Luke Davidson
ME 5659
HW #3

1.) 3.1) $y(t) = e^{-t} - \frac{1}{2}e^{-2t}$ $\quad x_1(0) = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$

$y(t) = -\frac{1}{2}e^{-t} - e^{-2t}$ $\quad x_2(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

what is $y(t)$ for $x_3(0) = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix}$

$$a[x_1(0)] + b[x_2(0)] = [x_3(0)]$$

$$\Rightarrow a\, y_1(t) + b\, y_2(t) = y_3(t).$$

$a - b = 2$

$\boxed{a = b+2}$

$0.5a + b = 0.5$

$0.5(b+2) + b = 0.5$

$1.5b = -0.5$

$\frac{3}{2} b = -\frac{1}{2}$

$b = -\frac{1}{2} \cdot \frac{2}{3} = -\frac{1}{3} = b$

$a = -\frac{1}{3} + \frac{6}{3} = \frac{5}{3} = a$

$$y_3(t) = \frac{5}{3}\left(e^{-t} - \frac{1}{2}e^{-2t}\right) - \frac{1}{3}\left(-\frac{1}{2}e^{-t} - e^{-2t}\right)$$

$$y_3(t) = \frac{5}{3}e^{-t} - \frac{5}{6}e^{-2t} + \frac{1}{6}e^{-t} + \frac{1}{3}e^{-2t}$$

$$y(t) = \frac{11}{6}e^{-t} - \frac{1}{2}e^{-2t} \quad @ \quad x(0) = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix}$$

2.) 3b) e^{At} for $\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$ by $\mathcal{L}$

$$e^{At} = \mathcal{L}^{-1}\left\{[sI - A]^{-1}\right\}$$

$$\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} s+1 & -1 & 0 \\ 0 & s+1 & 1 \\ 0 & -1 & s+1 \end{bmatrix}$$

$$\det(sI - A) = s+1\left(\det\begin{bmatrix} s+1 & 1 \\ -1 & s+1 \end{bmatrix}\right) - (-1)\left(\det\begin{bmatrix} 0 & 1 \\ 0 & s+1 \end{bmatrix}\right) + 0$$

$$= (s+1)\left((s+1)^2 + 1\right) = \left(s^2 + 2s + 1 + 1\right)(s+1) = (s+1)(s^2 + 2s + 2)$$

$$= s^3 + 2s^2 + 2s + s^2 + 2s + 2$$

$$= s^3 + 3s^2 + 4s + 2$$

$$\mathrm{adj}(A) = C_{ij}^{T}$$

$$C_{ij} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \times M_{ij}$$

$$M_{ij} = \begin{bmatrix} (s+1)^2 + 1 & 0 & 0 \\ -(s+1) & (s+1)^2 & -(s+1) \\ -1 & s+1 & (s+1)^2 \end{bmatrix}$$

$$\begin{bmatrix} (s+1)^2 + 1 & 0 & 0 \\ s+1 & (s+1)^2 & s+1 \\ -1 & -(s+1) & (s+1)^2 \end{bmatrix}$$

$$\mathrm{adj} = \begin{bmatrix} (s+1)^2 + 1 & s+1 & -1 \\ 0 & (s+1)^2 & -(s+1) \\ 0 & s+1 & (s+1)^2 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{adj}{det} = \begin{bmatrix} (s+1)^2+1 & s+1 & -1 \\ 0 & (s+1)^2 & -(s+1) \\ 0 & s+1 & (s+1)^2 \end{bmatrix} \frac{1}{(s+1)((s+1)^2+1)}$$

$$= \begin{bmatrix} \frac{1}{s+1} & \frac{1}{((s+1)^2+1)} & \frac{-1}{(s+1)((s+1)^2+1)} \\ 0 & \frac{s+1}{(s+1)^2+1} & \frac{-1}{(s+1)^2+1} \\ 0 & \frac{1}{(s+1)^2+1} & \frac{s+1}{(s+1)^2+1} \end{bmatrix} \Longrightarrow e^{At} = \mathcal{L}^{-1}\left[\quad\right]$$

$$\frac{(s+1)^2}{(s+1)^2\left((s+1)+\frac{1}{s+1}\right)} = \frac{1}{(s+1)+\frac{1}{s+1}} = \frac{s+1}{(s+1)^2+1}$$

$$e^{At} = \begin{bmatrix} e^{-t} & e^{-t}\sin(t) & e^{-t}(\cos(t)-1) \\ 0 & e^{-t}\cos(t) & -e^{-t}\sin(t) \\ 0 & e^{-t}\sin(t) & e^{-t}\cos(t) \end{bmatrix}$$

```matlab
% Luke Davidson
% ME 5659
% HW3 Q2

clc;
clear all;
close all;

t_total = [1 2 4];

fprintf('-------------------- B & C --------------------\n\n')

for i = 1:3
    fprintf('e^At for t = %d\n',t_total(i))
    t = t_total(i);
    eat = [exp(-t) exp(-t)*sin(t) -exp(-2*t)*sin(t);0 exp(-t)*cos(t) -exp(-t)*sin(t);0 exp(-t)*sin(t) exp(-t)*cos(t)];
    disp(eat)
end

fprintf('\n-------------------- D --------------------\n\n')

A = [-1 1 0;0 -1 -1;0 1 -1];

fprintf('e^A2:\n')
disp(exp(A*2))
fprintf('\n')
fprintf('(e^A)^2:\n')
disp(exp(A).^2)
```

```
-------------------- B & C --------------------

e^At for t = 1
    0.3679    0.3096   -0.1139
         0    0.1988   -0.3096
         0    0.3096    0.1988

e^At for t = 2
    0.1353    0.1231   -0.0167
         0   -0.0563   -0.1231
         0    0.1231   -0.0563

e^At for t = 4
    0.0183   -0.0139    0.0003
         0   -0.0120    0.0139
         0   -0.0139   -0.0120


-------------------- D --------------------

e^A2:
    0.1353    7.3891    1.0000
    1.0000    0.1353    0.1353
    1.0000    7.3891    0.1353


(e^A)^2:
    0.1353    7.3891    1.0000
    1.0000    0.1353    0.1353
    1.0000    7.3891    0.1353
```

3) c) $|A - \lambda I| = 0$

$$\begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ -3 & -4-\lambda \end{bmatrix}$$

$$\det \begin{bmatrix} -\lambda & 1 \\ -3 & -4-\lambda \end{bmatrix} = -\lambda(-4-\lambda) - (-3)(1)$$

$$= 4\lambda + \lambda^2 + 3 = \lambda^2 + 4\lambda + 3$$

$$\lambda^2 + 4\lambda + 3 = 0$$
$$(\lambda + 1)(\lambda + 3) = 0$$

$$\lambda = -1, -3$$ eigvalues

① $x_2 = \lambda x_1$

② $-3x_1 - 4x_2 = \lambda x_2$

for $\lambda = -1$

$\boxed{x_2 = -x_1}$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \xrightarrow{\text{unit}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$-3x_1 - 4x_2 = -x_2$$
$$-3x_1 = 3x_2$$
$$x_1 = -x_2$$

for $\lambda = -3$

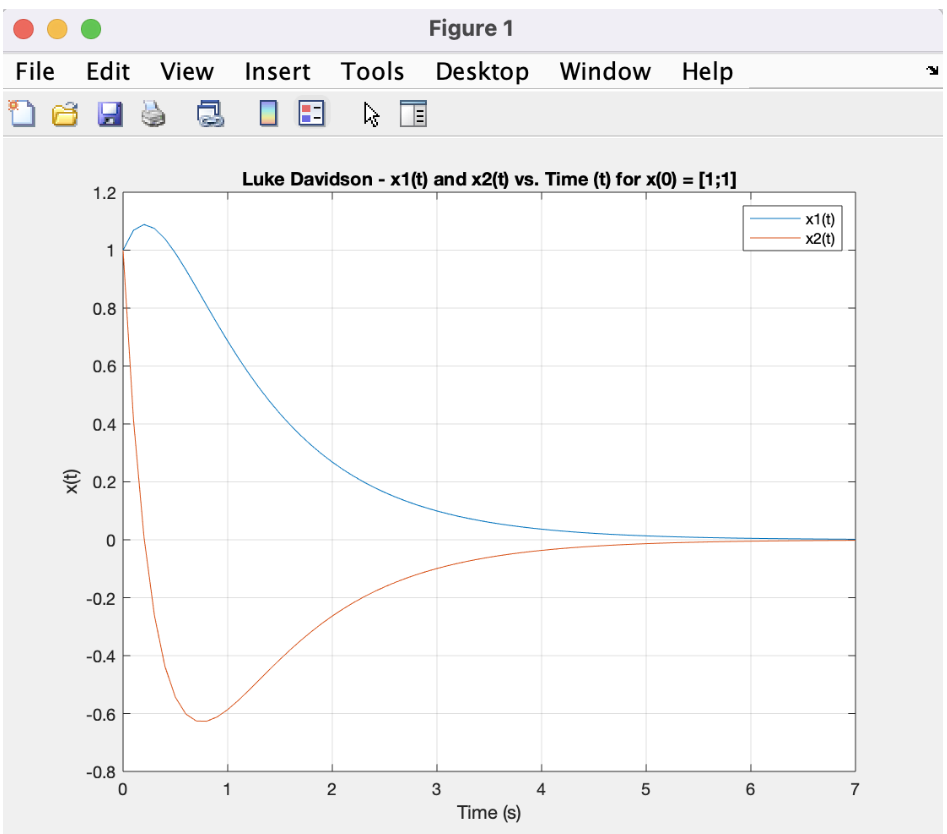
$$x_2 = -3x_1 \qquad \begin{bmatrix} -1 \\ 3 \end{bmatrix} \xrightarrow{\text{unit}} \begin{bmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$$

$$\text{eig vectors} = \begin{array}{cc} \lambda_1 & \lambda_2 \\ \end{array}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{10}} \\ \frac{-1}{\sqrt{2}} & \frac{3}{\sqrt{10}} \end{bmatrix}$$

confirmation and plots in attached MATLAB

```matlab
% Luke Davidson
% ME 5659
% HW3 Q3

clc;
clear all;
close all;

A = [0 1; -3 -4];
xzero = [1;1];
syms s;

sI_Ainv = (s*eye(2)-A)^(-1);
total_x = ilaplace(sI_Ainv)*xzero;

% disp(total_x(1,1));
% disp(total_x(2,1));

t = [0:0.1:7];

x1 = 2*exp(-t)-exp(-3*t);
x2 = 3*exp(-3*t)-2*exp(-t);

figure()
plot(t,x1,t,x2)
title('Luke Davidson - x1(t) and x2(t) vs. Time (t) for x(0) = [1;1]')
xlabel('Time (s)')
ylabel('x(t)')
legend('x1(t)','x2(t)')
grid on

figure()
plot(x1,x2)
title('Luke Davidson - x2(t) vs. x1(t) for x(0) = [1;1]')
xlabel('x1(t)')
ylabel('x2(t)')
grid on

[V,D] = eig(A);
fprintf('-------------------- C --------------------\n')
fprintf('eigenvalues:\n')
disp(D)
fprintf('eigenvectors:\n')
disp(V)

total_x_eig1 = ilaplace(sI_Ainv)*V(:,1);
total_x_eig2 = ilaplace(sI_Ainv)*V(:,2);

x1_eig1 = (2^(1/2)*((3*exp(-t))/2-exp(-3*t)/2))/2-(2^(1/2)*(exp(-t)/2-exp(-3*t)/2))/2;
x2_eig1 = (2^(1/2)*(exp(-t)/2-(3*exp(-3*t))/2))/2-(2^(1/2)*((3*exp(-t))/2-(3*exp(-3*t))/2))/2;

x1_eig2 = (3*10^(1/2)*(exp(-t)/2-exp(-3*t)/2))/10-(10^(1/2)*((3*exp(-t))/2-exp(-3*t)/2))/10;
x2_eig2 = (10^(1/2)*((3*exp(-t))/2-(3*exp(-3*t))/2))/10-(3*10^(1/2)*(exp(-t)/2-(3*exp(-3*t))/2))/10;

figure()
plot(t,x1_eig1,t,x2_eig1)
title('Luke Davidson - x1(t) and x2(t) vs. Time (t) for x(0) = eigenvector at eigenvalue = -1')
xlabel('Time (s)')
ylabel('x(t)')
legend('x1(t)','x2(t)')
grid on

figure()
plot(t,x1_eig2,t,x2_eig2)
title('Luke Davidson - x1(t) and x2(t) vs. Time (t) for x(0) = eigenvector at eigenvalue = -3')
```

```
xlabel('Time (s)')
ylabel('x(t)')
legend('x1(t)','x2(t)')
grid on

figure()
plot(x1_eig1,x2_eig1)
title('Luke Davidson - x2(t) vs. x1(t) for x(0) = eigenvector at eigenvalue = -1')
xlabel('x1(t)')
ylabel('x2(t)')
grid on

figure()
plot(x1_eig2,x2_eig2)
title('Luke Davidson - x2(t) vs. x1(t) for x(0) = eigenvector at eigenvalue = -3')
xlabel('x1(t)')
ylabel('x2(t)')
grid on

-------------------- C --------------------
eigenvalues:
    -1     0
     0    -3

eigenvectors:
    0.7071   -0.3162
   -0.7071    0.9487

>>
```



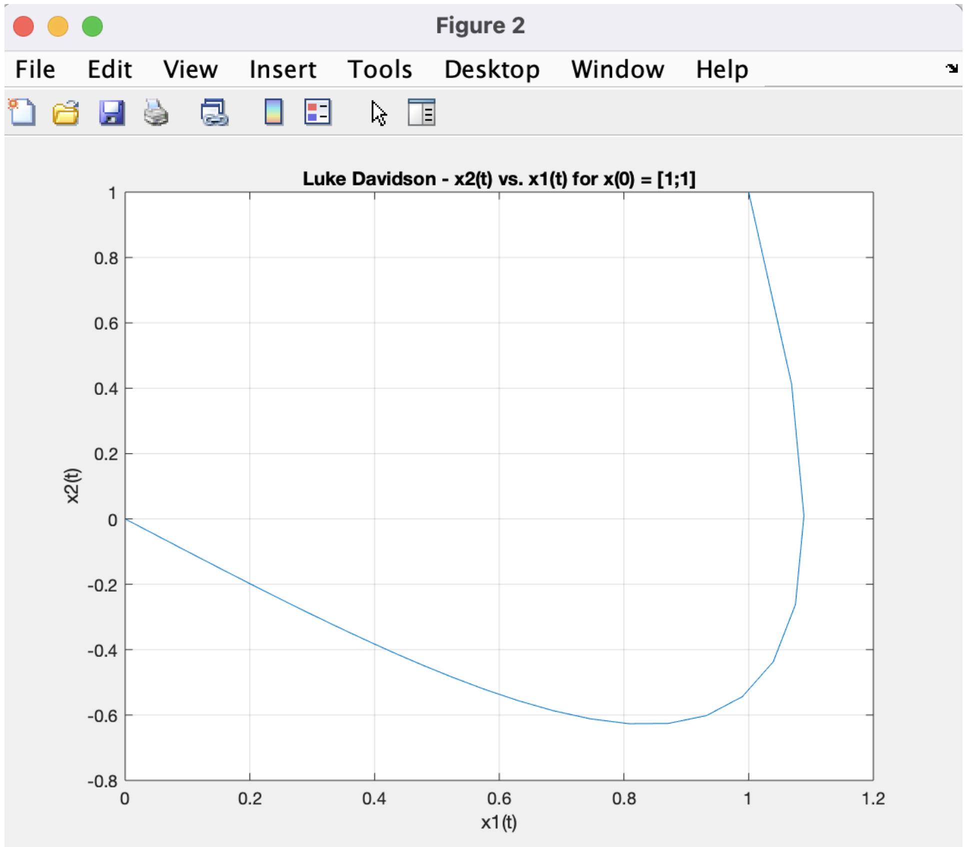


Figure 2
File Edit View Insert Tools Desktop Window Help
Luke Davidson - x2(t) vs. x1(t) for x(0) = [1;1]
x2(t)
x1(t)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
0.2
0.4
0.6
0.8
1
1.2

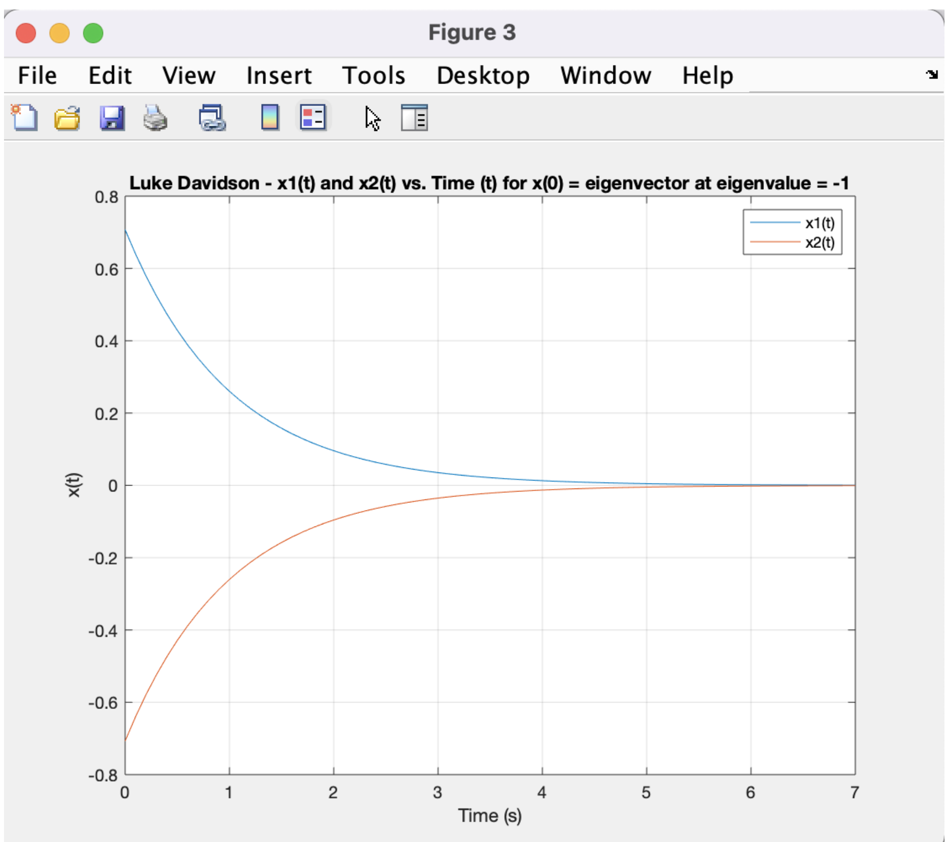


Figure 3
File Edit View Insert Tools Desktop Window Help
Luke Davidson - x1(t) and x2(t) vs. Time (t) for x(0) = eigenvector at eigenvalue = -1
x1(t)
x2(t)
x(t)
Time (s)
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
1
2
3
4
5
6
7

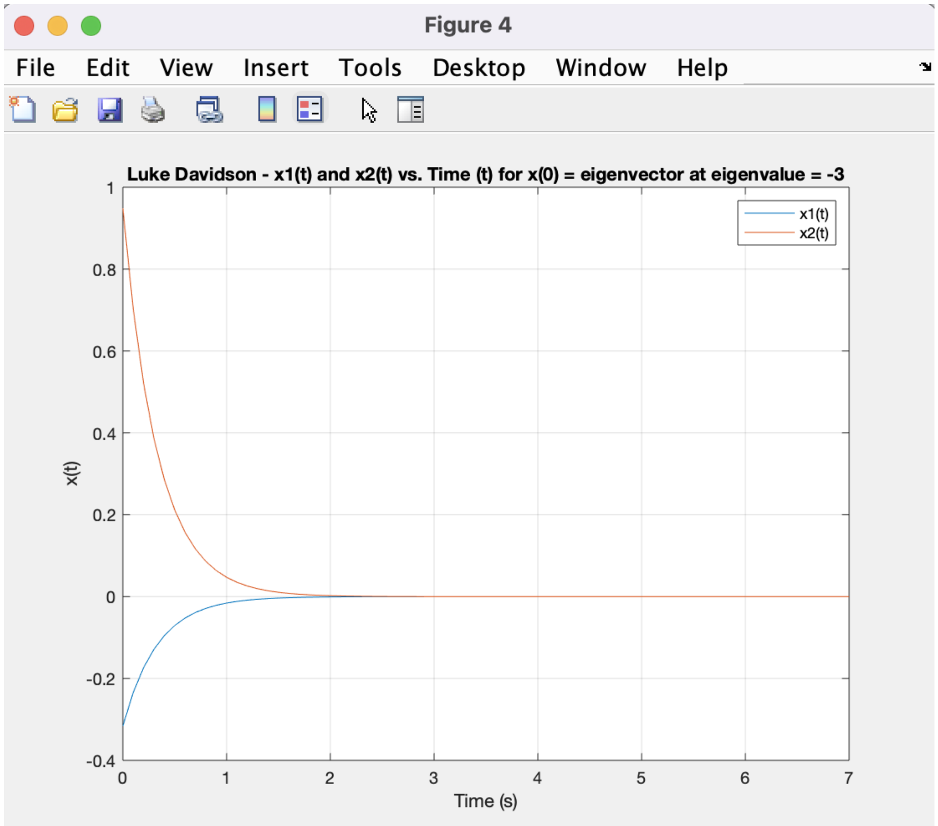


Figure 4
File Edit View Insert Tools Desktop Window Help
Luke Davidson - x1(t) and x2(t) vs. Time (t) for x(0) = eigenvector at eigenvalue = -3
x1(t)
x2(t)
x(t)
Time (s)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
1
2
3
4
5
6
7

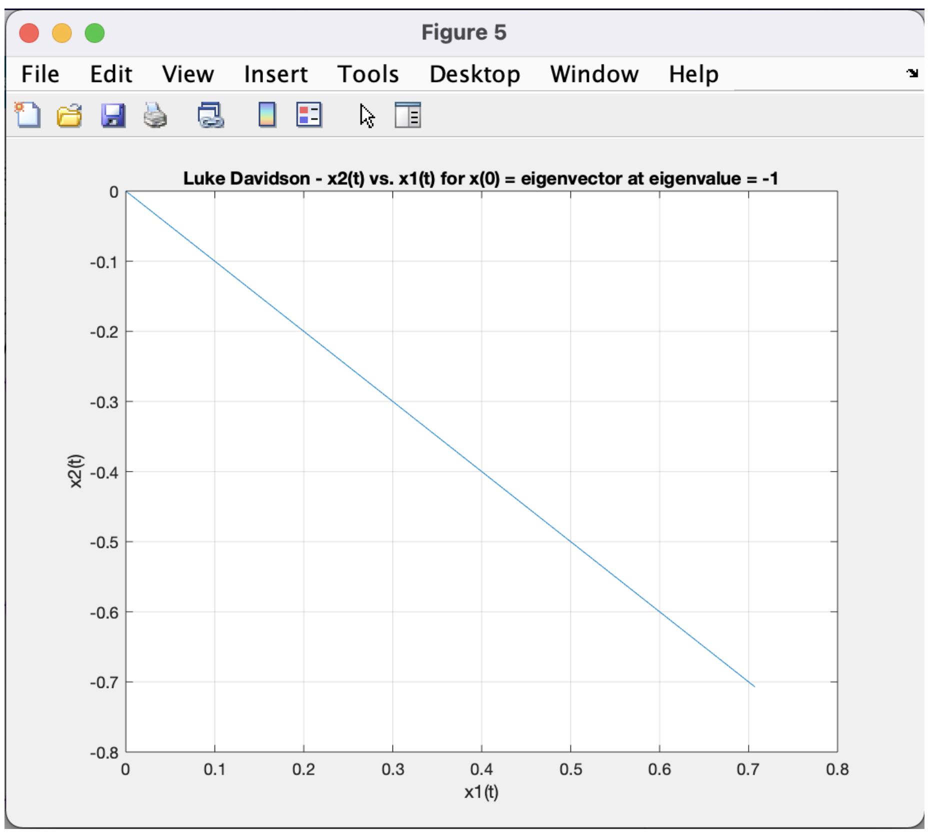


Figure 5
File Edit View Insert Tools Desktop Window Help
Luke Davidson - x2(t) vs. x1(t) for x(0) = eigenvector at eigenvalue = -1
x2(t)
x1(t)
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8

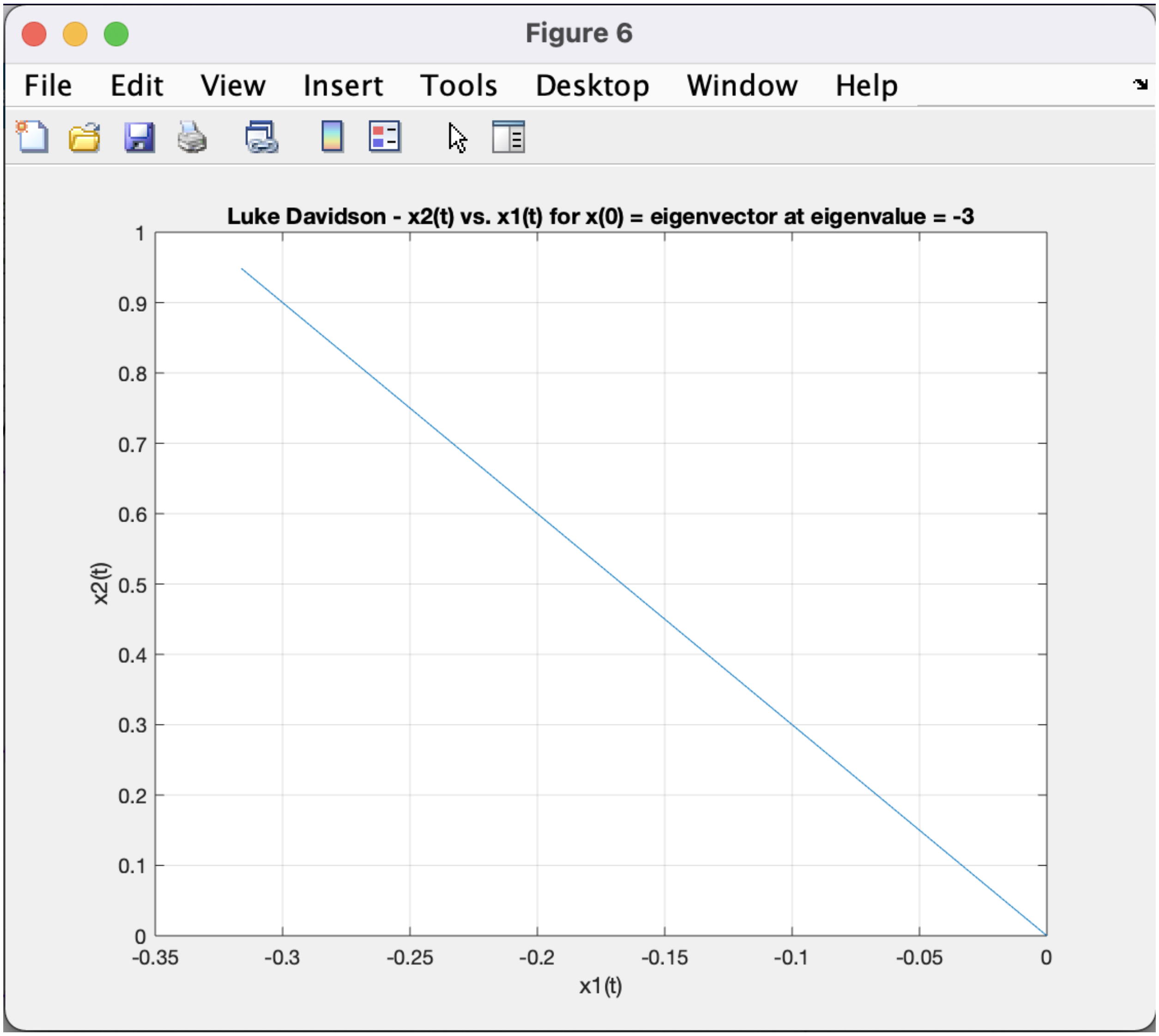


Figure 6
File Edit View Insert Tools Desktop Window Help
Luke Davidson - x2(t) vs. x1(t) for x(0) = eigenvector at eigenvalue = -3
x2(t)
x1(t)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.35
-0.3
-0.25
-0.2
-0.15
-0.1
-0.05
0

4.) 3.10)

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

equation 3.19

$$H(s) = C(sI - A)^{-1} B + D$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{s}{s^2-1} & \frac{1}{s^2-1} \\ \frac{1}{s^2-1} & \frac{s}{s^2-1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{s}{s^2-1} & \frac{1}{s^2-1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Longrightarrow H(s) = \frac{1}{s^2-1}$$

ODE

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_1 + u \longrightarrow \ddot{x}_1 = x_1 + u \longrightarrow s^2 X(s) = X(s) + U(s)$$

$$y = x_1 \longrightarrow Y(s) = X(s)$$

$$(s^2 - 1) X(s) = U(s)$$

$$(s^2 - 1) Y(s) = U(s)$$

$$H(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2-1}$$

```
% Luke Davidson
% ME 5659
% HW3 Q4

clc;
clear all;
close all;

A = [0 1;1 0];
B = [0;1];
C = [1 0];
D = [0];

[n,d] = ss2tf(A,B,C,D);

TF = tf(n,d)
```

```
TF =

     1
  -------
  s^2 - 1

Continuous-time transfer function.

>>
```

5.) $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$(\det)\ A - \lambda I = 0$

$\lambda^2 + 3\lambda + 2 = 0$

$(\lambda+2)(\lambda+1) = 0$

$\boxed{\lambda = -1, -2}$

$$e^{At} = I + At + \frac{A^2 t^2}{2!} + \dots$$

$$= \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

$A^k = C_0(k) I + C_1(k) A$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(C_0(k) I + C_1(k) A \right)$$

$$= I \underbrace{\sum_{k=0}^{\infty} \frac{t^k}{k!} C_0(k)}_{b_0(t)} + A \underbrace{\sum_{k=0}^{\infty} C_1(k)}_{b_1(t)}$$

$$= I\, b_0(t) + b_1(t) A$$

$\Rightarrow e^{\lambda t} = b_0(t) + b_1(t)\lambda$

$e^{-2t} = b_0(t) - 2 b_1(t)$

$e^{-t} = b_0(t) - b_1(t)$

$b_0(t) = e^{-2t} + 2 b_1(t)$

$e^{-t} = e^{-2t} + 2 b_1(t) - b_1(t)$

$\boxed{e^{-t} - e^{-2t} = b_1(t)}$

$b_0(t) = e^{-2t} + 2e^{-t} - 2e^{-2t}$

$\boxed{b_0(t) = -e^{-2t} + 2e^{-t}}$

$$e^{At} = -e^{-2t} + 2e^{-t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e^{-t} - e^{-2t} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$\begin{bmatrix} -e^{-2t} + 2e^{-t} & 0 \\ 0 & -e^{-2t} + 2e^{-t} \end{bmatrix} + \begin{bmatrix} 0 & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -3e^{-t} + 3e^{-2t} \end{bmatrix}$$

$2e^{-2t} - e^{-t}$

ANS. ON BACK.

$$e^{At} = \begin{bmatrix} -e^{-2t} + 2e^{-t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & 2e^{-2t} - e^{-t} \end{bmatrix}$$