Gravitational Aharonov-Bohm effect: Supplementary Materials

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Introduction

This document provides the mathematical background and the derivation of the content displayed in [1]. It is first aimed at students who wish to follow along the article. We first demonstrate the Aharonov-Bohm effect with Feynman's path integral approach. The rest of the document is mostly about describing the properties of the metrics discussed in the article. We calculate the curvatures and the energy-momentum tensors. We demonstrate that the exterior metrics are indeed flat as described in the paper, and we describe the gravitational source by analyzing the energy-momentum tensors.

Aharonov-Bohm effect

Following for example [2] or [4], one can highlight the Aharonov-Bohm effect using the path integral approach. Using that formalism one can express the time evolution of a generic wave function $\psi(x,t)$ using the propagator, or kernel, $K(x,t_1;x',t_0)$, yielding

$$\psi(x,t_1) = \int_{-\infty}^{+\infty} K(x,t_1;x',t_0)\psi(x',t_0)dx' \tag{1}$$

In Feynman's formalism this propagator can be expressed as

$$K(x,t_1;x',t_0) = \sum_p W_p \exp\left\{\frac{iS_p(t_1,t_0)}{\hbar}\right\} \tag{2}$$

where we sum over all possible paths, W_p being some weight function and S the classical action. Feynman showed that with this formulation, for a particle who was at position x_0 at time t_0, the probability amplitude to find it at position x_1 at time t_1 is given by:

$$\langle x_1,t_1|x_0,t_0\rangle = \int_{x_0}^{x_1} \mathscr{D}\left[x(t)\right] \exp\left\{\frac{iS(t_1,t_0)}{\hbar}\right\} \tag{3}$$

where $\mathscr{D}$ is a sort of infinite dimensional integral operator; it encapsulates all the mathematical complexity. The latter equation is known as the Feynman's path integral. To arrive at the Aharonov-Bohm effect, the relevant part is

$$\langle x_1,t_1|x_0,t_0\rangle \propto \sum_p \exp\left\{\frac{iS_p(t_1,t_0)}{\hbar}\right\} \tag{4}$$

We recall that the Lagrangian of a charged particle of mass m and charge q in an electromagnetic field is as follow:

$$\mathscr{L} = \frac{m}{2}\dot{\vec{x}}^2 + \frac{m}{2}\dot{\vec{x}}\cdot\vec{A} - qV$$

If we do not have any electromagnetic field, we go back to the original two slits experiment, and denote the action for this case as S_0. By turning on a magnetic field we change the action, such as :

$$S \to S_0 + \frac{q}{c}\int \vec{A}\cdot\dot{\vec{x}}dt = S_0 + \frac{q}{c}\int \vec{A}\cdot d\vec{l} \tag{5}$$

For the experiment we should consider the two different trajectories, the upper and lower ones. Using (4) for each of those trajectories and plugging in (5), we obtain:

$$\sum_{up}\exp\left\{S_0 + \frac{q}{c}\int \vec{A}\cdot d\vec{l}\right\} + \sum_{low}\exp\left\{S_0 + \frac{q}{c}\int \vec{A}\cdot d\vec{l}\right\} \tag{6}$$

The quantity we are interested in here is actually the difference in accumulated phases between the two paths. This phase shift is given by :

$$\beta = \exp\left(\frac{iq}{c}\int_{low}\vec{A}\cdot d\vec{l} - \frac{iq}{c}\int_{up}\vec{A}\cdot d\vec{l}\right) = \exp\left(\frac{iq}{c}\oint \vec{A}\cdot d\vec{l}\right) \tag{7}$$

Finally, using Stokes' theorem we can rewrite the exponent in (7) as :

$$\frac{iq}{c}\oint \vec{A}\cdot d\vec{l} = \frac{iq}{c}\oiint \vec{B}\cdot d\vec{S} = \frac{iq}{c}\Phi \tag{8}$$

Gravitational Fields

In this section we investigate in more details the metrics and their sources. In 4D, the Riemann tensor has 256 components. We use to compute those[1] the Python library **Pytearcat** [3].

Introduction

In order to compute the energy-momentum tensor, one uses Einstein's equation:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu} \tag{9}$$

here in relativistic units and with $R_{\mu\nu}$ Ricci's tensor, R Ricci's scalar and $g_{\mu\nu}$ the metric tensor. Ricci's tensor is a compressed version of Riemann's curvature tensor, giving the curvature of a manifold at any point, where the scalar R gives the global curvature of a manifold. In order to compute Ricci's tensor and Ricci's scalar, we first need to determine Riemann's tensor, for which we have :

$$R^l_{kji} = \partial_j\Gamma^l_{ki} - \partial_i\Gamma^l_{kj} + \Gamma^l_{kj} + \Gamma^m_{ki}\Gamma^l_{mj} - \Gamma^m_{kj}\Gamma^l_{mi} \tag{10}$$

[1] The numerical calculations are available at `https://github.com/lukbrb/academic-physics/blob/master/GravitationAharonovBohmEffect/TensorCalculations.ipynb`

where the $\Gamma^{\gamma}_{\alpha\beta}$ are Christoffel's symbols, given by:

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2g_{\gamma\gamma}} \left(\partial_\beta g_{\gamma\alpha} + \partial_\alpha g_{\sigma\beta} - \partial_\gamma g_{\alpha\beta}\right) \tag{11}$$

The procedure is therefore the following; from the metric we deduce the metric tensor, we then find the Christoffel's symbols and compute Riemann's tensor. Once this is done it is pretty straightforward to calculate Ricci's tensor

$$R_{\mu\nu} = \sum_\lambda {R^\lambda}_{\mu\nu\lambda} = {R^\lambda}_{\mu\nu\lambda}$$

and Ricci's scalar which is the trace of the latter tensor. We finally use Einstein's equation (9) to find the energy-momentum tensor.

Original metric

We start with the following metric :

$$ds^2 = dt^2 - A^2dr^2 - r^2d\phi^2 - A^2dz^2$$

for which the metric tensor has the form :

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -A^2 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -A^2 \end{bmatrix} \tag{12}$$

for such a metric Christoffel's symbols $\Gamma^{\alpha}_{\beta\gamma}$ are:

$$\Gamma^1_{22} = -\frac{r}{A^2} \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} \tag{13}$$

One can then show that for Riemann's curvature tensor $R^{\alpha}_{\beta\gamma\delta}$ all components are zero. Incidently, for Ricci's tensor $R_{\alpha\beta}$ and Einstein's tensor $G_{\alpha\beta}$ the components vanish as well. Since the energy-momentum tensor and Einstein's tensor are related[2] by

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \tag{14}$$

all components of $T_{\alpha\beta}$ vanish as well.

Metric of the primed system

We now consider the metric of the primed system $\mathscr{R}'$. It is given by

$$ds^2 = dt^2 - dr'^2 - r'^2d\phi'^2 - dz'^2$$

[2] In relativistic units.

for which the metric tensor is:

$$g'_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r'^2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{15}$$

The non-zero Christoffel's symbols are then:

$$\Gamma^1_{22} = -r' \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r'} \tag{16}$$

Once again for such a metric all components of Riemann's tensor are zero, and so are the components of the energy-momentum tensor. The space-time is indeed flat.

We now consider the interior metric :

$$ds^2 = dt^2 - dr_1^2 - f^2(r_1) \cdot d\phi^2 - dz'^2$$

The metric tensor is given by

$$g^{int}_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -f^2(r) & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{17}$$

for which we calculate that the non-vanishing Christoffel's symbols are:

$$\Gamma^1_{22} = -f(r)\frac{d}{dr} \cdot f(r) \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{f(r)} \cdot \frac{d}{dr} f(r) \tag{18}$$

For Riemann's tensor we obtain this time that there are four non-vanishing components :

$$R^1_{212} = -R^1_{221} = -f(r) \cdot \frac{d^2}{dr^2} f(r) \tag{19}$$

$$R^2_{112} = -R^2_{121} = \frac{1}{f(r)} \cdot \frac{d^2}{dr^2} f(r) \tag{20}$$

To compute Einstein's tensor we then need Ricci's tensor and scalar R, given respectively by :

$$R_{11} = \frac{1}{f(r)} \cdot \frac{d^2}{dr^2} f(r) \qquad R_{22} = -f(r) \cdot \frac{d^2}{dr^2} f(r) \tag{21}$$

and $R = -2R_{11}$.

This yields for Einstein's tensor :

$$G_{00} = -G_{33} = -f(r) \cdot \frac{d^2}{dr^2} f(r) = Z^2 \tag{22}$$

Given (14), we find that indeed $T_{00} = -T_{33} = \dfrac{Z^2}{8\pi}$.

Metric due to the analogy with the vector potential $\vec{A}$

In this section we consider the direct analogy between the vector potential $\vec{A}$ and the vector $\vec{\gamma} \equiv (\gamma_{10}, \gamma_{20}, \gamma_{30})$, $\gamma_{\mu\nu}$ being the metric deviation; if the considered metric is given by $g_{\mu\nu}$ and a flat metric by $\eta_{\mu\nu}$, then

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \tag{23}$$

We first rewrite the vector potential $\vec{A}$ in cartesian coordinates :

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_z \end{bmatrix} = \begin{bmatrix} -\sin\theta \cdot \frac{\Phi}{2\pi\rho} \\ \cos\theta \cdot \frac{\Phi}{2\pi\rho} \\ A_z \end{bmatrix} = \begin{bmatrix} -\frac{y\sqrt{x^2+y^2}}{x^2+y^2} \cdot \frac{\Phi}{2\pi\rho} \\ \frac{x\sqrt{x^2+y^2}}{x^2+y^2} \cdot \frac{\Phi}{2\pi\rho} \\ A_z \end{bmatrix} \tag{24}$$

given that $\rho = \sqrt{x^2+y^2}$, we find that :

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \frac{1}{\rho^2} \begin{bmatrix} -\lambda y \\ \lambda x \\ 0 \end{bmatrix} \quad , \lambda = \frac{\Phi}{2\pi} \tag{25}$$

If we now associate $\vec{A}$ to $\vec{\gamma}$ and use (23), the metric $g_{\mu\nu}$ is given by :

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{-\Phi y}{\rho^2} & 0 & 0 & 0 \\ \frac{\Phi x}{\rho^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{26}$$

which yields the metric

$$ds^2 = 1 \cdot dt^2 - \frac{\Phi y}{2\pi\rho^2} dt \cdot dx - 1 \cdot dx^2 + \frac{\Phi x}{2\pi\rho^2} dt \cdot dx - 1 \cdot dy^2 - 1 \cdot dz^2$$

The non vanishing Christoffel's symbols for that metric are :

$$\Gamma^x_{ty} = \Gamma^x_{yt} = -\Gamma^y_{tx} = -\Gamma^y_{xt} = -\frac{\phi x^2}{(x^2+y^2)^2} - \frac{\phi y^2}{(x^2+y^2)^2} + \frac{\phi}{x^2+y^2} \tag{27}$$

and all the components of the Riemann tensor vanish. There is no curvature, the space-time described by this metric is indeed flat.

We now consider the interior metric :

$$ds^2 = dt^2 - (d\vec{r})^2 - \lambda \cdot \frac{\sin^2(\pi\rho/2a)}{\rho^2}(ydx - xdy)dt$$

for which the metric tensor is given by :

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{\sin(\pi\rho/2a)}{2\pi\rho^2}\Phi y & -1 & 0 & 0 \\ \frac{\sin(\pi\rho/2a)}{2\pi\rho^2}\Phi x & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{28}$$

the Christoffel's symbols by:

$$\Gamma^x_{ty} = \Gamma^x_{yt} = -\Gamma^y_{xt} = -\Gamma^y_{tx} = \frac{\pi \sin(\pi\rho/a)}{4a\rho} \tag{29}$$

Dowker finds that only two components of the energy-momentum tensor are non-vanishing :

$$\frac{1}{x}T_{xt} = \frac{1}{y}T_{yt} = -\frac{\pi}{a}\cos\frac{\pi\rho}{a} + \frac{1}{\rho}\sin\frac{\pi\rho}{a} \tag{30}$$

Using two different libraries I however find different results, that can be found in the dedicated notebook. The aim of this paper is not to deep dive into general relativity, thus no calculations will be pursued yet to determine which results are correct. What has to be retained is that a phase does arise due to the gravitational field, despite the fact that the gravitational sources considered here are hard to realize physically.

Bibliography

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