

# Calculus II

MAT187 Student Slides

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## Exercise 1

Consider the plot of the complex numbers  $p_1, p_2, p_3, p_4$  in the complex plane.



- 1.1 For which complex numbers is the real part greater than the imaginary part?
- 1.2 Which complex number has the smallest *modulus/absolute value*?
- 1.3 Which complex number has the largest *argument*? Is your answer at all ambiguous?

## Exercise 2

Consider the plot of the complex number  $p$  in the complex plane.



- 2.1 Sketch the complex number  $2p$ .
- 2.2 Sketch the complex number  $p^2$ .
- 2.3 Sketch the complex numbers  $p^n$  for  $n = 3, 4, \dots$ . Will your answer depend on  $r$ ?
- 2.4 Use the geometry of the complex plane to find  $\sqrt{i}$ . Express your answer in both polar and rectangular form.

### Exercise 3

Consider the equation

$$z^3 = -1 \tag{1}$$

3.1 Find a solution to Equation (1).

3.2 If  $z = re^{i\theta}$  is a solution to Equation (1), what conditions must  $r$  and  $\theta$  satisfy? Justify your conclusions.

3.3 Find all solutions to Equation (1).

## Exercise 4

For each situation, decide whether *least squares* curve fitting or *polynomial interpolation* would be more appropriate.

- 4.1 You are modelling the arch used in the construction of a particular Roman aqueduct. You have collected several hundred data points of height of the arch vs. distance from the base of the aqueduct.
- 4.2 You are creating a function to govern the brightness of a light which will be used for signalling a computer. There are three different brightnesses that must be achieved exactly and the transition between those brightnesses must be smooth.
- 4.3 You are given exact data points from a lab and told that the data was created with a 4th degree polynomial. You are asked to find the coefficients of the polynomial.

## Exercise 5

A baseball is thrown on the moon. You are trying to find the function

- $h(t)$ , the height (in meters) of the baseball above the moon's surface at time  $t$  (in seconds).

You collected the following data

$t$	$h(t)$
1	4
2	3.8
3	2

- 5.1 What degree polynomial would best model  $h$ ?
- 5.2 Use polynomial interpolation to find  $h$ .
- 5.3 Find the maximum height of the baseball above the moon's surface.
- 5.4 What would change (if anything) if you were given 4 data points?

## Exercise 6

While developing a robotics control system, you find the need for a function  $f$  which satisfies the following properties:

(i)  $f(0) = -1$  and  $f(1) = 2$

(ii)  $f'(0) = -1$  and  $f'(1) = 2$

Your friend suggests that you could use the following polynomial to come up with  $f$ :

$$L_1(x) = -(x-1)$$

$$L_2(x) = x$$

$$S_1(x) = (x-1)^2x$$

$$S_2(x) = (x-1)x^2$$

6.1 Can Lagrange interpolation be used to directly find  $f$ ? Explain.

6.2 Complete the following table

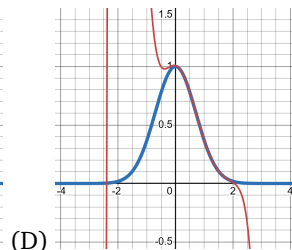
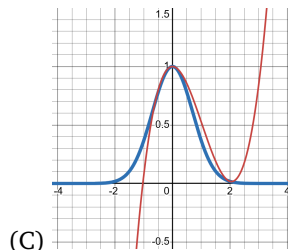
$g$	$g(0)$	$g(1)$	$g'(0)$	$g'(1)$
$L_1$				
$L_2$				
$S_1$				
$S_2$				

6.3 Use  $L_1$ ,  $L_2$ ,  $S_1$ , and  $S_2$  to find a polynomial satisfying the properties of  $f$ .

6.4 Explain how Lagrange interpolation can be generalized to allow finding a polynomial that passes through particular points and takes on particular derivatives at those points.

## Exercise 7

7.1 For each polynomial approximation of the bell curve, is the approximation best at 0, best on the interval  $[-2, 2]$ , or best on the interval  $[0, 2]$ .



7.2 Based on the pictures, which polynomial(s) do you think come from a Taylor approximation?



## Exercise 8

The function  $f$  satisfies

$$f(0) = 1 \quad f'(0) = 0 \quad f''(0) = -2$$

$$f'''(0) = 0 \quad f''''(0) = 12$$

8.1 Write down  $T_4$ , the 4th degree Taylor approximation to  $f$  centered at 0.

8.2 Use Desmos to compare the graph of  $T_4$  with the graphs

of  $g_1$ ,  $g_2$ ,  $g_3$ , and  $g_4$ . Which of the  $g$ 's do you think is most likely equal to  $f$ ?

(a)  $g_1(x) = e^{-|x|}$

(b)  $g_2(x) = e^{-x^2}$

(c)  $g_3(x) = \frac{1}{1+x^2}$

(d)  $g_4(x) = \frac{1}{1+(2x)^4}$

## Exercise 9

A bee is flying back and forth along a window sill trying to escape from your living room.

The bee's position at time  $t$  along the window sill is given by  $r(t)$ .

You know that a first-order Taylor approximation to  $r(t)$  at time  $t = 2$  is

$$A_1(t) = 3(t - 2) + 1$$

- 9.1 Estimate the position of the bee on the window sill at time 2.1. Is your answer exact or approximate?
- 9.2 Estimate the velocity of the bee at time 2.1. Is your answer exact or approximate?
- 9.3 Are there any times you can compute the *exact* position of the bee?
- 9.4 Are there any times you can compute the *exact* velocity?
- 9.5 What is your best estimate for the acceleration of the bee at time 2.1?

## Exercise 10

A bee is flying back and forth along a window sill trying to escape from your living room.

The bee's position at time  $t$  along the window sill is given by  $r(t)$ .

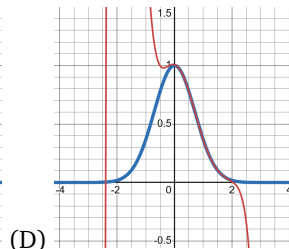
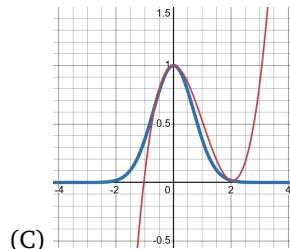
You know that a second-order Taylor approximation to  $r(t)$  at time  $t = 2$  is

$$A_2(t) = 2(t - 2)^2 + 3(t - 2) + 1$$

- 10.1 Estimate the position of the bee on the window sill at time 2.1. Is your answer exact or approximate?
- 10.2 Estimate the velocity of the bee at time 2.1. Is your answer exact or approximate?
- 10.3 Are there any times you can compute the *exact* position of the bee?
- 10.4 Are there any times you can compute the *exact* velocity?
- 10.5 What is your best estimate for the acceleration of the bee at time 2.1?

## Exercise 11

Based on the pictures, which polynomial approximations of the bell curve do you think are *Taylor* polynomials?



## Exercise 12

Let  $f(x) = e^x$  and let  $P_n(x)$  be the  $n$ th Taylor approximation to  $f$  centered at 0. In particular

$$P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

Let  $R_n(x)$  be the (signed) error in  $P_n(x)$ .

12.1 Find  $R_3(1.5)$  (you may use a calculator).

12.2 What is the largest value of  $R_3(x)$  when  $0 \leq x \leq 2$ ?

12.3 Is there a value of  $x$  for which  $R_3(x) = 0$ ? What does

this say about  $P_3$ ?

12.4 Given that  $|f^{(5)}(x)| \leq 8$  when  $x \in [0, 2]$ , find an upper bound for  $R_4(x)$  that

(a) works for a fixed  $x \in [0, 2]$

(b) works simultaneously for all  $x \in [0, 2]$

12.5 Given what you know from the previous part(s), can you bound  $R_n(x)$ ?

### Exercise 13

Let  $f$  be an infinitely differentiable function, and let  $P_n$  be a Taylor polynomial for  $f$  of degree  $n$  centered at  $a$ .

We approximate  $f(x) \approx P_n(x)$ . Which of the following affect the size of the error in  $P_n(x)$  (i.e., the magnitude of  $R_n(x)$ )?

- (A) The degree of  $P_n$ , i.e.,  $n$ .
- (B) The magnitude of  $f(a)$ , i.e.,  $|f(a)|$ .
- (C) The magnitudes of the derivatives of  $f$  at  $a$ , i.e., the size of  $|f'(a)|$ ,  $|f''(a)|$ , etc..
- (D) The distance from  $a$  that you are approximating at, i.e., the size of  $|x - a|$ .

## Exercise 14

Use Desmos to conjecture about the following questions.

<https://www.desmos.com/calculator/nrru5n0gqq>

- 14.1 True/False? When approximating  $\sin(x)$  using Taylor polynomials centered at  $x = 0$ , higher degree polynomials will approximate  $\sin(2)$  better.
- 14.2 True/False? When approximating  $\tan(x)$  using Taylor polynomials centered at  $x = 0$ , higher degree polynomials will approximate  $\tan(2)$  better.
- 14.3 True/False? When approximating  $f(x) = \frac{1}{1+x^2}$  using Taylor polynomials centered at  $x = 0$ , higher degree polynomials will approximate  $f(2)$  better.
- 14.4 Make a conjecture about the relationship between the degree of your Taylor approximation and the accuracy of its values. Does this contradict what you know from Taylor's remainder theorem?

## Exercise 15

Consider the function  $f(x) = \frac{1}{2}x^2 + 1$  and the value

$$I = \int_0^3 f(x) dx.$$

15.1 Make three sketches: one where the left-endpoint rule is used to approximate  $I$ , one where the right-endpoint rule is used, and one where the trapezoid rule is used. (Use at least three intervals.)

15.2 For the left-endpoint, right-endpoint, and trapezoid rules, which will give over estimates of  $I$  and which will give underestimates? Will any give an exact value?

15.3 Consider the following estimates of  $I$ :

- $E_1 = 8.6875$
- $E_2 = 6.4375$
- $E_3 = 7.5625$

Each estimate comes from using the same partition.

Which estimates come from a left-endpoint approximation, a right-endpoint approximation, and a trapezoid approximation?

*Hint: you know calculus!*

15.4 (Homework) Will the midpoint rule produce an over or under estimate of  $I$ ?



## Exercise 16

In a classic problem, you are trying to find the volume of a wine barrel. Let  $r(\ell)$  represent the radius of the barrel  $\ell$  cm from the base. The total length of the barrel is 80cm.

You know the volume of the barrel can be computed exactly by

$$\int_0^{80} \pi[r(\ell)]^2 d\ell.$$

You have measured the barrel in several places and gotten the following data

$r(0)$	$r(20)$	$r(40)$	$r(60)$	$r(80)$
12.8	21.2	22.7	21.4	13.4

- 16.1 Make a sketch of the barrel's profile. Make a second sketch of  $\pi[r(\ell)]^2$ .

- 16.2 Based on your sketch, do you think using a trapezoid approximation will produce an over or under estimate for the volume?

- 16.3 Use a trapezoid approximation to estimate the volume of the barrel.

- 16.4 Use a Simpson's approximation to estimate the volume of the barrel.

*Reminder:* if  $p$  is a quadratic polynomial,

$$\int_a^b p(x) dx = \frac{b-a}{6} \left( p(a) + 4p\left(\frac{a+b}{2}\right) + p(b) \right)$$

- 16.5 The exact (rounded) volume of the barrel is  $104384\text{cm}^3$ . What approximation method was most accurate? Why?

## Exercise 17

For this question, the domain of integration will be 0 to 5 and you will be using a uniform partition with 5 pieces.

- 17.1 Draw a function where the left endpoint approximation is an *under estimate*.
- 17.2 Draw a function where the right endpoint approximation is an *under estimate*.
- 17.3 Draw a function where the trapezoid approximation is an *under estimate*.
- 17.4 Draw a function where the midpoint approximation is an *under estimate*.

## Exercise 18



The graph above is of the function  $f$ . Marked on the graph are the intervals  $A = [0, 2]$ ,  $B = [2, 4]$ , and  $C = [4, 8]$ . We

are interested in the quantity  $I = \int_0^8 f(x) dx$ .

- 18.1 On each interval, identify whether left/right/mid-point/trapezoid approximations will produce an
- (a) Underestimate
  - (b) Overestimate
  - (c) Cannot be determined
- 18.2 Is there any interval where you're confident that Simpson's rule would produce an over/under estimate?
- 18.3 Come up with a strategy (i.e., a choice of integration method for each interval) that gives the best possible upper and lower bounds for  $I$ .

## Exercise 19

Given an interval  $[a, b]$  the midpoint-rule (with one interval) says to use  $(b - a)f(0.5a + 0.5b)$  as an estimate for

$$I = \int_a^b f(x) dx.$$

A *biased* midpoint rule with bias  $\alpha \in [0, 1]$  uses  $(b - a)f(\alpha a + (1 - \alpha)b)$  as an estimate for  $I$ .

- 19.1 Is there a bias  $\alpha$  so that with a single partition,  $\int_0^1 x^2 dx$  is *perfectly* approximated? If so, what is the bias?
- 19.2 Is there a bias  $\alpha$  so that with *two* partitions,  $\int_0^1 x^2 dx$  is *perfectly* approximated? If so, what is the bias?
- 19.3 How do your biases compare with the standard midpoint rule?

## Exercise 20

- 20.1 Explain to your table: What is the difference between a sequence and a series?
- 20.2 How can you produce a sequence from a series?
- 20.3 How can you produce a series from a sequence?
- 20.4 Give an example of a bounded sequence that when summed produces an unbounded series.

## Exercise 21

Define the sequence  $a_n$  by  $a_n = \sin(\pi n)$  and the function  $f$  by  $f(x) = \sin(\pi x)$ .

21.1 Find  $\lim_{n \rightarrow \infty} a_n$ , if it exists.

21.2 Find  $\lim_{x \rightarrow \infty} f(x)$ , if it exists.

21.3 What is the difference between a sequence and a function.

## Exercise 22

Define

$$a_n = \frac{4+n}{2+n} \quad b_n = \frac{(-1)^n}{n^2}$$

for  $n \geq 1$ .

22.1 If  $a_n$  and  $b_n$  define sequences, what values can  $n$  take on? (E.g., any number in  $\mathbb{R}$ , any number in  $\mathbb{Z}$ , etc.)

22.2 Make a plot of  $a_n$  vs.  $n$  and  $b_n$  vs.  $n$ .

22.3 Which sequences (out of  $a_n$  and  $b_n$ ) are (i) bounded above, (ii) bounded below, (iii) strictly increasing, (iv) strictly decreasing, (v) alternating.

22.4 Define  $c_n = a_{n-1} + b_{2n}$  for  $n \geq 2$ . Find a formula for  $c_n$ .

22.5 Based on your answer to Part 3, will  $c_n$  be bounded above or below? Neither?

22.6 Find  $\lim_{n \rightarrow \infty} c_n$ .

## Exercise 23

Let  $a_n$  (for  $n \geq 1$ ) be a sequence and define

$$S_n = \sum_{i=1}^n a_i.$$

Let  $S_\infty = \lim_{n \rightarrow \infty} S_n$ .

23.1 Which of the following statements must be true?

- (a) If  $|a_n| \geq 1$  for all  $n$ , then  $S_n$  converges.
- (b) If  $|a_n| \leq 1$  for all  $n$ , then  $S_n$  converges.
- (c) If  $|S_n| \geq 1$  for all  $n$ , then  $a_n$  diverges.
- (d) If  $|S_n| \leq 1$  for all  $n$ , then  $a_n$  diverges.
- (e) If  $a_n \rightarrow 0$  then  $S_n$  converges.

23.2 If you switch *converges*  $\leftrightarrow$  *diverges*, which statements change their truth value? (I.e., switch from being true to false or false to true.)



## Exercise 24

Consider the function  $f(x) = 1/x$ , the sequence  $a_n = 1/n$  and the sequence of partial sums  $S_n = \sum_{i=1}^n a_i$ .

In this question we want to get bounds on the *series*

$$\sum_{i=1}^{\infty} a_i$$

24.1 Use  $\Sigma$ -notation to write down a formula for the left-endpoint approximation of  $\int_1^n \frac{1}{x} dx$  using a partition whose intervals are width 1.

24.2 Use  $\Sigma$ -notation to write down a formula for the right-endpoint approximation of  $\int_1^n \frac{1}{x} dx$  using a partition whose intervals are width 1.

24.3 Use the actual value of  $\int_1^n \frac{1}{x} dx$  to give upper and lower bounds for  $S_n$ .

24.4 Does  $S_n$  converge or diverge? Explain.

## Exercise 25

Consider the function  $f(x) = 1/x^2$ , the sequence  $a_n = 1/n^2$  and the sequence of partial sums  $S_n = \sum_{i=1}^n a_i$ .

In this question we want to get bounds on the series

$$\sum_{i=1}^{\infty} a_i$$

25.1 Use  $\Sigma$ -notation to write down a formula for the left-endpoint approximation of  $\int_1^n \frac{1}{x^2} dx$  using a partition whose intervals are width 1.

25.2 Use  $\Sigma$ -notation to write down a formula for the right-endpoint approximation of  $\int_1^n \frac{1}{x^2} dx$  using a partition whose intervals are width 1.

25.3 Use the actual value of  $\int_1^n \frac{1}{x^2} dx$  to give upper and lower bounds for  $S_n$ .

25.4 Does  $S_n$  converge or diverge? Explain.

25.5 Conjecture about the convergence of  $\sum_{i=1}^{\infty} i^\alpha$  for  $\alpha > 0$ .

Can you justify your answer by comparing with known integrals?

## Exercise 26

Let

$$a_n = \frac{1}{\sqrt{n}} \quad b_n = \frac{1}{n^3} \quad c_n = e^{-n} \quad d_n = e^{-n^2}$$

and consider the corresponding sequences of partial sums  $A_n$ ,  $B_n$ ,  $C_n$ , and  $D_n$ . (I.e.,  $A_n = \sum_{i=1}^n a_i$ , etc.)

26.1 Use a comparison with known integrals to decide the convergence of  $A_n$ ,  $B_n$ ,  $C_n$ .

26.2 Can you decide the convergence of  $D_n$  using a comparison to a known integral? Explain.

## Exercise 27

Consider the function  $f(x) = \sin(x)$ .

27.1 Write down  $T_k(x)$ , the  $k$ th Taylor approximation to  $f$  centered at 0. You may use “ $\dots$ ” notation or  $\Sigma$ -notation.

27.2 Write down, using  $\Sigma$ -notation,  $T(x)$ , the Taylor series for  $f$  centered at 0.

27.3 In general a Taylor series may be written as  $\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ , where  $a_n$  is a sequence. Find  $a_n$  in this case.

27.4 Let  $R_k(x) = f(x) - T_k(x)$ . Find an expression for  $R_k(x)$  using Taylor's Remainder Theorem. Use your expression to find an upper bound for  $|R_k(x)|$  (Hint: your bound may depend on  $x$ ).

27.5 Using the fact that for any  $\alpha \in \mathbb{R}$ ,  $\lim_{n \rightarrow \infty} \frac{\alpha^n}{n!} = 0$ , find  $\lim_{k \rightarrow \infty} R_k(x)$ .

27.6 For which  $x$  is  $f(x) = T(x)$ ? Justify your answer.

## Exercise 28

Consider the function  $g(x) = \frac{1}{1-x}$ . The  $k$ th Taylor approximation of  $g$  centered at 0 is

$$T_k(x) = \sum_{i=0}^k x^i$$

and the remainder  $R_k(x) = g(x) - T_k(x)$  satisfies

$$|R_k(x)| \leq \frac{1}{1-x} \left( \frac{x}{1-x} \right)^{k+1}$$

when  $x \geq 0$  and

$$|R_k(x)| \leq x^{k+1}$$

when  $x < 0$ .

28.1 For which  $x$  is  $\lim_{k \rightarrow \infty} R_k(x) = 0$ ?

28.2 Let  $T(x)$  be the Taylor series for  $g$  centered at 0. For which  $x$  can you guarantee that  $g(x) = T(x)$ ?

28.3 Use the following Desmos link to numerically answer the question: for which  $x$  does  $g(x) = T(x)$ ?

<https://www.desmos.com/calculator/yi4qczkxqn>

28.4 Does your answer to the previous part contradict Taylor's remainder theorem?

## Exercise 29

Let  $f(x) = \sin(x)$  and let

$$T(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

be the Taylor series for  $f$  centered at 0. We know that  $f(x) = T(x)$  for all  $x \in \mathbb{R}$ .

29.1 Find a series representation for  $g_1(x) = f(2x)$  (without computing any derivatives).

29.2 Find a series representation for  $g_2(x) = f(x^2)$ .

29.3 Use WolframAlpha to integrate  $g_2$ . Does WolframAlpha's solution make sense?

29.4 Compute  $g_3(x) = \int g_2(x) dx$  by integrating your series for  $g_2(x)$  term by term. What should you do with the constants of integration?

29.5 For which  $x$  do you expect  $g_3(x)$  to be valid? Explain.

29.6 When would it be advantageous to integrate a Taylor series term by term instead of integrating the original function? Explain.

## Exercise 30

Let  $f(x) = \frac{1}{1-x}$  and let

$$T(x) = \sum_{n=0}^{\infty} x^n$$

be the Taylor series for  $f$  centered at 0. We know that  $f(x) = T(x)$  for all  $x \in (-1, 1)$ .

30.1 Find a series representation for  $g_1(x) = f(2x)$  (without computing any derivatives).

30.2 For which  $x$  do you expect your series for  $g_1(x)$  to be valid (i.e. to equal  $f(2x)$ )? Explain.

30.3 Find a series representation for  $g_2(x) = f(x^2)$ .

30.4 Compute  $g_3(x) = \int g_2(x) dx$  by integrating your series for  $g_2(x)$  term by term.

30.5 For which  $x$  do you expect  $g_3(x)$  to be valid? Explain.

## Exercise 31

The function  $f$  has a Taylor series centered at 0 of the form

$$T(x) = -\frac{1}{2} + \frac{x}{3} - \frac{x^2}{4} + \frac{x^3}{5} - \frac{x^4}{6} + \cdots.$$

31.1 Express  $T$  using  $\Sigma$ -notation.

31.2 Find a series representation for  $f'(x)$  and  $\int f(x) dx$ .

31.3 Modify the following Desmos link and make a conjecture: for which values of  $x$  is  $f(x) = T(x)$ ?

<https://www.desmos.com/calculator/try63qzvo5>

31.4 Based on your conjecture, for which values of  $x$  should your series for  $f'(x)$  and  $\int f(x) dx$  be valid?



Recall

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

for all  $x \in \mathbb{R}$ .

Let  $f(x) = \cos(\sqrt{x})$ .

32.1 Write down a Taylor series,  $T$ , for  $f$ .

*Hint: you don't need to take any derivatives.*

32.2 Find  $f^{(6)}(0)$ .

32.3 For what  $x$  is  $T(x) = f(x)$ ? Explain.

32.4 Using Desmos, make a conjecture: for which values of  $x$  does your series converge?

<https://www.desmos.com/calculator/try63qzvo5>

32.5 Let  $T$  be a Taylor series for an unknown function  $g$ . If  $T$  converges at a value  $x_0$ , must it be true that  $T(x_0) = g(x_0)$ ? Explain.

## Exercise 33

The sequence  $a_n$  is defined by  $a_0 = 10$  and

$$\frac{a_{n+1}}{a_n} = \frac{1}{4}$$

Define  $S_n = \sum_{i=0}^n a_i$  and  $S = \lim_{n \rightarrow \infty} S_n$ .

33.1 Find an expression for  $a_n$ .

33.2 Is  $S_n$  bounded? Explain.

33.3 Compute  $S$ .

$$\text{Recall: } \sum_{i=0}^n \alpha^i = \frac{1 - \alpha^{n+1}}{1 - \alpha}$$

## Exercise 34

Recall the sequence  $a_n$  from Exercise 33 defined by  $a_0 = 10$  and  $\frac{a_{n+1}}{a_n} = \frac{1}{4}$ .

Consider the unknown, positive, sequence  $b_n$ . You know that  $b_0 = 5$  and  $\frac{b_{n+1}}{b_n} < \frac{1}{5}$ .

34.1 Do you have enough information to write down an expression for  $b_n$ ?

34.2 Which (if any) of the following relationships must hold

for all  $n$ ?

$$a_n < b_n \quad b_n < a_n \quad a_n = b_n$$

Justify your answer.

34.3 Consider the series  $\sum_{n=0}^{\infty} b_n$ . Does the series converge?

Justify your answer by comparing with a known series.

34.4 If you were told that, actually,  $b_0 = 100$ , would that change your answer to the previous part?

## Exercise 35

The ratio test states for a sequence  $c_n$  if

$$\lim_{n \rightarrow \infty} \frac{|c_{n+1}|}{|c_n|} < 1$$

then  $\sum_{n=0}^{\infty} c_n$  converges.

Recall the sequence  $a_n$  from Exercise 33 defined by  $a_0 = 10$  and  $\frac{a_{n+1}}{a_n} = \frac{1}{4}$ .

You know the following about the positive sequence  $d_n$ :

$$\lim_{n \rightarrow \infty} \frac{d_{n+1}}{d_n} = \rho < \frac{1}{5}.$$

for all  $n$ ?

$$a_n < d_n \quad d_n < a_n \quad a_n = d_n$$

35.2 Which (if any) of the following relationships *eventually* hold (i.e. hold for all sufficiently large  $n$ )?

$$a_n < d_n \quad d_n < a_n \quad a_n = d_n$$

Justify your answer.

35.3 Justify, without the ratio test, whether  $\sum_{n=0}^{\infty} d_n$  converges.

35.4 Prove the ratio test.

**Theorem (Ratio Test).** If  $c_n$  is a sequence and

$$\lim_{n \rightarrow \infty} \frac{|c_{n+1}|}{|c_n|} = \rho$$

then  $\sum_{n=0}^{\infty} c_n$

- converges if  $\rho < 1$
- diverges if  $\rho > 1$
- could converge or diverge if  $\rho = 1$

36.1 The Ratio Test talks about the convergence of  $\sum_{n=0}^{\infty} c_n$ .

Does it also apply to sums that don't start at  $n = 0$ ? Explain.

36.2 Apply the ratio test to determine the convergence of the following series:

(a)  $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

(b)  $\sum_{n=1}^{\infty} \frac{8^n}{(-2)^{n+1}n}$

(c)  $\sum_{n=1}^{\infty} \frac{1}{n}$

## Exercise 37

The Taylor series for  $f(x) = \frac{1}{1-2x}$  is

$$T(x) = \sum_{n=0}^{\infty} 2^n x^n$$

37.1 Apply the ratio test to  $T(x)$ . Does  $T(x)$  converge? Does your answer depend on  $x$ ?

37.2 Let  $G(x)$  be the Taylor series for  $g(x) = e^x$ . Apply the ratio test to  $G(x)$ . Does your answer depend on  $x$ ?

37.3 Write down the largest (open) interval of convergence and the radius of convergence for  $T$  and  $G$ .

## Exercise 38

The Taylor series for  $h(x) = \frac{1}{1-x}$  centered at  $a > 1$  is

$$H(x) = \sum_{n=0}^{\infty} \frac{(x-a)^n}{(1-a)^{n+1}}.$$

38.1 Find the largest (open) interval of converge and radius of convergence for  $H$ .

38.2 Graph  $h$ . Just looking at the graph, can you determine whether a Taylor series for  $h$  should have an infinite or finite radius of converge?

**Theorem (Integration by Parts).** If  $f(x)$  and  $g(x)$  are differentiable functions, then

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

39.1 The integration by parts formula comes from reversing one of the differentiation rules (e.g., chain rule/product rule/quotient rule). Which rule does the integration by parts formula come from?

39.2 Let  $h_1(x) = x \sin x$ .

- (a) For  $h_1$ , write down all the ways to divide it into a product of “parts”  $f$  and  $g'$  so that  $h_1 = f \cdot g'$ .
- (b) Pick the decomposition into parts that you think will be most useful and integrate  $h_1$ .

39.3 Let  $h_2(x) = x^3 e^{x^2}$ .

- (a) For  $h_2$ , write down all the ways to divide it into a product of “parts”  $f$  and  $g'$  so that  $h_2 = f \cdot g'$ .
- (b) Pick the decomposition into parts that you think will be most useful and integrate  $h_2$ .



**Theorem (Integration by Parts).** If  $f(x)$  and  $g(x)$  are differentiable functions, then

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

40.1 Use integration by parts to find  $\int e^x \sin x dx$ .

Hint: *if at first you don't succeed, try, try again.*

40.2 Use integration by parts to find  $\int_1^2 \ln x dx$ .

Hint: *sometimes  $g$  is hiding in plain sight.*

## Exercise 41

We would like to compute

$$F(\theta) = \int \sin^2(\theta) d\theta$$

41.1 (Review) Use integration by parts to find  $F(\theta)$ .

Hint: *The identity  $1 = \cos^2 \theta + \sin^2 \theta$  may reduce your workload.*

41.2 Use the trig identity  $\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$  to find  $F(\theta)$ .

41.3 Find  $\int \cos^2(\theta) d\theta$  using any method you like.

## Exercise 42

Let  $f(x) = \sqrt{1-x^2}$  and consider

$$I = \int_0^1 f(x) dx$$

$g(\theta)$  so that

$$\int_0^1 f(x) dx = \int_?^? g(\theta) d\theta$$

Find the function  $g$  and the bounds for the new integral (i.e. fill in the ?'s).

42.1 If we define a change of variables  $x = \sin \theta$ , what would  $dx$  equal?

42.3 Find  $I$ .

42.2 Apply the substitution  $x = \sin \theta$  to get a new function

42.4 Graph  $f$ . What shape does the graph make? Use your knowledge of geometry to find  $I$ .

## Exercise 43

In Exercise 42 we computed

$$I = \int_0^1 \sqrt{1-x^2} \, dx = \int_0^{\pi/2} \cos^2 \theta \, d\theta$$

- 43.1 Explain why the bounds changed from  $[0, 1]$  to  $[0, \pi/2]$ .
- 43.2 Would it be okay to change the bounds from  $[0, 1]$  to  $[2\pi, 5\pi/2]$ ?
- 43.3 Would it be okay to change the bounds from  $[0, 1]$  to  $[0, 5\pi/2]$ ?
- 43.4 Compute  $I$  using the substitution  $x = \cos \theta$ . Pay close attention to make sure you use the correct bounds.

## Exercise 44

Let  $f(x) = \frac{\sqrt{9-x^2}}{x^2}$  and consider

$$F(x) = \int f(x) dx$$

Using a substitution of  $x = 3 \sin \theta$ , we arrive at

$$F(x) = -\frac{\cos \theta}{\sin \theta} - \theta + C.$$

44.1 Find an expression for  $F(x)$  that involves only  $x$ .

44.2 Are there restrictions on domain of  $x$  for which your answer makes sense?

44.3 The domain of  $\arcsin$  is  $[-1, 1]$ . Does this change your restrictions on the domain of  $x$ ?

## Exercise 45

Let  $f(x) = \sqrt{\cos x}$  and consider

$$I = \int_0^{\sqrt{2}} f(x) \, dx$$

You would like to find  $I$ .

45.1 Use WolframAlpha to find an anti-derivative of  $f$ . Does WolframAlpha give you a useful answer?

45.2 Using a 2<sup>nd</sup> degree Taylor approximation for  $\cos$ , write down an integral that will approximate  $I$ .

45.3 Find an approximation for  $I$ .

45.4 Use a 2<sup>nd</sup> degree Taylor approximation for  $f$  to approximate  $I$ .

45.5 Desmos claims  $I \approx 1.15686930348$ . Which of your estimates is more accurate?

## Exercise 46

Let

$$f(x) = \frac{1}{x^2 - 1} \quad \text{and} \quad g(x) = \frac{A}{x - 1} + \frac{B}{x + 1}.$$

46.1 You would like to know if there are constants  $A$  and  $B$  such that  $f(x) = g(x)$  for all  $x$ .

Set up a system of linear equations (or a matrix equation) which has a solution if and only if there are constants  $A$  and  $B$  that make  $f(x) = g(x)$  for all  $x$  (in the domain of  $f$ ).

46.2 Find  $A$  and  $B$ , if possible.

46.3 Compute  $\int f(x) dx$  using any method of your choice.

## Exercise 47

Let

$$f(x) = \frac{1}{(x-1)x^2} \quad \text{and} \quad g(x) = \frac{A}{x-1} + \frac{B}{x}.$$

47.1 You would like to know if there are constants  $A$  and  $B$  such that  $f(x) = g(x)$  for all  $x$ .

Set up a system of linear equations (or a matrix equation) which has a solution if and only if there are constants  $A$  and  $B$  that make  $f(x) = g(x)$  for all  $x$  (in the domain of  $f$ ).

47.2 Find  $A$  and  $B$ , if possible.

47.3 Let  $h(x) = \frac{A}{x-1} + \frac{B}{x} + \frac{C}{x^2}$ . Can you find constants  $A$ ,  $B$ , and  $C$  so that  $f(x) = h(x)$  for all  $x$  (in the domain of  $f$ )? If so, do it.

47.4 Compute  $\int f(x) dx$  using any method of your choice.



## Exercise 48

We know  $\int \frac{1}{x^2+1} dx = \arctan x + C$ . However, we can also use partial fraction decomposition over the complex numbers to integrate.

Let  $f(x) = \frac{1}{1+x^2}$  and  $g(x) = \frac{A}{1+ix} + \frac{B}{1-ix}$ .

48.1 Find  $A$  and  $B$  so that  $f(x) = g(x)$  for all  $x$  (in the domain of  $f$ ).

48.2 Compute  $\int f(x) dx$  using your result from the previous part.

*However:* use  $\ln x$  as the antiderivative of  $\frac{1}{x}$  rather than  $\ln|x|$ .

48.3 Use the fact that  $\ln(re^{i\theta}) = \ln r + i\theta$  to simplify your answer.

## Exercise 49

An *improper integral* formalizes the concept of the area under an “infinite” curve.

Suppose  $f$  is a bounded function and let  $I = \int_0^{\infty} f(x)dx$ .

49.1 Write down a formal definition of  $I$ .

49.2 Compute, using the definition,  $\int_0^{\infty} \frac{1}{(x+1)^2} dx$

49.3 Compute, using the definition,  $\int_0^{\infty} \frac{1}{(x+1)} dx$

## Exercise 50

Let  $f(x) = \frac{x}{x^2 + 1}$

In this question, we will try to compute

$$Q = \int_{-\infty}^{\infty} f(x) dx$$

50.1 Graph  $f$ . Make a guess on what you think the “total area under the curve” (i.e.  $Q$ ) should be.

50.2 Find  $\int f(x) dx$

50.3 Compute  $\lim_{N \rightarrow \infty} \int_{-N}^N f(x) dx$ . Should your result be

equal to  $Q$ ?

50.4 Should  $\lim_{N \rightarrow \infty} \int_{-N}^{2N} f(x) dx$  correspond to  $Q$ ? Compute it and compare with the previous part.

50.5 Compute  $A = \int_0^{\infty} f(x) dx$  and  $B = \int_{-\infty}^0 f(x) dx$ .

50.6 By the properties of integrals, we must have

$$Q = A + B.$$

Do the properties of integrals hold for this improper integral? What does this say about  $Q$ ?

## Exercise 51

The moral of improper integral is:

*Wherever an infinity might appear, take a separate limit.*

51.1 Rewrite  $\int_{-\infty}^{\infty} \frac{x}{x^2 + 1} dx$  using limits of definite integrals.

51.2 Let  $g(x) = \frac{1}{x^{1/3}}$  and consider  $I = \int_{-8}^{27} g(x) dx$ .

- (a) Identify all regions where  $\int g(x) dx$  could produce infinities.
- (b) Rewrite  $I$  using limit(s) of definite integrals.
- (c) Find  $I$ .

## Exercise 52

$$\text{Let } f(x) = \frac{\ln|x|}{x^4 + 1}$$

$$J = \int_{-\infty}^{\infty} f(x) dx$$

52.1 Rewrite  $J$  using limit(s) of definite integrals.

52.2 Consider the functions

$$b_1(x) = \ln x \quad b_2(x) = \frac{\ln x}{2}$$

$$b_3(x) = \frac{x}{x^4 + 1} \quad b_4(x) = 0$$

(a) Let  $A = \int_0^1 f(x) dx$ . Find upper and lower bounds for  $A$  by comparing with the appropriate  $b_i$  functions.

(b) Let  $B = \int_1^{\infty} f(x) dx$ . Find upper and lower bounds for  $B$ .

52.3 Use your results from the previous part to find upper and lower bounds for  $J$ .

## Exercise 53

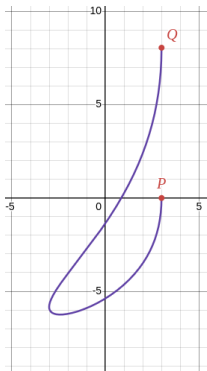
Consider the functions

$$\begin{array}{lll} A(t) = t & B(t) = t^2 & C(t) = t^{1/2} \\ D(t) = 2t & E(t) = 2t^2 & F(t) = 2t^{1/2} \end{array}$$

- 53.1 Consider the parametric equations  $x(t) = A(t)$  and  $y(t) = D(t)$ .  
Graph, by hand,  $(x, y)$  for  $t \in [0, 4]$ .
- 53.2 Consider the parametric equations  $x(t) = B(t)$  and  $y(t) = A(t)$ .  
Graph, by hand,  $(x, y)$  for  $t \in [0, 4]$ .
- 53.3 Identify all possible assignments of  $x(t) = ??$  and  $y(t) = ??$  (where ?? come from the functions above) so that the graph of  $(x, y)$  for  $t \in [0, 4]$  is a line segment.
- 53.4 Out of your examples above, which example produces the *longest* line segment?

## Exercise 54

Shown is the graph of  $\begin{cases} x(t) = 3 \cos t \\ y(t) = t^2 - 5t \end{cases}$  for  $t \in [0, 2\pi]$ .

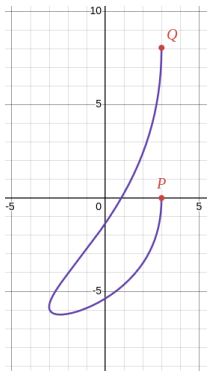


The parametric equations describe the position of a particle at time  $t$ .

- 54.1 Is the particle moving from  $P$  to  $Q$  or  $Q$  to  $P$ ? Explain.
- 54.2 At what time(s) is the particle moving up *and* to the right?
- 54.3 At what time(s) is the particle moving parallel to the  $x$ -axis?
- 54.4 Find the tangent line to the particles path at time  $t = \pi/2$ .

## Exercise 55

Shown is the graph of  $\begin{cases} x(t) = 3 \cos t \\ y(t) = t^2 - 5t \end{cases}$  for  $t \in [0, 2\pi]$ .



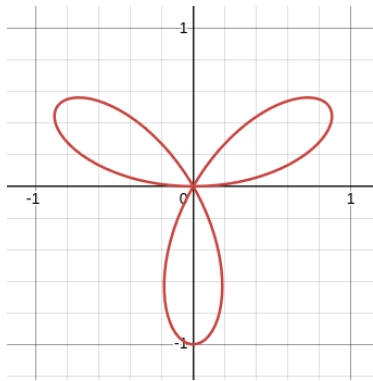
The parametric equations describe the position of a particle at time  $t$ .

- 55.1 Find a parameterization for a particle that traces the same path, but starts at  $Q$  and ends at  $P$ .
- 55.2 Find a parameterization so that the particle finishes its journey in  $\pi$  seconds instead of  $2\pi$  seconds.
- 55.3 Is there a parameterization so that the particle finishes its journey in  $\pi$  seconds but starts its journey at the same speed as the original particle? If such a parameterization exists, how would you come up with it?



## Exercise 56

Shown is the graph of  $r(\theta) = \sin(3\theta)$  in polar coordinates for  $\theta \in [0, \pi]$ .



The curve models the boundary of a propeller.

- 56.1 The blade in the first quadrant achieves a maximum length at an angle of  $\theta = \pi/6$ . Find the rectangular coordinates of the tip of the blade in the first quadrant.
- 56.2 Find parametric equations  $(x(t), y(t))$  that trace out the propeller.
- 56.3 Find the tangent line to the propeller when  $\theta = 0$  and when  $\theta = \pi/3$ .
- 56.4 We know that the propeller is contained in a circle with area  $\pi$ .

Come up with a better upper bound for the area of the propeller.

## Exercise 57

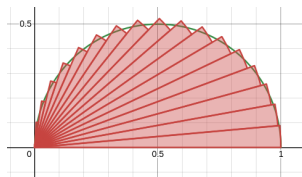
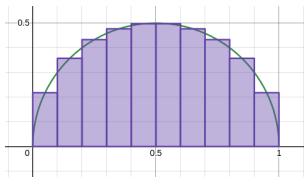
The same semi-circle can be described in polar coordinates by

$$r(\theta) = \cos \theta \quad \text{with} \quad \theta \in [0, \pi/2]$$

or in rectangular coordinates by

$$y(x) = \sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2} \quad \text{with} \quad x \in [0, 1].$$

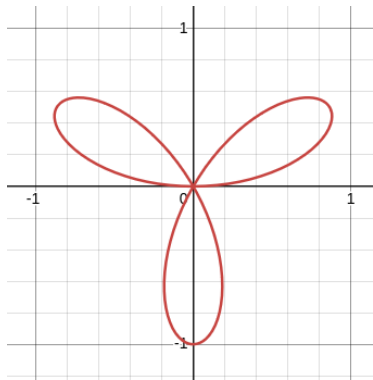
Below are two ways to divide up the semicircle to approximate its area: one with rectangles and one with circular *sectors*.



- 57.1 Write down a Riemann sum that approximates the area using *rectangles*. Use  $\Delta x$  as the width of a rectangle.
- 57.2 Write down a Riemann sum that approximates the area using *sectors*. Use  $\Delta \theta$  as the sector angle.
- 57.3 Take limits of your previous two Riemann sums to find integrals that represent the *exact* area of the semicircle.  
*Do not evaluate your integrals.*
- 57.4 Which integral would you rather do?

## Exercise 58

Shown is the graph of  $r(\theta) = \sin(3\theta)$  in polar coordinates for  $\theta \in [0, \pi]$ .

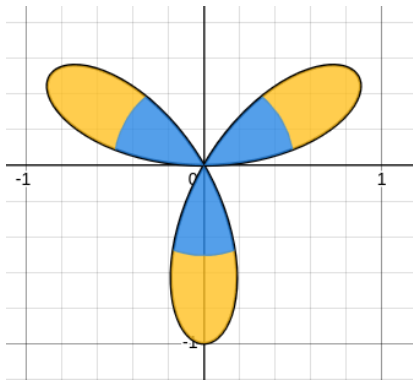


The curve models the boundary of a propeller.

- 58.1 The first propeller blade is traced out for  $\theta \in [0, \pi/3]$ . Set up a Riemann sum that approximates the area of the first propeller blade.
- 58.2 Set up an integral that will give the *exact* area of the first propeller blade. Then, find the area of the first propeller blade.
- 58.3 Find the total area of the propeller.

## Exercise 59

Shown is the graph of  $r(\theta) = \sin(3\theta)$  in polar coordinates for  $\theta \in [0, \pi]$ .



The curve models the boundary of a propeller. The parts of the propeller within distance  $1/2$  of the origin are painted blue. The rest is painted yellow.

- 59.1 Consider the propeller blade in the first quadrant. At what angle does the yellow paint start to appear? At what angle does it disappear?
- 59.2 Set up an integral that will give the amount of yellow paint needed for the first blade.
- 59.3 Set up an expression with integral(s) that will give the amount of blue paint needed for the first blade.
- 59.4 Find the amounts of each paint needed to paint the whole propeller.