Gravitational Aharonov-Bohm effect

Lucas Barbier

Department of Physics, Linnéuniversitetet, Sweden

May 2022

Abstract

The Aharonov-Bohm effect was predicted by Aharanov and Bohm [1] in 1959, and shortly after confirmed by experimentation [3, 11, 12]. In this paper, we study the gravitational analogue of this magnetic effect, and determine to what extent the analogy can be pursued. We show that given particular metrics, a phase shift can be identified.

1 Adiabatic theorem and Berry phases

We will start by introducing the adiabatic approximation, a key factor in the establishment of Berry phases. This theorem stipulates that a physical system is maintained in its instantaneous eigenstate if a given perturbation acts on it sufficiently slow, and if there is a significant difference between the associated eigenvalue and the Hamiltonian spectrum. This means that if external conditions are gradually modified, a quantum system can adapt its functional form. However, if the modifications are too rapid, the adaptation cannot develop.

The Berry phase arises in the case of a general Hamiltonian, that we usually write $h(R)$. R is a family of parameters characterizing the environment. The phase is usually given as[2]:

$$\gamma(t) = \int_{R(0)}^{R(t)} \vec{A}^n(\vec{R}) \cdot d\vec{R}$$

with $\vec{A}^n(\vec{R}) = i \langle n; R|\vec{\nabla}|n; R\rangle$.

2 The Aharonov-Bohm effect

First, we consider the second experiment that Aharonov and Bohm proposed in their original article [1] (see Fig. 1). A ray of electrons is divided into two beams, each of them passing on a different side of a solenoid. The latter one contains an internal magnetic field $\vec{B}$, but a vanishing external field. The beams are

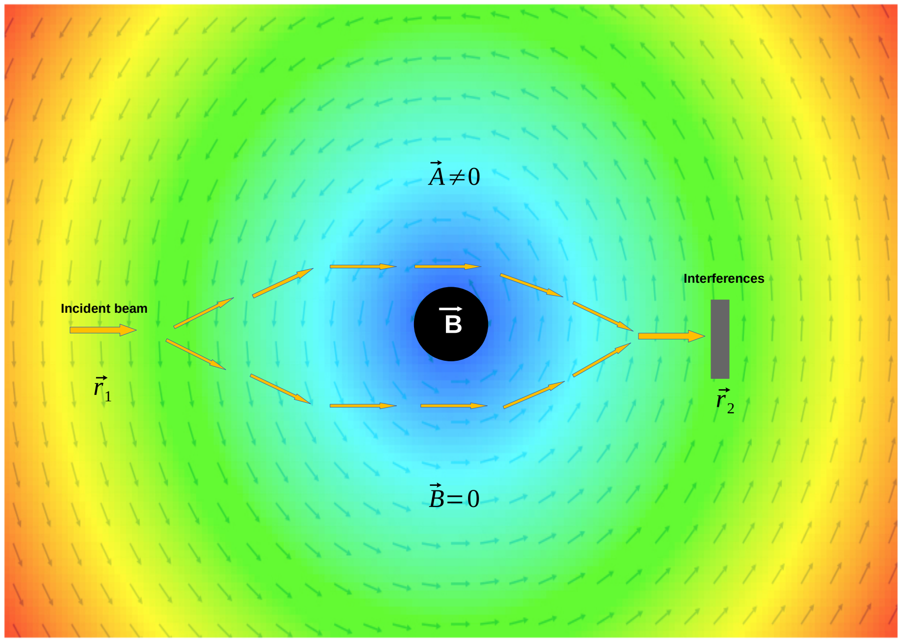

Figure 1: The Aharonov-Bohm Experiment. An incident electron beam is split in two, circles around a solenoid with interior magnetic field $\vec{B}$, and is recombined. We can observe the interference fringes due to the presence of the vector potentiel $\vec{A}$ on the screen in r_2. The vector potential $\vec{A}$ is illustrated.

thereafter recombined, and we can observe the interference fringes on a screen. It is also assumed that the passage of electrons is sufficiently slow that the fields are perceived by the particles as static. This implies that there are no energy transitions due to the varying fields[7].

Given the magnetic field, the magnetic vector potential $\vec{A}$ takes the form $\vec{A} = \frac{\phi}{2\pi r}\hat{\theta}$[1] outside the solenoid. The wave will interact with $\vec{A}$ but not with $\vec{B}$. The electron enters and leaves a zone where $\vec{A}$ is non-zero, it is evident that $\vec{A}$ is perceived by the particle as time-dependent. The latter experiences an electric field $\frac{\partial A}{\partial ct}$, and therefore a magnetic field, that is however neglectable.

We now consider the closest point of the trajectory from the solenoid. For the lower component of the wave, $\vec{A}$ points in the same direction as the wave's momentum. The configuration being antisymmetric, the opposite is observed for the upper component. One can deduce here that depending on the direction of $\vec{A}$, a phase shift is introduced to each one of the components. In one case $\vec{A}$ slows the wave's phase, and in the other it advances it. The presence of $\vec{A}$ has created a phase change β, that we can observe on the screen placed in r_2. One calculates that

[1]We choose $\hat{\theta}$ to be counterclockwise.

$$\beta = \frac{q}{\hbar c}\left(\int_{r_1}^{r_2} d\vec{r}\cdot\vec{A}_{lower} - \int_{r_1}^{r_2} d\vec{r}\cdot\vec{A}_{upper}\right) = \frac{q}{\hbar c}\oint d\vec{r}\cdot\vec{A}$$

The region surrounding being simply connected, it is possible to reduce and change the shape of the contour. Finally, it yields that

$$\beta = \frac{q}{\hbar c}\int_0^{2\pi} A_\theta \cdot r d\theta = \frac{q\Phi}{\hbar c} \tag{1}$$

The Aharonov-Bohm effect has been introduced[2] by plural geometric considerations. It has also been shown that the induced phase is of a geometrical nature; a phase derived by a line integral in parameter space. The Aharonov-Bohm phase belongs to the family of Berry phases.

3 Gravitational Analogue

We would like to find a gravitational analogue to the magnetic effect outlined above. We wish to have a restricted zone containing a gravitational potential that would affect the passage of a beam of electrons. Gravitationally, we conceptualize this situation by a curved space-time in a limited area, but flat around. Marder[8] showed that such a space-time was possible, creating a non-trivial gravitational field. First, we define this space-time and its source, then the phase shift created by the source on a spinor's wavefunction is calculated.

3.1 Metric, space-time and gravitational field

The analogy for the metric for the exterior gravitational field of a static infinite cylinder may be expressed in this form:

$$ds^2 = c^2 r^{2C} dt^2 - r^{2(1-C)} d\phi^2 - A^2 r^{-2C(1-C)}(dr^2 + dz^2)$$

A and C depend on the distribution of the source. For physical reasons, C has to be small and positive, or zero. For simplicity we choose $C = 0$. Also, we now use relativistic units, setting $c = 1$. These changes yield the following metric:

$$ds^2 = dt^2 - r^2 d\phi^2 - A^2(dr^2 + dz^2) \tag{2}$$

If we set $r' = Ar$, $z' = Az$, $\phi' = \phi/A$, the metric then becomes:

$$ds^2 = dt^2 - r'^2 d\phi'^2 - dr'^2 - dz'^2 \tag{3}$$

It is already clear that if C vanishes and $A \neq 1$, the metric describes a locally flat space-time but that the gravitational field is non-trivial. Indeed, even though we find a Minkowskian metric $diag(-1, 1, r^2, 1)$, ϕ' ranges from 0 to $2\pi/A$ only,

[2] A more detailed derivation of the phenomenon using Feynman's path integral approach can be found in the supplementary content.

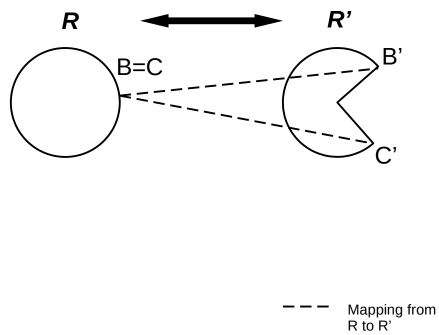

Figure 2: Mapping between $\mathscr{R}$ and $\mathscr{R}'$. If $A \neq 1$, the mapping creates a "forbidden" zone, modifying the connectedness of the space.

leaving an inaccessible area. If we consider the mapping between the two coordinate systems $\mathscr{R}$ and $\mathscr{R}'$ to be homeomorphic, we have to identify the points of a conical 2-surface. In order to precise that phenomenon, let us imagine two points $B = (r, \phi = \alpha, z)$ and $C = (r, \phi = \alpha + 2\pi, z)$ in $\mathscr{R}$, as shown on Fig. 2. It is evident that the two points are the same.

Let us now transfer those points onto $\mathscr{R}'$. We have $B' = (r', \alpha/A, z')$ and $C = (r', \alpha + 2\pi/A, z')$. In $\mathscr{R}'$ the two points are no longer the same. In the mapping from one system to another, going from a single point to two, a conical 2-surface is left inaccessible. It appears that we have a flat space-time with a peculiar connectedness. This topological particularity recalls the situation described in section 1.

By a thought experiment, we now wish to demonstrate that the metric (3) describes a non-trivial gravitational field.

Let us imagine a ray of photons emitted from a light source S. We now work in the system $\mathscr{R}'$. The photons travel in straight lines towards the source G of the gravitational field. Once the photons pass by G, they continue to propagate in straight lines until the points B' and C'. From the perspective of an observer in $\mathscr{R}$, the beams emitted from a common source take different paths, circle around G, and meet again in $B = C$. We recall that these two points are the same in $\mathscr{R}$. In the original system $\mathscr{R}$, the rays have been bent around the source G.

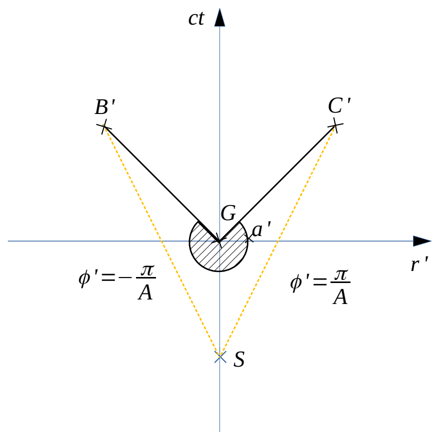

Figure 3: Photons traveling close to the gravitational source along the lines $\phi = \pm - \frac{\pi}{A}$ in $\mathscr{R}'$.

These paths are independent of the particle's velocity, and from this we can imagine an analogous experiment with electrons. In a relativistic frame, this highlights the existence of a gravitational field, whose properties depend on A.

To perfectly describe the gravitational field, we now want to describe the aforementioned gravitational source. We first re-introduce the angular coordinate $\phi = A\phi'$, and rewrite $r' = r_1 + K$, K being a positive constant. This transforms the exterior metric to

$$ds^2 = dt^2 - dr^2 - A^{-2}(r_1 + K)d\phi^2 - dz'^2$$

For the interior metric, Marder[8] finds:

$$ds^2 = dt^2 - dr_1^2 - f^2(r_1)d\phi^2 - dz'^2 \tag{4}$$

with $f = Z^{-1}\sin(Zr)$, where $Z = \frac{\sqrt{A^2-1}}{K+a_1}$ and $r_1 = a_1$ representing the boundary.

We calculate that the Riemann curvature tensor has only one independent component:

$$R_{1212} = -f\frac{d^2f}{dr_1^2} \tag{5}$$

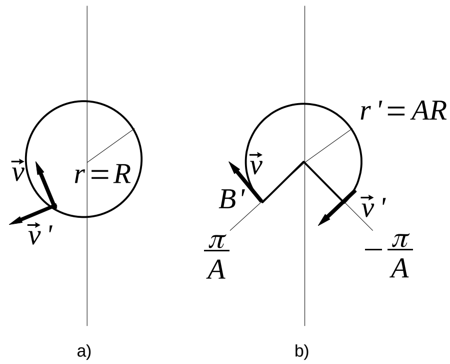

Figure 4: Contour around the z-axis in each coordinate systems. The vector $\vec{v}$ is parallely propagated around the circuit in the clockwise direction. For $A \neq 1$, the difference between the paths in a) ($\mathscr{R}$) and b) ($\mathscr{R}'$) appears clearly when comparing the final vectors $\vec{v'}$.

and that the only two non vanishing components of the energy-momentum tensor are:

$$T_3^3 = T_0^0 = \frac{Z^2}{8\pi}. \tag{6}$$

The stress and energy densities are equal but positive.

3.2 Gravitational phase shift

We now show that by parallely propagating a spinor around a path encircling the z-axis, we can expose a phase change. As one can observe on Fig. 4, the paths are dependent on the coordinate system. Indeed, the angle difference in $\mathscr{R}'$ does not allow a full circulation, leaving the propagation incomplete. Inevitably, we observe a difference between the two vectors at the end. To quantify it, we consider the path BC (circle $z = const$, $r = R$, $t = const$) from Fig. 3. By propagating a spinor along this path, Dowker [5] shows that this spinor acquires a phase:

$$\beta = \exp\left\{ i \frac{2\pi}{A'} \vec{z} \cdot \vec{J} \right\} \tag{7}$$

with $A' = \frac{(A-1)}{A}$.

By means of geometry we showed that a shift does indeed arise from the presence of a gravitational field meeting the conditions of the Ahoronov-Bohm model. We would like to generalize and compute the phase shift using the metric describing our field. This will also help us to carry out the analogy in a more obvious way.

Let us consider an experiment similar to the one first introduced. The inaccessible zone here is the consequence of the gravitational source. It appears clearly that the topology is the same in both configurations. Similarly, we can reshape the contour integral around the source of the field, and compute an integral of the form:

$$\frac{\alpha}{\hbar} \oint \vec{V} \cdot d\vec{r} \tag{8}$$

for any vector $\vec{V}$ and constant α depending on the properties of the exterior field considered. To compute the phase shift in this case, we first rewrite the exterior metric in the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu} \tag{9}$$

where $\eta_{\mu\nu}$ represents a flat metric and $|\gamma_{\mu\nu}| \ll 1$ is the metric deviation. For a spinless particle of mass m in the field, the Hamiltonian is [4]

$$H = \frac{1}{2m}(\vec{p} - m\vec{\gamma})^2 + \frac{m}{2}\gamma_{00} \quad , \vec{\gamma} \equiv (\gamma_{10}, \gamma_{20}, \gamma_{30}) \tag{10}$$

In this case, one might identify the vector $\vec{V}$ and constant α to the metric deviation vector $\vec{\gamma}$ and the mass m, which transforms (8) to

$$\beta = \frac{m}{\hbar} \oint_{\mathscr{C}} \gamma_\mu \cdot dx^\mu \tag{11}$$

We observe that for any $\gamma_\mu \neq 0$ a phase-shift similar to that of the magnetic Aharonov-Bohm effect arises. If we are now to consider the general case of a particle of mass m and spin j, the general expression for a phase shift due to a weak gravitational field is given by [6, 10]

$$\beta = \frac{1}{4} \int_\tau R_{\mu\nu\alpha\beta} J^{\alpha\beta} d\tau^{\mu\nu} + \frac{m}{\hbar} \oint_{\mathscr{C}} \gamma_\mu dx^\mu \tag{12}$$

where τ is the surface bounded by the closed path $\mathscr{C}$. The J are the generators of the Lorentz group and are related to the Ji by $J^{ij} = \epsilon^{ijk} J_k$. In the present case since the space-time is flat all the γ_μ are zero and the last integral vanish, while the first yields (7), if for $\mathscr{C}$ we choose a circuit around $z = 0$, and for τ the curvature given by (5).

These results demonstrate a clear gravitational analogue to the Aharanov-Bohm effect. The particles move in a flat space-time, but still experience the effects of a gravitational field. Two effects stand out. The first is, perhaps more anecdotic and linked to the nature of the field, the distortion of the particles trajectories. The second is the analogue we were looking for. However, it is important to note that given the properties of the field neither of these phases are of quantum nature.

3.3 Direct Analogy

The phase outlined is of "classical" nature. Particularly, the constant $\hbar$ does not appear in the final expression (7). Dowker highlights[6] that analogies between gravity and electromagnetism are double; we can assimilate the couple charge-potential to the mass-metric one, or the couple charge-field and spin-curvature. Comparing equations (1) and (12), these analogies naturally appear.

We have so far treated the second case, but we now consider the first analogy and show that a gravitational Aharonov-Bohm effect can also emerge from it. We come back on the comparison between equations (1) and (12) and, particularly the second integral of the latter. Doing so, we can identify the correspondences between the mention coupled : $\begin{cases} q & \leftrightarrow m \\ \vec{A} & \leftrightarrow \vec{\gamma} \end{cases}$

Using that correspondence, we can translate any electromagnetic situation into its gravitational equivalent.

We now consider the magnetic vector potential $\vec{A}$ from Aharonov-Bohm's model, using this time cartesian coordinates. The metric corresponding to this potential is of the form :

$$ds^2 = dt^2 - (d\vec{r})^2 - \frac{\Phi}{2\pi\rho^2}(ydx - xdy)dt \quad , \rho = \sqrt{x^2 + y^2} \tag{13}$$

Christoffel's symbols all are zero except for $\Gamma^0_{ij} = \partial_i h_j$. We calculate that the curvature vanishes too. Nevertheless,

$$\vec{\gamma} = \left(-\frac{\Phi y}{2\pi\rho^2}, \frac{\Phi x}{2\pi\rho^2}, 0\right)$$

and from (12) one calculates that a phase shift

$$\beta = \frac{m\Phi}{\hbar} \tag{14}$$

appears. This change in phase is purely of quantum nature, and is the exact analogue[3] of (1).

Here, we show that a phase change of quantum nature due to a gravitational field appears. It is nonetheless necessary to verify the properties of the source,

[3] We should remember here that we are in relativistic units and that $c = 1$.

and its physical reality. We therefore start by prolonging the metric to the inside of the cylinder. Dowker shows that this interior metric is given by:

$$ds^2 = dt^2 - (d\vec{r})^2 - \Phi \cdot \frac{\sin \pi\rho/2a}{2\pi\rho}(ydx - xdy)dt$$

Given the energy-momentum tensor[4], this source has a vanishing energy density, and is hardly realizable physically. Marder discusses other sources, each of them at least as complicated to realize, if not physically impossible.

4 Phase comparison

Here, we wish to compare the two phase shifts. In the magnetic case the parameter space in which the Berry phase arises is the three-dimensional physical space. This phase shift arises because of the vector potential $\vec{A}$, in a zone where the magnetic field $\vec{B}$ is zero. The effect is non local, a feature shared with numerous quantum effects.

The gravitational phase is of geometrical nature as well. The parameter space is the four-dimensional space-time. In the present case the time-evolution is of classical nature, and one could argue that the parameter space can here be reduced to the three-dimensional physical space. A phase shift arises in a zone where the curvature vanishes. Likewise, the effect appears to be non local. This feature is surprising in general relativity, and is discussed in the conclusion.

5 Conclusion

Through Marder (1959) and Dowker (1967) it has been shown that a gravitational field can indeed entail a phase shift similar to that of the Aharonov-Bohm effect. Even though hardly realizable, the physical nature of the effect is present, and could one day be measured. Incidentally, roughly 55 years after Dowker's article, a gravitational Aharonov-Bohm effect has been observed and measured[9]. In the latter experiment the field considered is slightly different. Space is supposed to be flat, and the phase shift is introduced by the time dilation due to the gravitational field. Under those considerations, they measure a phase shift analogous to (11):

$$\beta = \frac{m}{\hbar}\int [V(x_1,t) - V(x_2,t)]$$

where V is the gravitational potential and if we neglect the sources of uncertainty.

These elements may seem anodyne, but they might highlight something more fundamental than a change in phase. This discovery from Aharonov and Bohm had upturned the world of physics by showing that the vector and scalar potentials are of fundamental importance in quantum theory. The effect highlighted

[4]The details of this calculation can be found in the supplementary material.

in this article indicates the importance of space-time's topology, from which emerges a real physical phase. Incidentally, in his 1959 article, Marder wrote that in a purely geometrical theory such as general relativity, the topology of space-time is also important. It might seem that this theory is not solely local. This idea is also discussed by Overstreet et al. [9]. As they state, their measurement is to be distinguished from other gravitational measurements on quantum systems by the nonlocal nature of the phase shift.

The phases examined in this paper are geometrical, but share totally different origins. No information on interactions between quantum systems and gravity can be deduced from this work, but the possible nonlocality of gravity remains to be explored.

Acknowledgements

The author would like to thank Maître Rachel Anderson-Deas for her revision of the text and her fine suggestions to make it more concise and readable.

References

[1] Y. Aharonov and D. Bohm. Significance of electromagnetic potentials in the quantum theory. *Phys. Rev.*, 115:485–491, Aug 1959.

[2] Carlo Canali. Lectures notes in advanced quantum mechanics, Spring 2022.

[3] R. G. Chambers. Shift of an electron interference pattern by enclosed magnetic flux. *Phys. Rev. Lett.*, 5:3–5, Jul 1960.

[4] Bryce S. DeWitt. Superconductors and gravitational drag. *Phys. Rev. Lett.*, 16:1092–1093, Jun 1966.

[5] J. S. Dowker and Yih P. Dowker. Interactions of massless particles of arbitrary spin. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 294(1437):175–194, 1966.

[6] J S Dowker and J A Roche. The gravitational analogues of magnetic monopoles. *Proceedings of the Physical Society*, 92(1):1–8, sep 1967.

[7] Henrik Johannesson. Lectures notes, aharonov-bohm effect and geometric phase, Fall 2014.

[8] L. Marder. Flat space-times with gravitational fields. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 252(1268):45–50, 1959.

[9] Chris Overstreet, Peter Asenbaum, Joseph Curti, Minjeong Kim, and Mark A. Kasevich. Observation of a gravitational aharonov-bohm effect. *Science*, 375(6577):226–229, 2022.

[10] Giorgio Papini. Spin–gravity coupling and gravity-induced quantum phases. *General Relativity and Gravitation*, 40(6):1117–1144, 2008.

[11] Akira Tonomura, Nobuyuki Osakabe, Tsuyoshi Matsuda, Takeshi Kawasaki, Junji Endo, Shinichiro Yano, and Hiroji Yamada. Evidence for aharonov-bohm effect with magnetic field completely shielded from electron wave. *Phys. Rev. Lett.*, 56:792–795, Feb 1986.

[12] Alexander van Oudenaarden, Michel H. Devoret, Yu. V. Nazarov, and J. E. Mooij. Magneto-electric aharonov–bohm effect in metal rings. *Nature*, 391(6669):768–770, Feb 1998.