Calculus II

MAT187 Student Slides

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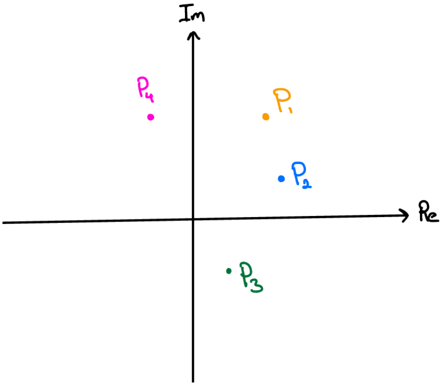
Exercise 1

Consider the plot of the complex numbers p_1, p_2, p_3, p_4 in the complex plane.



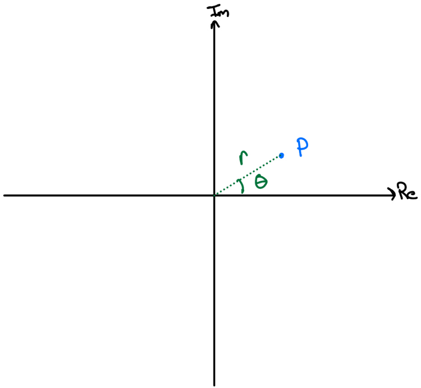
1.1 For which complex numbers is the real part grater than the imaginary part?

1.2 Which complex number has the smallest *modulus/absolute value*?

1.3 Which complex number has the largest *argument*? Is your answer at all ambiguous?

Exercise 2

Consider the plot of the complex number p in the complex plane.



2.1 Sketch the complex number $2p$.

2.2 Sketch the complex number p^2.

2.3 Sketch the complex numbers p^n for $n = 3, 4, \ldots$. Will your answer depend on r?

2.4 Use the geometry of the complex plane to find $\sqrt{i}$. Express your answer in both polar and rectangular form.

Exercise 3

Consider the equation

$$z^3 = -1 \tag{1}$$

3.1 Find a solution to Equation (1).

3.2 If $z = re^{i\theta}$ is a solution to Equation (1), what conditions must r and θ satisfy? Justify your conclusions.

3.3 Find all solutions to Equation (1).

Exercise 4

For each situation, decide whether *least squares* curve fitting or *polynomial interpolation* would be more appropriate.

4.1 You are modelling the arch used in the construction of a particular Roman aqueduct. You have collected several hundred data points of height of the arch vs. distance from the base of the aqueduct.

4.2 You are creating a function to govern the brightness of a light which will be used for signalling a computer. There are three different brightnesses that must be achieved exactly and the transition between those brightnesses must be smooth.

4.3 You are given exact data points from a lab and told that the data was created with a 4th degree polynomial. You are asked to find the coefficients of the polynomial.

Exercise 5

A baseball is thrown on the moon. You are trying to find the function

- $h(t)$, the height (in meters)of the baseball above the moon's surface at time t (in seconds).

You collected the following data

t	$h(t)$
1	4
2	3.8
3	2

5.1 What degree polynomial would best model h?

5.2 Use polynomial interpolation to find h.

5.3 Find the maximum height of the baseball above the moon's surface.

5.4 What would change (if anything) if you were given 4 data points?

Exercise 6

While developing a robotics control system, you find the need for a function f which satisfies the following properties:

(i) $f(0) = -1$ and $f(1) = 2$

(ii) $f'(0) = -1$ and $f'(1) = 2$

Your friend suggests that you could use the following polynomial to come up with f:

$$L_1(x) = -(x-1) \qquad L_2(x) = x$$

$$S_1(x) = (x-1)^2 x \qquad S_2(x) = (x-1)x^2$$

6.1 Can Lagrange interpolation be used to directly find f? Explain.

6.2 Complete the following table

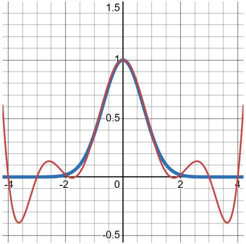
g	$g(0)$	$g(1)$	$g'(0)$	$g'(1)$
L_1				
L_2				
S_1				
S_2				

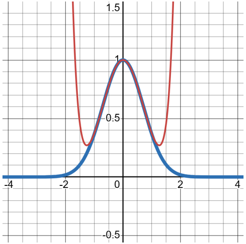
6.3 Use L_1, L_2, S_1, and S_2 to find a polynomial satisfying the properties of f.

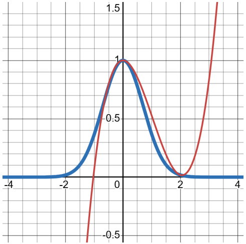
6.4 Explain how Lagrange interpolation can be generalized to allow finding a polynomial that passes through particular points and takes on particular derivatives at those points.

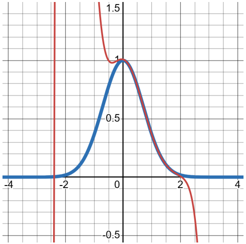
Exercise 7

7.1 For each polynomial approximation of the bell curve, is the approximation best at 0, best on the interval $[-2,2]$, or best on the interval $[0,2]$.

(A) 

(B) 

(C) 

(D) 

7.2 Based on the pictures, which polynomial(s) do you think come from a Taylor approximation?

Exercise 8

The function f satisfies

$$f(0) = 1 \qquad f'(0) = 0 \qquad f''(0) = -2$$

$$f'''(0) = 0 \qquad f''''(0) = 12$$

8.1 Write down T_4, the 4th degree Taylor approximation to f centered at 0.

8.2 Use Desmos to compare the graph of T_4 with the graphs of g_1, g_2, g_3, and g_4. Which of the g's do you think is most likely equal to f?

(a) $g_1(x) = e^{-|x|}$

(b) $g_2(x) = e^{-x^2}$

(c) $g_3(x) = \dfrac{1}{1+x^2}$

(d) $g_4(x) = \dfrac{1}{1+(2x)^4}$

Exercise 9

A bee is flying back and forth along a window sill trying to escape from your living room.

The bee's position at time t along the window sill is given by $r(t)$.

You know that a first-order Taylor approximation to $r(t)$ at time $t = 2$ is

$$A_1(t) = 3(t-2) + 1$$

9.1 Estimate the position of the bee on the window sill at time 2.1. Is your answer exact or approximate?

9.2 Estimate the velocity of the bee at time 2.1. Is your answer exact or approximate?

9.3 Are there any times you can compute the *exact* position of the bee?

9.4 Are there any times you can compute the *exact* velocity?

9.5 What is your best estimate for the acceleration of the bee at time 2.1?

Exercise 10

A bee is flying back and forth along a window sill trying to escape from your living room.

The bee's position at time t along the window sill is given by $r(t)$.

You know that a second-order Taylor approximation to $r(t)$ at time $t = 2$ is

$$A_2(t) = 2(t-2)^2 + 3(t-2) + 1$$

10.1 Estimate the position of the bee on the window sill at time 2.1. Is your answer exact or approximate?

10.2 Estimate the velocity of the bee at time 2.1. Is your answer exact or approximate?

10.3 Are there any times you can compute the *exact* position of the bee?

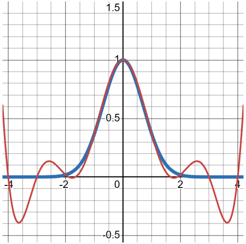
10.4 Are there any times you can compute the *exact* velocity?

10.5 What is your best estimate for the acceleration of the bee at time 2.1?

Exercise 11

Based on the pictures, which polynomial approximations of the bell curve do you think are *Taylor* polynomials?

(A)



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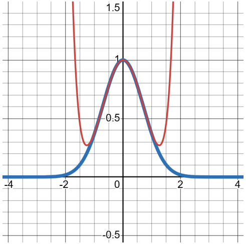

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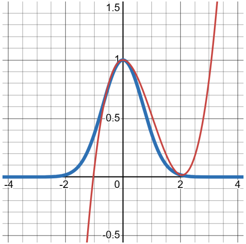

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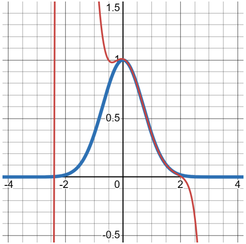


Exercise 12

Let $f(x) = e^x$ and let $P_n(x)$ be the nth Taylor approximation to f centered at 0. In particular

$$P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

Let $R_n(x)$ be the (signed) error in $P_n(x)$.

12.1 Find $R_3(1.5)$ (you may use a calculator).

12.2 What is the largest value of $R_3(x)$ when $0 \leq x \leq 2$?

12.3 Is there a value of x for which $R_3(x) = 0$? What does this say about P_3?

12.4 Given that $|f^{(5)}(x)| \leq 8$ when $x \in [0, 2]$, find an upper bound for $R_4(x)$ that

(a) works for a fixed $x \in [0, 2]$

(b) works simultaneously for all $x \in [0, 2]$

12.5 Given what you know from the previous part(s), can you bound $R_n(x)$?

Exercise 13

Let f be an infinitely differentiable function, and let P_n be a a Taylor polynomial for f of degree n centered at a.

We approximate $f(x) \approx P_n(x)$. Which of the following affect the size of the error in $P_n(x)$ (i.e., the magnitude of $R_n(x)$)?

(A) The degree of P_n, i.e., n.

(B) The magnitude of $f(a)$, i.e., $|f(a)|$.

(C) The magnitudes of the derivatives of f at a, i.e., the size of $|f'(a)|$, $|f''(a)|$, etc..

(D) The distance from a that you are approximating at, i.e., the size of $|x - a|$.

Exercise 14

Use Desmos to conjecture about the following questions.

`https://www.desmos.com/calculator/nrru5n0gqq`

14.1 True/False? When approximating $\sin(x)$ using Taylor polynomials centered at $x = 0$, higher degree polynomials will approximate $\sin(2)$ better.

14.2 True/False? When approximating $\tan(x)$ using Taylor polynomials centered at $x = 0$, higher degree polynomials will approximate $\tan(2)$ better.

14.3 True/False? When approximating $f(x) = \frac{1}{1+x^2}$ using Taylor polynomials centered at $x = 0$, higher degree polynomials will approximate $f(2)$ better.

14.4 Make a conjecture about the relationship between the degree of your Taylor approximation and the accuracy of its values. Does this contradict what you know from Taylor's remainder theorem?

Exercise 15

Consider the function $f(x) = \frac{1}{2}x^2 + 1$ and the value $I = \int_0^3 f(x)\,\mathrm{d}x$.

15.1 Make three sketches: one where the left-endpoint rule is used to approximate I, one where the right-endpoint rule is used, and one where the trapezoid rule is used. (Use at least three intervals.)

15.2 For the left-endpoint, right-endpoint, and trapezoid rules, which will give over estimates of I and which will give underestimates? Will any give an exact value?

15.3 Consider the following estimates of I:

- $E_1 = 8.6875$
- $E_2 = 6.4375$
- $E_3 = 7.5625$

Each estimate comes from using the same partition.

Which estimates come from a left-endpoint approximation, a right-endpoint approximation, and a trapezoid approximation?

Hint: you know calculus!

15.4 (Homework) Will the midpoint rule produce an over or under estimate of I?

Exercise 16

In a classic problem, you are trying to find the volume of a wine barrel. Let $r(\ell)$ represent the radius of the barrel ℓ cm from the base. The total length of the barrel is 80cm.

You know the volume of the barrel can be computed exactly by

$$\int_0^{80} \pi[r(\ell)]^2 \, d\ell.$$

You have measured the barrel in several places and gotten the following data

r(0)	r(20)	r(40)	r(60)	r(80)
12.8	21.2	22.7	21.4	13.4

16.1 Make a sketch of the barrel's profile. Make a second sketch of $\pi[r(\ell)]^2$.

16.2 Based on your sketch, do you think using a trapezoid approximation will produce an over or under estimate for the volume?

16.3 Use a trapezoid approximation to estimate the volume of the barrel.

16.4 Use a Simpson's approximation to estimate the volume of the barrel.

Reminder: if p is a quadratic polynomial,

$$\int_a^b p(x)\, dx = \frac{b-a}{6}\left(p(a) + 4p\left(\frac{a+b}{2}\right) + p(b)\right)$$

16.5 The exact (rounded) volume of the barrel is 104384cm^3. What approximation method was most accurate? Why?

Exercise 17

For this question, the domain of integration will be 0 to 5 and you will be using a uniform partition with 5 pieces.

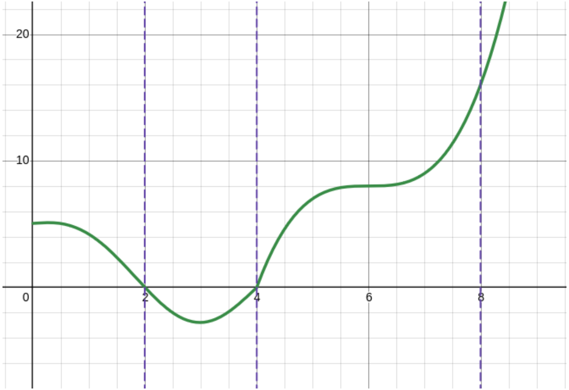
17.1 Draw a function where the left endpoint approximation is an *under estimate*.

17.2 Draw a function where the right endpoint approximation is an *under estimate*.

17.3 Draw a function where the trapezoid approximation is an *under estimate*.

17.4 Draw a function where the midpoint approximation is an *under estimate*.

Exercise 18



The graph above is of the function f. Marked on the graph are the intervals $A = [0,2]$, $B = [2,4]$, and $C = [4,8]$. We are interested in the quantity $I = \int_0^8 f(x)\,dx$.

18.1 On each interval, identify whether left/right/midpoint/trapezoid approximations will produce an

(a) Underestimate

(b) Overestimate

(c) Cannot be determined

18.2 Is there any interval where you're confident that Simpson's rule would produce an over/under estimate?

18.3 Come up with a strategy (i.e., a choice of integration method for each interval) that gives the best possible upper and lower bounds for I.

Exercise 19

Given an interval $[a,b]$ the midpoint-rule (with one interval) says to use $(b-a)f(0.5a+0.5b)$ as an estimate for $I = \int_a^b f(x)\,\mathrm{d}x$.

A *biased* midpoint rule with bias $\alpha \in [0,1]$ uses $(b-a)f(\alpha a+(1-\alpha)b)$ as an estimate for I.

19.1 Is there a bias α so that with a single partition, $\int_0^1 x^2\,\mathrm{d}x$ is *perfectly* approximated? If so, what is the bias?

19.2 Is there a bias α so that with *two* partitions, $\int_0^1 x^2\,\mathrm{d}x$ is *perfectly* approximated? If so, what is the bias?

19.3 How do your biases compare with the standard midpoint rule?

Exercise 20

20.1 Explain to your table: What is the difference between a sequence and a series?

20.2 How can you produce a sequence from a series?

20.3 How can you produce a series from a sequence?

20.4 Give an example of a bounded sequence that when summed produces an unbounded series.

Exercise 21

Define the sequence a_n by $a_n = \sin(\pi n)$ and the function f by $f(x) = \sin(\pi x)$.

21.1 Find $\lim_{n\to\infty} a_n$, if it exists.

21.2 Find $\lim_{x\to\infty} f(x)$, if it exists.

21.3 What is the difference between a sequence and a function.

Exercise 22

Define

$$a_n = \frac{4+n}{2+n} \qquad b_n = \frac{(-1)^n}{n^2}$$

for $n \geq 1$.

22.1 If a_n and b_n define sequences, what values can n take on? (E.g., any number in $\mathbb{R}$, any number in $\mathbb{Z}$, etc.)

22.2 Make a plot of a_n vs. n and b_n vs. n.

22.3 Which sequences (out of a_n and b_n) are (i) bounded above, (ii) bounded below, (iii) strictly increasing, (iv) strictly decreasing, (v) alternating.

22.4 Define $c_n = a_{n-1} + b_{2n}$ for $n \geq 2$. Find a formula for c_n.

22.5 Based on your answer to Part 3, will c_n be bounded above or below? Neither?

22.6 Find $\lim\limits_{n\to\infty} c_n$.

Exercise 23

Let a_n (for $n \geq 1$) be a sequence and define

$$S_n = \sum_{i=1}^{n} a_n.$$

Let $S_\infty = \lim_{n\to\infty} S_n$.

23.1 Which of the following statements must be true?

(a) If $|a_n| \geq 1$ for all n, then S_n converges.

(b) If $|a_n| \leq 1$ for all n, then S_n converges.

(c) If $|S_n| \geq 1$ for all n, then a_n diverges.

(d) If $|S_n| \leq 1$ for all n, then a_n diverges.

(e) If $a_n \to 0$ then S_n converges.

23.2 If you switch *converges* $\longleftrightarrow$ *diverges*, which statements change their truth value? (I.e., switch from being true to false or false to true.)

Exercise 24

Consider the function $f(x) = 1/x$, the sequence $a_n = 1/n$ and the sequence of partial sums $S_n = \sum_{i=1}^{n} a_i$.

In this question we want to get bounds on the *series*

$$\sum_{i=1}^{\infty} a_i$$

24.1 Use Σ-notation to write down a formula for the left-endpoint approximation of $\int_1^n \frac{1}{x}\,\mathrm{d}x$ using a partition whose intervals are width 1.

24.2 Use Σ-notation to write down a formula for the right-endpoint approximation of $\int_1^n \frac{1}{x}\,\mathrm{d}x$ using a partition whose intervals are width 1.

24.3 Use the actual value of $\int_1^n \frac{1}{x}\,\mathrm{d}x$ to give upper and lower bounds for S_n.

24.4 Does S_n converge or diverge? Explain.

Exercise 25

Consider the function $f(x) = 1/x^2$, the sequence $a_n = 1/n^2$ and the sequence of partial sums $S_n = \sum_{i=1}^{n} a_i$.

In this question we want to get bounds on the *series*

$$\sum_{i=1}^{\infty} a_i$$

25.1 Use Σ-notation to write down a formula for the left-endpoint approximation of $\int_1^n \frac{1}{x^2}\,\mathrm{d}x$ using a partition whose intervals are width 1.

25.2 Use Σ-notation to write down a formula for the right-endpoint approximation of $\int_1^n \frac{1}{x^2}\,\mathrm{d}x$ using a partition whose intervals are width 1.

25.3 Use the actual value of $\int_1^n \frac{1}{x^2}\,\mathrm{d}x$ to give upper and lower bounds for S_n.

25.4 Does S_n converge or diverge? Explain.

25.5 Conjecture about the convergence of $\sum_{i=1}^{\infty} i^{\alpha}$ for $\alpha > 0$. Can you justify your answer by comparing with known integrals?

Exercise 26

Let

$$a_n = \frac{1}{\sqrt{n}} \qquad b_n = \frac{1}{n^3} \qquad c_n = e^{-n} \qquad d_n = e^{-n^2}$$

and consider the corresponding sequences of partial sums A_n, B_n, C_n, and D_n. (I.e., $A_n = \sum_{i=1}^{n} a_n$, etc.)

26.1 Use a comparison with known integrals to decide the convergence of A_n, B_n, C_n.

26.2 Can you decide the convergence of D_n using a comparison to a known integral? Explain.

Exercise 27

Consider the function $f(x) = \sin(x)$.

27.1 Write down $T_k(x)$, the kth Taylor approximation to f centered at 0. You may use "$\cdots$" notation or Σ-notation.

27.2 Write down, using Σ-notation, $T(x)$, the Taylor series for f centered at 0.

27.3 In general a Taylor series may be written as $\displaystyle\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$, where a_n is a sequence. Find a_n in this case.

27.4 Let $R_k(x) = f(x) - T_k(x)$. Find an expression for $R_k(x)$ using Taylor's Remainder Theorem. Use your expression to find an upper bound for $|R_k(x)|$ (Hint: your bound may depend on x).

27.5 Using the fact that for any $\alpha \in \mathbb{R}$, $\displaystyle\lim_{n\to\infty} \frac{\alpha^n}{n!} = 0$, find $\displaystyle\lim_{k\to\infty} R_k(x)$.

27.6 For which x is $f(x) = T(x)$? Justify your answer.

Exercise 28

Consider the function $g(x) = \frac{1}{1-x}$. The kth Taylor approximation of g centered at 0 is

$$T_k(x) = \sum_{i=0}^{k} x^k$$

and the remainder $R_k(x) = g(x) - T_k(x)$ satisfies

$$|R_k(x)| \leq \frac{1}{1-x}\left(\frac{x}{1-x}\right)^{k+1}$$

when $x \geq 0$ and

$$|R_k(x)| \leq x^{k+1}$$

when $x < 0$.

28.1 For which x is $\lim_{k\to\infty} R_k(x) = 0$?

28.2 Let $T(x)$ be the Taylor series for g centered at 0. For which x can you guarantee that $g(x) = T(x)$?

28.3 Use the following Desmos link to numerically answer the question: for which x does $g(x) = T(x)$?

`https://www.desmos.com/calculator/yi4qczkxqn`

28.4 Does your answer to the previous part contradict Taylor's remainder theorem?

Exercise 29

Let $f(x) = \sin(x)$ and let

$$T(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

be the Taylor series for f centered at 0. We know that $f(x) = T(x)$ for all $x \in \mathbb{R}$.

29.1 Find a series representation for $g_1(x) = f(2x)$ (without computing any derivatives).

29.2 Find a series representation for $g_2(x) = f(x^2)$.

29.3 Use WolframAlpha to integrate g_2. Does WolframAlpha's solution make sense?

29.4 Compute $g_3(x) = \displaystyle\int g_2(x)\,\mathrm{d}x$ by integrating your series for $g_2(x)$ term by term. What should you do with the constants of integration?

29.5 For which x do you expect $g_3(x)$ to be valid? Explain.

29.6 When would it be advantageous to integrate a Taylor series term by term instead of integrating the original function? Explain.

Exercise 30

Let $f(x) = \frac{1}{1-x}$ and let

$$T(x) = \sum_{n=0}^{\infty} x^n$$

be the Taylor series for f centered at 0. We know that $f(x) = T(x)$ for all $x \in (-1, 1)$.

30.1 Find a series representation for $g_1(x) = f(2x)$ (without computing any derivatives).

30.2 For which x do you expect your series for $g_1(x)$ to be valid (i.e. to equal $f(2x)$? Explain.

30.3 Find a series representation for $g_2(x) = f(x^2)$.

30.4 Compute $g_3(x) = \displaystyle\int g_2(x)\,dx$ by integrating your series for $g_2(x)$ term by term.

30.5 For which x do you expect $g_3(x)$ to be valid? Explain.

Exercise 31

The function f has a Taylor series centered at 0 of the form

$$T(x) = -\frac{1}{2} + \frac{x}{3} - \frac{x^2}{4} + \frac{x^3}{5} - \frac{x^4}{6} + \cdots.$$

31.1 Express T using Σ-notation.

31.2 Find a series representation for $f'(x)$ and $\displaystyle\int f(x)\,\mathrm{d}x$.

31.3 Modify the following Desmos link and make a conjecture: for which values of x is $f(x) = T(x)$?

`https://www.desmos.com/calculator/try63qzvo5`

31.4 Based on your conjecture, for which values of x should your series for $f'(x)$ and $\displaystyle\int f(x)\,\mathrm{d}x$ be valid?

Exercise 32

Recall

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

for all $x \in \mathbb{R}$.

Let $f(x) = \cos(\sqrt{x})$.

32.1 Write down a Taylor series, T, for f.

Hint: you don't need to take any derivatives.

32.2 Find $f^{(6)}(0)$.

32.3 For what x is $T(x) = f(x)$? Explain.

32.4 Using Desmos, make a conjecture: for which values of x does your series converge?

`https://www.desmos.com/calculator/try63qzvo5`

32.5 Let T be a Taylor series for an unknown function g. If T converges at a value x_0, must it be true that $T(x_0) = g(x_0)$? Explain.

Exercise 33

The sequence a_n is defined by $a_0 = 10$ and

$$\frac{a_{n+1}}{a_n} = \frac{1}{4}$$

Define $S_n = \sum_{i=0}^{n} a_i$ and $S = \lim_{n\to\infty} S_n$.

33.1 Find an expression for a_n.

33.2 Is S_n bounded? Explain.

33.3 Compute S.

Recall: $\sum_{i=0}^{n} \alpha^i = \frac{1-\alpha^{n+1}}{1-\alpha}$

Exercise 34

Recall the sequence a_n from Exercise 33 defined by $a_0 = 10$ and $\dfrac{a_{n+1}}{a_n} = \dfrac{1}{4}$.

Consider the unknown, positive, sequence b_n. You know that $b_0 = 5$ and $\dfrac{b_{n+1}}{b_n} < \dfrac{1}{5}$.

34.1 Do you have enough information to write down an expression for b_n?

34.2 Which (if any) of the following relationships must hold for all n?

$$a_n < b_n \qquad b_n < a_n \qquad a_n = b_n$$

Justify your answer.

34.3 Consider the series $\sum_{n=0}^{\infty} b_n$. Does the series converge? Justify your answer by comparing with a known series.

34.4 If you were told that, actually, $b_0 = 100$, would that change your answer to the previous part?

Exercise 35

The ratio test states for a sequence c_n if

$$\lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} < 1$$

then $\sum_{n=0}^{\infty} c_n$ converges.

Recall the sequence a_n from Exercise 33 defined by $a_0 = 10$ and $\frac{a_{n+1}}{a_n} = \frac{1}{4}$.

You know the following about the positive sequence d_n: $\lim_{n\to\infty} \frac{d_{n+1}}{d_n} = \rho < \frac{1}{5}$.

35.1 Which (if any) of the following relationships must hold for all n?

$$a_n < d_n \qquad d_n < a_n \qquad a_n = d_n$$

35.2 Which (if any) of the following relationships *eventually* hold (i.e. hold for all sufficiently large n)?

$$a_n < d_n \qquad d_n < a_n \qquad a_n = d_n$$

Justify your answer.

35.3 Justify, without the ratio test, whether $\sum_{n=0}^{\infty} d_n$ converges.

35.4 Prove the ratio test.

Exercise 36

Theorem (Ratio Test). If c_n is a sequence and

$$\lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \rho$$

then $\sum_{n=0}^{\infty} c_n$

- converges if $\rho < 1$
- diverges if $\rho > 1$
- could converge or diverge if $\rho = 1$

36.1 The Ratio Test talks about the convergence of $\sum_{n=0}^{\infty} c_n$. Does it also apply to sums that don't start at $n = 0$? Explain.

36.2 Apply the ratio test to determine the convergence of the following series:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{8^n}{(-2)^{n+1} n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

Exercise 37

The Taylor series for $f(x) = \frac{1}{1-2x}$ is

$$T(x) = \sum_{n=0}^{\infty} 2^n x^n$$

37.1 Apply the ratio test to $T(x)$. Does $T(x)$ converge? Does your answer depend on x?

37.2 Let $G(x)$ be the Taylor series for $g(x) = e^x$. Apply the ratio test to $G(x)$. Does your answer depend on x?

37.3 Write down the largest (open) interval of convergence and the radius of convergence for T and G.

Exercise 38

The Taylor series for $h(x) = \frac{1}{1-x}$ centered at $a > 1$ is

$$H(x) = \sum_{n=0}^{\infty} \frac{(x-a)^n}{(1-a)^{n+1}}.$$

38.1 Find the largest (open) interval of converge and radius of convergence for H.

38.2 Graph h. Just looking at the graph, can you determine whether a Taylor series for h should have an infinite or finite radius of converge?

Exercise 39

Theorem (Integration by Parts). If $f(x)$ and $g(x)$ are differentiable functions, then

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x$$

39.1 The integration by parts formula comes from reversing one of the differentiation rules (e.g., chain rule/product rule/quotient rule). Which rule does the integration by parts formula come from?

39.2 Let $h_1(x) = x\sin x$.

(a) For h_1, write down all the ways to divide it into a product of "parts" f and g' so that $h_1 = f \cdot g'$.

(b) Pick the decomposition into parts that you think will be most useful and integrate h_1.

39.3 Let $h_2(x) = x^3 e^{x^2}$.

(a) For h_2, write down all the ways to divide it into a product of "parts" f and g' so that $h_2 = f \cdot g'$.

(b) Pick the decomposition into parts that you think will be most useful and integrate h_2.

Exercise 40

Theorem (Integration by Parts). If $f(x)$ and $g(x)$ are differentiable functions, then

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x$$

40.1 Use integration by parts to find $\displaystyle\int e^x \sin x\,\mathrm{d}x$.

Hint: *if at first you don't succeed, try, try again.*

40.2 Use integration by parts to find $\displaystyle\int_1^2 \ln x\,\mathrm{d}x$.

Hint: *sometimes g is hiding in plain sight.*

Exercise 41

We would like to compute

$$F(\theta) = \int \sin^2(\theta)\,\mathrm{d}\theta$$

41.1 (Review) Use integration by parts to find $F(\theta)$.

Hint: *The identity* $1 = \cos^2\theta + \sin^2\theta$ *may reduce your workload.*

41.2 Use the trig identity $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$ to find $F(\theta)$.

41.3 Find $\displaystyle\int \cos^2(\theta)\,\mathrm{d}\theta$ using any method you like.

Exercise 42

Let $f(x) = \sqrt{1-x^2}$ and consider

$$I = \int_0^1 f(x)\,\mathrm{d}x$$

42.1 If we define a change of variables $x = \sin\theta$, what would $\mathrm{d}x$ equal?

42.2 Apply the substitution $x = \sin\theta$ to get a new function $g(\theta)$ so that

$$\int_0^1 f(x)\,\mathrm{d}x = \int_?^? g(\theta)\,\mathrm{d}\theta$$

Find the function g and the bounds for the new integral (i.e. fill in the ?'s).

42.3 Find I.

42.4 Graph f. What shape does the graph make? Use your knowledge of geometry to find I.

Exercise 43

In Exercise 42 we computed

$$I = \int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = \int_0^{\pi/2} \cos^2\theta\,\mathrm{d}\theta$$

43.1 Explain why the bounds changed from $[0,1]$ to $[0,\pi/2]$.

43.2 Would it be okay to change the bounds from $[0,1]$ to $[2\pi, 5\pi/2]$?

43.3 Would it be okay to change the bounds from $[0,1]$ to $[0, 5\pi/2]$?

43.4 Compute I using the substitution $x = \cos\theta$. Pay close attention to make sure you use the correct bounds.

Exercise 44

Let $f(x) = \dfrac{\sqrt{9-x^2}}{x^2}$ and consider

$$F(x) = \int f(x)\,\mathrm{d}x$$

Using a substitution of $x = 3\sin\theta$, we arrive at

$$F(x) = -\frac{\cos\theta}{\sin\theta} - \theta + C.$$

44.1 Find an expression for $F(x)$ that involves only x.

44.2 Are there restrictions on domain of x for which your answer makes sense?

44.3 The domain of arcsin is $[-1, 1]$. Does this change your restrictions on the domain of x?

Exercise 45

Let $f(x) = \sqrt{\cos x}$ and consider

$$I = \int_0^{\sqrt{2}} f(x)\,\mathrm{d}x$$

You would like to find I.

45.1 Use WolframAlpha to find an anti-derivative of f. Does WolframAlpha give you a useful answer?

45.2 Using a 2^{nd} degree Taylor approximation for cos, write down an integral that will approximate I.

45.3 Find an approximation for I.

45.4 Use a 2^{nd} degree Taylor approximation for f to approximate I.

45.5 Desmos claims $I \approx 1.15686930348$. Which of your estimates is more accurate?

Exercise 46

Let

$$f(x) = \frac{1}{x^2 - 1} \qquad \text{and} \qquad g(x) = \frac{A}{x-1} + \frac{B}{x+1}.$$

46.1 You would like to know if there are constants A and B such that $f(x) = g(x)$ for all x.

Set up a system of linear equations (or a matrix equation) which has a solution if and only if there are constants A and B that make $f(x) = g(x)$ for all x (in the domain of f).

46.2 Find A and B, if possible.

46.3 Compute $\displaystyle\int f(x)\,\mathrm{d}x$ using any method of your choice.

Exercise 47

Let

$$f(x) = \frac{1}{(x-1)x^2} \qquad \text{and} \qquad g(x) = \frac{A}{x-1} + \frac{B}{x}.$$

47.1 You would like to know if there are constants A and B such that $f(x) = g(x)$ for all x.

Set up a system of linear equations (or a matrix equation) which has a solution if and only if there are constants A and B that make $f(x) = g(x)$ for all x (in the domain of f).

47.2 Find A and B, if possible.

47.3 Let $h(x) = \frac{A}{x-1} + \frac{B}{x} + \frac{C}{x^2}$. Can you find constants A, B, and C so that $f(x) = h(x)$ for all x (in the domain of f)? If so, do it.

47.4 Compute $\displaystyle\int f(x)\,\mathrm{d}x$ using any method of your choice.

Exercise 48

We know $\int \frac{1}{x^2+1}\,\mathrm{d}x = \arctan x + C$. However, we can also use partial fraction decomposition over the complex numbers to integrate.

Let $f(x) = \frac{1}{1+x^2}$ and $g(x) = \frac{A}{1+ix} + \frac{B}{1-ix}$.

48.1 Find A and B so that $f(x) = g(x)$ for all x (in the domain of f).

48.2 Compute $\displaystyle\int f(x)\,\mathrm{d}x$ using any method of your choice.

However: use $\ln x$ as the antiderivative of $\frac{1}{x}$ rather than $\ln|x|$.

48.3 Use the fact that $\ln(re^{i\theta}) = \ln r + i\theta$ to simplify your answer.

Exercise 49

An *improper integral* formalizes the concept of the area under an "infinite" curve.

Suppose f is a bounded function and let $I = \displaystyle\int_0^\infty f(x)\mathrm{d}x$.

49.1 Write down a formal definition of I.

49.2 Compute, using the definition, $\displaystyle\int_0^\infty \frac{1}{(x+1)^2}\mathrm{d}x$

49.3 Compute, using the definition, $\displaystyle\int_0^\infty \frac{1}{(x+1)}\mathrm{d}x$

Exercise 50

Let $f(x) = \dfrac{x}{x^2+1}$

In this question, we will try to compute

$$Q = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

50.1 Graph f. Make a guess on what you thing the “total area under the curve” (i.e. Q) should be.

50.2 Find $\displaystyle\int f(x)\,\mathrm{d}x$

50.3 Compute $\displaystyle\lim_{N\to\infty}\int_{-N}^{N} f(x)\,\mathrm{d}x$. Should your result be equal to Q?

50.4 Should $\displaystyle\lim_{N\to\infty}\int_{-N}^{2N} f(x)\,\mathrm{d}x$ correspond to Q? Compute it and compare with the previous part.

50.5 Compute $A = \displaystyle\int_{0}^{\infty} f(x)\,\mathrm{d}x$ and $B = \displaystyle\int_{-\infty}^{0} f(x)\,\mathrm{d}x$.

50.6 By the properties of integrals, we must have

$$Q = A + B.$$

Do the properties of integrals hold for this improper integral? What does this say about Q?

Exercise 51

The moral of improper integral is:

Wherever an infinity might appear, take a separate limit.

51.1 Rewrite $\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,\mathrm{d}x$ using limits of definite integrals.

51.2 Let $g(x) = \frac{1}{x^{1/3}}$ and consider $I = \displaystyle\int_{-8}^{27} g(x)\,\mathrm{d}x$.

(a) Identify all regions where $\displaystyle\int g(x)\,\mathrm{d}x$ *could* produce infinities.

(b) Rewrite I using limit(s) of definite integrals.

Exercise 52

Let $f(x) = \dfrac{\ln|x|}{x^4+1}$

$$J = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

52.1 Rewrite J using limit(s) of definite integrals.

52.2 Consider the functions

$$b_1(x) = \ln x \qquad b_2(x) = \frac{\ln x}{2}$$

$$b_3(x) = \frac{x}{x^4+1} \qquad b_4(x) = 0$$

(a) Let $A = \displaystyle\int_0^1 f(x)\,\mathrm{d}x$. Find upper and lower bounds for A by comparing with the appropriate $b_?$ functions.

(b) Let $B = \displaystyle\int_1^{\infty} f(x)\,\mathrm{d}x$. Find upper and lower bounds for B.

52.3 Use your results from the previous part to find upper and lower bounds for J.

Exercise 53

Consider the functions

$$A(t) = t \qquad B(t) = t^2 \qquad C(t) = t^{1/2}$$
$$D(t) = 2t \qquad E(t) = 2t^2 \qquad F(t) = 2t^{1/2}$$

53.1 Consider the parametric equations $x(t) = A(t)$ and $y(t) = D(t)$.

Graph, by hand, (x, y) for $t \in [0, 4]$.

53.2 Consider the parametric equations $x(t) = B(t)$ and $y(t) = A(t)$.

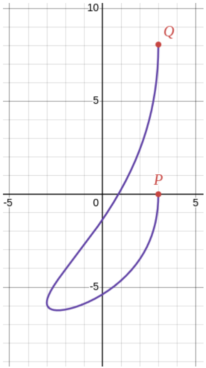
Graph, by hand, (x, y) for $t \in [0, 4]$.

53.3 Identify all possible assignments of $x(t) = ??$ and $y(t) = ??$ (where $??$ come from the functions above) so that the graph of (x, y) for $t \in [0, 4]$ is a line segment.

53.4 Out of your examples above, which example produces the *longest* line segment?

Exercise 54

Show is the graph of $\begin{cases} x(t) = 3\cos t \\ y(t) = t^2 - 5t \end{cases}$ for $t \in [0, 2\pi]$.



The parametric equations describe the position of a particle at time t.

54.1 Is the particle moving from P to Q or Q to P? Explain.

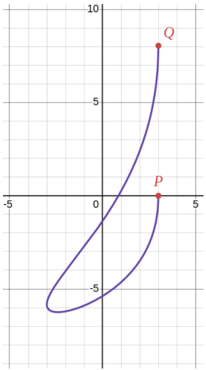
54.2 At what time(s) is the particle moving up *and* to the right?

54.3 At what time(s) is the particle moving parallel to the x-axis?

54.4 Find the tangent line to the particles path at time $t = \pi/2$.

Exercise 55

Show is the graph of $\begin{cases} x(t) = 3\cos t \\ y(t) = t^2 - 5t \end{cases}$ for $t \in [0, 2\pi]$.



The parametric equations describe the position of a particle at time t.

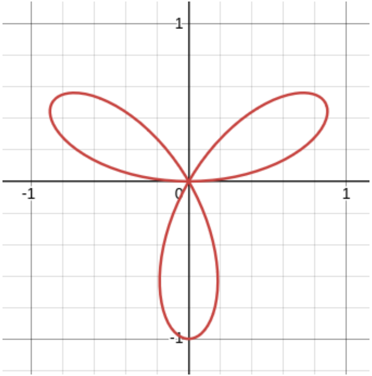
55.1 Find a parameterization for a particle that traces the same path, but starts at Q and ends at P.

55.2 Find a parameterization so that the particle finishes its journey in π seconds instead of 2π seconds.

55.3 Is there a parameterization so that the particle finishes its journey in π seconds but starts its journey at the same speed as the original particle? If such a parameterization exists, how would you come up with it?

Exercise 56

Show is the graph of $r(\theta) = \sin(3\theta)$ in polar coordinates for $\theta \in [0, \pi]$.



The curve models the boundary of a propeller.

56.1 The blade in the first quadrant achieves a maximum length at an angle of $\theta = \pi/6$. Find the rectangular coordinates of the tip of the blade in the first quadrant.

56.2 Find parametric equations $(x(t), y(t))$ that trace out the propeller.

56.3 Find the tangent line to the propeller when $\theta = 0$ and when $\theta = \pi/3$.

56.4 We know that the propeller is contained in a circle with area π.

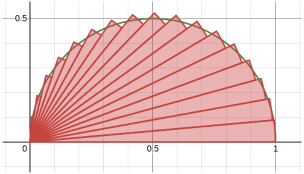
Come up with a better upper bound for the area of the propeller.

The same semi-circle can be described in polar coordinates by

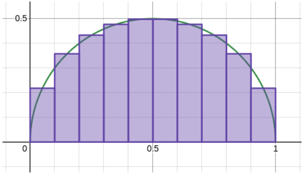
$$r(\theta) = \cos\theta \quad \text{with} \quad \theta \in [0, \pi/2]$$

or in rectangular coordinates by

$$y(x) = \sqrt{\frac{1}{4} - (x - \frac{1}{2})^2} \quad \text{with} \quad x \in [0, 1].$$



Below are two ways to divide up the semicircle to approximate its area: one with rectangles and one with circular *sectors*.



57.1 Write down a Riemann sum that approximates the area using *rectangles*. Use Δx as the width of a rectangle.

57.2 Write down a Riemann sum that approximates the area using *sectors*. Use $\Delta\theta$ as the sector angle.

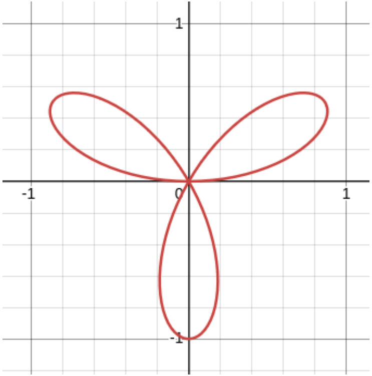
57.3 Take limits of your previous two Riemann sums to find integrals integrals to represent the *exact* area of the semicircle.

Do not evaluate your integrals.

57.4 Which integral would you rather do?

Exercise 58

Show is the graph of $r(\theta) = \sin(3\theta)$ in polar coordinates for $\theta \in [0, \pi]$.



The curve models the boundary of a propeller.

58.1 The first propeller blade is traced out for $\theta \in [0, \pi/3]$

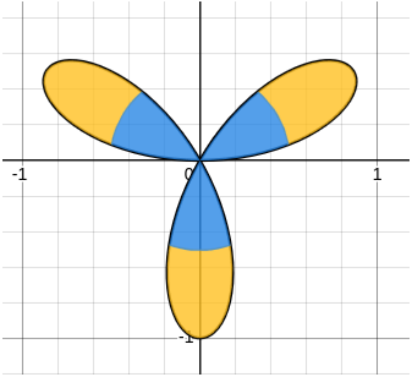
Set up a Riemann sum that approximates the area of the first propeller blade.

58.2 Set up an integral that will give the *exact* area of the first propeller blade. Then, find the area of the first propeller blade.

58.3 Find the total area of the propeller.

Exercise 59

Show is the graph of $r(\theta) = \sin(3\theta)$ in polar coordinates for $\theta \in [0, \pi]$.



The curve models the boundary of a propeller. The parts of the propeller within distance $1/2$ of the origin are painted blue. The rest is painted yellow.

59.1 Consider the propeller blade in the first quadrant. At what angle does the yellow paint start to appear? At what angle does it disappear?

59.2 Set up an integral that will give the amount of yellow paint needed for the first blade.

59.3 Set up an expression with integral(s) that will give the amount of blue paint needed for the first blade.

59.4 Find the amounts of each paint needed to paint the whole propeller.