NEGATIVE CONDUCTANCE
IN DRIVEN JOSEPHSON JUNCTIONS*

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We investigate an optimal regime of negative-valued conductance, emerging in a resistively and capacitively shunted Josephson junction, which is driven simultaneously by both, a time-periodic (ac) and a constant (dc) current. We analyze the current-voltage characteristics in the regime of absolute negative conductance. We additionally explore the stability of the negative response with respect to the ac-current frequency.

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1. Introduction

The problem of far-from-equilibrium transport in periodic systems continues to attract considerable attention during the last decade. Research in this field provides an important contribution to the foundations of thermodynamics and statistical physics. For example, the interplay of nonlinearity, dissipation, fluctuations and external driving in the presence of chaotic dynamics can lead to a number of unusual scenarios of dynamical behavior of Brownian particles. The most prominent example is a Brownian motor system [1, 2]. Despite all the interesting features revealed in the last years like Brownian ratchet transport [3], current reversals [4], noise-induced phase transitions and the general fact that noise can play constructive role in many cases [5], there exist still unexplored areas of dynamical phenomena awaiting to become disclosed. Such a rich plethora of transport behaviors emerges when inertia effects start to dominate the transport characteristics [6]. As an

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example we recall the prominent phenomenon of anomalous response due to external driving which can appear in relatively simple systems such as in an ac-driven and dc-biased Josephson junction [7]. For the systems in thermodynamic equilibrium the input–output relationship is in accordance with the linear response theory (LRT). Yet there are circumstances where LRT holds perfectly well also far away from equilibrium [8]. A good example is the over-damped classical [9] and quantum [10] transport of Brownian particles in a washboard potential: the velocity is an increasing function of the small external static force (positive mobility). However, this intuitive or "normal" situation where the effect follows the cause may change radically when the system is subjected to several forcing degrees of freedom. There exists a broad variety of physical systems which can exhibit "anomalous" behavior. One of them is the already mentioned ratchet effect, which is caused by a symmetry breaking and a source of non-equilibrium forcing. When the static force affects the time-periodically driven massive Brownian particle moving in spatially periodic structures [11], it can respond with a negative differential mobility (NDM) [12, 13] or even with an absolute negative mobility (ANM) [7]. In terms of an electric transport, one can observe an absolute negative conductance (ANC) when upon an increase of the static voltage bias, starting out from zero, a current is induced in the opposite direction. This situation was experimentally confirmed in p-modulation-doped multiple quantum-well structures [14] and semiconductor super-lattices [15]. ANC (ANM) was also studied theoretically for ac-dc-driven tunneling transport [16] and in the dynamics of cooperative Brownian motors [17], for Brownian transport in systems of a complex topology [18] and in some stylized, multi-state models with state-dependent noise [19], to name but a few.

In this paper, we continue our prior studies of the same system detailed in Refs. [7, 20, 21] and analyze the optimal regime of negative-valued conductance. In Section 2, we present the details of the model of the resistively and capacitively shunted Josephson junction. In Section 3, we study novel aspects of current-voltage characteristics in the optimal regime while putting special emphasis on the frequency dependence of the ac-driving source on the anomalous response.

2. Stewart–McCumber model

We explain the dynamics of the Josephson junction in terms of the well known Stewart–McCumber model [22] in which the current through the junction is a sum of a Josephson supercurrent characterized by the critical current I_0, a normal ohmic current characterized by the resistance R and a displacement current accompanied with a capacitance C. The Johnson noise plays the role of the thermal equilibrium noise which is associated

with the resistance R. The quasi-classical dynamics of the phase difference $\phi = \phi(t)$ between the macroscopic wave functions of the Cooper electrons on both sides of the junction is described by the following equation [22]

$$\left(\frac{\hbar}{2e}\right)^2 C\ddot{\phi} + \left(\frac{\hbar}{2e}\right)^2 \frac{1}{R}\dot{\phi} = -\frac{\hbar}{2e}I_0 \sin(\phi) + \frac{\hbar}{2e}I_{\mathrm{d}}$$
$$+ \frac{\hbar}{2e}I_{\mathrm{a}}\cos(\Omega t + \phi_0) + \frac{\hbar}{2e}\sqrt{\frac{2k_{\mathrm{B}}T}{R}}\,\xi(t)\,. \quad (1)$$

The dot denotes differentiation with respect to time t, I_{d} and I_{a} are the amplitudes of the applied dc and ac, respectively, Ω denotes the angular frequency of the ac driving source and ϕ_0 defines the initial phase value. The parameter k_{B} is the Boltzmann constant and T stands for temperature of the system. The ubiquitous thermal equilibrium fluctuations are modeled by δ-correlated Gaussian white noise $\xi(t)$ of zero mean and unit intensity.

The dimensionless form of this equation then reads [7, 20, 21]:

$$\ddot{x} + \gamma\dot{x} = -2\pi\sin(2\pi x) + f + a\cos(\omega s + \phi_0) + \sqrt{2\gamma D}\,\Gamma(s)\,. \quad (2)$$

Here $x = \phi/2\pi$ and the dot denotes differentiation with respect to the dimensionless time $s = t/\tau_0 = \omega_0 t$, where the characteristic time $\tau_0 = 1/\omega_0 = 2\pi\sqrt{\hbar C/2eI_0}$ and ω_0 is the plasma frequency [7]. Other dimensionless parameters after rescaling assume the form: friction coefficient $\gamma = \tau_0/RC$; the amplitude and the angular frequency of the ac-current are denoted by $a = 2\pi I_{\mathrm{a}}/I_0$ and $\omega = \Omega\tau_0$, respectively. The dimensionless bias load stands for $f = 2\pi I_{\mathrm{d}}/I_0$, the rescaled zero-mean Gaussian white noise $\Gamma(s)$ possesses the auto-correlation function $\langle\Gamma(s)\Gamma(u)\rangle = \delta(s - u)$, the noise intensity $D = k_{\mathrm{B}}T/E_{\mathrm{J}}$ and the Josephson coupling energy is $E_{\mathrm{J}} = (\hbar/2e)I_0$. The actual stationary averaged voltage reads

$$V = \frac{\hbar\omega_0}{2e}v\,, \qquad v = \langle\dot{x}\rangle\,, \quad (3)$$

where the stationary dimensionless averaged voltage is $v = \langle\dot{x}\rangle$. The brackets denote an average over the initial conditions, over all realizations of the thermal noise and the long time limit over one cycle of the external ac-driving.

3. Negative conductance

It often proves useful to use the mechanical analog of the system (2), being an inertial Brownian particle which performs a one-dimensional random motion in the periodic potential $U(x) = -\cos(2\pi x)$. The phase is the

equivalent of the space coordinate of the Brownian particle and the currents play the role of driving forces on the particle [13, 23]. Because Eq. (2) is equivalent to a set of three ordinary differential equations, the phase space of (2) is three-dimensional. In consequence, the deterministic ($D = 0$) non-linear system (2) typically may exhibit chaotic properties [22]. In general, the deterministic dynamics ($D = 0$) exhibits a very rich behavior. One can detect periodic, as well as quasi-periodic and chaotic solutions depending on the system's parameters. Switching on finite thermal noise one thereby activates the diffusive dynamics where stochastic escape events among existing attractors become possible. Moreover, the particle can now visit any part of the phase space and proceed within some finite time interval by following closely any of the existing stable or unstable orbits.

Assuming a zero dc ($f = 0$), the motion is unbiased and symmetric in the ac-amplitude. The stationary averaged velocity (or voltage) in this case then must be zero. The simplest option to destroy this symmetry is to apply a dc-bias, *i.e.* we set $f \neq 0$. It breaks the reflection symmetry $x \rightarrow -x$ of the potential and in turn allows the averaged velocity (voltage) to assume non-zero values, which typically assume the same sign as f. Any deviation from this rule is counterintuitive. As we have shown previously, a Josephson junction exhibits many exciting features, including absolute negative conductance (ANC) [7, 20], negative differential conductance (NDC), negative-valued nonlinear conductance (NNC) or the reentrant effect of the negative conductance [21]. In mechanical, particle-like motion terms, it corresponds to a negative mobility of the Brownian particle.

In a related study [21], we found an optimal regime of the negative conductance in the parameter space $\{a, \omega, \gamma\}$. This regime is located around the values $a \in (12, 21), \omega \in (6.5, 7.5)$ and $\gamma \in (0.9, 1.4)$. This regime seems to be optimal in the sense that the negative conductance is most profound in a relatively large domain with relatively large values of the dimensionless voltage. The occurrence of a negative conductance may be governed by various mechanisms. In some regimes negative response is induced solely by thermal equilibrium fluctuations, *i.e.* the effect is absent for vanishing thermal fluctuations. Yet in other regimes, ANC can occur in the noiseless, deterministic system while a finite temperature either destroys the effect or diminishes its strength [7, 21]. Nevertheless, both situations have their origin in the noise-free ($D = 0$) structure of the stable and unstable orbits. To be more concrete, let us consider a particular set of parameters, namely, $a = 19.5, \gamma = 1.2$ and $\omega = 6.9$. For this set of parameters, the deterministic behavior can be understood if one studies the structure of the underlying attractors and the corresponding basins of attraction. In the upper panel of Fig. 1, we show the long-time averaged voltage. It turns out that the system possesses four attractors: two transporting in the positive direction, $v = 1.1$
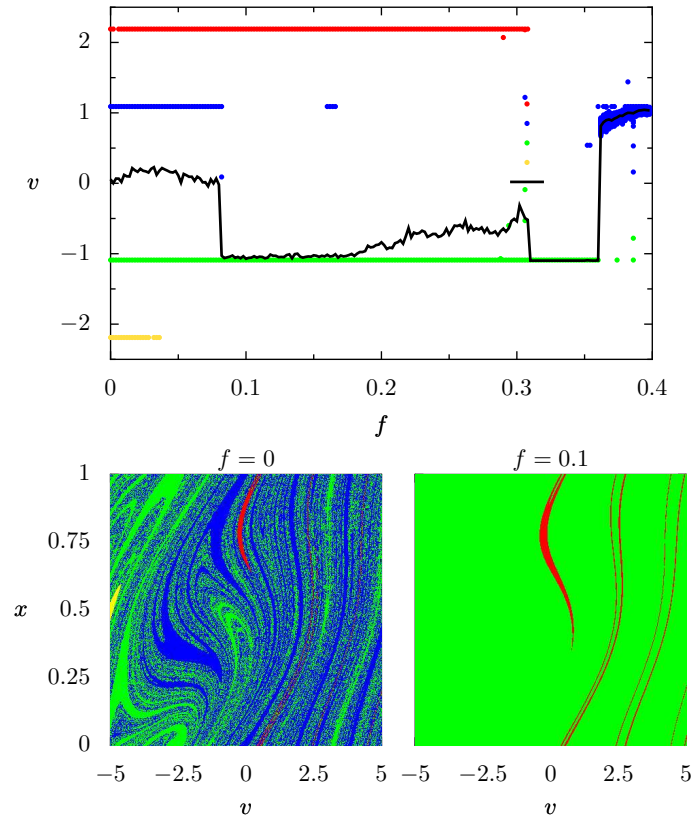
Fig. 1. (color online) Upper panel: colored lines represent the time averaged velocity in the deterministic case $D = 0$. The four colors correspond to the four attractors $v = -2.2, -1.1, 1.1, 2.2$. The black line denotes the long-time velocity (voltage), averaged over initial conditions (position, velocity and phase). In the lower panel, the corresponding basins of attraction are depicted: yellow and green mark regimes where transport occurs in the negative direction (*i.e.* for $v = -2.2$ and $v = -1.1$, respectively); blue and red mark regimes for which the attractor is transporting in the positive direction (*i.e.* for $v = 1.1$ and $v = 2.2$, respectively). The averaged velocity (voltage) for $f = 0.1$ is determined by the structure of the basins of attraction rather than by the attractor itself: Although the positively transporting attractor (red) possesses a larger velocity v, its basin of attraction is much smaller than the one transporting into negative direction. Thus, the contribution of this smaller basin to the total transport remains small. The presence of noise does not markedly change this characteristics (see Fig. 3 for $f = 0.1$). The remaining parameters read: $a = 19.5$, $\gamma = 1.2$, $\omega = 6.9$. The basins of attraction are shown for the initial phase value at $\phi_0 = -\pi/2$.

and $v = 2.2$, and two transporting in the negative direction $v = -1.1$ and $v = -2.2$. Notably, if one performs the average over all initial conditions (position, velocity and initial phase) with a corresponding, non-weighted uniform distribution then the resulting voltage v behaves in the way depicted by the black curve. The zero voltage for the case $f = 0$ follows from symmetry arguments, see in Fig. 2. The right panel in this figure can be obtained
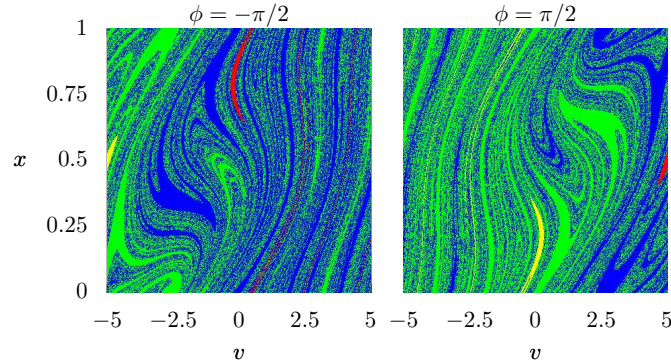

Fig. 2. (color online) An example of two corresponding sets of basins of attraction for the deterministic dynamics, $D = 0$ at vanishing bias $f = 0$. The left part is exactly the same as the left part in the lower panel of Fig. 1. The right part corresponds to the transformed initial conditions $(x, v, \phi_0) \to (-x, -v, \phi_0 + \pi)$. We recall that the system is periodic with period 1. Noting the present symmetry one obtains zero voltage after averaging over all initial conditions.

from the left panel by the transformation $(x, v, \phi_0) \to (-x, -v, \phi_0 + \pi)$ or, what is easier visible, rotation about the origin (0,0) by the angle π. In other words, for each trajectory transporting to the right direction there exists its partner transporting exactly in the opposite direction. So, both contributions cancel each other and thus all averages of the voltage over symmetrical distributions in phase space become zero, yielding $v = 0$. When a finite bias is applied, this symmetry becomes broken, now typically yielding a non-vanishing velocity v. A small positive bias results in this regime in a positive voltage (normal response). A further increase of the dc-current destabilizes two attractors, namely, $v = -2.2$ and $v = 1.1$, and for example at $f = 0.1$ there exist only two attractors: $v = -1.1$ and $v = 2.2$, see Fig. 1. By taking the arithmetic average of these two numbers one could expect that the voltage v would be positive (and thus no ANC is expected). However, if one performs the averaging over all initial uniformly distributed initial conditions then the voltage v is negative and its value is very close to $v = -1.1$. To explain this result one has to inspect the basins of attraction of orbits with $v = -1.1$ and $v = 2.2$. It turns out that for the positively transporting

attractor $v = 2.2$, the basin of attraction covers a much smaller area than the basin of attraction for the negative transporting attractor $v = -1.1$. This is depicted with the right lower panel of Fig. 1: The green (light) regions are much larger than the red (dark) regions. As a consequence, directed transport is dominated by orbits which carry a negative voltage v.
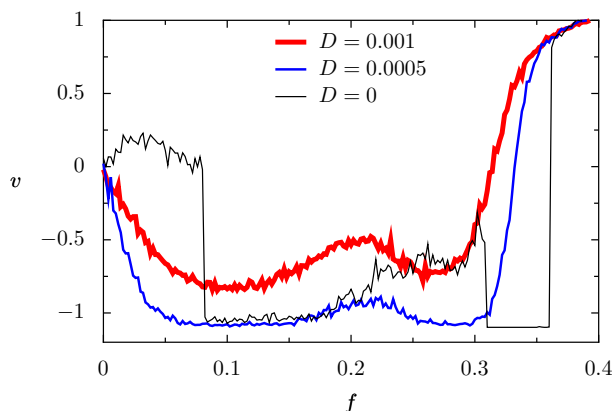

Fig. 3. (color online) The dimensionless stationary averaged voltage v as a function of the dc-bias current f. The black (thin) line corresponds to the deterministic ($D = 0$) dynamics, the blue (intermediate-thick) line corresponds to a dimensionless temperature of $D = 0.005$, while the red (thick) line corresponds to $D = 0.001$. For a non-zero temperature, absolute negative conductance (or absolute mobility) occurs. The remaining fixed parameters read: $a = 19.5, \gamma = 1.2, \omega = 6.9$. For small values of the dc-current $f < 0.05$, the temperature changes dramatically the $v - f$ characteristics.

The influence of temperature is presented in Fig. 3, where we depict the dimensionless stationary averaged voltage v *versus* the dimensionless dc-current $f \in [0, 0.4]$ for three selected values of the noise intensity $D = k_B T / E_J = 0, 0.001, 0.0005$. We clearly detect that at non-zero temperature ($D > 0$) and for a small positive dc-current f, the voltage v is negative. This case illustrates the phenomenon of absolute negative conductance (ANC). Let us emphasize the constructive role of noise here: In the noiseless, deterministic case ($D = 0$) and for small bias, the system response behaves normal: The averaged voltage is positive for a positive dc-current (normal transport behavior). One can notice that an increase of the temperature typically diminishes the effect. The above ANC effect can be explained by the fact that the dynamics of the system located close to the bifurcation point, if perturbed by thermal noise, takes place in the region of the phase space where the stable attractor is emerging beyond the bifurcation point.
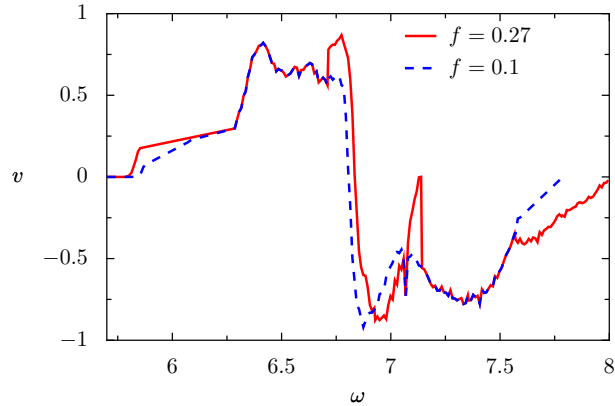
Fig. 4. (color online) The stationary, averaged voltage is depicted *versus* the ac frequency for $D = 0.001$ at two dc-currents $f = 0.1$ and $f = 0.27$. The other parameters are as in Fig. 3.

Finally, the impact of the frequency of the ac-current on the average voltage is depicted in Fig. 4 for two different values of the dc-current: $f = 0.1$ (dashed blue line) and $f = 0.27$ (solid red line). We see that up to the value of $\omega \simeq 5.8$ the average voltage remains zero. A further increase of angular driving frequency leads to finite transport with a positive voltage. For $\omega \simeq$ 6.7 the voltage suddenly drops and crosses over into a negative average value, thus representing a negative-valued conductance. This situation remains essentially for the remaining regime of frequencies up to $\omega \simeq 8$. One can also observe that in some regions of ω, the voltage stays close to the same value for both, $f = 0.1$ and $f = 0.27$, *cf.* Fig. 4 and the regime around $\omega \approx 6.5$ or $\omega \approx 7.3$. It means that in the current–voltage characteristics one could observe Shapiro steps [24]. Please note that for both scenarios with $f = 0.1$ and $f = 0.27$ the variation of the voltage with increasing angular frequency depicts a qualitatively robust similar behavior, despite the fact that the deterministic dynamics can behave quite different (not shown).

In summary, we put forward an analysis of the negative conductance occurring in the system of a resistively and capacitively shunted Josephson junction. For this phenomenon to occur it is necessary that two driving sources operate simultaneously, namely an ac and a dc source. We have related the deterministic dynamics with its stable and unstable orbits to the normal and anomalous response of the junction to the external stimuli. We are confident that this very regime of ANC can be successfully tested with an experiment involving a single Josephson junction, and the presented setup will be stable within small variations of any of the structural parameters.

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