Mathematics 2 - Part 2 - Interior Point Method

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Introduction

This is the report of the first homework of second part of the Mathematics 2 course, where we show implement and test the interior point method and

All the test executions that can be used to reproduce results that are shown in continuation can be found in notebook *Homework - Interior Point Method.ipynb*.

A documented Interior point method implementation in Python can be found in *InteriorPointMethodMinimizer.py* file.

Program that runs in the interior

Interior point method was implemented in Python. We first point out some of the differences with the method proposed in the given article.

- Because the big M method can return numbers that are bigger than the floating-point numbers that Python supports, we reduce the numbers in the input arrays: A, b and c. This is used only for calculations of auxiliary big variables like M, W and so on.

- The proposed δ in the article did not return any feasible solution. Thus, we replaced the proposed δ with $\delta' \in [0.80, 0.95]$ and in each iterative improvement μ_i is multiplied directly with δ'.

- Because some of the matrices that are used to calculate steps k, f and h are sometimes not invertible, we use the pseudo-inverse of a matrix instead of the actual inverse. This is likely a consequence of a minor bug, probably due to some problems with float-point precision.

- While we check whether or not the matrix has full rank, we do not transform the matrix to full rank because the bread problem includes both lower and upper bounds thus forming a matrix with duplicated rows, where one of the duplicated pair rows is just negated.

A man does not live by bread alone

In Table 1 we see the optimal solution to the given problem calculated by the commercial LP solver and our own implementation of the interior point method. We note that the solutions are interestingly close. The solution of the problem proposes that we add vegetable oil to our bread-only diet.

Food	**Commercial solver**	**Our implementation**
Potatoes	0.0	0.0
Bread	6.23	6.81
Milk	0.0	0.0
Eggs	0.0	0.0
Yoghurt	0.0	0.0
Vegetable oil	0.59	0.52
Beef	0.0	0.0
Strawberries	0.0	0.0

Table 1. A man does not live by bread alone optimal food distribution.

	Commercial solver	**Our implementation**
Cost (cents/100g)	148.83	160.23

Nutritional values	**Commercial solver**	**Our implementation**
Carbohydrates (CH)	299.04	326.78
Protein (PR)	68.53	74.89
Fat (FT)	90.0	86.32
Energy (EN)	2200.0	2298.15

Table 2. A man does not live by bread alone results.

Table 2 shows results of the optimal solutions calculated by the commercial LP solver and our own implementation of the interior point method. Results show that we came very close to the optimal solution. The cost differs for approximately 12 cents/100g and the provided solution is between the given bounds of nutritional values, that is, with such diet we can survive both according to the commercial LP solver and our very own implementation of the interior point method.

Analytic center

Show that there exists $x \in \Phi$ so that for all $i \in I$ we have $s(x)_i > 0$.

We know that Φ is the set of all feasible solutions. Then x is a feasible solution, which means that for all i, $s(x)_i =$

$b_i - (Ax)_i$. Because $(Ax)_i < b_i$ by definition, we know that $s(x)_i > 0$ for all i.

Show that the analytic center optimization problem is equivalent to a strictly convex optimization problem.

The analytic center solves the problem $max_x \prod_{i \in I} s(x)_i$, which is equivalent to minimizing the negative problem. If we go one step further and take the logarithm of the function we are trying to minimize we get $f(x) = -\sum_{i \in I} log(s(x)_i)$ and we know that $s(x) = b - Ax$. To show that f is convex, we need to show that its Hessian is positive semi-definite.

$$\begin{aligned} \frac{df(x)}{dx} &= \sum_{i \in I} \frac{(A)_i}{b_i - (Ax)_i} \\ \frac{d^2 f(x)}{d^2 x} &= \sum_{i \in I} \frac{(A)_i^T (A)_i}{(b_i - (Ax)_i)^2} \end{aligned} \tag{1}$$

Because $(b_i - (Ax)_i > 0)$ and $(A)_i^T A_i > 0$ due to definition of dot product, the second derivative is strictly positive, hence the Hessian is PD and the function strongly convex.

Show that the analytic center is unique.

Because of the part we just proved, that is, the analytic center optimization problem being equivalent to a strictly convex optimization problem, we know that the analytic center is unique. A strictly convex function has exactly one minimum, which is in our case the analytic center.

Let a be a positive real. Find the analytic centre of system in Equation 2.

$$\begin{aligned} -x_1 &\leq 0 \\ -x_2 &\leq 0 \\ ax_1 + x_2 &\leq 0 \end{aligned} \tag{2}$$

The system in Equation 2 can be rewritten to:

$$\begin{aligned} f(x_1, x_2) &= x_1 \cdot x_2 \cdot (1 - (ax_1 + x_2)) \\ \frac{df(x_1, x_2)}{dx_1} &= 0, 1 - 2ax_1 - x_2 = 0 \\ \frac{df(x_1, x_2)}{dx_2} &= 0, 1 - ax_1 - 2x_2 = 0 \end{aligned} \tag{3}$$

From Equations 3 it follows that $x_1^* = \frac{1}{3a}$ and that $x_2^* = \frac{1}{3}$.

Find the analytic center of the system containing the following four linear inequalities in Equation 4.

$$\begin{aligned} -x_1 &\leq 0 \\ -x_2 &\leq 0 \\ x_1 + x_2 &\leq 1 \\ x_1 + x_2 &\leq 1 \end{aligned} \tag{4}$$

The system in Equations 4 can be rewritten to:

$$\begin{aligned} f(x_1, x_2) &= x_1 \cdot x_2 \cdot (1 - (ax_1 + x_2))^2 \\ \frac{df(x_1, x_2)}{dx_1} &= 0, x_2 = 1 - 3x_1 \\ \frac{df(x_1, x_2)}{dx_2} &= 0, x_1 = 1 - 3x_2 \end{aligned} \tag{5}$$

From Equation 5 it follows that $x_1^* = \frac{1}{4}$ and that $x_2^* = \frac{1}{4}$.