Calculus II

MAT187 Student Slides

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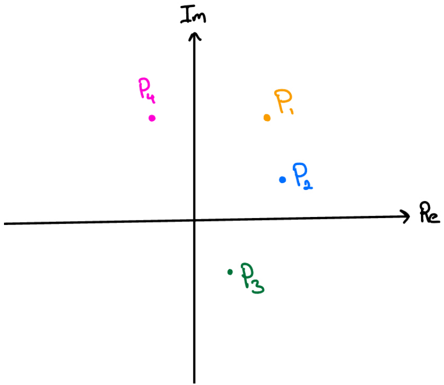
Exercise 1

Consider the plot of the complex numbers p_1, p_2, p_3, p_4 in the complex plane.



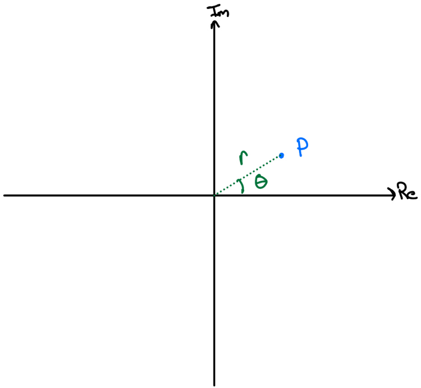
1.1 For which complex numbers is the real part grater than the imaginary part?

1.2 Which complex number has the smallest *modulus/absolute value*?

1.3 Which complex number has the largest *argument*? Is your answer at all ambiguous?

Exercise 2

Consider the plot of the complex number p in the complex plane.



2.1 Sketch the complex number $2p$.

2.2 Sketch the complex number p^2.

2.3 Sketch the complex numbers p^n for $n = 3, 4, \ldots$. Will your answer depend on r?

2.4 Use the geometry of the complex plane to find $\sqrt{i}$. Express your answer in both polar and rectangular form.

Exercise 3

Consider the equation

$$z^3 = -1 \tag{1}$$

3.1 Find a solution to Equation (1).

3.2 If $z = re^{i\theta}$ is a solution to Equation (1), what conditions must r and θ satisfy? Justify your conclusions.

3.3 Find all solutions to Equation (1).

Exercise 4

For each situation, decide whether *least squares* curve fitting or *polynomial interpolation* would be more appropriate.

4.1 You are modelling the arch used in the construction of a particular Roman aqueduct. You have collected several hundred data points of height of the arch vs. distance from the base of the aqueduct.

4.2 You are creating a function to govern the brightness of a light which will be used for signalling a computer. There are three different brightnesses that must be achieved exactly and the transition between those brightnesses must be smooth.

4.3 You are given exact data points from a lab and told that the data was created with a 4th degree polynomial. You are asked to find the coefficients of the polynomial.

Exercise 5

A baseball is thrown on the moon. You are trying to find the function

- $h(t)$, the height (in meters)of the baseball above the moon's surface at time t (in seconds).

You collected the following data

t	$h(t)$
1	4
2	3.8
3	2

5.1 What degree polynomial would best model h?

5.2 Use polynomial interpolation to find h.

5.3 Find the maximum height of the baseball above the moon's surface.

5.4 What would change (if anything) if you were given 4 data points?

While developing a robotics control system, you find the need for a function f which satisfies the following properties:

(i) $f(0) = -1$ and $f(1) = 2$

(ii) $f'(0) = -1$ and $f'(1) = 2$

Your friend suggests that you could use the following polynomial to come up with f:

$$L_1(x) = -(x-1) \qquad L_2(x) = x$$

$$S_1(x) = (x-1)^2 x \qquad S_2(x) = (x-1)x^2$$

6.1 Can Lagrange interpolation be used to directly find f? Explain.

6.2 Complete the following table

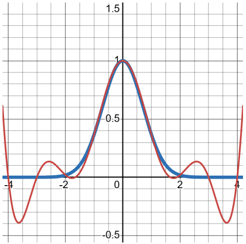
g	$g(0)$	$g(1)$	$g'(0)$	$g'(1)$
L_1				
L_2				
S_1				
S_2				

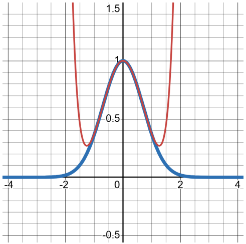
6.3 Use L_1, L_2, S_1, and S_2 to find a polynomial satisfying the properties of f.

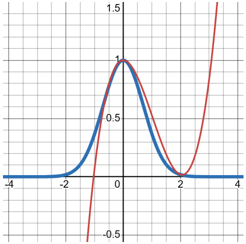
6.4 Explain how Lagrange interpolation can be generalized to allow finding a polynomial that passes through particular points and takes on particular derivatives at those points.

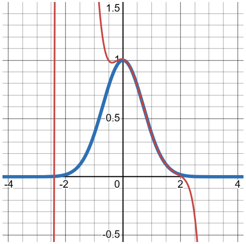
Exercise 7

7.1 For each polynomial approximation of the bell curve, is the approximation best at 0, best on the interval $[-2,2]$, or best on the interval $[0,2]$.

(A) 

(B) 

(C) 

(D) 

7.2 Based on the pictures, which polynomial(s) do you think come from a Taylor approximation?

Exercise 8

The function f satisfies

$$f(0) = 1 \qquad f'(0) = 0 \qquad f''(0) = -2$$

$$f'''(0) = 0 \qquad f''''(0) = 12$$

8.1 Write down T_4, the 4th degree Taylor approximation to f centered at 0.

8.2 Use Desmos to compare the graph of T_4 with the graphs of g_1, g_2, g_3, and g_4. Which of the g's do you think is most likely equal to f?

(a) $g_1(x) = e^{-|x|}$

(b) $g_2(x) = e^{-x^2}$

(c) $g_3(x) = \dfrac{1}{1+x^2}$

(d) $g_4(x) = \dfrac{1}{1+(2x)^4}$

Exercise 9

A bee is flying back and forth along a window sill trying to escape from your living room.

The bee's position at time t along the window sill is given by $r(t)$.

You know that a first-order Taylor approximation to $r(t)$ at time $t = 2$ is

$$A_1(t) = 3(t - 2) + 1$$

9.1 Estimate the position of the bee on the window sill at time 2.1. Is your answer exact or approximate?

9.2 Estimate the velocity of the bee at time 2.1. Is your answer exact or approximate?

9.3 Are there any times you can compute the *exact* position of the bee?

9.4 Are there any times you can compute the *exact* velocity?

9.5 What is your best estimate for the acceleration of the bee at time 2.1?

Exercise 10

A bee is flying back and forth along a window sill trying to escape from your living room.

The bee's position at time t along the window sill is given by $r(t)$.

You know that a second-order Taylor approximation to $r(t)$ at time $t = 2$ is

$$A_2(t) = 2(t-2)^2 + 3(t-2) + 1$$

10.1 Estimate the position of the bee on the window sill at time 2.1. Is your answer exact or approximate?

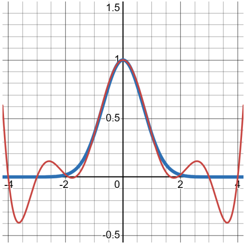
10.2 Estimate the velocity of the bee at time 2.1. Is your answer exact or approximate?

10.3 Are there any times you can compute the *exact* position of the bee?

10.4 Are there any times you can compute the *exact* velocity?

10.5 What is your best estimate for the acceleration of the bee at time 2.1?

Based on the pictures, which polynomial approximations of the bell curve do you think are *Taylor* polynomials?

(A)


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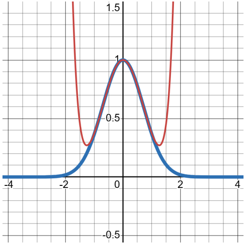

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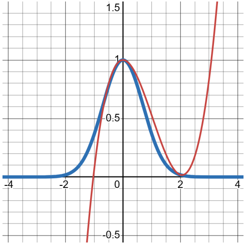

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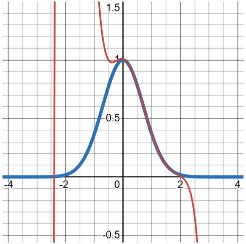

Exercise 12

Let $f(x) = e^x$ and let $P_n(x)$ be the nth degree Taylor approximation to f centered at 0. In particular

$$P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

Let $R_n(x)$ be the (signed) error in $P_n(x)$.

12.1 Find $R_3(1.5)$ (you may use a calculator).

12.2 What is the largest value of $R_3(x)$ when $0 \leq x \leq 2$?

12.3 Is there a value of x for which $R_3(x) = 0$? What does this say about P_3?

12.4 Given that $|f^{(5)}(x)| \leq 8$ when $x \in [0, 2]$, find an upper bound for $R_4(x)$ that

(a) works for a fixed $x \in [0, 2]$

(b) works simultaneously for all $x \in [0, 2]$

12.5 Given what you know from the previous part(s), can you bound $R_n(x)$?

Exercise 13

Let f be an infinitely differentiable function, and let P_n be a a Taylor polynomial for f of degree n centered at a.

We approximate $f(x) \approx P_n(x)$. Which of the following affect the size of the error in $P_n(x)$ (i.e., the magnitude of $R_n(x)$)?

(A) The degree of P_n, i.e., n.

(B) The magnitude of $f(a)$, i.e., $|f(a)|$.

(C) The magnitudes of the derivatives of f at a, i.e., the size of $|f'(a)|$, $|f''(a)|$, etc..

(D) The distance from a that you are approximating at, i.e., the size of $|x-a|$.

Exercise 14

Use Desmos to conjecture about the following questions.

`https://www.desmos.com/calculator/nrru5n0gqq`

14.1 True/False? When approximating $\sin(x)$ using Taylor polynomials centered at $x = 0$, higher degree polynomials will approximate $\sin(2)$ better.

14.2 True/False? When approximating $\tan(x)$ using Taylor polynomials centered at $x = 0$, higher degree polynomials will approximate $\tan(2)$ better.

14.3 True/False? When approximating $f(x) = \frac{1}{1+x^2}$ using Taylor polynomials centered at $x = 0$, higher degree polynomials will approximate $f(2)$ better.

14.4 Make a conjecture about the relationship between the degree of your Taylor approximation and the accuracy of its values. Does this contradict what you know from Taylor's remainder theorem?

Exercise 15

Consider the function $f(x) = 4x^2 + 1$ and the value $I = \int_{-1}^{3} f(x)\,\mathrm{d}x$.

15.1 Make three sketches: one where the left-endpoint rule is used to approximate I, one where the right-endpoint rule is used, and one where the trapezoid rule is used.

15.2 For the left-endpoint, right-endpoint, and trapezoid rules, which will give over estimates of I and which will give underestimates? Will any give an exact value?

15.3 Consider the following estimates of I:

- $E_1 = 53.797$
- $E_2 = 32.505$
- $E_3 = 43.151$

Which estimates come from a left-endpoint approximation, a right-endpoint approximation, and a trapezoid approximation?

Hint: you know calculus!

Exercise 16

In a classic problem, you are trying to find the volume of a wine barrel. Let $r(\ell)$ represent the radius of the barrel ℓ cm from the base. The total length of the barrel is 80cm.

You know the volume of the barrel can be computed exactly by

$$\int_0^{80} \pi[r(\ell)]^2\, d\ell.$$

You have measured the barrel in several places and gotten the following data

h(0)	h(20)	h(40)	h(60)	h(80)
12.8	21.2	22.7	21.4	13.4

16.1 Make a sketch of the barrel's profile.

16.2 Based on your sketch, do you think using a trapezoid approximation will produce an over or under estimate for the volume?

16.3 Use a trapezoid approximation to estimate the volume of the barrel.

16.4 Use a Simpson's approximation to estimate the volume of the barrel.

Reminder: if p is a quadratic polynomial,

$$\int_a^b p(x)\, dx = \frac{b-a}{6}\left(p(a) + 4p\left(\frac{a+b}{2}\right) + p(b)\right)$$

16.5 The exact (rounded) volume of the barrel is 104384cm^3. What approximation method was most accurate? Why?