Overview of Interpolation Using Radial Basis Functions the and Runge Phenomena

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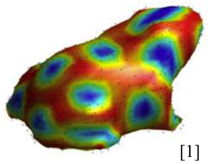
Introduction

Interpolation is a mathematical tool that is used to fill in the gaps between observed data points. This technique can be helpful when estimating the value of data points that are not explicitly observed. Interpolation techniques can be used to construct brain surfaces, create elevation maps from limited measurement sites, and even analyze the color patterns of a frog.



[1]

Our research specifically focuses on using radial basis functions (RBFs) to interpolate the data. Radial basis functions use a weighted system to solve for the value of unknown values. In this poster, we will discuss the process of interpolating data using RBFs, how to choose the best interpolant, and phenomena that occur when creating the interpolant.

Runge Phenomenon

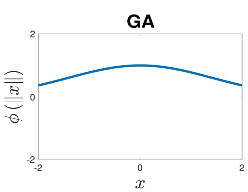
The Runge Phenomenon is classically observed as oscillations of interpolants at the edges of the interval when using higher-ordered polynomials for the interpolant. It is a common occurrence when interpolating using equidistant points.

Radial Basis Functions

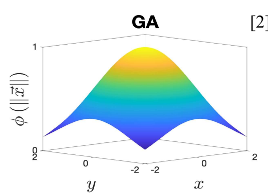
There are many different radial basis functions (ϕ) to choose from such as the multiquadric (MQ), inverse multiquadric, and Gaussian (GA) RBFs. The input into RBFs is $r = \|\vec{x} - \vec{x}_i\|$, or the Euclidean distance between two points.

$$\text{MQ: } \phi(r) = \sqrt{1 + (\epsilon r)^2}$$

$$\text{GA: } \phi(r) = e^{(-\epsilon r)^2}$$

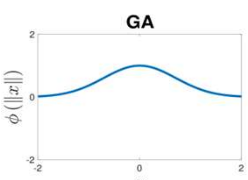


$1D$; $\epsilon = 0.5$

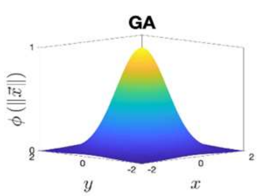


[2]

$2D$; $\epsilon = 0.5$



$1D$; $\epsilon = 1$



$2D$; $\epsilon = 1$

Interpolation with Radial Basis Functions

RBFs are used in interpolation to create a collocation matrix that can be used to solve for $\vec{\lambda}$ weights. The collocation matrix is a matrix of distances between points that are used as inputs for the RBF. Each vector within the collocation matrix chooses a different point as the center of the RBF and calculates the distance from that point to the rest using the Euclidean norm. We then multiply the inverse of this matrix by the true function values to solve for the $\vec{\lambda}$ weights.

$$A\vec{\lambda} = \vec{f} \;\rightarrow\; \vec{\lambda} = A^{-1}\vec{f}$$

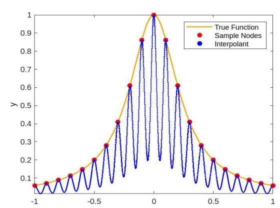
$$\vec{\lambda} = [\lambda_1, \lambda_2, \dots, \lambda_N]\,; \quad \vec{f} = [f(\vec{x}_1), f(\vec{x}_2), \dots, f(\vec{x}_N)\,]$$

$$A = \begin{bmatrix} \phi\|\vec{x}_1 - \vec{x}_1\| & \phi\|\vec{x}_1 - \vec{x}_2\| & \dots & \phi\|\vec{x}_1 - \vec{x}_N\| \\ \phi\|\vec{x}_2 - \vec{x}_1\| & \phi\|\vec{x}_2 - \vec{x}_2\| & \dots & \phi\|\vec{x}_2 - \vec{x}_N\| \\ \vdots & \vdots & & \vdots \\ \phi\|\vec{x}_N - \vec{x}_1\| & \phi\|\vec{x}_N - \vec{x}_2\| & \dots & \phi\|\vec{x}_N - \vec{x}_N\| \end{bmatrix}$$

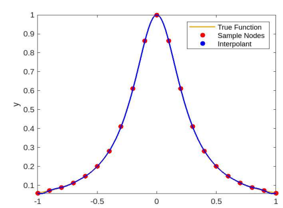
These $\vec{\lambda}$ weights are then applied to a larger collocation matrix that includes unknown points to create an interpolant from the original sample nodes.

1D Computational Results

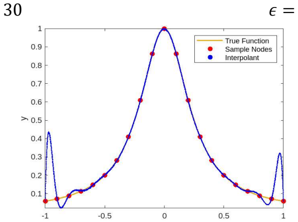
We first varied the shape parameter to observe its impact on the interpolant. Below, we recreated interpolants from [3] using MATLAB. In this example, we used the GA to create the collocation matrix and calculated equispaced 21 interpolation points in between the sample nodes over the interval $[-1,1]$. Note that the Runge Phenomena can be observed at the edge of the interpolant with $\epsilon = 1.6$. The true function we are interpolating is $f(x) = \frac{1}{1+16x^2}$.



$\epsilon = 30$



$\epsilon = 3.5$



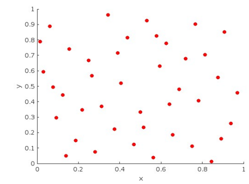
$\epsilon = 1.6$

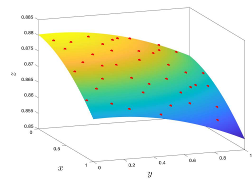
The inaccuracy of higher shape parameters, such as $\epsilon = 30$, can be shown by the intense oscillation patterns between observed data points. However, too low of a shape parameter causes the Runge Phenomenon to appear at the edges of the interval. The shape parameter must be tuned for any problem. One way mathematicians accomplish this is by finding the shape parameter that produces an interpolant with the lowest error.

2D Computational Results

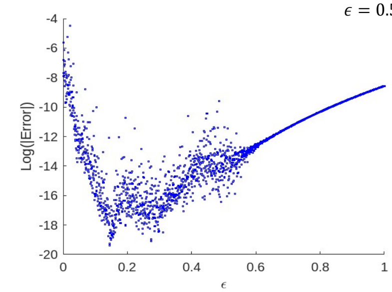
Below, we have recreated a plot from [4] that shows how varying the shape parameter impacts the error of the interpolant. The green scattered nodes on the left are the sample nodes that we will use to interpolate the data. Note that because our nodes have an x and y value, this is a 2D problem. The error is calculated from subtracting the resulting interpolation points from the true function value given by the function

$$f(x,y) = \frac{59}{67\;\left(x+\frac{1}{7}\right)^2+\left(y-\frac{1}{11}\right)^2}$$





$\epsilon = 0.55$



Note that as the shape parameter increases past a threshold ($\epsilon \approx 0.55$), the error of the interpolant steadily increases. Furthermore, when the shape parameter decreases below $\epsilon = 0.17$, we observe that the error rapidly increases. Mathematicians use these graphs when the true function is known to identify an ideal shape parameter for the problem.

Conclusion and Future Work

RBFs are a tool that can be used to accurately interpolate discrete data sets. The shape parameter of a RBF needs to be tuned to create an accurate interpolant. Lower shape parameters provide more accuracy, until the Runge Phenomenon is observed and rapidly increases the error.

In the future, we plan to research minimizing the Runge Phenomena's impact in RBF interpolation. We also plan to research cross validation methods as way of estimating the accuracy of an interpolant. This would be particularly useful when interpolating an unknown function.

Acknowledgments and References

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[1] Piret, C. (2012) The orthogonal gradients method: A radial basis functions method for solving partial differential equations on arbitrary surfaces. *Journal of Computational Physics*. 231(14): 4662-2675

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[3] Fornberg, B. & Zuev, J. (2007). The Runge phenomenon and spatially variable shape parameters in RBF Interpolation. *Computers & Mathematics with Applications*, *54*(3), 379–398.

[4] Flyer, N. & Fornberg B. (2010) Radial basis functions: Developments and applications to planetary scale flows. *Computer & Fluids 46(1)*, 23-32.