Statistical Physics
The Dog-Flea Model

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Arrow of time in the flea universe: Two flea infested dogs are lying next to each other. The fleas hop from one dog to the other. Each flea has a name. The dynamics of the fleas is determined by a flea God who loves to play dice with this flea universe. It generates a random number, after a certain time step, between 1 and N (where N is the number of fleas). Depending on the number, It calls out the name of the corresponding flea, which is compelled to jump from the current dog it is inhabiting to the other. Starting with a given distribution of fleas, these 'stochastic' dynamics will evolve the microstate (a precise description of which flea infests which dog) and the macrostate (how many fleas on a given dog).

(a) Is this stochastic dynamics time-symmetric?

(b) Given a macrostate corresponding to m fleas on a dog (and $N - m$ on the other), calculate the entropy of this macrostate. For what macrostate is this entropy maximum?

(c) Write an equation for the probability $P_n(m)$ that after n time steps there are m fleas infesting a dog (where m can take values 0,1,2,..,N), in terms of the corresponding probabilies at the end of $n - 1$ steps. Show that the relation can be written in the form

$$P_n = TP_{n-1} \tag{1}$$

where T is an $(N + 1) \times (N + 1)$ matrix and P_n is a column vector with entries $P_n = (P_n(0), P_n(1), .., P_n(N))$. Identify the entries of the matrix and find a general expression for T_{ij}. Compute the sum of entries of any column of the matrix T. What is this equal to?

(d) Assume that after waiting 'long enough', an equilibrium macrostate is reached. What should be the property of the equilibrium probability distribution $P_{eq}(m)$? Determine this distribution. Does this distribution agree with the principle of equal a-priori probability for all microstates?

(e) Simulate this system on the computer, starting with a configuration in which all the fleas infest one dog. Evolve the system by generating random numbers, and plot (i) the number of fleas on the two dogs and (ii) the entropy of the system as functions of time. Finally, once equilibrium is reached, collect data to plot a histogram of the number of fleas on any one dog. Does this agree with the equilibrium distribution computed analytically?