Gauss circle problem

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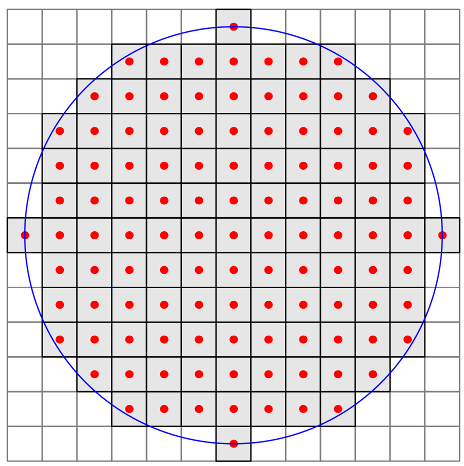
1 Introduction

We're interested in the behaviour of the number of integral points inside circles of various radii centered at the origin. That is, if our circle's radius is r, we want to count the size of $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x^2 + y^2 \leq r^2\}$. We shall denote this by $N(r)$.

Intuitively speaking, the size of the set should be well approximated by the area of the circle, as we can associate a unit square to each lattice point, centered at that point.



The area of the circle is πr^2, so we expect $N(r)$ to be approximately equal to that. More precisely, we can write

$$N(r) = \pi r^2 + E(r),$$

where $E(r)$ can be viewed as a kind of *error term* incurred by our approximation. The main problem then becomes to give an upper bound to $|E(r)|$.

2 First attempts

Geometrically, the error term arises from the squares intersected by the disk's boundary. So a simple way to bound it is to notice that all these squares must lie completely inside the circle of radius $r + \sqrt{2}$, and completely outside of one with radius $r - \sqrt{2}$. It follows that

$$\pi r^2 - 2\pi\sqrt{2}r + 2\pi = \pi(r - \sqrt{2})^2 \leq N(r) \leq \pi(r + \sqrt{2})^2 = \pi r^2 + 2\pi\sqrt{2}r + 2\pi,$$

and therefore $|E(r)| \leq 2\pi\sqrt{2}r + 2\pi$. These constants of course aren't optimal, but there's a bigger problem: our estimate was very weak. It turns out that there's a lot of cancellation among the excess areas of the squares, and what we've done is comparable to bounding $|1 - 1 + 1 - 1 + 1 - \cdots - 1|$ with the triangle inequality. We've proven that $|E(r)|$ *grows* at most as quickly as r, up to some constant factor. Let's introduce a notation which makes stating this growth rate a bit simpler.

Definition 2.1 (Big O notation)**.** For a real or complex valued function f and a real valued function g, both defined on the positive reals, we say that $f(x) = O(g(x))$ if there exists a constant C, such that for all large enough real x, we have $|f(x)| \leq Cg(x)$. (It would be more logical to call $O(g(x))$ the set of all such f, but this notation is more convenient, despite it perhaps being initially confusing.)

So, we've shown that $E(r) = O(r)$, or $N(r) = \pi r^2 + O(r)$. The conjectured true rate of growth, however, is $E(r) = O(r^{1/2+\varepsilon})$ for all $\varepsilon > 0$. (The state of the art appears to be $E(r) = O(r^{3014/4875} \ln r)$ by Shaneson [1], proven with advanced techniques combined with very intricate estimates). We'll content ourselves with proving a weaker bound, $E(r) = O(r^{2/3+\delta})$ for any $\delta > 0$. This, however, still requires some very interesting and non-obvious methods from Fourier analysis.

We begin by introducing the necessary tools.

3 Preliminaries

Fourier analysis concerns itself with studying functions by decomposing them into sums of waves of various frequencies. The basic idea is that any sufficiently nice function can be expressed as a sum (or integral) of complex exponential functions, which can be much easier to analyse. We call this decomposition the *Fourier transform* of the function.

We will not be concerning ourselves with convergence issues, because we will be applying these theorems only to bounded, compactly supported (piecewise) smooth functions.

Definition 3.1 (Fourier transform)**.** The Fourier transform of a function $f\colon \mathbb{R}^n \to \mathbb{C}$ is defined to be a function $\hat{f}\colon \mathbb{R}^n \to \mathbb{C}$, equal to

$$\hat{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^n} f(\mathbf{t}) e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{t}}\, d\mathbf{t},$$

where $\boldsymbol{\xi} \cdot \mathbf{t}$ represents the usual scalar product.

This integral is well-defined provided f decreases sufficiently quickly, which it will. f is often called the *time domain* representation of the function, while $\hat{f}$ is called the *frequency domain* representation.

A useful property of the Fourier transform, which illustrates the power of working in the frequency domain, is the following:

Theorem 3.1 (Convolution theorem)**.** *Define the convolution $f * g$ of two functions $f, g\colon \mathbb{R}^n \to \mathbb{C}$ as the integral*

$$(f * g)(\boldsymbol{t}) = \int_{\mathbb{R}^n} f(\boldsymbol{\tau}) g(\boldsymbol{t} - \boldsymbol{\tau})\, d\boldsymbol{\tau}.$$

This operation is commutative. Also, the Fourier transform of $f * g$ is the product of the Fourier transforms of f and g.

Proof.

$$\begin{aligned}
\widehat{(f * g)}(\boldsymbol{\xi}) &= \int_{\mathbb{R}^n} (f * g)(\mathbf{t}) e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{t}}\, d\mathbf{t} = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(\boldsymbol{\tau}) g(\mathbf{t} - \boldsymbol{\tau})\, d\boldsymbol{\tau} \right) e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{t}}\, d\mathbf{t} = \\
&= \int_{\mathbb{R}^n} f(\boldsymbol{\tau}) \left(\int_{\mathbb{R}^n} g(\mathbf{t} - \boldsymbol{\tau}) e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{t}}\, d\mathbf{t} \right) d\boldsymbol{\tau} = \int_{\mathbb{R}^n} f(\boldsymbol{\tau}) \left(\int_{\mathbb{R}^n} g(\mathbf{t}) e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{t}} e^{-2\pi i \boldsymbol{\xi}\cdot\boldsymbol{\tau}}\, d\mathbf{t} \right) d\boldsymbol{\tau} = \\
&= \left(\int_{\mathbb{R}^n} f(\boldsymbol{\tau}) e^{-2\pi i \boldsymbol{\xi}\cdot\boldsymbol{\tau}}\, d\boldsymbol{\tau} \right) \left(\int_{\mathbb{R}^n} g(\mathbf{t}) e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{t}}\, d\mathbf{t} \right) = \hat{f}(\boldsymbol{\xi}) \cdot \hat{g}(\boldsymbol{\xi}).
\end{aligned}$$

□

The complicated (but, as we shall later see, intuitively nice) notion of a convolution becomes a simple multiplication in the frequency domain.

Another extremely useful result is the Poisson summation formula, which directly relates sums of a function over the integer lattice to sums of its Fourier transform over the same lattice.

Theorem 3.2 (Poisson summation formula)**.** *Provided $f\colon \mathbb{R}^n \to \mathbb{C}$ is nice enough, the following holds:*

$$\sum_{\boldsymbol{k} \in \mathbb{Z}^n} f(\boldsymbol{k}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^n} \hat{f}(\boldsymbol{k}).$$

Proof. To prove this, we'll use basic notions of *Fourier series.* The idea is that sufficiently nice periodic functions can be expressed as a sum of waves whose frequencies are integer multiples of the base frequency. This is very similar to the Fourier transform, except that instead of a continuous spectrum of frequencies, we only have discrete ones.

Begin by defining the function $g\colon \mathbb{R}^n \to \mathbb{C}$ by

$$g(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} f(\mathbf{k}+\mathbf{x}).$$

Clearly, this function is periodic: changing $\mathbf{x}$ by any vector in $\mathbb{Z}^n$ preserves $g(\mathbf{x})$.

$$g(\mathbf{x}) = \sum_{\boldsymbol{\xi}\in\mathbb{Z}^n} c_{\boldsymbol{\xi}} e^{2\pi i\boldsymbol{\xi}\cdot\mathbf{x}}, \text{where } c_{\boldsymbol{\xi}} = \int_{[0,1]^n} g(\mathbf{t})e^{-2\pi i\boldsymbol{\xi}\cdot\mathbf{t}}\,d\mathbf{t}.$$

Since f is nice enough, g is too, so the Fourier series converges. We can now derive the following expression for the Fourier coefficients $c_{\boldsymbol{\xi}}$:

$$\begin{aligned} c_{\boldsymbol{\xi}} &= \int_{[0,1]^n} g(\mathbf{t})e^{-2\pi i\boldsymbol{\xi}\cdot\mathbf{t}}\,d\mathbf{t} = \int_{[0,1]^n} \sum_{\mathbf{k}\in\mathbb{Z}^n} f(\mathbf{k}+\mathbf{t})e^{-2\pi i\boldsymbol{\xi}\cdot\mathbf{t}}\,d\mathbf{t} = \sum_{\mathbf{k}\in\mathbb{Z}^n}\left(\int_{[0,1]^n} f(\mathbf{k}+\mathbf{t})e^{-2\pi i\boldsymbol{\xi}\cdot\mathbf{t}}\,d\mathbf{t}\right) = \\ &= \sum_{\mathbf{k}\in\mathbb{Z}^n}\left(\int_{[0,1]^n+\mathbf{k}} f(\mathbf{t})e^{-2\pi i\boldsymbol{\xi}\cdot(\mathbf{t}-\mathbf{k})}\,d\mathbf{t}\right) = \sum_{\mathbf{k}\in\mathbb{Z}^n}\left(\int_{[0,1]^n+\mathbf{k}} f(\mathbf{t})e^{-2\pi i\boldsymbol{\xi}\cdot\mathbf{t}}\,d\mathbf{t}\right) = \int_{\mathbb{R}^n} f(\mathbf{t})e^{-2\pi i\boldsymbol{\xi}\cdot\mathbf{t}}\,d\mathbf{t} = \\ &= \hat{f}(\boldsymbol{\xi}). \end{aligned}$$

Inserting this into the Fourier series expansion, we get

$$g(\mathbf{x}) = \sum_{\boldsymbol{\xi}\in\mathbb{Z}^n} \hat{f}(\boldsymbol{\xi})e^{2\pi i\boldsymbol{\xi}\cdot\mathbf{x}},$$

and upon specializing to $\mathbf{x} = \mathbf{0}$ it follows that

$$\sum_{\mathbf{k}\in\mathbb{Z}^n} f(\mathbf{k}) = \sum_{\boldsymbol{\xi}\in\mathbb{Z}^n} \hat{f}(\boldsymbol{\xi}).$$

□

4 The Power of Fourier Analysis

We're now ready to apply the Poisson summation formula to our initial problem. We'll want to take $f_r\colon \mathbb{R}^2 \to \mathbb{C}$ to be such that $f(\mathbf{t})$ is 1 when $|\mathbf{t}| \le r$ and 0 otherwise, because then $N(r) = \sum_{\mathbf{k}\in\mathbb{Z}^2} f_r(\mathbf{k})$. Applying the formula, we see that the sum is equal to $\sum_{\boldsymbol{\xi}\in\mathbb{Z}^2} \hat{f}_r(\boldsymbol{\xi})$. Because $\hat{f}_r(\mathbf{0}) = \int_{\mathbb{R}^2} f_r(\mathbf{t})\,d\mathbf{t} = \pi r^2$, that term will represent the main growth rate of $N(r)$, so we will have to bound the remaining terms somehow.

$$N(r) = \sum_{\mathbf{k}\in\mathbb{Z}^2} f_r(\mathbf{k}) = \sum_{\boldsymbol{\xi}\in\mathbb{Z}^2} \hat{f}_r(\boldsymbol{\xi}) = \pi r^2 + \sum_{\boldsymbol{\xi}\in\mathbb{Z}^2\setminus(0,0)} \hat{f}_r(\boldsymbol{\xi})$$

In order to do that, we should first evaluate $\hat{f}_r(\boldsymbol{\xi})$. Luckily, we can ask a physicist friend, who tells us it's in fact equal to $\frac{r}{|\boldsymbol{\xi}|}J_1(2\pi r|\boldsymbol{\xi}|)$, where J_1 is the Bessel function of the first kind of order 1. He goes on to tell us that this function has a known asymptotic form:

$$J_1(x) = \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{3\pi}{4}\right) + O(x^{-3/2}),$$

which we could use to bound our error term. In fact, we'd like to ignore the oscillations of the cosine function (since they seem quite hard to deal with), and use only the estimate $J_1(x) = O(x^{-1/2})$. From this we obtain $\hat{f}_r(\boldsymbol{\xi}) = \frac{r}{|\boldsymbol{\xi}|}J_1(2\pi r|\boldsymbol{\xi}|) = \frac{r}{|\boldsymbol{\xi}|}O(r^{-1/2}|\boldsymbol{\xi}|^{-1/2}) = O(r^{1/2}|\boldsymbol{\xi}|^{-3/2})$.

$$\sum_{\boldsymbol{\xi}\in\mathbb{Z}^2\setminus(0,0)} \hat{f}_r(\boldsymbol{\xi}) = \sum_{\boldsymbol{\xi}\in\mathbb{Z}^2\setminus(0,0)} \hat{f}_r(\boldsymbol{\xi}) = O\left(r^{1/2}\sum_{\boldsymbol{\xi}\in\mathbb{Z}^2\setminus(0,0)} |\boldsymbol{\xi}|^{-3/2}\right)$$

Oops, that sum doesn't converge. As it turns out (and this is relatively simple to prove by comparing the sum to an integral), $\sum_{\mathbf{k}\in\mathbb{Z}^2\setminus(0,0)} |\mathbf{k}|^{-p}$ converges if and only if $p > 2$.

Looks like we'll have to be more clever. This next step might look very unmotivated, so buckle up. We will *smooth* our sum, replacing the sharp cutoff at $|\mathbf{k}| = r$ with a smoother one. This will have the effect of making $\hat{f}_r(\mathbf{k})$ much smaller (in general, smoother functions have smaller Fourier transforms, which makes intuitive sense, as you'd expect to need fewer high-frequency waves to form a function with less steep slopes), so the transformed sum will actually converge. This will come at a cost of making the original sum slightly inaccurate, but it will turn out to be worth it.

To describe this smoothing process more rigorously we'll use convolutions. As mentioned previously, they behave very nicely with respect to the Fourier transform. We need to convolve f_r with a smooth function that is 0 outside of some disk (say with radius ε) and integrates to 1. Such functions are called "mollifiers" and can be quite easily proven to exist by construction; call one such function $\eta_\varepsilon\colon \mathbb{R}^2 \to \mathbb{C}$. Then it can be seen that $(f_r * \eta_\varepsilon)(\mathbf{t})$ is equal to 1 for $|\mathbf{t}| \le r - \varepsilon$, 0 for $|\mathbf{t}| \ge r + \varepsilon$, and smoothly interpolates between the two in the annulus between the two circles. Thus, to estimate $N(r)$, we can take the circles with radii $r \pm \varepsilon$, which ensure that we only overcount or undercount points in the annulus between the two circles and our desired circle of radius r. This way we get both a lower and an upper bound:

$$\sum_{\mathbf{k}\in\mathbb{Z}^2} (f_{r-\varepsilon} * \eta_\varepsilon)(\mathbf{k}) \le N(r) \le \sum_{\mathbf{k}\in\mathbb{Z}^2} (f_{r+\varepsilon} * \eta_\varepsilon)(\mathbf{k}),$$

so it indeed suffices to analyse $\sum_{\mathbf{k}\in\mathbb{Z}^2} (f_r * \eta_\varepsilon)(\mathbf{k})$. Applying the Poisson summation formula and the convolution theorem, we get

$$\sum_{\mathbf{k}\in\mathbb{Z}^2} (f_r * \eta_\varepsilon)(\mathbf{k}) = \sum_{\boldsymbol{\xi}\in\mathbb{Z}^2} \hat{f}_r(\boldsymbol{\xi})\hat{\eta}_\varepsilon(\boldsymbol{\xi}).$$

The key point is that η_ε is in a class of functions known as the Schwartz space, meaning that it and all its derivatives decrease like $O(|\boldsymbol{\xi}|^{-p})$ for all real p. An important property of this space is that the Fourier transform maps it to itself, so $\hat{\eta}_\varepsilon$ is in it too. Because $\eta_\varepsilon(\mathbf{t}) = \eta_1(\frac{\mathbf{t}}{\varepsilon})/\varepsilon^2$, we get $\hat{\eta}_\varepsilon(\boldsymbol{\xi}) = \hat{\eta}_1(\varepsilon\boldsymbol{\xi}) = O(\varepsilon^{-p}|\boldsymbol{\xi}|^{-p})$.

$$\sum_{\boldsymbol{\xi}\in\mathbb{Z}^2} \hat{f}_r(\boldsymbol{\xi})\hat{\eta}_\varepsilon(\boldsymbol{\xi}) = \pi r^2 + \sum_{\boldsymbol{\xi}\in\mathbb{Z}^2\setminus(0,0)} \hat{f}_r(\boldsymbol{\xi})\hat{\eta}_\varepsilon(\boldsymbol{\xi}) = \pi r^2 + O\left(r^{1/2}\varepsilon^{-p} \sum_{\boldsymbol{\xi}\in\mathbb{Z}^2\setminus(0,0)} |\boldsymbol{\xi}|^{-3/2-p}\right)$$

Now, taking any $p > \frac{1}{2}$ makes the sum converge, so it is $O(1)$ and we achieve the powerful bound

$$\sum_{\boldsymbol{\xi}\in\mathbb{Z}^2} \hat{f}_r(\boldsymbol{\xi})\hat{\eta}_\varepsilon(\boldsymbol{\xi}) = \pi r^2 + O(r^{1/2}\varepsilon^{-p}).$$

Finally, to account for the error introduced by smoothing, we use the inequalities we established earlier:

$$\pi r^2 + O(r\varepsilon + r^{1/2}\varepsilon^{-p}) = \pi(r-\varepsilon)^2 + O(r^{1/2}\varepsilon^{-p}) = \sum_{\mathbf{k}\in\mathbb{Z}^2} (f_{r-\varepsilon} * \eta_\varepsilon)(\mathbf{k}) \le N(r),$$

and

$$N(r) \le \sum_{\mathbf{k}\in\mathbb{Z}^2} (f_{r+\varepsilon} * \eta_\varepsilon)(\mathbf{k}) = \pi(r+\varepsilon)^2 + O(r^{1/2}\varepsilon^{-p}) = \pi r^2 + O(r\varepsilon + r^{1/2}\varepsilon^{-p}).$$

From this it follows that $N(r) = \pi r^2 + O(r\varepsilon + r^{1/2}\varepsilon^{-p})$. We want to choose an ε that minimizes the error term. Let's parametrize it with r^{-t}, so we need to pick $t > 0$ such that the maximum of the exponents in $O(r^{1-t} + r^{1/2+pt})$ is minimized. This happens when $1 - t = \frac{1}{2} + pt$, so $t = \frac{1}{2(p+1)}$ and the optimal error term is $O(r^{1-1/(2p+2)})$ for any $p > 1/2$. Clearly it's best to take p as small as possible, which results in

$$N(r) = \pi r^2 + O(r^{2/3+\delta})$$

for arbitrary $\delta > 0$.

References

[1] Julius L. Shaneson. *Estimates on Lattice Points in the Circle*. 2014. arXiv: 1409.2446 [math.NT]. URL: https://arxiv.org/abs/1409.2446.