a nearby local minimum. If the new tour was not as good as the champion tour at that point,

the algorithm would decide whether or not to keep it as a starting point for the next perturbation by using a simulated annealing cooling schedule. Subsequent implementations, however, generally discard any new tour that does not improve on the current champion and always perturb the champion (Applegate et al. 1999, 2000; Johnson 1990; Johnson and McGeoch 1997).

The method for perturbing the tour varies from implementation to implementation but generally involves a so-called "double-bridge" move that exchanges four edges. This has the advantages of being simple and compact, and the move cannot be undone by standard implementations of the LK algorithm. Applegate et al. (2000) test some different perturbation strategies and conclude that generally those that do not alter the cost too greatly are to be preferred over completely random kicks.

In this paper we used the chained Lin-Kernighan (CLK) implementation of Applegate et al. (1999) because a research version was easily accessible. However, as stated previously, the multilevel procedure can, in principle, be used with any iterative refinement scheme, and had a version of Johnson and McGeoch's (1997) "production mode iterated LK" algorithm been available, we would have tried that, too.

3.3.1. Fixed Edges. The only change we needed to make to the implementation of the CLK algorithm was to ensure that none of the fixed edges were exchanged. We enforced this by altering the subroutine that calculated edge lengths between a given pair of cities to return a large negative value whenever it was asked to evaluate the length of a fixed edge. The reasoning was that such a value would make any fixed edge so unattractive for being exchanged that the code would never flip it. It should also mean that such edges are kept well away from searches looking for candidate long edges to replace. In practice, we found that in all our experiments no such edges were ever flipped; however it is possible that, with a good knowledge of the LK code, a more efficient implementation might be found by simply blocking (at a high level) any fixed edges from ever being considered.

4. EXPERIMENTAL RESULTS

We have tested the multilevel strategy with a summary and somewhat inefficient implementation of the matching and coarsening techniques built around a well-engineered, highly optimised, and very efficient research implementation of the chained Lin-Kernighan algorithm (which also handles input of the TSP instance and output of the final tour). The LK software is contained in an optimisation package written by Applegate et al. (1999) and known as `concorde`. The version we have been using is `co991215.tar.gz` ⟨http://www.keck.caam.rice.edu/concorde/download.html⟩ and we very gratefully acknowledge its authors for making this code available, hence saving many months of work in the preparation of this paper.

The multilevel code wrapper that we have written around `concorde` is called `sierra`, both to reflect the nature of the multilevel paradigm that ascends and descends through a mountain-like structure of problems, and also because it was the name of a car reputedly popular amongst sales representatives in the 1990s. To give an idea of the ease of implementation, `sierra` is written in C and contains less than 1,000 lines of code. Whilst we acknowledge that it is somewhat inefficient (see §5.2 for possible improvements), because the vast majority of the execution time is generally spent in the execution of the CLK algorithm (and this is an inherent feature of this algorithm rather than a fault of `concorde`), we do not believe that a more efficient version would significantly improve the results.

The tests were carried out on a DEC Alpha machine with a 466 MHz CPU and 1 Gbyte of memory. For each instance and each code configuration we ran three tests with different random seed values.

4.1. Test Suite

This paper was written in part for the 8th DIMACS implementation challenge ⟨http://www.research.att.com/~dsj/chtsp/⟩, which is aimed at characterising approaches to the TSP, (Johnson and McGeoch 2002). As such, we have used the test suite of TSP problem instances supplied there. These are in four groups:

(I) All 34 symmetric instances of 1,000 or more vertices from `TSPLIB` ⟨http://www.iwr.uni-heidelberg.de/iwr/comopt/software/TSPLIB95/⟩, a collection of sample TSP instances, including some from real-life applications, compiled by Reinelt (1991, 1995).

(II) 26 randomly generated instances with uniformly distributed vertices. These range in size from 1,000 to 10,000,000 vertices, going up in size gradations of $\sqrt{10}$ and were constructed by Johnson, Bentley, and McGeoch specifically to study asymptotic behaviour in tour finding heuristics.

(III) 23 randomly generated instances with randomly clustered vertices. These range in size from 1,000 to 100,000 and have the same origin and purpose as (II), although clustered examples such as these are generally considered to be more difficult to solve.

(IV) seven randomly generated instances with the distances specified by a matrix. These range in size from 1,000 to 10,000 and again have the same origin as (II).

Of these examples we have omitted the eight instances, one from `TSPLIB` plus all seven from category IV, which specify the problem as the upper triangular part of an $N \times N$ matrix of intercity distances (rather than Euclidean distance). This is because our grid-based matching algorithm is unable to handle instances that do not have an associated coordinate system. However, this is not an intrinsic problem of the multilevel paradigm, and in §5.1 we suggest a possible alternative.

We have also omitted the two largest instances (with 3,162,278 and 10,000,000 vertices) from the uniformly distributed random category (II), because they were too large to run on our test platform.

For readers to make their own analysis of the test data, in Table 4 we give the Held-Karp lower bound for each instance, the optimal tour length (if known), and the execution time, T_{LK}, for the Lin-Kernighan algorithm averaged over three runs. For each instance the name indicates the problem size, e.g., fl1577 has 1,577 cities, C10k has 10,000 cities, and E1M has 1 million cities. The HKLB figures were downloaded from the DIMACS implementation challenge webpage and originally calculated using concorde (Applegate et al. 1999). Optimal tour lengths were also downloaded from the same webpage and were either originally calculated using concorde (categories II and III) or, for the TSPLIB instances (category I), were obtained from TSPLIB (Reinelt 1991, 1995).

4.2. A Worked Example

Before describing the large-scale tests and analysing the general trends of the results, we demonstrate the nature of the multilevel CLK algorithm (MLCLK) by looking in detail at one particular example, the instance usa13509 from TSPLIB. The example is illustrative, but we have not examined the complete set of tests in enough detail to know whether it is truly representative. Figure 5 shows the output from sierra (with some intermediate lines removed) as the problem is coarsened from 13,509 vertices down to 2 and then back out to 13,509. As can be seen, there are 20 levels (numbered from 0 to 19), and the coarsening rate is around 1.6, i.e., the problem size, which we define to be the number of free edges (which, in turn, is the number of vertices minus the number of fixed edges), shrinks by a factor of approximately 1.6 at every level.

The tour is initialised on the coarsest level, and the optimisation commences on the next level down. The tour length figures shown are those at the end of a given level; the tour length at the beginning of the next level is the same as this figure. At each level the CLK algorithm is allowed N kicks or restarts where N is the problem size (the number of free edges) at that level. For example, on level 3 the CLK algorithm is allowed 3,061 kicks. In the following sections we refer to this configuration as MLC^{N}LK.

Figure 5. An example of the sierra output.

level	vertices	fixed edges	free edges	tour length	cumulative time
	13509		13509		1.07
1	13509	5494	8015		1.51
2	9005	4063	4942		1.75
3	5801	2740	3061		1.88
4	3659	1769	1890		1.95
5	2247	1100	1147		1.99
14	26	13	13		2.04
15	16	8	8		2.04
16	10	5	5		2.04
17	6	3	3		2.04
18	4	2	2		2.04
19	2	1	1	25498974	2.04
18	4	2	2	25498974	2.04
17	6	3	3	25303476	2.04
16	10	5	5	25303476	2.04
15	16	8	8	24912931	2.05
14	26	13	13	24829781	2.06
5	2247	1100	1147	22659916	10.41
4	3659	1769	1890	22147276	18.17
3	5801	2740	3061	21551965	33.58
2	9005	4063	4942	20866993	65.23
1	13509	5494	8015	20283439	133.19
	13509		13509	20025663	273.34

Tour Length: 20025663
Total Running Time: 273.39

In terms of runtime, notice that the problem input combined with the coarsening and initialisation take a total of only 2.04 seconds out of 273.39. In fact, for this configuration over half the time is spent in the 13,509 CLK iterations on the final level.

With regard to the tour quality, it is very interesting to compare these results with the standard CLK algorithm run on the same instance and allowed $2N$ kicks (i.e., 27,018). This configuration, C^{2N}LK, has almost the same runtime (in fact 276.77 seconds) and nearly the same final tour length. Most interestingly, the MLC^{N}LK configuration has only achieved a tour length of 20,283,439 after 133.19 seconds when it starts on the final C^{N}LK refinement step. The C^{2N}LK configuration, on the other hand, surpasses this value after just 179 kicks and 4.07 seconds (reaching a tour of length 20,253,398). However, C^{2N}LK then spends a further 271 seconds to reach its final tour quality of 20,026,251 whilst MLC^{N}LK marginally surpasses this figure in just 140 seconds. We take this as evidence that backs up our speculation about the smoothing of the cost function so that the final refinement step of MLC^{N}LK is searching a more profitable region of the solution space. Thus, even though the tour quality is not exceptional at the start of the final CLK refinement, the final set of fixed edges released for optimisation are generally the shortest, and typically the CLK algorithm finds these the easiest to optimise.

4.3. Parameter Settings

As described in §3, the multilevel CLK algorithm has very few modifiable parameters. One is the average number of vertices, n, in each coarsening grid cell, which in turn determines the grid spacing. After some initial testing, as mentioned in §3.1.3, we used $n = 10$ for all of the tests.

A second, more important parameter which perhaps deserves more thorough testing is the relative number of kicks or restarts that the CLK algorithm is allowed at each level (relative as compared to the number of kicks on other levels). In the experiments described below we set it to a user-specified fraction of the problem size N (the number of free edges) at that level. Thus if the user picks $N/10$ then at each level it is allowed 1/10 of the problem size for that level. However, it could be argued that the algorithm should be allowed a greater proportion of kicks on the upper levels (especially since the problem sizes are so small and hence optimisation so fast) in order to better explore the solution space. On the other hand, it could equally be argued that the lower levels should be favoured even more than they already are because they represent the original problem more closely. We have not properly investigated this issue save for some incomplete testing using the same strategy

as above but redefining the problem size (and hence the number of kicks) as the number of vertices (which is typically around double the number of free edges). Experiments with this configuration provided marginally better results than those for MLC^{N}LK but, since both the penultimate and final refinement steps include all the vertices, took much longer to run and we concluded that it was not a worthwhile investment of time.

4.4. (Chained) Lin-Kernighan Benchmarks

In Table 5, columns 2–4, we present example benchmark results from the `concorde` implementation of the CLK algorithm that uses an algorithm known as *Quick-Boruvka* to generate the initial tour (Applegate et al. 2000). For the chained variant, it is important to realise that, in common with many optimisation algorithms, the more time it is allowed, the better the solution it may be able to find (although with a rapid tail off as the algorithm starts to approach its quality limit). An important parameter, therefore, is the amount of optimisation allowed, which can be specified as a time limit in seconds or, as we have used here, the number of kicks or restarts that the algorithm is given. We have expressed this as a factor of N, the problem size, and so for example we use C$^{N/10}$LK to denote the configuration that allows $N/10$ kicks (and we can also then refer to LK as C^{0}LK). The figures in Table 5 are for C^{2N}LK configuration but more extensive results for LK, C$^{N/10}$LK and C^{N}LK can be found in Walshaw (2000).

The results in Table 5 are laid out as follows. For the TSPLIB instances (category I) we report the results for each example averaged over three runs with different seed values. However, for the randomly generated instances we average the results for each class so that asymptotic analysis is easier. For example, the row labelled "`E31k (2)`" contains average values over the three runs for the two instances with 31,000 vertices, `E31k.0` and `E31k.1` (i.e., this row is averaged over a total of six runs). For each different algorithmic configuration and each instance we then present the percentage excess over the HKLB and, in the next column, the percentage excess over the optimum tour length (if known). We also give the ratio of average runtime for the instance and configuration over the average runtime for the LK algorithm for the same instance (the T_{LK} figures in Table 4).

Finally, for each configuration, at the bottom of the table we average all the results; the HKLB and runtime results are averaged over all three runs and all 80 instances whilst the optimal excess figures are averaged over all three runs and those 59 instances for which an optimal tour is known.

4.5. A Comparison of CLK and MLCLK

Denoting the multilevel versions of the LK and CLK code as MLLK and MLCLK, Table 5, columns 5–7, contains an example detailed listing of the results from the MLC^{N}LK configuration. Again, more extensive results (for MLLK, MLC$^{N/10}$LK and MLC$^{N/2}$LK) can be found in Walshaw (2000). Recall from §4.2 that for each MLC^{m}LK variant, the number of kicks or restarts, m, refers to the problem size (the number of free edges) for the particular level and not the original problem size.

Table 1. A summary of C^{m}LK and MLC^{m}LK results.

Configuration	Average % Excess		T/T_{LK}
	HKLB	opt	
MLC^{N}LK	1.028	0.250	77.366
MLC$^{N/10}$LK	1.389	0.622	10.388
C^{N}LK	1.504	0.754	41.062
C$^{N/10}$LK	2.094	1.369	5.418
MLLK	2.538	1.746	2.537
LK	3.884	3.098	1.000

In order to compare the overall results, Table 1 contains a summary of different variants by just presenting the overall averages sorted in order of tour quality. Note again that here the average percentage excess above optimal is calculated over the 59 instances for which an optimal tour is known whilst the HKLB excess and runtime results are averaged over all 80 instances. First, we can immediately see from the table that each MLC^{m}LK result is better than the corresponding C^{m}LK. This might not be too surprising because each multilevel variant takes longer than the C^{m}LK counterpart and includes a complete C^{m}LK run (albeit with different starting conditions). However, notice that although the quality measures are sorted in order, the timings are not, and impressively MLC$^{N/10}$LK actually achieves higher quality results than C^{N}LK and is nearly four times faster.

Looking at the figures in more detail, there is actually a remarkable consistency. For each value of m ($= 0, N/10, N$), the MLC^{m}LK configuration cuts the percentage excess over the HKLB by about a third compared to C^{m}LK. Furthermore, for those instances where an optimal tour is known, MLC^{N}LK cuts the percentage excess over the optimal tour length by two-thirds compared with C^{N}LK (in other words C^{N}LK is three times further from the optimum).

In order to achieve these improvements, MLC^{N}LK and MLC$^{N/10}$LK require only roughly twice the runtime of C^{N}LK and C$^{N/10}$LK, respectively. Any additional runtime is regrettable, but this twofold increase compares very favourably with the near 40-fold average increase required (for these instances) to go from LK to C^{N}LK.

The additional time overhead for MLLK compared to LK is also of a similar order, about 2.54. However, MLLK has greater relative overheads than, say, MLC^{N}LK, where most of the execution time is spent in CLK iterations. Therefore, we feel that this 2.54 figure is more susceptible to the improvements suggested in §5.2 and might well be considerably reduced by a more efficient implementation.

In fact, it is not too difficult to give an approximate justification why the multilevel strategy should add this factor of two. Suppose, first of all, that the CLK algorithm were of $O(N)$ in execution time. In fact, we know that it is greater than this, but Johnson and McGeoch (1997)

Table 2. A summary of results comparing the quality achieved for similar runtimes.

	Average % Excess				Average % Excess		
Configuration	HKLB	opt	T/T_{LK}	Configuration	HKLB	opt	T/T_{LK}
LK	3.884	3.098	1.000	MLLK	2.538	1.746	2.537
$C^{N/10}$LK	2.094	1.369	5.418	$MLC^{N/20}$LK	1.611	0.858	6.687
$C^{N/5}$LK	1.854	1.123	9.558	$MLC^{N/10}$LK	1.389	0.622	10.388
C^{N}LK	1.504	0.754	41.062	$MLC^{N/2}$LK	1.100	0.325	40.429
C^{2N}LK	1.428	0.670	79.241	MLC^{N}LK	1.028	0.250	77.366

conclude that for instances of up to 1 million cities it is subquadratic. Now suppose that the multilevel coarsening manages to halve the problem size at every step. Again, we know that this is an upper bound, and in practice it is actually somewhat less than this (e.g., 1/1.6 in the worked example, §4.2), but experience indicates that typically this is not too far off. Let T_O be the time for CLK to run on a given instance of size N and T_M the time to coarsen and contract it. The assumption on the coarsening rate gives us a series of problems of size $N, N/2, \ldots, N/N$ whilst the assumption on CLK having linear runtime gives the total runtime for MLCLK as $T_M + T_O/N + \cdots + T_O/2 + T_O$. Again, experience (and §4.2) indicate that $T_M \ll T_O$ and so we can neglect it giving a total runtime of $T_O/N + \cdots + T_O/2 + T_O = 2T_O$, i.e., MLCLK takes twice as long as CLK to run. Of course, the fact that the CLK algorithm is actually superlinear and that the coarsening rate is less than 2 serve to neutralise each other in some way. Also, the final CLK run of the MLCLK algorithm is likely already to have a very good starting tour, which means that it should run even faster than when used as a standalone. Nonetheless, this factor of two is a good rule of thumb. Finally, note that if the multilevel procedure were to be combined with an $O(N^2)$ or even $O(N^3)$ algorithm, then this analysis comes out even better for the multilevel overhead because the final refinement step would require an even larger proportion of the total.

With this factor of two in mind we then decided to compare C^mLK against $MLC^{m/2}$LK using $m = 0, N/10, N/5, N, 2N$, reasoning that for each pair their runtimes should be approximately equivalent. Detailed figures are given in Walshaw (2000), but we summarise the comparison in Table 2. As can be seen, the runtime assumptions were fairly good, particularly for larger m, and for such pairs, C^mLK and $MLC^{m/2}$LK, the average runtime figures are extremely close. Meanwhile the quality improvement imparted by the multilevel process is again fairly consistent with both multilevel variants cutting the percentage excess over the HKLB by over a quarter (actually 26–27%).

These tests also illustrate something further. It could be argued that the multilevel scheme MLC^mLK works better than C^mLK because for a given problem instance it is allowed more kicks in total (even though at the coarser levels the kicks are applied to a problem with large numbers of fixed edges). However, assuming the coarsening rate of 2 again then MLC^mLK is allowed approximately $2m$ kicks. Hence, the fact that MLC^mLK produces results 25% better than C^{2m}LK (at least for $m = N/2, N$) demonstrates that the multilevel procedure is adding some extra quality.

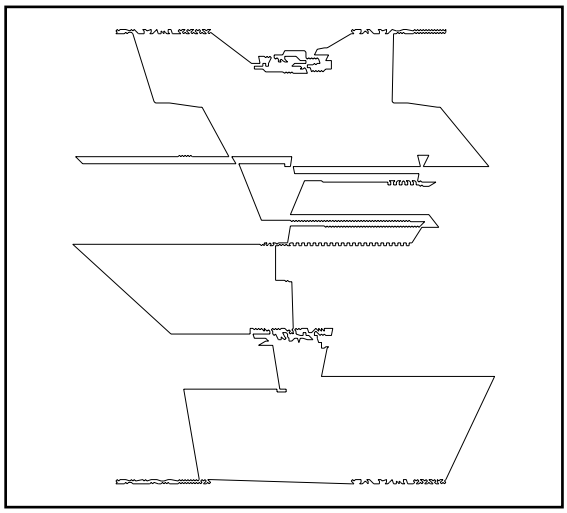
Figure 7(a) illustrates these results graphically by plotting the average percentage excess over the HKLB against average normalised runtime. The runtime factor of 2 is illustrated by the fact that each point of the $MLC^{m/2}$LK curve is almost directly below underneath that for the C^mLK curve. More importantly Figure 7(a) clearly shows the dramatic improvement in the asymptotic convergence of the solution quality.

Finally, by good fortune the average quality of $MLC^{N/10}$LK, an excess of 1.389 over the HKLB, is almost the same as (in fact marginally better than) that of C^{2N}LK, an excess of 1.422, allowing us to make a comparison based on how much time each variant requires to achieve the same quality. In fact for this set of benchmark instances $MLC^{N/10}$LK is a factor of 7.4 faster than C^{2N}LK on average, an impressive improvement.

4.5.1. Results on Subclasses of the Test Suite. The above analysis is based on averaged results over all 80 test instances and has thrown up some interesting consistencies. However, looking in more detail at the individual instances in Table 5, the results are a little less clear. First, comparing C^{2N}LK and MLC^NLK for the uniformly distributed random instances Enk for $n = 1, 3, 10, 31, 100, 316$, and E1M, it is clear that the multilevel process contributes only a little to the quality. On the other hand, for the clustered random examples, Cnk for $n = 1, 3, 10, 31, 100, 316$ those which the CLK algorithm finds more difficult to optimise, the multilevel strategy significantly enhances the tour quality. This is even more striking for the real-life instances fl1400, fl1577 and fl3795 from TSPLIB. Figure 6 illustrates an optimal tour for the fl1577 instance that shows some of the dense clustering (this optimal tour was found by an MLC^NLK variant in around 16 seconds). CLK variants all have great difficulty with these examples. This is also remarked on in Applegate et al. (1999) and Johnson and McGeoch (1997). Indeed, Neto (1999) has suggested improvements in the CLK algorithm to make it *cluster-aware*, and yet the multilevel variants find very good solutions. This is true especially if one considers the percentage excess over the optimal solution, which in these cases are further away from the HKLB than average.

To explore this dependency on different classes of problem instance a little further, we split the test suite into its three subclasses: TSPLIB instances (class I in §4.1),

Figure 6. An optimal tour of the fl1577 instance.



random instance with uniform distribution (class II), and random instances with clustered distribution (class III). The averaged results are shown in Table 3 (in the same format as Table 2) and illustrated graphically in Figures 7(b), 7(c), and 7(d).

As can be seen, this subdivision is highly illuminating. The TSPLIB instances with their wide range of examples, some from real-life applications, mirror fairly closely the results over the entire test suite (Figure 7(a)) with the multilevel version achieving considerably better asymptotic convergence. For the uniformly distributed random instances, however, the picture is completely different. The multilevel performance is actually slightly worse initially, presumably because of the additional runtime overhead, although the asymptotic convergence looks to be about the same. Finally, of the randomly clustered examples (Figure 7(d)), the ability of the multilevel framework to aid the convergence is at its most dramatic.

It seems likely that this is because the multilevel algorithm is good at regarding clusters as a single entity, a megacity, as it were. In a high-quality tour, a cluster is typically going to have only one inbound edge and one outbound. The algorithm can thus concentrate on getting these longer edges correct when it has a much simpler coarse representation of the problem and then sort out the tour details within the cluster later on.

These results also give a salutary warning to practitioners about the dependency of the computational experiments on the test suite. If we had just considered random uniformly distributed instances (as some existing papers on the TSP and other combinatorial optimisation problems do), we could not have demonstrated that the multilevel framework offered any significant advantages. Clearly, then, algorithms need to be tested on as broad a range of examples as possible and preferably on a suite that includes real-life instances so that theoretical benefits are tested for, and hence can be realised in, practical applications.

5. SUMMARY AND FUTURE RESEARCH

We have described and tested a multilevel approach to the Travelling Salesman Problem. The approach has been derived from first principles; by examining existing examples of the multilevel paradigm in action and extracting generic techniques, we have been able to apply it to a completely different problem. The resulting multilevel algorithm is shown to enhance considerably the quality of tours for both the Lin-Kernighan and Chained Lin-Kernighan algorithms, in combination the TSP champion heuristics for nearly 30 years. We speculate that this is because the multilevel process samples the solution space and smooths the objective function and is thus able to get closer to the global minimum. In this sense we regard the multilevel paradigm as a type of optimisation accelerator, which we have used here in combination with the (C)LK algorithm rather than as a specific enhancement to (C)LK alone.

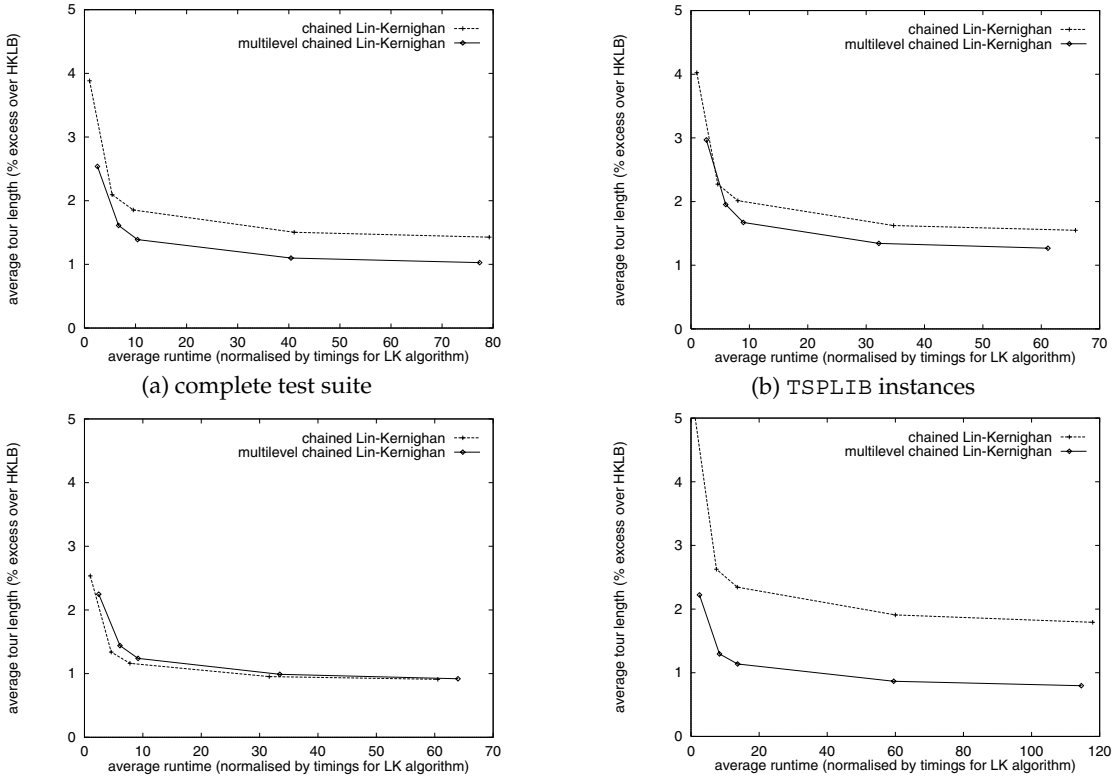
For the instances and code configurations tested here, the highlights of the results are:

- For three different C^mLK configurations (for $m = 0, N/10, N$) the multilevel procedure cuts the percentage excess over the HKLB by around one-third in return for a modest twofold increase in runtime.

Table 3. A summary of results on subsets of the test suite.

	TSPLIB			Random Uniform Instances			Random Clustered Instances		
	Average % Excess			Average % Excess			Average % Excess		
Configuration	HKLB	opt	T/T_{LK}	HKLB	opt	T/T_{LK}	HKLB	opt	T/T_{LK}
LK	4.024	3.231	1.000	2.535	1.762	1.000	5.090	4.178	1.000
$C^{N/10}$LK	2.274	1.414	4.579	1.338	0.653	4.588	2.626	1.998	7.486
$C^{N/5}$LK	2.014	1.144	8.028	1.162	0.473	7.789	2.345	1.732	13.599
C^{N}LK	1.624	0.729	34.698	0.953	0.255	31.652	1.909	1.302	60.011
C^{2N}LK	1.549	0.652	65.864	0.910	0.212	60.523	1.793	1.164	117.965
MLLK	2.969	2.055	2.633	2.248	1.456	2.451	2.223	1.436	2.490
$MLC^{N/20}$LK	1.954	1.035	5.948	1.440	0.754	6.088	1.296	0.620	8.372
$MLC^{N/10}$LK	1.672	0.742	8.964	1.239	0.542	9.184	1.138	0.471	13.687
$MLC^{N/2}$LK	1.343	0.404	32.183	0.989	0.294	33.458	0.866	0.201	59.535
MLC^{N}LK	1.268	0.329	61.138	0.919	0.212	63.980	0.796	0.134	114.617

Figure 7. Plot of convergence behavior for different subsets of the test suite.



(a) complete test suite (b) TSPLIB instances

(c) random uniform instances (d) random clustered instances

• For those instances where an optimal tour is known, MLC^{N}LK cuts the percentage excess over the optimal tour length by two-thirds compared with C^{N}LK. In other words, C^{N}LK is three times further from the optimum.

• In two cases ($m = N/2, N$), given approximately the same amount of execution time, a multilevel configuration MLC^{m}LK cut the percentage excess over the HKLB by more than one-fourth compared with a single level configuration C^{2m}LK.

• Alternatively, in order to achieve the same quality of tour, the C^{2N}LK configuration took more than seven times as long as MLC$^{N/10}$LK.

• The multilevel versions tend to do significantly better on the harder, clustered problems that the LK and CLK algorithms have the most difficultly with.

However, the warning in §4.5.1 about the dependency of the results on the test suite applies, and we certainly would not claim that they could necessarily be repeated for a completely different set of problem instances.

We conclude that the multilevel strategy can be a powerful tool in the solution of the TSP and that the multilevel paradigm can be successfully applied to yet another combinatorial optimisation problem. One major piece of work for further research, therefore, is to apply the multilevel paradigm to further combinatorial optimisation problems and examine the results. See also Walshaw (2001c, 2002) for further thoughts on this project.

5.1. Variants and Extensions

In terms of the multilevel CLK scheme, apart from optimisation of parameter settings (see §4.3), we suspect that the algorithm might benefit from further research into matching strategies. For example, one could build a Delaunay triangulation of the vertices (by constructing a Voronoi diagram as in Fortune 1987) or some other form of neighbour graph and allow matching only along its edges. Alternatively, one could use any tour construction heuristic initially (such as the greedy algorithm) and force the coarsening to match only those pairs of vertices that are adjacent in this tour.

This tour construction suggestion might also be a good way to address instances where the intercity distances are specified by a matrix rather than by Euclidean distance (e.g., see §4.1). It would certainly be more efficient than a naïve but straightforward matching procedure such as picking vertices at random, searching all $N-1$ edge lengths to find the closest pair of neighbours, and matching with one of them.

5.2. Efficiency

As mentioned in §4, the `sierra` implementation of the multilevel techniques is not as efficient as it could be, and it is certainly possible that further work could give

Table 4. Baseline results: HKLB, optimal tour lengths, and T_{LK}.

TSPLIB				Random Instances			
Instance	Held-Karp Lower Bound	Optimal Tour Length	T_{LK}	Instance	Held-Karp Lower Bound	Optimal Tour Length	T_{LK}
dsj1000	18546976.916667	18659688	0.28	E1k.0	23183212.500000	23360648	0.19
pr1002	256765.916667	259045	0.18	E1k.1	22839567.526190	22985695	0.18
u1060	222650.875000	224094	0.20	E1k.2	22858725.666667	23023351	0.20
vm1084	236162.416667	239297	0.17	E1k.3	23002034.500000	23143748	0.21
pcb1173	56351.000000	56892	0.19	E1k.4	22542848.833333	22698717	0.17
d1291	50208.578571	50801	0.19	E1k.5	23057465.166667	23192391	0.21
rl1304	249093.833333	252948	0.21	E1k.6	23166620.333333	23349803	0.19
rl1323	265814.500000	270199	0.20	E1k.7	22666814.125000	22879091	0.19
nrw1379	56396.208333	56638	0.23	E1k.8	22795477.333333	23025754	0.18
fl1400	19783.000000	20127	0.51	E1k.9	23215285.062500	23356256	0.18
u1432	152535.000000	152970	0.23	E3k.0	40348236.083333	40634081	0.70
fl1577	21886.000000	22249	0.30	E3k.1	40046054.229167	40315287	0.66
d1655	61549.250000	62128	0.25	E3k.2	40006528.375000	40303394	0.68
vm1748	332060.722222	336556	0.30	E3k.3	40318840.516667	40589659	0.68
u1817	56688.250000	57201	0.24	E3k.4	40462881.041667	40757209	0.68
rl1889	311704.500000	316536	0.31	E10k.0	71362276.444048		2.52
d2103	79307.000000	80450	0.26	E10k.1	71565485.443519		2.53
u2152	63858.062500	64253	0.27	E10k.2	71351794.654167		2.48
u2319	234215.000000	234256	0.65	E31k.0	126474847.479514		9.26
pr2392	373489.666667	378032	0.38	E31k.1	126647285.326389		9.15
pcb3038	136587.500000	137694	0.50	E100k.0	224330692.072818		39.75
fl3795	28477.250000	28772	0.79	E100k.1	224241788.835713		39.40
fnl4461	181568.833333	182566	0.78	E316k.0	398582616.458995		164.63
rl5915	556848.833333	565530	0.93	E1M.0	708703512.913668		611.80
rl5934	548470.550000	556045	0.90	C1k.0	11325839.750000	11387430	0.31
pla7397	23126463.916667	23260728	1.42	C1k.1	11330835.500000	11376735	0.33
rl11849	913980.291667	923288	2.33	C1k.2	10809149.125000	10855033	0.33
usa13509	19851463.750000	19982859	3.51	C1k.3	11823905.791667	11886457	0.38
brd14051	467127.622222		2.88	C1k.4	11433764.375000	11499958	0.37
d15112	1564880.029167	1573084	3.87	C1k.5	11328719.175000	11394911	0.34
d18512	642116.826389		3.93	C1k.6	10092637.145833	10166701	0.33
pla33810	65705438.125000		6.90	C1k.7	10602996.291667	10664660	0.37
pla85900	141806385.000000		21.38	C1k.8	11566101.750000	11605723	0.45
				C1k.9	10835951.222222	10906997	0.42
				C3k.0	19080350.708333	19198258	1.24
				C3k.1	18901572.291667	19017805	1.21
				C3k.2	19410947.104167	19547551	1.25
				C3k.3	19001115.625000	19108508	1.23
				C3k.4	18757584.625000	18864046	1.20
				C10k.0	32782155.221284		4.36
				C10k.1	32958945.515201		4.26
				C10k.2	32926889.150397		4.34
				C31k.0	59169192.695278		16.12
				C31k.1	58840096.446393		16.44
				C100k.0	103916253.916802		67.11
				C100k.1	104663040.340307		67.79
				C316k.0	185576666.639071		283.56

improvements in execution time. In particular, during the coarsening intercity distances are calculated "on the fly" as required, and it is likely that the use of caching techniques such as those used by `concorde` (Applegate et al. 1999) would improve efficiency. Perhaps a more important enhancement would be closer integration of `sierra` and `concorde`. In particular, `concorde` is called as a self-contained subroutine that creates and then deletes all of its internal data structures every call. A better strategy would be for `sierra` to maintain them itself and allow `concorde` to reuse them. Finally, as mentioned in §3.3.1, a more efficient method for the blocking of fixed edges might prove beneficial.

It should be stressed here that all these improvement suggestions fall on the `sierra` implementation rather than on the `concorde` package, which could not anticipate them. How much difference they might make is impossible to say. However, as mentioned above, because the vast majority of the code execution time is spend in CLK iterations, it is doubtful that, apart from the MLLK configuration, such efficiency enhancements would significantly improve runtimes further.

Table 5. $C^{2N}LK$ and $MLC^{N}LK$ results.

	$C^{2N}LK$			$MLC^{N}LK$		
	Average % Excess			Average % Excess		
Instance	HKLB	opt	T/T_{LK}	HKLB	opt	T/T_{LK}
dsj1000	0.960	0.350	66.571	0.888	0.279	68.500
pr1002	1.295	0.404	41.333	1.375	0.484	42.833
u1060	0.866	0.217	49.250	0.859	0.209	46.450
vm1084	1.598	0.267	58.471	1.433	0.105	53.588
pcb1173	1.303	0.340	35.789	1.420	0.456	38.105
d1291	2.548	1.352	42.368	1.513	0.329	36.789
rl1304	2.361	0.802	50.429	1.643	0.095	36.143
rl1323	2.282	0.622	44.950	1.876	0.223	42.500
nrw1379	0.595	0.166	39.522	0.610	0.181	43.000
fl1400	2.411	0.661	107.039	1.867	0.126	85.824
u1432	0.661	0.375	47.696	0.709	0.423	46.043
fl1577	6.139	4.408	94.300	1.768	0.108	62.500
d1655	1.359	0.415	38.520	1.460	0.515	38.920
vm1748	1.483	0.128	60.233	1.619	0.262	57.700
u1817	1.792	0.880	36.083	1.544	0.633	33.708
rl1889	2.179	0.620	64.258	1.958	0.402	52.968
d2103	2.428	0.973	67.462	2.300	0.847	55.731
u2152	1.278	0.656	38.593	1.266	0.643	37.037
u2319	0.165	0.147	93.369	0.211	0.194	68.431
pr2392	1.739	0.516	43.947	1.744	0.521	46.421
pcb3038	1.124	0.311	44.980	1.109	0.296	48.700
fl3795	2.928	1.873	105.519	1.576	0.535	70.861
fnl4461	0.731	0.181	52.936	0.753	0.202	59.013
rl5915	2.197	0.629	71.387	1.789	0.227	57.204
rl5934	2.004	0.615	72.656	1.746	0.360	65.244
pla7397	0.860	0.278	83.261	0.862	0.280	85.366
rl11849	1.360	0.339	86.828	1.250	0.230	78.373
usa13509	0.878	0.215	92.610	0.850	0.187	95.293
brd14051	0.716		79.118	0.670		87.163
d15112	0.685	0.160	83.388	0.721	0.195	87.842
d18512	0.646		74.427	0.667		88.682
pla33810	0.933		100.939	1.039		99.441
pla85900	0.626		105.277	0.763		101.195
E1k (10)	0.935	0.197	43.508	0.957	0.219	44.696
E3k (5)	0.948	0.242	55.722	0.905	0.199	56.333
E10k (3)	0.859		78.782	0.897		85.774
E31k (2)	0.841		85.232	0.851		93.276
E100k (2)	0.868		80.345	0.878		89.660
E316k (1)	0.905		77.352	0.930		86.707
E1M (1)	0.862		94.025	0.879		97.009
C1k (10)	1.407	0.865	88.428	0.645	0.107	89.212
C3k (5)	2.385	1.761	107.851	0.804	0.189	105.544
C10k (3)	1.924		160.293	0.919		150.946
C31k (2)	1.897		170.333	1.028		155.499
C100k (2)	1.875		157.504	0.989		152.745
C316k (1)	1.914		153.123	1.034		147.030
Average	1.428	0.670	79.241	1.028	0.250	77.366

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