CS580 Homework 9

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Question 1.

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a We can first observe the following:

(a) set A = A $\cap$ B + A $\setminus$ B.
(b) set B = A $\cap$ B + B $\setminus$ A.
(c) A $\cup$ B = A $\cap$ B + A $\setminus$ B + B $\setminus$ A.

If we define the function f'(S,T) to be the number of edges from S to T and S and T are disjoint, we can see that

(a) f(A) = f'(A, V $\setminus$ A)
(b) f(B) = f'(B,V $\setminus$ B)
(c) f(A $\cap$ B) = f'(A $\cap$ B, V $\setminus$ (A $\cap$ B))
(d) f(A $\cup$ B) = f'(A $\cup$ B, V $\setminus$ (A $\cup$ B))

We break down each of the above to prove the function f is submodular.

f'(A, V $\setminus$ A) = f'(A $\cap$ B, V $\setminus$ (A $\cup$ B)) + f'(A $\cap$ B, B $\setminus$ A) + f'(A $\setminus$ B, B $\setminus$ A) + f'(A $\setminus$ B, V $\setminus$ (A $\cup$ B))

f'(B, V $\setminus$ B) = f'(A $\cap$ B, V $\setminus$ (A $\cup$ B)) + f'(A $\cap$ B, A $\setminus$ B) + f'(B $\setminus$ A, A $\setminus$ B) + f'(B $\setminus$ A, V $\setminus$ (A $\cup$ B))

f'(A $\cup$ B, V $\setminus$ (A $\cup$ B)) = f'(A $\cap$ B, V $\setminus$ (A $\cup$ B)) + f'(A $\setminus$ B,V $\setminus$ (A $\cup$ B)) + f'(B $\setminus$ A, V $\setminus$ (A $\cup$ B))

f'(A $\cap$ B, V $\setminus$ (A $\cap$ B)) = f'(A $\cap$ B, V $\setminus$ (A $\cup$ B)) + f'(A $\cap$ B, A $\setminus$ B) + f'(A $\cap$ B, B $\setminus$ A)

From the above, we can see that f(A) + f(B) - f(A $\cup$ B) - f(A $\cap$ B) = f'(A, V $\setminus$ A) + f'(B,V $\setminus$ B) - f'(A $\cap$ B, V $\setminus$ (A $\cap$ B)) - f'(A $\cup$ B, V $\setminus$ (A $\cup$ B)) = f'(A $\setminus$ B, B $\setminus$ A) + f'(B $\setminus$ A, A $\setminus$ B) $\geq$ 0.

Hence, f(A) + f(B) $\geq$ f(A $\cup$ B) + f(A $\cap$ B) and f is submodular b We prove both directions

(a) if G is Eulerian, f is symmetric:
if G is Eulerian, we know every vertex has the same number of outgoing edges as incoming edges. We prove f is symmetric using induction on the size of S

Base case: when |S| = 1, meaning there is only 1 vertex in the set, f(S) is the number of outgoing edges and f(V $\setminus$ S) is the number of edges going from V $\setminus$ S to S, which is also the number of incoming edges of S. Since G is Eulerian, f(S) = f(V $\setminus$ S)

Induction Hypothesis: when |S| = k $\leq$ |V| - 1, f(S) = f(V $\setminus$ S)

Inductive Step: when |S| = k + 1, we added a new vertex v to the previous set S'. We define the function f'(S,T) to be the number of edges from S to T and S and T are disjoint. Hence f(S) = f(S') + f'(v,V $\setminus$ S) - f'(S',v) and f(V $\setminus$ S) = f(V $\setminus$ S') + f'(V $\setminus$ S,v) - f'(v,S'). Since f'(v,V $\setminus$ S) - f'(S',v) = f'(V $\setminus$ S,v) - f'(v,S'), we can conclude f(S) = f((V $\setminus$ S). Hence, f is symmetric

(b) if f is symmetric, G is Eulerian If f is symmetric, we know f(S) = f(V $\setminus$ S). If |S| = 1, S contains only one vertex v and f(S) is the number of outgoing edges of v and f(V $\setminus$ S) is the number of incoming edges to v. Hence, the number of outgoing edges is the same as the number of outgoing edges for a single vertex. This is true for all vertices, hence G is Eulerian

c From question 2, we know if G is Eulerian, we have f(A) = f(V $\setminus$ A) and f(B) = f(V $\setminus$ B). Since f(V $\setminus$ A) $\geq$ f(B $\setminus$ A) and f(V $\setminus$ B) $\geq$ f(A $\setminus$ B), we know f(A) + f(B) $\geq$ f(B $\setminus$ A) + f(A $\setminus$ B)

Question 2.

Suppose (S, T) is a valid minimum (s,t)-cut of G, where $s \in S$ ans $t \in T$. Then u either belongs to S or T. Suppose $u \in S$. In this case, (S, T) is also a minimum cut for (u, t). Therefore λ(s, t) = λ(u, t). Suppose λ(s, u) > λ(u, t), then the goal becomes to prove λ(s, t) >= λ(u, t), which is true because λ(s, t) = λ(u, t). Suppose λ(s, u) $\leq$ λ(u, t), then the goal becomes to prove λ(s, t) >= λ(s, u). Since λ(s, t) = λ(u, t), the goal is equivalent to proving λ(u, t) >= λ(s, u), which is the hypothesis we assumed for this case. Therefore, λ(s, t) >= min λ(s, u), λ(u, t) . The same proof logic can be used to prove the $u \in T$ case.