Continuous-Time Financial Economics

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Abstract

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Part I
Mathematical Preliminaries

1 Basic Analysis and Probability

We start with the notion of a topological vector space. This is the "right" mathematical objet to consider if we want to start with as general a framework as possible. Each agent in the economy acts by selecting some optimal consumption plan in a *consumption space*. Since the consumption space can consist of scalars, vectors, random variables, stochastic processes, and even more complicated objects, it is convenient to start from some general tools and concepts that are applicable to general consumption spaces. *The theory of topological vector spaces* gives us the relevant framework. We then start dealing more specifically with uncertainty by reviewing some basic concepts from probability theory. Finally, we define some topological vector spaces of functions that we will later take as the consumption space.

1.1 Vector Spaces

1.2 Topologies

1.3 Topological Vector Spaces

1.4 Normed Vector Spaces

1.5 Measure Spaces and Probability Spaces

Let Ω be the sample space with typical element $\omega \in \Omega$. 2^Ω will denote the power set of Ω, which is the collection of all possible subsets of Ω. Our goal is to develop the objects used to characterize uncertainty in economics. To this end, we first define the notion of a σ-field.

Definition 1. A set ("collection") $\mathcal{F} \subset 2^\Omega$ is a σ-field on Ω if

(i) $\mathcal{F} \neq \emptyset$

(ii) $A \in \mathcal{F} \implies A^c \in \mathcal{F}$ ("closed under complementation")

(iii) $\{A_i\}_{i=1}^\infty \subset \mathcal{F} \implies \cup_{i=1}^\infty A_i \in \mathcal{F}$ ("closed under countable unions")

The pair $(\Omega, \mathcal{F})$ is said to be a measurable space.

Remark 2. The definition is standard. Note that the trivial σ-field is given by $\mathcal{F} = \{\emptyset, \Omega\}$.

- A measurable space does not need a measure to be defined. You can assign a measure on it to get a measure space (hence the name measurable).
- We can also verify that for any σ-field $\mathcal{F}$ on Ω, $\emptyset, \Omega \in \mathcal{F}$, and that $\{A_i\}_{i=1}^\infty \subset \mathcal{F} \implies (\cap_{i=1}^\infty A_i) \in \mathcal{F}$.
- We will use σ-field to model information, in the sense that it will denote the set of events that agents can discern as true or not.

Definition 3. A filtration $F = \{\mathcal{F}_t\}_{t\in[0,T]}$ is a non-decreasing sequence of sub σ-fields of $\mathcal{F}$; that is

$$s \leq t \implies \mathcal{F}_s \subset \mathcal{F}_t$$

Remark 4. (Taken from StackExchange) Here's an interpretation for the meaning of a filtration: A filtration $\{\mathcal{F}_t\}$ contains *any information that could possibly be asked and answered for the considered random process* at time t.

Consider a single dice throw. Before the throw, all you know is that the result will be a "1 or 2 or ... or 6." In set notation, this corresponds to $\Omega_1 = \{1, 2, 3, 4, 5, 6\}$, which is the full set of outcomes. Moreover, you can ask and answer the silly question "does nothing happen," which corresponds to the empty set. So the filtration before the throw is $\mathcal{F}_0 = \{\emptyset, \Omega_1\}$.

After the throw, you get a single outcome $\omega_1 \in \Omega_1$. Now you can answer all kinds of questions, like "is the result a 4?" which corresponds to $\{4\}$; "Is the result an odd number", which coresponds to $\{1, 3, 5\}$; or "is the result larger than 4?" corresponding to $\{5, 6\}$. and so on.

You see, all the possible information after one throw (all that can be asked and answered) is contained in the power set 2^{Ω_1} of Ω_1.

Example 5. (Figure?) In the example, we have that

$$\mathcal{F}_t = \begin{cases} \{\emptyset, \Omega\} & t = 0 \\ \{\emptyset, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4, \omega_5\}, \Omega\} & t = 1/2 \\ 2^{\Omega} & t = 1 \end{cases}$$

This is meant to capture the gradual revealing of information over time.

Example 6. ("Two coin throw, ordered") Consider two coins thrown sequentially. Write the set of possible outcomes of a single coin throw as $\Omega_1 = \{H, T\}$. The filtration before both throws, after one throw, after both throws, is repectively modeled as

$$\begin{aligned}
\mathcal{F}_0 &= \{\emptyset, \Omega_2\} \\
\mathcal{F}_1 &= 2^{\Omega_1} \times \Omega_1 = \{\emptyset, \{H\}, \{T\}, \{H, T\}\} \times \{H, T\} \\
&= \{\emptyset, \{HH, HT\}, \{TH, TT\}, \{HH, HT, TH, TT\}\} \\
\mathcal{F}_2 &= 2^{\Omega_2} \\
&= \left\{ \begin{array}{c} \emptyset, \\ \{HH\}, \{HT\}, \{TH\}, \{TT\} \\ \{HH, HT\}, \{HH, TH\}, \{HH, TT\}, \{HT, TH\}, \{HT, TT\}, \{TH, TT\} \\ \{HH, HT, TH\}, \{HH, HT, TT\}, \{HH, TH, TT\}, \{HH, HT, TH\} \\ \{HH, HT, TH, TT\} = \Omega_2 \end{array} \right\}
\end{aligned}$$

Before any throw, all you can ask and answer is that the result will be in $\Omega_2 = \Omega_1 \times \Omega_1$, so $\mathcal{F}_0 = \{\emptyset, \Omega_2\}$. After the first throw, you can only answer any of the questions related to the first throw (as in the one-dice example), and nothing related to the second (e.g. "Is the first throw heads"). So the filtration is $\mathcal{F}_1 = 2^{\Omega_1} \times \Omega_1$. After both throws, you can answer any questions.

Check that at each t, $\mathcal{F}_t$ is a σ-field. Also note that $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2$. Thus we have a filtration.

Now we define a measure. This is a generalized concept of length (for intervals in $\mathbb{R}$), area in $\mathbb{R}^2$, and volume in higher dimensions of $\mathbb{R}^n$.

Definition 7. A measure μ on $(\Omega, \mathcal{F})$ is a map

$$\mu : \mathcal{F} \to [0, +\infty]$$

that is countably additive:

$$\{A_i\}_{i=1}^{\infty} \subset \mathcal{F} \text{ and } A_i \cap A_j = \emptyset \quad \forall i \neq j \implies \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

Furthermore, if $\mu(\Omega) = 1$, then μ is called a P-measure.

We will call the triple $(\Omega, \mathcal{F}, \mu)$ a measure space. If the measure is a P-measure, we will call it a probability space, $(\Omega, \mathcal{F}, P)$.

Now we define the concept of a measurable function, which we will see later is exactly what a random variable is.

Definition 8. A function $f : \Omega \to \mathbb{R}$ is measurable if the level sets

$$\{\omega \in \Omega : f(\omega) \leq \alpha\} \in \mathcal{F}$$

for all $\alpha \in \mathbb{R}$.

Example 9. Here's an example of a non-measurable function. Take the measurable space $(\Omega, \mathcal{F})$ where

$$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$$
$$\mathcal{F} = \{\emptyset, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4, \omega_5\}, \Omega\}$$

Then consider the following function $f : \Omega \to \mathbb{R}$ where

$$f(\omega) = \begin{cases} 1 & \text{if } \omega \in \{\omega_1, \omega_2\} \\ 2 & \text{if } \omega = \omega_3 \\ 3 & \text{if } \omega \in \{\omega_4, \omega_5\} \end{cases}$$

Before comparing level set, already notice that you would need to separately identify ω_3 in the function. Indeed, for $\alpha = 2$, the level set is $\{\omega_1, \omega_2, \omega_3\} \notin \mathcal{F}$. Thus f is not measurable.

Remark 10. Often in financial economics we will be confronted with the notion of time-t measurability. In the basic example, to be time-t measurable, the choice of consumption must be the same in $\{\omega_1, \omega_2\}$, and similarly the same among $\{\omega_3, \omega_4, \omega_5\}$ as the agent cannot distinguish between them. This is an information constraint with respect to the time-t filtration.

Remark 11. Why are things defined this way?

(i) Part of the motivation for using this structure is that we talk about the probability of subsets of Ω, not elements. If you have a finite set Ω, then of course you can just define a probability in Ω by defining the probability of each element of Ω. This however breaks down when you have an uncountable set – for instance for a uniform probability in $[0, 1]$ the probability of a set $\{x\}$ is zero for every $x \in [0, 1]$. We want to make sure we cover instances like this (and analyze them rigorously), so we need the extra power given by the additional mathematical structure.

(ii) We do know the size of certain sets like intervals.

(iii) The sets for which we do have a probability defined is the family $\mathcal{F}$. Those are the "measurable" sets. It is not always possible to extend the measure to the whole family of subsets of Ω, so we are happy to limit the domain of our measure to some class $\mathcal{F}$ of subsets of Ω. This is one reason why the P-measure is a function $\mu : \mathcal{F} \to [0, 1]$.

(iv) A measurable function $f : \Omega \to X$ transports the probabiliity in $(\Omega, \mathcal{F})$ to a probability in (X, Σ) (often $X = \mathbb{R}$ the reals and $\Sigma = \mathcal{B}$ the Borel σ-field). Think, for instance, when you are betting on an outcome of a coin: heads you win a dollar, tails you lose a dollar. Thinking about this leads to the next point.

(v) We want $f^{-1}(I)$ to be measurable. For instance, it might happen that we want the probabilities to be defined at least for the intervals. That is, given an interval $I \subset \mathbb{R}$, we want $f^{-1}(I)$ to have a probability associated with it.

1.6 Independence

Besides observing the evolution of an economy over time, there is a second way we might acquire information about the value of $\omega \in \Omega$. Let X be a random variable. Suppose rather than being told the value of ω, we are told only about the value of $X(\omega)$. This resolves some questions we may ask about the true state ω (but in most cases, not all). It resolves certain "sets". For instance, if we know $X(\omega)$, then we know if $\omega \in \{X \leq 0\}$[1] (yes if $X(\omega) \leq 0$ and no if $X(\omega) > 1$). In fact, every set of the form $\{X \in B\}$ where B is a subset of $\mathbb{R}$, is resolved. This idea is captured in the σ-field generated by a random variable.

[1] A short-hand for $\{\omega \in \Omega : X(\omega) \leq 0\}$

Definition 12. (σ-field generated by X) Let X be a random variable defined on a nonempty sample space Ω. The σ-algebra generated by X, denoted by $\sigma(X)$, is the collection of all subsets of Ω of the form $\{X \in B\}$, where B ranges over the Borel subsets of $\mathbb{R}$. That is

$$\sigma(X) = \{A \subset \Omega : A = \{\omega \in \Omega : X(\omega) \in B \text{ where } B \in \mathcal{B}(\mathbb{R})\}\}$$

Here $\{X \in B\}$ is just a short-hand for $\{\omega \in \Omega : X(\omega) \in B\}$.

Intuitively, an event E is in this σ-field if and only if we can determine whether E is true or false just by observing the random variable.

Let's develop the notion of independence. In contrast to the concept of measurability of a random variable, we need a probability measure to talk about independence. Consequently, independence can be affected by changes of probability measure; measurability is not.

Definition 13. ("Independence") Let $(\Omega, \mathcal{F}, P)$ be a probability space. We say that:

(i) Two sets $A, B \in \mathcal{F}$ are independent if $P(A \cap B) = P(A) \cdot P(B)$. That is, knowing the outcome ω of a random experiment A does not change our estimation of the probability that it is in B.

(ii) Two sub-σ-fields $\mathcal{G}, H$ of $\mathcal{F}$ are indepndent if

$$P(A \cap B) = P(A) \cdot B(B) \quad \forall A, \in \mathcal{G}, \forall B \in \mathcal{H}$$

(iii) Two random variables X, Y on $(\Omega, \mathcal{F}, P)$ are independent if the sigma-algebras they generate $\sigma(X)$ and $\sigma(Y)$ are independent:

$$P\{X \in C \text{ and } Y \in D\} = P(X \in C) \cdot P(Y \in D)$$

for all $C, D \in \mathcal{B}(\mathbb{R})$. That is, if we observe $X(\omega)$ (and not ω itself), then our estimation of the distribution of Y is the same as when we did not know the value of $X(\omega)$.

(iv) A random variable X is independent of the sigma-algegra $\mathcal{G}$, if $\sigma(X)$ and $\mathcal{G}$ are independent.

1.7 Expectation and the Lebesgue Integral

To develop the notion of an expectation of a random variable (which is a Lebesgue integral), we need to first define some of its building blocks.

Definition 14. An indicator function is defined as

$$\mathbb{I}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

If $A \in \mathcal{F}$, then this is a measurable function.

Definition 15. A random variable X is simple if there exists a finite partition $\{A_j\}_{j=1}^n$ of Ω with $A_j \in \mathcal{F}$ for all j and real numbers $\{x_j\}_{j=1}^n$ such that

$$X(\omega) = \sum_{j=1}^{n} x_j \mathbb{I}_{A_j}(\omega)$$

We that that X is a simple function.

Take as given some measure space $(\Omega, \mathcal{F}, \mu)$ and measurable function f over this measure space. We want to define

$$\int_{\omega \in \Omega} f(\omega)\, d\mu(\omega)$$

To do this, we start with simple functions and then generalize.

Definition 16. The Lebesgue integral

$$\int_{\omega\in\Omega} X(\omega)\, d\mu(\omega)$$

is defined as follows:

(i) If X is simple, we have $X(\omega) = \sum_{j=1}^{n} x_j \mathbb{I}_{A_j}(\omega)$ and we can define

$$\int_{\omega\in\Omega} X(\omega)\, d\mu(\omega) = \sum_{i=1}^{n} x_i \mu(A_i)$$

(ii) If X is a non-negative measurable function, then

$$\int_{\omega\in\Omega} X(\omega)\, d\mu(\omega) = \sup\left\{\int_{\omega\in\Omega} \hat{X}(\omega)\, d\mu(\omega) : \hat{X} \leq X \text{ is simple}\right\}$$

where the supremum is taken over all possible partitinos and simple functions. Note that this supremum can be infinite.

(iii) If X is an arbitrary measurable function then

$$\int_{\omega\in\Omega} X(\omega)\, d\mu(\omega) = \int_{\omega\in\Omega} X^{+}(\omega)\, d\mu(\omega) - \int_{\omega\in\Omega} X^{-}(\omega)\, d\mu(\omega)$$

(iv) In the extreme cases,

$$\int_{\omega\in\Omega} X(\omega)\, d\mu(\omega) = \begin{cases} +\infty & \text{if } \int_{\Omega} X^{+} d\mu = +\infty, \int_{\Omega} X^{-} d\mu < \infty \\ -\infty & \text{if } \int_{\Omega} X^{+} d\mu < \infty, \int_{\Omega} X^{-} d\mu = \infty \\ \text{does not exist} & \text{if } \int_{\Omega} X^{+} d\mu = \int_{\Omega} X^{-} d\mu = \infty \end{cases}$$

and the third case is called an indeterminate form.

Remark 17. In the above, we used X^{+} and X^{-}. If X is a functional (i.e. real-valued function) on Ω,

$$X^{+}(\omega) = \max\{0, X(\omega)\}$$
$$X^{-}(\omega) = \max\{0, -X(\omega)\}$$

Then it follows as a consequence that

$$X(\omega) = X^{+}(\omega) - X^{-}(\omega)$$
$$|X(\omega)| = X^{+}(\omega) + X^{-}(\omega)$$

In addition, since $|X| = X^{+} + X^{-}$, if the integral of $|X|$ is finite, then both the integrals of X^{+} and X^{-} are required to be finite. This is exactly the notino of integrability we will use.

Definition 18. A measuarable function X is Lebesuge integrable if

$$\int_{\omega\in\Omega} |X(\omega)|\, d\mu(\omega) < \infty$$

1.8 Conditional Expectation

In dynamic models, information about the true state of the world is often gradually revealed over time. Therefore, agents will form expectations over time by conditioning on progressively larger information sets. This leads to the notion of a conditional expectation.

Say we have two random variables X and Y, and want to assess $E[X|Y = y]$ which conditions on some information set generated by observing y. In other words, we want $E[X]$ given some information of Y. Again, its rigorous development starts out using simple functions.

Definition 19. Let $(\Omega, \mathcal{F}, P)$ be some probability space, and X, Y be simple two random variables so that

$$X(\omega) = \sum_{i=1}^{n} x_i \mathbb{I}_{A_i}(\omega)$$
$$Y(\omega) = \sum_{j=1}^{m} y_j \mathbb{I}_{B_j}(\omega)$$

where again both $\{A_i\}_{i=1}^n$ and $\{B_j\}_{j=1}^m$ form some finite partitions of Ω with each $A_i, B_j \in \mathcal{F}$. Then, the conditional expectation is defined as

$$\begin{aligned} E(X|Y = y_j) &= \sum_{i=1}^{n} x_i P(X = x_i | Y = y_j) \\ &= \sum_{i=1}^{n} x_i \frac{P(X = x_i, Y = y_j)}{P(y = y_j)} \\ &= \sum_{i=1}^{n} x_i \frac{P(\omega \in (A_i \cap B_j))}{P(\omega \in B_j)} \end{aligned}$$

Remark 20. Note this object depends on the value y_j takes, which is again dependent on ω. Thus, it is a random variable, and we can write it as such: $Z = E[X|Y]$ or

$$\begin{aligned} Z(\omega) &= E[X|Y = Y(\omega)] \\ &= \sum_{j=1}^{m} E[X|Y = y_j] \mathbb{I}_{B_j}(\omega), \end{aligned}$$

again assuming that both X and Y are simple functions. We previously defined what the conditional expectation is for $Y = y_j$– this defines it for a general $Y = Y(\omega)$.

The above random variable has the following property: $\forall j = 1, \cdots, m$,

$$\begin{aligned} E\left[Z\mathbb{I}_{B_j}\right] &= E[X|Y = y_j] P(Y = y_j) \\ &= \sum_{i=1}^{n} x_i P(X = x_i, Y = y_j) \\ &= E\left[X\mathbb{I}_{B_j}\right] \\ &= \int_{\omega \in \Omega} X(\omega) \mathbb{I}_{B_j}(\omega) \, d\mu(\omega) \end{aligned}$$

That is, the expected value of Z multiplied by the indicator function is the same as the expected value of X multiplied by that indicator function.

Remark 21. Here's another way to think about conditional expectations. Let $\mathcal{F} = \mathcal{F}^Y$, i.e. the smallest σ-field on which Y is measurable (remember Y is still a simple function). Then,

$$\mathcal{F}^Y = \left\{ \bigcup_{j \in J} B_j : J \subset \{1, \cdots, m\} \right\}$$

So for example say $m = 3$, so that Y can only take three different values over Ω, y_1,y_2, and y_3. Take the corresponding partition of Ω to be $\{B_j\}_{j=1}^3$. First it's a partition so $\cup_{j=1}^3 B_j = \Omega$ and $B_i \cap B_k = \emptyset$ for $i \neq k$. Then, in this case $\mathcal{F}^Y = \{\emptyset, B_1, B_2, B_3, B_1 \cup B_2, B_2 \cup B_3, B_1 \cup B_3, \Omega = B_1 \cup B_2 \cup B_3\}$. You can and should check that for any $m > 0$ this is itself a σ-field.

Back to conditional expectations. This means that $\forall G \in \mathcal{F}^Y$,

$$E[Z\mathbb{I}_G] = E[X\mathbb{I}_G]$$

as it holds for $G = B_j's$, it holds for their unions (which are also in $\mathcal{F}^Y$). Thus

$$E\left[(Z - X)\,\mathbb{I}_G\right] = 0$$

What can we make of this? This says that Z is the best approximation of X using a function measurable in the information set generated by Y . Z is the best guess of X if you just consider the information in Y.

Definition 22. Let $\mathcal{G}$ be a sub σ-field of $\mathcal{F}$. Then, $Z = E\left[X|\,\mathcal{G}\right]$ is defined as the (unique) random variable that is $\mathcal{G}$-measurable with

$$E\left[Z\mathbb{I}_A\right] = E\left[X\mathbb{I}_A\right], \quad \forall A \in \mathcal{G}$$

Remark 23. Couple of comments about this $\mathcal{G}$ sub σ-field and the conditional expectation:

(i) If $\mathcal{G} = \{\emptyset, \Omega\}$, then $E\left[X|\,\mathcal{G}\right] = E\left[X\right]$. Here $\mathcal{G}$ gives no (additional) information about X, as a random variable must take constant values (i.e. it's a constant function) to be measurable.

(ii) If $X \in \mathcal{G}$, then $E\left[X|\,\mathcal{G}\right] = E\left[X\right]$. Here $X \in \mathcal{G}$ means X is measurable on $\mathcal{G}$. That is, knowing $\mathcal{G}$ gives you the value of X.

(iii) If we have σ-fields $\mathcal{G}_1 \subset \mathcal{G}_2$, then

$$\begin{aligned} E\left[E\left(X|\,\mathcal{G}_2\right)|\,\mathcal{G}_1\right] &= E\left[E\left(X|\,\mathcal{G}_1\right)|\,\mathcal{G}_2\right] \\ &= E\left[X|\,\mathcal{G}_1\right] \end{aligned}$$

As $E\left(X|\,\mathcal{G}_2\right)$ is a random variable, you can take its expectation. This is known as the law of iterated expectations.

Definition 24. ("Conditional Expectation") Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $\mathcal{G}$be a sub-sigma-algebra of $\mathcal{F}$, and let X be a random variable that is either nonnegative or integrable. The conditional expectation of X given $\mathcal{G}$, denoted $E\left[X|\,\mathcal{G}\right]$, is any random variable that satisfies:

(i) (Measurability) $E\left[X|\,\mathcal{G}\right]$ is $\mathcal{G}$-measurable, and

(ii) (Partial averaging)

$$\int_A E\left[X|\,\mathcal{G}\right](\omega)\,dP(\omega) = \int_A X(\omega)\,dP(\omega), \quad \forall A \in \mathcal{G}$$

If $\mathcal{G}$ is the sigma-algebra generated by some other random variable W (i.e. $\mathcal{G} = \sigma(W)$), we generally write $E\left[X|\,W\right]$ rather than $E\left[X|\,\sigma(W)\right]$.

Remark 25. The first property guarantees that, although the estimate of X based on the information in $\mathcal{G}$ is itself a random variable, the value of the estimate $E\left[X|\,\mathcal{G}\right]$ can be determined from the information in $\mathcal{G}$. It captures the fact that the estimate of $E\left[X|\,\mathcal{G}\right]$ of X is *based on the information in* $\mathcal{G}$. Among the "atoms" of $\mathcal{G}$, the conditional expectation $E\left[X|\,\mathcal{G}\right]$ is constant .

The second property ensures that $E\left[X|\,\mathcal{G}\right]$ is indeed an estimate of X. It gives the same averages as X over all the sets in $\mathcal{G}$. If $\mathcal{G}$ has many sets, which provide a fine resolution of the uncertainty inherent in ω, then this partial-averaging property over the "small" sets in $\mathcal{G}$ says that $E\left[X|\,\mathcal{G}\right]$ is a good estimator of X. If $\mathcal{G}$ has only a few sets, this partial-averaging property guarantees only that $E\left[X|\,\mathcal{G}\right]$ is a crude estimate of X.

This random variable satisfying the conditions above (1) always exists by the Random-Nikodym Theorem; and (2) is unique up to an equivalence class (i.e. if Y, Z satisfy both conditions, then $Y = Z$ almost surely).

1.9 Radon-Nikodym Derivative

Here we review Radon-Nikodym derivative, which is used for the operation of changing measures. In financial economics, this is used to get to risk-neutral measures from physical measures (and vice versa). To apply the central theorem (called the Radon-Nikodym Theorem), which asserts the existence of the Radom-Nikodym derivative, we need the notion of absolute continuity. We turn to that definition now.

Let P, Q be probability measures on the measurable space $(\Omega, \mathcal{F})$.

Definition 26. A measure Q is absolutely continuous with respect to another measure P if

$$\forall A \in \mathcal{F}, P(A) = 0 \implies Q(A) = 0$$

which we denote $Q \ll P$. If $Q \ll P$ and $P \ll Q$, then we write $Q \sim P$.

Remark 27. The intuition here is that both measures "agree" on the sets of measure zero (or events of zero probability). If $Q \ll P$, then whichever events P assigns a zero probability to, so does Q.

Remark 28. Suppose you just have the measure space $(\Omega, \mathcal{F}, P)$ and would like to construct some other measure Q such that $Q \ll P$. There is actually a simple way to do this. Let X be any random variable with $X(\omega) \geq 0$ almost surely and $E^P[X] = 1$. Then,

$$Q(A) = E^P[X \mathbb{I}_A] = \int_{\omega \in A} X(\omega)\, dP(\omega), \quad \forall A \in \mathcal{F}$$

is a probability measure with $Q \ll P$. Check that this a probability measure on $(\Omega, \mathcal{F})$ and that P is absolutely continuous with respect to Q.

The next theorem states that (1) this non-negative X exists; and (2) Q must have this form.

Theorem 29. *("Radon-Nikodym") Let P and Q be two probability measures with $Q \ll P$. Then there exists a random variable $X \geq 0$ and $E^P(X) = 1$ such that*

$$Q(A) = E^P[X \mathbb{I}_A]$$

and we write $X = \frac{dQ}{dP}$.

Remark 30. Several comments are in order.

(i) Two random variables X and Y are in an equivalence class if they only differ on sets of measure zero.

(ii) If $P \sim Q$ then both $\frac{dP}{dQ}$ and $\frac{dQ}{dP}$ exist and $\frac{dP}{dQ} = \left(\frac{dQ}{dP}\right)^{-1}$ almost surely, and in this case $\frac{dQ}{dP} > 0$ a.s.

(iii) For any random variable X,

$$\begin{aligned} E^Q[X] &= \int_{\omega \in \Omega} X(\omega)\, dQ(\omega) = \int_{\omega \in \Omega} X(\omega) \frac{dQ}{dP}(\omega)\, dP(\omega) \\ &= E^P\left[X \frac{dQ}{dP}\right] \end{aligned}$$

(iv) Conditional Bayes Rule:

$$E^Q[X|\mathcal{G}] = \frac{E^P\left[X \frac{dQ}{dP}\middle|\mathcal{G}\right]}{E^P\left[\frac{dQ}{dP}\middle|\mathcal{G}\right]}$$

(Proof of Conditional Bayes Rule)

Example 31. An example is $\frac{dP}{dQ} = \frac{q(\omega)}{p(\omega)}$ for finite states $\{\omega_i\}_{i=1}^n$.

1.10 L^p Spaces

We now introduce some normed vector spaces of functions that will be often used in our subsequent analysis.

Definition 32. Let $(\Omega, \mathcal{F}, P)$ be a probability space. For any $p \in [0, \infty)$, a random variable X is said to belong to the space $L^p(\Omega, \mathcal{F}, P)$ if

$$E|X|^p < \infty$$

A random variable X is said to belong to the spaec $L^\infty(\Omega, \mathcal{F}, P)$ if

$$\text{ess sup}\, |X(\omega)| < \infty$$

where

$$\text{ess sup}\, |X(\omega)| = \inf\{\alpha \in \mathbb{R} : |X(\omega)| < \alpha \text{ a.s.}\}$$

denotes the *essential supremum* of X.

2 Stochastic Processes

2.1 Continuous-Time Stochastic Processes

Let $[0, T]$ for $T < \infty$. A filtered probability space is $(\Omega, \mathcal{F}, F, P)$ where $F = \{\mathcal{F}_t\}_{t \in [0,T]}$ is a filtration with $\mathcal{F}_T = \mathcal{F}$. Note that P does not change, and represents the common knowledge of beliefs.

Definition 33. A stochastic process X is a family of random variables $X = \{X_t\}_{t \in [0,T]}$ indexed by time.

Remark 34. We can view this as a mapping $X : \Omega \times [0, T] \to \mathbb{R}$ with $X(\omega, t)$ denoting the value of process at time t in state ω.

- Fix a state $\omega \in \Omega$. Then, $\{X(\omega, t) : t \in [0, T]\}$ is called a sample path of X, and is denoted by $X(\omega, \cdot)$.
- For each $t \in [0, T]$, $\{X(\omega, t) : \omega \in \Omega\}$ is a random variable itself, and is denoted by X_t.

If we want to model variables that agents choose or observe with respect to t, then we need measurability with respect to $\mathcal{F}_t$ and well as $\mathcal{F}$:

Definition 35. A stochastic process $X = \{X_t\}_{t \in [0,T]}$ is adpated to the filtration $F = \{\mathcal{F}_t\}_{t \in [0,T]}$ if $X_t \in \mathcal{F}_t$ for each $t \in [0, T]$. That is, we need each X_t at each point in time to be measurable with respect to $\mathcal{F}_t$.

Remark 36. Fubini's theorem allows us to change the order of integration

$$\int_0^T E(X_t)\, dt = E\left[\int_0^T X(s, \omega)\, ds\right]$$

This says that the integral of the expected value of each random variable X_t is the same as the expected value over all integrated sample paths.

To use Fubini's theorem, we need the condition of progressively measurable. Adaptedness is not enough! Note every adapted process with left- or right-continuous paths is progressively measurable. The notion of predictability is even stronger. To summarize,

$$\text{adaptedness} \prec \begin{array}{c}\text{progressively}\\ \text{measurable}\end{array} \prec \text{predictability}$$

in the order of increasing strength. Predictability matters for processes with jumps. Every continuous-time adapted process that is left continuous is a predictable process.

2.2 Stopping Times

Now we define a particular type of random variable called a stopping time. It is a random variable whose value is interpreted as the time at which a given stochastic process exhibits a certain behavior of interest. It is not only important in applications, but also in mathematical proofs where it is often used to "tame the continuum of time."

Definition 37. A stopping time iss a map $\tau : \Omega \to [0, T] \cup \{+\infty\}$ such that

$$\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t, \quad \forall t \in [0, T]$$

Say you have some event, which you typically don't know when occurs, but that can/will occur some time in the future. The time that this event occurs is random, and it is a stopping time if, at any point in time, you know whether the event has occured or not. Generally speaking, if it requires information about the future as well as the present and the past, it is *not* a stopping time.

This object is fairly general. All that's required of mapping τ is that you know whether or not it has occurred at time-t. Thus a random variable τ is a stopping time if it chooses its value based on the information up to time τ (e.g. path of an asset price up to time τ). For instance, if X_t is the stock price, then the time that X_t hits its maximum on the interval $[0, T]$ is not a stopping time, because you also need future information to make this call.

Definition 38. Let X_t be a stochastic process and τ a stopping time. The stoppped stochastic process X^τ is defined by

$$X^\tau(\omega, t) = X(\omega, t \wedge \tau(\omega))$$

2.3 Martingales

Here we focus our attention to a special class of stochastic processes called a martingale.

Definition 39. A stochastic process X with $E^P[|X_t|] < \infty$ for all $t \in [0, T]$ is called:

(i) A martingale if $\forall s \geq t$, $E^P[X_s | \mathcal{F}_t] = X_t$

(ii) A submartingale if $\forall s \geq t$, $E^P[X_s | \mathcal{F}_t] \geq X_t$

(iii) A martingale if $\forall s \geq t$, $E^P[X_s | \mathcal{F}_t] \leq X_t$

A martingale captures the idea of a stochastic process with no drift; a submartingale a process with positive drift; and a supermartingale a process with negative drift. Note that if X is a submartingale, then $-X$ is a supermartingale.

Next we present the optional sampling theorem. It is a powerful and extremely useful result about stopping times for martingales and submartingales. The theorem has many forms and applies to a wide class of stochastic processes.

Roughly, the theorem says that if a process X is a (sub)martingale and T is a stopping time, then the stopped process $X(T \wedge t)$ is also a (sub)martingale. Moreover, the exepcted value of the stopped process at any date t (is bounded below by) is equal to the expectation of the initial value $X(0)$, and (is bounded above by) is equal to the expected value of the unstopped process at t.

Theorem 40. *("Optional Sampling Theorem") Let X be a right-continuous submartingale and τ a stopping time with $t \leq \tau \leq T$ almost surely. Then,*

$$E[X_\tau | \mathcal{F}_t] \geq X_t$$

If X is a marginale, then

$$E[X_\tau | \mathcal{F}_t] = X_t$$

almost surely.

The definition of a martingale can be "localized" by using a sequence of stopping times.

Definition 41. A process X is a *local martingale* if there exists a sequence of stopping times $\{\tau_n\}$ with $\tau_n \uparrow T$ almost surely such that each of the stopped process X^{τ_n} is a martingale.

Note we write $\tau_n \uparrow T$ if

$$\tau_1(\omega) \leq \tau_2(\omega) \leq \cdots \leq \tau_n(\omega) \leq T, \quad \forall \omega \in \Omega$$

In discrete time, a local martingale and a martingale are the same. In continuous-time this is not true.

Next theorem gives us a way to characterize (right-continuous) supermartingales

Theorem 42. *("Doob-Meyer Decomposition") Let X be a right-continuous supermartingale. Then X admits a unique decomposition of the form*

$$X = M - A$$

where M is a local martingale and A is a predictable increasing process with $A_0 = 0$ and $E[A_T] < \infty$.

2.4 Brownian Motion

Definition 43. A continuous, adapted stochastic process w is a standard Brownian Motion if

(i) $w_0 = 0$

(ii) $\forall s, t$ with $0 \leq t \leq s \leq T$, we have that

$$w_s - w_t \sim N(0, s-t)$$

and independent of $\mathcal{F}_t$.

Remark 44. Some comments:

- Note that w is a martingale as $E^P[w_s - w_t | \mathcal{F}_t] = 0$. In fact, w is a L^P-martingale: The process is a martingale and the value of the process at each time is in L^P,

$$E^P[|w_t|^p] = E^P[|w_t - w_0|^p] < \infty$$

  as $t - 0 \sim N(0, t)$.

- w has stationary, independent increments. Stationary, as its distributional property only depends on the length of time.

- In discrete time, a stochastic process with stationary and independent increments is a random walk.

Definition 45. A stochastic process X is a generalized Brownian motion if

$$X_t = X_0 + \mu t + \sigma w_t$$

for some $X_0, \mu, \sigma \in \mathbb{R}$ and w_t a standard Brownian motion.

σ is the volatility coefficient, and controls how much the process oscillates. If $\sigma > 1$ it amplifies; if $\sigma < 1$ it damps. μ controls the trend. If $\mu > 0$ it has a positive drift (i.e. a submartingale) and if $\mu < 0$ it has a negative drift (i.e. a supermartingale). X_0 gives the intitial value, which was previous always zero for the standard Brownian motion.

As a short hand, we can also write

$$dX_t = \mu dt + \sigma dw_t$$

which is called the **differential form**. In discrete time, Δw is a random variable; in continuous time dt is not a random variable.

We said that a Brownian motion has stationary and independent increments; the next proposition states the converse.

Proposition 46. *If X is a continuous stochastic process with stationary and independent increments, then X is a generalized Brownian Motion.*

What is the intuitive reason behind this? For discrete t, this is not true. $X_{t+1}-X_t = \Delta X$ can take any distribution (e..g Binominal, etc.). For continuous t, the increment must take the normal distribution by the Central Limit Theorem. For any finite interval $[t, t+s]$, we can cut up the interval into infinitely many little increments that are each i.i.d. Taking the whole interval then becomes the Brownian motion.

Definition 47. A n-dimensional Brownian motion is a vector-valued process

$$w = (w_1, \cdots, w_n)^\top$$

where w_i is a Brownian motion for each $i \in \{1, \cdots, n\}$and w_i independent of $w_j$for all $j \neq i$.

2.5 Stochastic Integration.

Suppose we are interest ed in the object

$$I_t^\theta = \int_0^t \theta(s)\, dw(s)$$

We can think of $\theta(\cdot)$ as some trading strategy with w representing stock price. Then, the above object is also a random variable with

$$I_t^\theta(\omega) = \int_0^t \theta(\omega, s)\, dw(\omega, s), \quad \forall \omega \in \Omega$$

Once you fix a $\omega \in \Omega$, the above becomes a Riemann-Stieltjes integral. But for the RS integral to be well-defined, you need continuity and finite variation of at least one function at each point (i.e. either $\theta(\omega, \cdot)$ or $w(\omega, \cdot)$). And this is what we will check.

Proposition 48. *Let w be a standard Brownian motion. Then*

$$\lim_{n\to\infty} \sum_{i=0}^{2^n-1} \left| w\left(\frac{i+1}{2^n}t\right) - w\left(\frac{i}{2^n}t\right)\right| = +\infty, \quad a.s.$$

that is, it has infinite total variation almost surely. On the other hand,

$$\lim_{n\to\infty} \sum_{i=0}^{2^n-1} \left| w\left(\frac{i+1}{2^n}t\right) - w\left(\frac{i}{2^n}t\right)\right|^2 = t, \quad a.s.$$

it has a quadratic variation equal to the length of the interval.

It what follows we will aim to understand what class of stochastic processes $\theta(\cdot)$ we can define the stochastic integral. We will start with the most restrictive case, and seek to expand the class of processes. Here's a road map:

$$\begin{matrix}\text{processes of} \\ \text{finite variation}\end{matrix} \subset \mathcal{H}_0^2 \subset \mathcal{H}^2 \subset \mathcal{L}^2$$

We will define everything soon.

Recall for a Riemann-Stieltjes integral to be well-define, we need continuity and finite variation. If we take θ to be of finite variation, then this is sufficient to guarantee the stochastic integral is well-defined. However, this is far too restrictive for the setting we would be interested in.

Definition 49. The set of proesses $\mathcal{H}^2$ is

$$\mathcal{H}^2 := \left\{\theta : \theta \text{ is adapted and } E\left[\int_0^T |\theta_t|^2\, dt\right] < \infty\right\}$$

The $\mathcal{H}^2$ norm is

$$\|\theta_t\|_{\mathcal{H}^2} = \left(E\left[\int_0^T \theta_t^2 dt\right]\right)^{1/2}$$

Definition 50. A stochastc process θ is simple if there exists a finite partition $0 = t_0 < t_1 < \cdots < t_N = T$ of $[0,T]$ and random variables $\{\theta_j\}_{j=0}^{N}$with each $\theta_j \in \mathcal{F}_{t_{j-1}}$ for each j such that

$$\theta(t) = \begin{cases} \theta_0 & \text{if } t = 0 \\ \theta_j & \text{if } t \in (t_{j-1}, t_j] \end{cases}$$

Note the analogy of simple processes to simple functions. Sample paths of simple processes are step functions!

Definition 51. The set of simple processes is called $\mathcal{H}_0^2$:

$$\mathcal{H}_0^2 = \left\{\theta \in \mathcal{H}^2 : \theta \text{ is simple}\right\}$$

If $\theta \in \mathcal{H}_0^2$, then it has finite variation and our integral is defined:

$$\begin{aligned} I^\theta(\omega, t) &= \int_0^t \theta(\omega, s)\, dw(\omega, s) \\ &= \sum_{j=1}^{N} \theta_j(\omega)\left[w(\omega, t_j \wedge t) - w(\omega, t_{j-1} \wedge t)\right] \end{aligned}$$

To see what the minimum operators are doing, notice you only consider the intervals up to the last increment that includes t. So if $t \in (t_{k-1}, t_k]$ for some k, then the last term in the summation is

$$\cdots + \theta_k(\omega)\left[w(\omega, t) - w(\omega, t_{k-1})\right] + \underbrace{\sum_{j=k+1}^{N} \theta_j(\omega)\left[w(\omega, t) - w(\omega, t)\right]}_{=0}$$

Proposition 52. *(Properties of stochastic integral with $\mathcal{H}^2$ integrands) Let $\theta, \theta_1, \theta_2 \in \mathcal{H}_0^2$. Then,*

(i) I^θ is adpated to F.

(ii) I^θ is continuous.

(iii) $I^\theta$is a L^2-martingale

(iv) $E\left[\left(\int_0^t \theta_1(s)\, dw_s\right)\left(\int_0^t \theta_2(s)\, dw_s\right)\right] = E\left[\int_0^t \theta_1(s)\,\theta_2(s)\, dw_s\right]$

Remark 53. As a side remark, why is it that $\mathcal{H}^2 \subset \mathcal{L}^2$? This is because

$$E|X| < \infty \implies X < \infty \text{ almost surely}$$

But the other way around is not true. The show this, take some random variable X with finite expectation, $E|X| < \infty$. By Markov's inequality,

$$tP(X \geq t) \leq E[X] < \infty$$

for any $t \geq 0$. Take the limit as $t \to \infty$, the left-hand side remains finite, so it must be that

$$0 = \lim_{t\to\infty} P(X \geq t) = P(X = \infty)$$

or $X < \infty$ almost surely.

However, a random variable being finite almost surely does not imply finite moment. Define Y to be a random variable that is equal to 2^n with probability 2^{-n} for positive integer n. Then,

$$E[X] = \sum_{n=1}^{\infty} 2^{-n} \cdot 2^n = \sum_{n=1}^{\infty} 1 = \infty$$

Now we would like to stretch the definition. How do we go from $\mathcal{H}_0^2$ (the space of simple processes) to $\mathcal{H}^2$? We do this by an approximation property: Any process in $\mathcal{H}^2$ can be approximated arbitrarily closely by a sequence of $\mathcal{H}_0^2$ processes where the convergence is in $\mathcal{H}^2$-norm.

Proposition 54. *If $\theta \in \mathcal{H}^2$, then there exists a sequence $\{\theta_k\}_{k=1}^{\infty} \subset \mathcal{H}_0^2$ such that $\|\theta_k - \theta\|_{\mathcal{H}^2} \to 0$ as $k \to \infty$.*

In addition, we have that L^2-space is complete, so that

$$\lim_{k\to\infty} I_t^{\theta_k} = I_t^{\theta}$$

Notice that this is no longer a path-by-path integral, but instead it is a limit in L^2. The previous properties also hold for $\mathcal{H}^2$ as well as $\mathcal{H}_0^2$.

Now we define an even larger space of stochastic processes:

$$\mathcal{L}^2 = \left\{ \theta : \theta \text{ is adapated and } \int_0^T |\theta_t|^2 \, dt < \infty \text{ a.s.} \right\}$$

where $\mathcal{H}^2 \subset \mathcal{L}^2$ because $X < \infty$ a.s. is less restrictive than $E(|X|) < \infty$ a.s.

How do we make the next step and move to $\mathcal{L}^2$ for the integrands of stochastic integrals? We do this using stopping times. Suppose $\theta \in \mathcal{L}^2$. Define

$$\tau_m = \inf\left\{ t \in [0,T] : \int_0^t |\theta_s|^2 \, ds \geq m \right\} \wedge T$$

which is a stopping time. Roughly, τ_m is the first time the time-path integral of the squared process reaches m. Also define

$$\theta^n(t,\omega) = \theta(t,\omega)\, \mathbb{I}\{t \in [0,\tau_n]\} \in \mathcal{H}^2$$

$\theta^n(t,\omega)$ is some process that we define in relation to our process $\theta \in \mathcal{L}^2$. It is in $\mathcal{H}^2$ for any n because

$$E\left[\int_0^T |\theta_t^n|^2 \, dt\right] = E\left[\int_0^{\tau_n} |\theta_t|^2 \, dt\right] \leq n < \infty$$

where the inequality holds with probability 1. Thus, no matter what the n value is, $\theta^n \in \mathcal{H}^2$.

Now we take $n \to \infty$, and as we do that, we have $\tau_n \uparrow T$, and $I_t^{\theta} = \lim_{n\to\infty} I_t^{\theta_n}$. As you do this, we have the following important property:

Theorem 55. *If $\theta \in \mathcal{L}^2$, then $I_t^{\theta} = \int_0^T \theta_s dw_s$ is a local martingale. If $\theta \in \mathcal{H}^2$, then I_t^{θ} is a L^2 martingale.*

Other properties of the proposition also go through:

Proposition 56. *(Properties of stochastic integral with $\mathcal{L}^2$ integrands) Let $\theta, \theta_1, \theta_2 \in \mathcal{L}^2$. Then,*

(i) I^{θ} is adpated to F.

(ii) I^{θ} is continuous.

(iii) I^{θ} is a local martingale.

Extending to n-dimensions, say we have

$$w = (w_1, \cdots, w_n)$$
$$\theta = (\theta_1, \cdots, \theta_n)$$

then as a short-hand notation we write

$$\int_0^t \theta_s^{\top} dw_s = \sum_{i=1}^{n} \int_0^t \theta_i(s)\, dw_i(s)$$

Note in the above we had that $\theta \in \mathcal{L}^2$ implies that $\int_0^t \theta(s)\, dw(s)$ is a local martingale, where we can define

$$X_t = X_0 + \int_0^t \theta(s)\, dw(s)$$

Writing it this way does not affect increments. The following martingale representation theorem says something different: If we have a local martingale X_t then it must be of the form above!

Theorem 57. *(Martingale Representation Theorem) Let*

- *w be a n-dimensional Brownian motion*
- *F^w be the filtration generated by w*
- *X be a local martingale adapted to F^w*

Then there exists $\theta \in \mathcal{L}^2$ such that

$$X_t = X_0 + \int_0^t \theta_s^\top dw_s$$

This is an extremely useful theorem, and will be used repeatedly. The conditions are important, and the second bulletpoint is especially easy to miss when we we have changes of measures. For instance if w is a P-measure Brownian motion, and w^* is a Q-measure Brownian motion, then to make this representation for some local martingale X in Q, the stochastic integral above must be with respect to w^*. Make sure you are taking the right filtration.

For strictly positive local martingales, we have another characterization.

Corollary 58. *Let w, F^w be as above, and X a strictly positive local martingale adapted to F^w. Then, there exists $\theta \in \mathcal{L}^2$ such that*

$$X_t = X_0 \exp\left(\int_0^t \theta_s^\top dw_s - \frac{1}{2}\int_0^t |\theta_s|^2\, ds\right)$$

2.6 Ito Processes

We might think generalized Brownian motinos are too restrictive. In particular, the drift and volatility terms μ and σ are constant in

$$X_t = X_0 + \mu t + \sigma w_t$$

We want to think about time-varying drift and volatility terms.

Definition 59. A stochastic process X is an Ito process if there exist stochastic processes μ and σ, and some $X_0 \in \mathbb{R}$ such that

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s^\top dw_s$$

where $\sigma \in \mathcal{L}^2$ $\left(\int_0^T |\sigma_t|^2\, dt < \infty\right)$ and $\mu \in \mathcal{L}$ $\left(\int_0^T |\mu_t|\, dt < \infty\right)$.

To think about when the above integrals are well-defined, look at $\int_0^t \mu_s ds$ for instance. It is a well-defined path by path (ω by ω) Lebesgue integral, when each path is Lebesgue integrable. In differential form we write

$$dX_t = \mu_t dt + \sigma_t^\top dw_t$$

which is just a short hand notation. Here the μ_t and σ_t terms are explicitly given, and defines X explicitly!

But we can also define it implicitly, letting drift and volatility depend on X_t, which is called a **diffusion**:

$$X_t = X_0 + \int_0^t \mu(X_s, s)\,ds + \int_0^t \sigma(X_s, s)^\top dw_s$$

What is this object? Consider the case where $\sigma_t = 0$. Then,

$$dX_t = \mu(X_t, t)\,dt \iff \frac{dX_t}{dt} = \mu(X_t, t)$$

which is a first order differential equation (deterministic). Thus,

$$dX_t = \mu(X_t, t)\,dt + \sigma(X_t, t)^\top dw_t$$

is a stochastic differential equation. A solution has the form

$$X_t = f(t, X_0, \omega)$$

Let's look at an example.

Example 60. Geometric Brownian motion has

$$dX_t = \mu X_t dt + \sigma X_t dw_t$$

where

$$\begin{aligned} \mu(t, X_t) &= \mu X_t \\ \sigma(t, X_t) &= \sigma X_t \end{aligned}$$

form the stochastic differential equation. The solution is an explicit expression of X_t:

$$X_t = X_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma w_t\right\}$$

Remark 61. To be clear, diffusions are more restrictive: a diffusion is a special case of a general Ito process. This is because μ and σ only depend on ω via contemporaneous values of X_t.

The next step is to look at functions of Ito processes X_t such as

$$Y_t = f(X_t, t)$$

What kind of process is Y_t given the function $f \in C^{2,1}$ (that is, twice continuously differentiable with respect to X_t, and once so in t). Then, Y_t is yet another Ito process. This is the characterization given by the famous Ito's Lemma.

Theorem 62. *(Ito's Lemma) Let X be an Ito process and $f \in C^{2,1}$. Then,*

$$Y_t = f(X_t, t)$$

is an Ito process and has the following form:

$$\begin{aligned} Y_t &= f(X_0, 0) + \int_0^t \left[f_x(X_s, s)\,dX_s + f_t(X_s, s)\,ds + \frac{1}{2} f_{xx}(X_s, s)\,|\sigma_s|^2\,ds\right] \\ &= f(X_0, 0) + \int_0^t \left[f_x(X_s, s)\,\mu_s + f_t(X_s, s) + \frac{1}{2} f_{xx}(X_s, s)\,|\sigma_s|^2\right] ds \\ &+ \int_0^t f_x(X_s, s)\,\sigma_s^\top dw_s \end{aligned}$$

where the drift term $\mu_s^Y = f_x(X_s, s)\,\mu_s + f_t(X_s, s) + \frac{1}{2} f_{xx}(X_s, s)\,|\sigma_s|^2$ and $\sigma_s^Y = f_x(X_s, s)\,\sigma_s^\top$.

This says the if $Y_t = f(X_t, t)$, then Y_t is equal to the initial value and the sum of incremental changes over time. The above is kind of confusing. Notationally compact way to do this is below.

Theorem 63. *(Ito's Lemma : Cookbook formula) Say you know dX_t and $Y_t = f(X_t, t)$. Here's how you get dY_t:*

$$dY_t = f_t dt + f_x dX_t + \frac{1}{2} f_{xx} (dX_t)^2$$

which is more intuitive and resembles a second order Taylor expansion.

Let's extend to the multi-dimensional case, keeping close track of the dimensions. You have an n-dimensional Brownian motion, and m-dimensional Ito process X_t with

$$X = (X_1, \cdots, X_m)^\top$$

where each X_i is an Ito process for $i = 1, \cdots, m$. Compactly, we can write

$$X(t) = X(0) + \int_0^t \mu(s)\, ds + \int_0^t \sigma(s)\, dw(s)$$

(note last integral has matrix multiplication in it) for some m-dimensional, progressive process μ and some $m \times n$-dimensional, predictable process σ that satisfy

$$\int_0^T |\mu(t)|\, dt < \infty \text{ a.s.}$$
$$\int_0^T |\sigma(t)|^2\, dt < \infty \text{ a.s.}$$

where $|\sigma|^2 = \text{tr}\left(\sigma\sigma^\top\right)$, the trace of a square matrix. Here's a multi-dimensional Ito's formula.

Theorem 64. *(Multidimensional Ito's Lemma) Let X be an m-dimensional Ito process with the above representation, and let $f : \mathbb{R}^m \times [0, T] \to \mathbb{R}$ be twice continuously differentiable in its first m arguments and continuously differentiable in its last argument. Then $f(X_t, t)$ is also an Ito process and can be expressed explicity as*

$$\begin{aligned}
f(X_t, t) &= f(X_0, 0) + \int_0^t f_X(X_s, s)^\top dX_s + \int_0^t \left(\frac{1}{2} tr\left[f_{xx}(X_s, s)\,\sigma_s\sigma_s^\top\right] + f_t(X_s, s)\right) ds \\
&= f(X_0, 0) + \int_0^t f_X(X_s, s)^\top \sigma_s dw_s \\
&\quad + \int_0^t \left(\frac{1}{2} tr\left[f_{xx}(X_s, s)\,\sigma_s\sigma_s^\top\right] + f_X(X_s, s)^\top \mu_s + f_t(X_t, s)\right) ds
\end{aligned}$$

for all $t \in [0, T]$, almost surely.

The above formula is a little formidable, very general (in terms of applicable dimensions), and practically useless. Especially, what is this $tr\left[f_{xx}(X_s, s)\,\sigma_s\sigma_s^\top\right]$ term? f_{xx} is a $m \times m$ dimensional matrix (that looks like a Hessian of second partial derivatives excluding the time dimension), $\sigma_s\sigma_s^\top$ is some $m \times m$ square matrix. All this is doing is collecting some terms off the second order partial derivative terms of the Taylor expansion, making sure to collect the volatility terms along the way.

Again I provide a "cookbook" formula for actual problem solving:

Theorem 65. *(Multidimensional Ito's Lemma: Cookbook) With some conditions as above,*

$$df(X_t, t) = f_t dt + \sum_{k=1}^m f_k dX_t^k + \frac{1}{2}\sum_{j=1}^m \sum_{j=1}^m f_{jk}\left(dX_t^j\right)\left(dX_t^k\right)$$

For instance with $f\left(X_t, Y_t, Z_t, t\right)$,

$$\begin{aligned} df &= f_t dt + f_x dX_t + f_y dY_t + f_z dZ_t \\ &+ \frac{1}{2} f_{xx} \left(dX_t\right)^2 + \frac{1}{2} f_{yy} \left(dY_t\right)^2 + \frac{1}{2} f_{zz} \left(dZ_t\right)^2 \\ &+ f_{xy} \left(dX_t\right)\left(dY_t\right) + f_{yz} \left(dY_t\right)\left(dZ_t\right) + f_{xz} \left(dX_t\right)\left(dZ_t\right) \end{aligned}$$

Exactly in the same way one would Taylor expand.

2.7 Quadratic Variation and Covariation

Now let us characterize two objects that one should know when studying stochastic calculus. When we take arbitrarily fine partitions over the time path of an Ito processes and sum up the (vertical) squared increments, we get a random obect.

Proposition 66. *Let X be such that $X_t = X_0 + \int_0^t \mu_s^X ds + \int_0^t \left(\sigma_s^X\right)^\top dw_s$. Then,*

$$\lim_{n\to\infty} \sum_{i=0}^{2^n-1} \left[X\left(\frac{i+1}{2^n} t\right) - X\left(\frac{i}{2^n} t\right) \right]^2 = \int_0^t \left|\sigma^X(s)\right|^2 ds$$

which we write $[X,X]_t$. This is called the quadratic variation of the Ito proces X. If you have another process Y such that $Y_t = Y_0 + \int_0^t \mu_s^Y ds + \int_0^t \left(\sigma_s^Y\right)^\top dw_s$, then

$$\lim_{n\to\infty} \sum_{i=0}^{2^n-1} \left[X\left(\frac{i+1}{2^n} t\right) - X\left(\frac{i}{2^n} t\right) \right] \left[Y\left(\frac{i+1}{2^n} t\right) - Y\left(\frac{i}{2^n} t\right) \right] = \int_0^t \left(\sigma_t^X\right)^\top \left(\sigma_t^Y\right) ds$$

which we write $[X,Y]_t$. This is the quadratic covariation of X and Y.

Now we present a characterization of continuous, local martingales. The following says that if each dimension of a multidimensional continuous local martingale has pairwise zero quadratic covariation (and unit quadratic variation), then it must be a standard brownian motion. It takes a local martingale, and tells you what you need to say further that it is a standard Brownian motion.

Proposition 67. *(Levy's Characterization) If w is a continuous, local martingale with $w(0) = 0$, and $[w_i, w_j](t) = \delta_{i,j} t$, then w is a standard Brownian motion. $\delta_{i,j}$ is the Kronecker delta with*

$$\delta_{i,j} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

This may make you wonder, what is the quadratic variation of Brownian motion $w = (w_1, \cdots, w_n)$? We can write

$$w_i(t) = \int_0^t e_i^\top dw_s = \int_0^t dw_s^i$$

where e_i is the unit vector with 1 in the ith position, and zero elsewhere. Now we can write the quadratic covariation by picking out the volatility terms:

$$[w_i, w_j](t) = \int_0^t e_i^\top e_j ds = \begin{cases} 0 & \text{if } i \neq j \\ t & \text{if } i = j \end{cases}$$

2.8 Girsanov's Theorem

Okay, now we know Ito's Lemma, which says that if $f \in C^{2,1}$, then it maps Ito processes to Ito processes. Here we ask a different question, related to the change of measure. Let $(\Omega, \mathcal{F}, F, P)$ and

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dw_s$$

.

If X_t is an Ito process under P, then is X_t an Ito process under an equivalent measure Q? (so $P \sim Q$)

To investigate, we need to ask, did our definition of the Ito process X_t depend on P? It did, through the Brownian motion w_t. And the issue here is that a Brownian motion under P may not be a Brownian motion under Q, because the distribution of increments may change. What may be normal under P may not be normally distributed under Q.

Then, X_t is an Ito process under Q if there exist μ_s^*, σ_s^*, and a Q-measure Brownian motion w_s^* such that

$$X_t = X_0 + \int_0^t \mu_s^* ds + \int_0^t \sigma_s^* dw_s^*$$

Girsanov's theorem answers the above question in the affirmative, and tells you how to compute the relevant objects, drift, volatility, and the Radon-Nikodym derivative.

Theorem 68. *(Girsanov's Theorem) Let*

- *w be a n-dimensional Brownian motion under P*
- *$F = F^w$ the natural filtration*
- $X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s^\top dw_s$
- *Q an equivalent probability measure, $Q \sim P$.*

Then, there exists $\theta \in \mathcal{L}^2$ such that

$$\frac{dQ}{dP} = \exp\left(\int_0^T \theta_t^\top dW_t - \frac{1}{2}\int_0^T |\theta_t|^2 dt\right)$$

and

$$X_t = X_0 + \int_0^t \underbrace{(\mu_s + \sigma_s^\top \theta_s)}_{\equiv \mu_s^*} ds + \int_0^t \sigma_s^\top dw_s^*$$

where w_s^* is a Q-Brownian motion,

$$w_t^* = w_t - \int_0^t \theta_s ds$$

The volatility term does not change.

Before the proof I quickly present a lemma, which we will use often in the future.

Lemma 69. *$\xi_t = E^P[X|\mathcal{F}_t]$ is a martingale.*

Proof. Let $s \geq t$. Then,

$$E^P[\xi_s|\mathcal{F}_t] = E^P\left[E^P\left[\frac{dQ}{dP}\middle|\mathcal{F}_s\right]\middle|\mathcal{F}_t\right] = E^P\left[\frac{dQ}{dP}\middle|\mathcal{F}_t\right] = \xi_t$$

Thus, ξ_t is a martingale under P. □

Now I prove Girsanov's Theorem.

Proof. Let $Q \sim P$. By Radon-Nikodym Theorem, there exists a random variable called the Radon Nikodym derivative $\frac{dQ}{dP}$ with $E^P\left[\frac{dQ}{dP}\right] = 1$ and $\frac{dQ}{dP} > 0$ almost surely. Define

$$\xi_t = E^P\left[\frac{dQ}{dP}\middle|\mathcal{F}_t\right]$$

By the lemma, it is a martingale that's also strictly positive because it is a Radon Nikodym derivative. By the previously corollary following the Martingale Representation Theorem, there then exists a $\theta \in \mathcal{L}^2$ such that

$$\xi_t = \exp\left(\int_0^t \theta_s^\top dW_s - \frac{1}{2}\int_0^t |\theta_s|^2 \, ds\right)$$

In particular,

$$\frac{dQ}{dP} = \xi_T = \exp\left(\int_0^T \theta_t^\top dW_t - \frac{1}{2}|\theta_t|^2 \, dt\right)$$

as $\xi_T = E^P\left[\frac{dQ}{dP}\middle|\mathcal{F}_T\right] = \frac{dQ}{dP}$ as $\mathcal{F}_T = \mathcal{F}$ and $\frac{dQ}{dP} \in \mathcal{F}$. Note for any sigma algebra $\mathcal{G}$, if $X \in \mathcal{G}$, then $E[X|\mathcal{G}] = X$.

Now let's just define

$$w_t^* = w_t - \int_0^t \theta_s ds$$

Then,

$$\begin{aligned}
X_t &= X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s^\top dw_s \\
&= X_0 + \int_0^t \left(\mu_s + \sigma_s^\top \theta_s\right) ds + \int_0^t \sigma_s^\top \left(\underbrace{dw_s - \theta_s ds}_{=dw_s^*}\right)
\end{aligned}$$

We would like to show that w^* is a Brownian motion under Q. To do this remember Levy's Characterization. We want to know that:

- w^* has continuous sample paths, which it is if w_t is continuous and $\int_0^t \theta_s ds$ is continuous
- w_t^* is a local martingale with the "right" covariation structure.

I claim the w^* is a local Q-martingale if and only if ξw^* is a local P-martingale.

Apply Ito's Lemma to ξ_t to obtain

$$\begin{aligned}
\xi_t &= 1 + \int_0^t e^{X_s} dX_s + \frac{1}{2}\int_0^t e^{X_s} |\theta_s|^2 \, ds \\
&= 1 + \int_0^t e^{X_s}\left(\theta_s^\top dW_s - \frac{1}{2}|\theta_s|^2 \, ds\right) + \frac{1}{2}\int_0^t e^{X_s}|\theta_s|^2 \, ds \\
&= 1 + \int_0^t \xi_s \theta_s^\top dW_s
\end{aligned}$$

How about ξw^*? Again, by Ito's Lemma (Product Rule)[2],

$$\begin{aligned}
\xi_t w_t^* &= \int_0^t \xi_s dw_s^* + \int_0^t w_s^* d\xi_s + [\xi, w^*]_t \\
&= \int_0^t \xi_s \left(dw_s - \theta_s ds\right) + \int_0^t w_s^* \left(\xi_s \theta_s^\top dw_s\right) + [\xi, w^*]_t
\end{aligned}$$

How do we get the last covariation term? We need the volatilities of both ξ and w^*. Recall

$$w_t^* = w_t + \int_0^t \theta_s ds = \int_0^t I_n dw_s + \int_0^t \theta_s ds$$

[2] Ito's Product rule is

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + [X, Y]_t$$

where $[X, Y]_t$ covariation is the product of the volatilities.

so that volatility is the identity matrix I_n. Thus, the covariation is

$$[\xi, w^*]_t = \int_0^t I\xi_s\theta_s ds$$

Now we are ready to write out the full expression:

$$\begin{aligned}
\xi_t w_t^* &= \int_0^t \xi_s \left(dw_s - \theta_s ds\right) + \int_0^t w_s^* \left(\xi_s \theta_s^\top dw_s\right) + \int_0^t I\xi_s\theta_s \\
&= \int_0^t \xi_s \left(1 + w_s^*\theta\right)^\top dw_s
\end{aligned}$$

The integrand $\xi_s \left(1 + w_s^*\theta\right)^\top \in \mathcal{L}^2$ and hence $\xi_t w_t^*$ is a local martingale.

We are almost done. To apply Levy's Characterization, we need to know the covariation structure:

$$[w^*, w^*]_t = [w, w]_t = I_n t$$

as the dt term does not affect volatility. Hence w_t^* is a Q Brownian motion. $\square$

It has been required that a process is (1) a continuous local martingale; and (2) has the right covariation structure. We have shown both in the above.

Part II
Arbitrage and Martingales

3 The Martingale Property

3.1 Fundamental Theorem of Asset Pricing

Consider a simple event tree economy with 2 dates, n stocks, and 1 riskless bond. Let the number of states be finite with $\{\omega_i\}_{i=1}^S$ denoting the outcomes. What are the price processes? At time $t = 0$, the bond price is B_0, the stock price $S_k(0)$ for $k = 1, \cdots, n$. At time $t = T$, bond prices could be one of $\{B(T, \omega_i)\}_{i=1}^S$ and price of stock k be $\{(S_k(T, \omega_i))_{k=1}^n\}_{i=1}^S$. Asume that $B > 0$ almost everywhere.

Now we characterize the notion of no arbitrage in this simple economy, with the goal of generalizing later.

Theorem 70. *There is no arbitrage if and only if there exist* $\{\phi(\omega_i)\}_{i=1}^n$ *such that*

(i) $\phi(\omega_i) > 0$ *for all* $i = 1, \cdots, n$*; and*

(ii) Initial price is the weighted average of future payoffs, i.e.

$$B_0 = \sum_{i=1}^n B(T, \omega_i)\,\phi(\omega_i)$$
$$S_k(0) = \sum_{i=1}^n S_k(T, \omega_i)\,\phi(\omega_i)$$

Harrison-Kreps then address the following question: "Do the results carry over to infinite horizon and infinite states?" As it stands, the answer if no, as for example the price of bond would 'blow up' to infinity if each state price was strictly positive because you would be summing over an infinite number of states.

Remark 71. (State price density) Definfe a random variable π by

$$\pi(\omega) = \frac{\phi(\omega)}{p(\omega)}$$

with $p(\omega)$ denoting the probability of outcome $\omega \in \Omega$. Then, $\pi > 0$ and

$$B_0 = \sum_{i=1}^n B(T, \omega_i)\,\pi(\omega_i)\,p(\omega_i) = E^P[\pi B(T)]$$
$$S_k(0) = E^P[\pi S_k(T)]$$

as one might have seen in a previous course in financial economics. $\pi(\omega)$ is then a "state-price density" because it represents prices per unit of probability. Keep in mind this equivalent representation as we proceed.

Remark 72. (Risk-neutral measure) Alternatively, define

$$q(\omega) = \pi(\omega)\frac{B(T, \omega)}{B_0}p(\omega)$$

Then $q > 0$ and we have that

$$\sum_{i=1}^n q(\omega_i) = \sum_{i=1}^n \pi(\omega)\frac{B(T, \omega_i)}{B_0}p(\omega_i) = \frac{E(\pi B(T))}{B_0} = 1$$

Also note that $p \sim q$, and they are equivalent measures. Moreover,

$$\frac{S_k(0)}{B_0} = E^P\left[\frac{1}{B_0}\pi S_k(T)\right] = E^P\left[\frac{q}{p}\frac{S_k(T)}{B(T)}\right] = \sum_{i=1}^n \frac{q(\omega_i)}{p(\omega_i)}\frac{S_k(T, \omega_i)}{B(T, \omega_i)}p(\omega_i) = E^Q\left[\frac{S_k(T)}{B(T)}\right]$$

so that the "normalized" stock price is a martingale under Q- measure. Note $\frac{S_k(0)}{B_0}$ is the price of stock k expressed in units of numeraire bond at time 0, and $\frac{S_k(T)}{B(T)}$ is the same price at time T.

The original result was proved by Ross, and the year later, Harrison and Kreps showed that things extend to continuous-time and inifinite states. The focus is on characterizing the equivalence of the existence of the equivalent martingale measure, state price densities, and no arbitrage.

3.2 The Economy

Let us describe the environment.

Fix time interval $[0,T]$ for some $T<\infty$, and a filtered probability space $(\Omega,\mathcal{F},F,P)$. There are no restrictions on Ω, $F=\{\mathcal{F}_t\}_{t\geq 0}$ with $\mathcal{F}_0=\{\emptyset,\Omega\}$ the trivial sigma algebra, and $\mathcal{F}_T=\mathcal{F}$. There is a single consumption good (the numeraire) and consumption occurs only at time T. Consumption will be a random variable, and we will denote its space

$$\mathcal{C}=L^p\left(\Omega,\mathcal{F},P\right),\quad \text{for some } p\in[1,\infty)$$

The choice of p is a technical assumption with no real economic content. Harrison and Kreps assume L^2.

There are $(n+1)$ long-lived securities with state price process

$$(B,S_1,\cdots,S_n)$$

Here a long-lived security is one that is tradable at each $t\in[0,T]$. We assume here that this n-dimensional process is an Ito process. (Harrison-Kreps make a different assumption)

Assume that B is bounded above and below away from zero. Intuitively, note that B is bounded above (below) if the interest rate process is bounded below (above).

Assume for the stocks that

$$S_k\left(t\right)\in L^p\left(P\right),\quad \forall t\in\left[0,T\right],\forall k=1,\cdots,n$$

and for the moment that they do not pay dividends before T. In vector notation, we write $S=(S_1,\cdots,S_n)^\top$.

Now we define what a trading strategy is in this economy.

Definition 73. A trading strategy is a $(n+1)$-dimensional stochastic process (α,θ) where α is the holdings of bond and θ is the n-dimensional holdings of stock.

Definition 74. A trading strategy (α,θ) is self-financing if

$$\alpha_t B_t+\theta_t^\top S_t=\alpha_0 B_0+\theta_0^\top S_0+\int_0^t \alpha_s dB_s+\int_0^t \theta_s^\top dS_s-C_t$$

where C is a non-decreasing stochastic process with $C_0=0$.

A trading strategy is tight if the above holds with $C_t=0$ for all t.

Remark 75. What does the above mean?

In discrete time if $(\alpha_{t_i},\theta_{t_i})$ denotes the portfolio held from t_{i-1} to t_i, then (α,θ) is self-financing if

$$\alpha_{t_{i+1}}B_{t_i}+\theta_{t_{i+1}}^\top S_{t_i}+c_{t_i}=\alpha_{t_i}B_{t_i}+\theta_{t_i}^\top S_{t_i}$$

where $c_{t_i}\geq 0$. Now sum over the consumptions to denote cumulative consumption as $C_{t_i}=\sum_{j=1}^i c_{t_i}$. Then

$$\left(\alpha_{t_{i+1}}B_{t_i}+\theta_{t_{i+1}}^\top S_{t_i}\right)-\left(\alpha_{t_i}B_{t_{i-1}}+\theta_{t_i}^\top S_{t_{i-1}}\right)+\Delta C_{t_i}=\alpha_{t_i}B_{t_i}+\theta_{t_i}^\top S_{t_i}-\left(\alpha_{t_i}B_{t_{i-1}}+\theta_{t_i}^\top S_{t_{i-1}}\right)$$

implies

$$\Delta\left(\alpha_{t_{i+1}}B_{t_i}+\theta_{t_{i+1}}^\top S_{t_i}\right)+\Delta C_{t_i}=\alpha_{t_i}\Delta B_{t_i}+\theta_{t_i}^\top\Delta S_{t_i}$$

In continuous time, we could write this as

$$d\left(\alpha_t B_t + \theta_t^\top S_t\right) = \alpha_t dB_t + \theta_t^\top dS_t - dC_t$$

which is directly analogous to the discrete case.

Definition 76. A trading strategy (α, θ) is **admissible** if

- (α, θ) is tight
- (α, θ) is simple.

Or more generally, Lemma 2 holds (to be stated later)

Remark 77. Notice our goal here is to characterize the appropriate space of trading strategies (α, θ) to allow for.

- Having (α, θ) simple is fairly restrictive.
- When people first studied this problem, they were particularly worried about *doubling strategies*, which proceed as follows. Suppose you have a coin toss. You bet \$N, and earn 2N if you get heads, lose N if you get heads. Consider the following betting scheme. First borrow \$1 to play. If you win, you get \$2 and you stop. If you lose, you are now at -\$1, but you borrow \$2 to play again. If you win, you get \$4 and you stop. Only if you lose you continue in the same fashion.
- Given playing n times, probability of not winning at all is $\frac{1}{2^n}$ which goes to zero as $n \to +\infty$. Note however, this hinges on having no short sale constraints at all.
- Similarly, we can do doubling strategies wth stocks and bonds. In this case, we can trade an infinite number of times over a finite interval. This creates arbitrage.

Remark 78. Let Θ be the set of admissible trading strategies. Note it is a vector space (i.e. linear combinations of admissible trading strategies are admissible, etc.) Now we look at the marketed space.

3.3 The Marketed Space and the Price Functional

Definition 79. A consumption plan $x \in \mathcal{C}$ is **marketed** if $\exists\, (\alpha, \theta) \in \Theta$ such that

$$\alpha_T B_T + \theta_T^\top S_T = x$$

and we say that (α, θ) finances x.

Here marketed is the notion that we can take some admissible trading strategy to finance a consumption plan.

Let $\mathcal{M}$ be the space of markted consumption plans. If restricting to consumption plans that can be financed by admissible strategies leads us to the same set of consumption plans as before, then we have the notion of dynamic completeness.

Definition 80. The market is dynamically complete if $\mathcal{M} = \mathcal{C}$.

Definition 81. A **free lunch** is a trading strategy $(\alpha, \theta) \in \Theta$ such that

$$\alpha_0 B_0 + \theta_0^\top S_0 \leq 0 \quad \text{and} \quad \alpha_T B_T + \theta_T^\top S_T > 0$$

This is the familiar notion of "arbitrage". Note in the above, the terminal value of the portfolio $\alpha_T B_T + \theta_T^\top S_T$ is a random variable, so we need to be clear about what we mean when we say it is greater than zero.

Remark 82. Let X be a random variable. Then, we have two ways of putting strict inequalities on this object.

- $X > 0$ a.s.; or

- $X > 0$ (without the a.s.), which means either $X \geq 0$ almost surely or $X \neq 0$ if and only if $P(X > 0) > 0$.

Now we study the implications of having no free lunches.

Proposition 83. *If there are no free lunches, and $(\alpha_1, \theta_1), (\alpha_2, \theta_2) \in \Theta$ both finance the same $x \in \mathcal{M}$, then*

$$\alpha_1 B + \theta_1^\top S \quad \textit{and} \quad \alpha_2 B + \theta_2^\top S$$

are indistinguishable.

Two processes are indistinguishable if they have the same sample path with probability 1. Try to see why this is the case. If the above two processes (which are price processes ever diverge, then there will be a free lunch (which you can implement by buying cheap and shorting the expensive).

Definition 84. If there are no free lunches, the price functional $p_{\mathcal{M}}$ is defined on $\mathcal{M}$ by

$$p_{\mathcal{M}}(x) = \alpha_0 B_0 + \theta_0^\top S_0$$

where $(\alpha, \theta) \in \Theta$ finances x. Furthermore, $p_{\mathcal{M}}$ is linear and strictly positive.

Remark 85. Since under no free lunches, we can characterize "equivalent" trading strategies by the consumption plans they finance, we can in turn look at the time-0 value of these positions. This is what the price functional (over the space of marketed consumptions) maps to.

Note the price functional $p_{\mathcal{M}}$ is linear and strictly positive. Let $\lambda \in \mathbb{R}$ and $x_1, x_2 \in \mathcal{M}$. Then

$$(\alpha_1, \theta_1), (\alpha_2, \theta_2) \in \Theta \implies (\alpha_1 + \lambda \alpha_2, \theta_1 + \lambda_2 \theta_2) \in \Theta$$

because Θ is a vector space. Then we can write

$$\begin{aligned} p_{\mathcal{M}}(x_1 + \lambda x_2) &= (\alpha_1 + \lambda \alpha_2) B_0 + (\theta_1 + \lambda \theta_2)^\top S_0 \\ &= p_{\mathcal{M}}(x_1) + p_{\mathcal{M}}(x_2) \end{aligned}$$

And $p_{\mathcal{M}}$ is strictly positive as there are no free lunches.

Definition 86. A state-price density π is a random variable in $L^q_{++}(P)$ (where $\frac{1}{p} + \frac{1}{q} = 1$) such that

$$p_{\mathcal{M}}(x) = E^P(\pi x), \quad \forall x \in \mathcal{M}$$

where $L^q_{++}(P)$ implies $\pi > 0$ a.s.

Take any marketed consumption plan (i.e. one that can be financed by some trading strategy). The state-price density (which is a random variable) gives us an alternative way to characterize the price functional $p_{\mathcal{M}}$ via an expectations operator under P measure.

Theorem 87. *If there are no free lunches, and $\mathcal{M} = \mathcal{C}$ (hence dynamically complete) then there exists a unique state price density π.*

Proof. Recall that if there are no free lunches, we can define the strictly positive linear functional $p_{\mathcal{M}}$ on $L^p(P)$. Because $p_{\mathcal{M}}$ is a linear, strictly positive functional defined over the whole of $L^p(P)$, it is continuous. Then, by the Riesz Representation Theorem, there exists a $\pi \in L^q(P)$ such that

$$p_{\mathcal{M}}(x) = E^p(\pi x), \quad \forall x \in \mathcal{M}$$

However, the Riesz Representation Theorem doesn't tell you that this π is strictly positive. Now we show this.

Claim: $\pi > 0$ a.s.

Suppose not. Then, $\exists A \in \mathcal{F}$ such that $P(A) > 0$ and $\mathbb{I}_A \pi \leq 0$. That is, some non-zero measure set that it is negative (non-positive). Then, $\mathbb{I}_A > 0$ and

$$p_{\mathcal{M}}(\mathbb{I}_A) = E^P(\pi \mathbb{I}_A) \leq 0$$

which is a contradiction because $p_{\mathcal{M}}$ is strictly positive, mapping positive values to positive consumption plans. $\square$

3.4 Viability

So far our considerations have been in partial equilibrium, as price process is exogenous. If (α, θ) are "sensible", then we should be able to embed it into a general equilibrium framework. To do this, we need that (1) solution to the optimization problem exists; and (2) markets clear. Viability captures this idea.

Before we proceed, note that the most general way to represent preferences are through preference relations "$\precsim$". We also need endowments.

Harrison and Kreps take a shortcut; instead of defining an agent with preference relations and endowments, $(\precsim, x_0)$, they define it over *net trades*. Note that for two net trades X,Y and endowment x_0,

$$X \precsim Y \iff X + x_0 \precsim' Y + X_0$$

where the first preference relation $\precsim$ is over net trades and the second $\precsim'$ is over endowments and consumption.

Definition 88. Agents can be represented by a preference relation $\precsim$ over net trades such that

(i) $\forall y \in \mathcal{C}$: $\{x \in \mathcal{C} : x \precsim y\}$ is closed (lower semi-continuity)

(ii) $\forall y \in \mathcal{C}$: $\{x \in \mathcal{C} : y \precsim x\}$ is convex (convexity)

(iii) $\forall x, y \in \mathcal{C}$: $y > 0 \implies x \prec x + y$ (monotonicity)

This defines a preference relation over net trades.

Remark 89. If there exists a utility function U such that $x \precsim y \iff U(x) \leq U(y)$ then

(i) First condition is equivalent to U being l.s.c. (continuity is even stronger)

(ii) Second condition is equivalent to U being concave.

(iii) Third condition is equiavelent to U being increasing.

From here on, let A be the set of $\precsim$ satisfying all three conditions.

Definition 90. The economy is **viable** if there exists $\precsim \in A$ such that the following problem has a solution:

$$\boxed{\begin{array}{c}\text{Find } x^* \in \mathcal{M} \text{ with } p_{\mathcal{M}}(x^*) \leq 0 \text{ such that} \\ x^* \succsim x \quad \forall x \in \mathcal{M} \text{ with } p_{\mathcal{M}}(x) \leq 0\end{array}} \qquad (P^*)$$

Note the condition $p_{\mathcal{M}}(x) \leq 0$ is equivalent to saying the price of consumption is less than the price of the endowment, which is exactly the budget constraint. Also note that: (1) viability is stronger than no arbitrage; and (2) having an equilibrium implies viability holds.

Now we connect the preference relation over net trades to preference relations over consumption and endowments.

Lemma 91. *Suppose there exists $\precsim \in A$ such that (P^*) has a solution x^*. Define a new preference relation $\precsim'$ on $\mathcal{C}$ by*

$$x \succsim' y \iff x + x^* \succsim y + x^*$$

Then $\precsim' \in A$ and 0 solves (P^*) with $\precsim$ replaced with $\precsim'$.

Remark 92. Viability is equivalent to being able to support ta price process in general equilibrium. If you have a finite number of states, then viability is equivalent to no arbitrage.

The above lemma states that you can define a new preference relation "shifted" by the solution x^* of the original problem, in which case zero will solve the problem with the new preference relations.

Now we state a different way to characterize viability.

Theorem 93. *(Harrison and Kreps) The economy is viable if and only if there exists a strictly positive linear functional p on $\mathcal{C}$ such that*

$$p(x) = p_{\mathcal{M}}(x), \quad \forall x \in \mathcal{M}$$

The functional p is said to be an extension of $p_{\mathcal{M}}$ to $\mathcal{C}$.

Corollary 94. *The economy is viable if and only if there exists a state price density $\pi \in \Pi$.*[3]

The corollary follows by Riez Representation Theorem, $\exists \pi \in L^q_{++}$ such that

$$p(x) = E[\pi x], \quad \forall x \in \mathcal{C}$$

so that $p_{\mathcal{M}}(x) = p(x) = E(\pi x)$. Now we present a proof for the theorem of Harrison and Kreps.

Proof. ($\Longleftarrow$) Suppose there exists a strictly positive linear functional p. We would like to show that the economy is viable (i.e. that P^* has a solution). Set $x \succsim y$ if and only if $p(x) \geq p(y)$. First, we need to check that this is a legitimate preference relation by checking the three conditions.

(1) Closedness follows if the utility function "representing" the preference relation is continuous. Here, $p(\cdot)$ plays the role of the utility function, and it is continuous as it is strictly positive. (2) Convexity would require $p(\cdot)$ to be also concave. (3) Monotonicity follows by π being strictly positive.

I claim that 0 solves (P^*). Suppose not. Then there exists $x^* \in \mathcal{M}$ with $p_{\mathcal{M}}(x^*) \leq 0$ such that $x^* \succ 0$, meaning $p(x^*) > p(0) = 0$. This implies $p_{\mathcal{M}}(x^*) > 0$, which is a contradiction.

($\Longrightarrow$) Suppose the economy is viable. We would like to assert the existence of a strictly positive linear functional p on $\mathcal{C}$ that satisfies the stated conditions. Assume without loss of generality that 0 solves the problem (P^*). Let

$$A = \{x \in \mathcal{C} : x \succsim 0\}$$
$$B = \{x \in \mathcal{M} : p_{\mathcal{M}}(x) \leq 0\}$$

Then, A, B are convex.

To apply the separating hyperplane theorem in a setting with a finite number of states, we only need convex / disjoint / closedness. With an infinite number of states, we also need a nonempty interior for the theorem to apply.

I claim that $\mathrm{int}(A) \neq \emptyset$ and $\mathrm{int}(A) \cap B = \emptyset$. Take the interior of A which is

$$\mathrm{int}(A) = \{x \in \mathcal{C} : x \succ 0\} \neq \emptyset$$

Note that $\{x \in \mathcal{C} : x \succ 0\}$ is the complement of a closed set, so it's open. Monotonicity on L^p says this is just the set of positive random variables – and we know there's plenty of them. Now note

$$\mathrm{int}(A) \cap B = \emptyset$$

because if not, that would contradict the assumption that the problem has a solution.

We are now ready to apply the separating hyperplane theorem: $\exists \phi \in \mathcal{C}^*$ (i.e. the dual of $\mathcal{C}^*$, or the space of all linear functionals over $\mathcal{C}$) such that

$$\phi(x) \geq a \geq \phi(y), \quad \forall x \in A, \forall y \in B$$

First note that a can be at most one value. Since $0 \in A \cap B$, $a = 0$. This follows from linearity of $\phi(\cdot)$. Now let

$$p(x) = \frac{B_0}{\phi(B_T)} \phi(x)$$

Claim: ϕ is strictly positive.

[3] Reacall that if there are no free lunches and $\mathcal{M} = C$, then we have a state price density $\pi \in \Pi$.

To be sure that we are not dividing by zero, we need to check that ϕ is strictly positive[4]Let $x \geq 0$. Then $x \in A$, meaning $\phi(x) \geq 0$ by the separating condition. Let $x > 0$. Fix some $y \in \mathcal{C}$ such that $\phi(y) > 0$ (How do we know the existence of such y? If not, then $\phi(z) \leq 0$ for all z, and by linearity, $\phi(-z) = -\phi(z) \leq 0$, which means $\phi(z) = 0$ everywhere. This contradicts our separation assumption). Since $x \in \operatorname{int}(A)$, then $\exists \lambda > 0$ such that $x - \lambda y \in A$. This implies

$$
\begin{aligned}
0 \leq p(x - \lambda y) &= p(x) - \lambda p(y) \\
p(x) &\geq \lambda p(y) > 0
\end{aligned}
$$

since we assumed $p(y) > 0$.

Note that $p(\cdot)$ has the same sign as ϕ and will also separate the two sets. Thus p is linear and strictly positive, as it inherits the properties of $\phi(\cdot)$.

It remains to check the following claim.

Claim: $p(x) = p_{\mathcal{M}}(x), \quad \forall x \in \mathcal{M}$**.**

Fix $x \in \mathcal{M}$. Let $\lambda = \frac{p_{\mathcal{M}(x)}}{B_0}$ so that

$$
p_{\mathcal{M}}(x) = \lambda B_0 = \lambda p_{\mathcal{M}}(B_T)
$$

where first equality follows from definition of λ, the second from definition of $p_{\mathcal{M}}$ (i.e. price of final B_T is its initial value B_0). Now this means that given the linearity of $p_{\mathcal{M}}$,

$$
p_{\mathcal{M}}(x - \lambda B_T) = 0 = p_{\mathcal{M}}(\lambda B_T - x)
$$

Since we defined $B = \{x \in \mathcal{M} : p_{\mathcal{M}}(x) \leq 0\}$, both $x - \lambda B_T, \lambda B_T - x \in B$. Then the separation condition tells us that $\phi(x - \lambda B_T) = -\phi(\lambda B_T - x) \leq 0$ and $\phi(\lambda B_T - x) = -\phi(x - \lambda B_T) \leq 0$. So ϕ maps both of those quantities to zero, and thus

$$
p(\lambda B_T - x) = p(x - \lambda B_T) = 0 \implies p(x) = \lambda p(B_T)
$$

by linearity of p.

Note that $p(B_T) = \frac{B_0}{\phi(B_T)} \phi(B_T) = B_0$, so that

$$
p(x) = \lambda p(B_T) = \lambda B_0 = p_{\mathcal{M}}(x)
$$

Thus we conclude that our choice of $p(x)$ yields our originally intended result that $p(x) = p_{\mathcal{M}}(x)$. ☐

Remark 95. State price density is not necessarily unique if markets are not complete (in which case potentially $\mathcal{M} \neq \mathcal{C}$). We could have one state price density for each strictly positive $p(\cdot)$ that "agrees" with $p_{\mathcal{M}}(\cdot)$ on the marketed space.

3.5 Equivalent Martingale Measure

Here we develop the notion of an equivalent martingale measure that we could apply to price assets. It is a specific type of probability measure (that is equivalent to the physical measure P).

The bulk of this section is a single theorem that formulates the equivalent martingale measure in terms of the state price density, and vice versa.

Definition 96. A probability measure Q on $(\Omega, \mathcal{F})$ is an equivalent martingale measure if

(i) $Q \sim P$

(ii) $\frac{1}{B_T} \frac{dQ}{dP} \in L^q(P)$

[4]That is,

$$
\begin{aligned}
\phi(x) &> 0, \quad \forall x > 0 \\
\phi(x) &\geq 0, \quad \forall x \geq 0
\end{aligned}
$$

(iii) $S^* = \frac{S}{B}$ numeraire is a Q-martingale.

Let $\mathcal{Q}$ denote the set of all equivalent martingale measures. To recap, here's the definition of state price densities.

Definition 97. A state price density is a random variable π such that

(i) $\pi \in L^q_{++}(P)$

(ii) $p_{\mathcal{M}}(x) = E^P[\pi x], \quad \forall x \in \mathcal{M}$

Let Π denote the set of all state price densities. We establishe that there is a strong connection between state price densities and equivalent martingale measures.

Theorem 98. *There is a one-to-one correspondence between* $\mathcal{Q}$ *and* Π *given by*

$$\pi = \frac{B_0}{B_T}\frac{dQ}{dP} \quad \text{and} \quad \frac{dQ}{dP} = \frac{B_T}{B_0}\pi$$

Note viability implies a strong connection between $\Pi \iff \mathcal{Q}$.

Proof. There are several steps to this proof. I want show both that (1) if you start with a state price density (as defined above) π and write $\frac{dQ}{dP} = \frac{B_T}{B_0}\pi$, this results in an actual Radon-Nikodym derivative with equivalent martingale measure Q; and that (2) if you start with an equivalent martingale measure Q whose Radon-Nikodym derivative is $\frac{dQ}{dP}$, then writing $\pi = \frac{B_0}{B_T}\frac{dQ}{dP}$ results in a legitimate state price density.

I will show each.

(1) Suppose $\pi \in \Pi$, and write $\frac{dQ}{dP} = \frac{B_T}{B_0}\pi$. First we would like to check that this is a valid Radon-Nikodym derivative[5]. Given $\pi \in L^q_{++}(P)$, $\frac{B_T}{B_0}\pi > 0$ a.s., $E^P\left[\frac{B_T}{B_0}\pi\right] = \frac{1}{B_0}E^P[\pi B_T] = \frac{1}{B_0}p_{\mathcal{M}}(B_T) = 1$. So it satisfies the conditions we would like to have at a minimum.

Furthermore, define the Q measure by

$$Q(A) = E^P\left[\frac{B_T}{B_0}\pi \mathbb{I}_A\right], \quad \forall A \in \mathcal{F}$$

Defining the Q measure like this concludes the claim that $\frac{dQ}{dP} = \frac{B_T}{B_0}\pi$ is a Radon-Nikodym derivative of Q with respect to P. In fact we have that $Q \sim P$, and it holds that $\frac{1}{B_T}\frac{dQ}{dP} = \frac{\pi}{B_0} \in L^q(P)$ simply because $\pi \in L^q_{++}(P)$. To show that Q is an equivalent martingale measure, it remains to show that S^* is a martingale under Q.

Claim: S^* is a Q-martingale.

This requires

$$S^*_k(t) = E^Q[S^*_k(T)|\mathcal{F}_t]$$

Writing out what it means to be a conditional expectation, it is the random variable $S^*_k(t)$ such that

$$E^Q[S^*_k(t)\mathbb{I}_A] = E^Q[S^*_k(T)\mathbb{I}_A], \quad \forall A \in \mathcal{F}_t$$

Let's fix some t and k, and some $A \in \mathcal{F}_t$. Define the following trading strategy:

$$\begin{cases} \theta_k(\omega, s) & = \mathbb{I}_A(\omega) \times \mathbb{I}_{[t,T]}(t) \\ \alpha(\omega, s) & = -\theta_k(\omega, s) S^*_k(t) \\ \theta_j(\omega, s) & = 0, \quad \forall j \neq k \end{cases}$$

What does the above do?

[5] A random variable X is a RN derivative if

(i) $X \geq 0$ P-a.s. ($X > 0$ if $P \sim Q$)

(ii) $E^P[X] = 1$

(iii) $Q(A) = E^P[\mathbb{I}_A X], \quad \forall A \in \mathcal{F}$.

- At time t, if the A is true, you buy a stock.
- You finance the purchase selling bonds. Note to finance 1 unit of stock takes $S_k(t)$ dollars, which takes $\frac{S_k(t)}{B(t)}$ units of bonds to finance.
- Otherwise you do nothing.

Then, this strategy $(\alpha, \theta) \in \Theta$, as it is admissible / self-financing / tight. Now, note that

- Initial payment: $\alpha_0 B_0 + \theta_0^\top S_0 = 0$
- Terminal value: $\alpha_T B_T + \theta_T^\top S_T = \mathbb{I}_A [S_k(T) - S_k^*(t) B(T)]$ which is a random variable.

Now, we can use the price functional $p_{\mathcal{M}}$ which maps terminal values (or consumption plans) to initial payments of marketed consumption:

$$\begin{aligned}
0 &= p_{\mathcal{M}} (\mathbb{I}_A [S_k(T) - S_k^*(t) B(T)]) \\
&= E^P [\pi \mathbb{I}_A [S_k(T) - S_k^*(t) B(T)]] \\
&= B_0 E^P \left[\underbrace{\frac{B_T}{B_0} \pi}_{\equiv \frac{dQ}{dP}} \mathbb{I}_A \left[\frac{S_k(T)}{B_T} - S_k^*(t) \right] \right] \\
&= B_0 E^P \left[\frac{dQ}{dP} \mathbb{I}_A (S_k^*(T) - S_k^*(t)) \right] \\
&= B_0 E^Q [S_k^*(T) - S_k^*(t)]
\end{aligned}$$

(2) Now let $Q \in \mathcal{Q}$ be an equivalent martingale measure with $\frac{dQ}{dP}$ the accompanying RN derivative. Let $\pi = \frac{B_0}{B_T} \frac{dQ}{dP}$. Since $P \sim Q$, we have $\frac{dQ}{dP} > 0$ a.s., and thus $\pi \in L^q_{++}(P)$ as $B_0, B_T > 0$.

Claim: $\forall x \in \mathcal{M}, \quad p_{\mathcal{M}}(x) = E^P(\pi x)$.

I will present two lemmas to help prove this claim. Once we have those two, this is quite easy to show.

Lemma 99. *Let $(\alpha, \theta) \in \Theta$. Then,*

$$\alpha_T + \theta_T^\top S_T^* = \alpha_0 + \theta_0^\top S_0^* + \int_0^T \theta_s^\top dS_s^*$$

Intuitively, why is this true? It says "tightness" of a trading strategy is invariant to changes of numeraire. We have that

$$\alpha_t B_t + \theta_t^\top S_t = \alpha_0 B_0 + \theta_0^\top S_0 + \int_0^t \alpha_s dB_s + \int_0^t \theta_s^\top dS_s$$

if a trading strategy is tight. Applying $t = T$ gives the results we want.

Lemma 100. *If (α, θ) is simple and tight, then*

$$W^* = \alpha + \theta^\top S^*$$

is a Q martingale for all $Q \in \mathcal{Q}$.

Here, W^* is the portfolio value under the bond numeraire. This is actually the only place that we use the fact that (α, θ) is simple in Harrison, Kreps. Thus, if this lemma holds for all of Θ (and not just simple strategies), all of the results follow through for Θ.

Back to our earlier claim. Fix a consumption plan $x \in \mathcal{M}$ and let $(\alpha, \theta) \in \Theta$ finance x. Then,

$$E^P(\pi x) = E^P \left[\frac{B_0}{B_T} \frac{dQ}{dP} \left(\alpha_T B_T + \theta_T^\top S_T \right) \right]$$

since we began by defining $\pi = \frac{B_0}{B_T}\frac{dQ}{dP}$, and the trading strategy finances x. Now,

$$E^P\left[\frac{B_0}{B_T}\frac{dQ}{dP}\left(\alpha_T B_T + \theta_T^\top S_T\right)\right] = B_0 E^Q\left[\alpha_T + \theta_T^\top S_T^*\right] = B_0\left(\alpha_0 + \theta_0^\top S_0^*\right)$$

because $\alpha_t + \theta_t^\top S_t^* = W_t$ is a Q-martingale by the lemma. But multiplting out the B_0 term of course gives us

$$E^P(\pi x) = \alpha_0 B_0 + \theta_0^\top S_0 = p_{\mathcal{M}}(x)$$

which is exactly the cost of financing x. □

3.6 Dividend Paying Securities

For this section, we remove the assumption that stocks pay no dividends. We revise our notion of self-financing, equivalent martingale measure, etc. to account for this new feature.

Suppose stocks now pay dividends. With n stocks, let D be the $n-$dimensional cumulative dividend process. We want to be general enough to account for both: (1) lumpy dividends and (2) continuous streams of dividends. Then let G be the *gain* process

$$G = S + D$$

Let's revise the notion of self-financing strategies:

Definition 101. With dividends, a trading strategy (α, θ) is sef-financing if

$$\alpha_t B_t + \theta_t^\top S_t = \alpha_0 B_0 + \theta_0^\top S_0 + \underbrace{\int_0^t \alpha_s dB_s}_{\substack{\text{capital gains} \\ \text{from bonds}}} + \underbrace{\int_0^t \theta_s^\top (dS_s + dD_s)}_{\substack{\text{capital gains from} \\ \text{stocks + dividends}}} - C_t$$

Or in bond numeraire,

$$\alpha_t + \theta_t S_t^* = \alpha_0 + \theta_0 S_0^* + \int_0^t \theta dG_s^* - C_t^*$$

which we have shown in the problem set.

Now let's define everything in terms of the bond numeraire:

$$\begin{aligned} S_t^* &= \frac{S_t}{B_t} \\ D_t^* &= \int_0^t \frac{1}{B_s} dD_s \\ G_t^* &= S_t^* + D_t^* \end{aligned}$$

Notice that $D_t^* \neq \frac{D_t}{B_t}$ because this is a cumulative amount. We need to account for the entire stream.

We also have a new notion of equivalent martingale measure, where we require G^* to be a Q-martingale, instead of S^* as before:

Definition 102. With dividends, a probability measure Q on $(\Omega, \mathcal{F})$ is an equivalent martingale measure if

(i) $Q \sim P$

(ii) $\frac{1}{B_T}\frac{dQ}{dP} \in L^q(P)$

(iii) $G_t^* = S_t^* + D_t^*$ is a Q-martingale.

Here we have another "equivalence" result among viability, state-price density, and equivalent martingale measures, due to Huang.

Theorem 103. *(Huang 1985) The following statements are equivalent:*

(i) The economy is viable

(ii) There exists a state price density π

(iii) There exists a equivalent martingale measure Q

In the same paper, Huang presents the result that, in this setting, we can characterize the gain process under the bond numeraire G^* as an Ito process:

Theorem 104. *(Huang 1985) Suppose the economy is viable and the filtration is the natural filtration $F = F^w$ where w_t is the n-dimensional Brownian motion. Then, G_t^* is an Ito process.*

For illustrative purposes, here we present a "wrong" version of the proof. Try to see where it goes wrong.

"Since the economy is viable there exists an equivalent martingale measure $Q \sim P$. Because it is an equivalent martingale measure, under Q, G^* is a martingale. Let w^* be the Brownian motino under Q. By the martingale representation theorem, then $\exists \theta \in \mathcal{L}^2$ such that

$$G_t^* = G_0^* + \int_0^t \theta_s^\top dw_s^*$$

Hence G^* is a Ito process under Q (and hence under P)."

This is incorrect because to invoke the martingale representation theorem[6] under Q, we would have needed to look at the filtration $F = F^{w^*}$. G_t^* would have needed to be a (local) martingale adapted to F^{w^*}. So adjustments need to be made. Here's the right proof.

Proof. Since economy is viable, there exists a probability measure $Q \sim P$ such that G^* is a Q martingale. Now define the process

$$\xi_t = E^P \left[\left. \frac{dQ}{dP} \right| \mathcal{F}_t \right]$$

Then G^* is a Q-martingale if and only if ξG^* is a P martingale (this is fairly mechanical, but you should check).

Since our filtration that we're looking at is $F = F^w$, we apply the MRT to $\xi_t G_t^*$. By the martingale representation theorem, then there exists $\theta \in \mathcal{L}^2$ such that

$$\xi_t G_t^* = \xi_0 G_0^* + \int_0^t \theta_s^\top dw_s$$

Hence we know that $\xi_t G_t^*$ is an Ito process under Q due to Girsanov's Theorem (Girsanov also tells you what representation it has). But our question is whether G^* is an Ito.

ξ_t is a martingale because it's a "conditional expectation" process, and hence it's an Ito process by Martingale Representation theorem. $\xi_t > 0$ as it's a Radon-Nikodym derivative. Then, by Ito's lemma,

$$G^* = \frac{\xi G^*}{\xi} = f(\xi G^*, \xi)$$

is an Ito process as well. $\square$

[6] Recall that the martingale representation theorem is:

Theorem 105. *(Martingale Representation Theorem) Let*

- *w be a n-dimensional Brownian motion*
- *F^w be the filtration generated by w*
- *X be a local martingale adapted to F^w*

Then there exists $\theta \in \mathcal{L}^2$ such that

$$X_t = X_0 + \int_0^t \theta_s^\top dw_s$$

Part III

Complete Markets: Hedging and Pricing Contingent Claims

For this part, we focus on a particular application, the Black-Scholes economy, so that we can derive sharper results. We also try to better study the space of trading strategies that are possible within this framework, specifically trying to rule out doubling strategies.

We then move on the pricing contingent claims in this economy. With the martingale approach, we can reduce this task to evaluating an integral (as opposed to solving a PDE). As an application, we look at pricing European call options.

4 The Black-Scholes Economy

4.1 The Economy

First we introduce the setup:

- $[0, T]$ with $T < \infty$ is the time interval
- $(\Omega, \mathcal{F}, F, P)$ where $F = F^w$ and w_t is the 1-dimensional Brownian motion
- $\mathcal{C} = L^2(P)$ is the space of consumption plans. Agent only consumes at the end.
- Single consumption good.
- Two securities:
  - A bond: $dB_t = rB_t dt$ for $r \in \mathbb{R}$ [7]
  - A stock: $dS_t = \mu S_t dt + \sigma S_t dw_t$ (where $\sigma \neq 0$, $\mu, \sigma \in \mathbb{R}$)[8] Stock pays dividend at rate $\delta \geq 0$

Given the price processes, the solutions are

$$
\begin{aligned}
B_t &= B_0 e^{rt} \\
S_t &= S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma w_t\right\} \\
D_t &= \int_0^t \delta S_s ds
\end{aligned}
$$

With the change of numeraire (to the bond numeraire), these become

$$
\begin{aligned}
S_t^* &= S_0^* \exp\left\{\left(\mu - \frac{\sigma^2}{2} - r\right) t + \sigma w_t\right\} \\
D_t^* &= \int_0^t B_s^{-1} dD_s = \int_0^t \delta S_s^* ds
\end{aligned}
$$

and in addition the gain process $G_t^* = S_t^* + D_t^*$. The stochastic differential equation that it satisfies is

$$
\begin{aligned}
dG_t^* &= dS_t^* + dD_t^* \\
&= \left[S_t^* (\mu - r) dt + S_t^* \sigma dw_t\right] + S_t^* \delta dt \\
&= S_t^* (\mu + \delta - r) dt + S_t^* \sigma dw_t
\end{aligned}
$$

[7]Think of this like a money market account, not a coupon bond (with value 1 at the end). r is the continuously compounded interest rate, and is constant.

[8]Note if $\sigma = 0$, then immediately $\mu = r$ by no arbitrage.

where dS_t^* is just like $dS_t$with the mean $(\mu - r)$ instead of μ.

As for the filtration, the information contained in the stock price is the same as that contained in the Brownian motion, so

$$F = F^w = F^S$$

This would not necessarily be true if there was an additional random component driving the stock prices.

4.2 Viability

Recall that an economy being viable is equivalent to: existence of state price density; and also equivalent to existence of an equivalent martingale measure.

We construct an equivalent martingale measure to show that the Black-Scholes economy is viable. This provides an explicit representation of the unique equivalent martingale measure in this economy.

Theorem 106. *Let* $\kappa = -\frac{\mu+\delta-r}{\sigma}$ *and* $\xi_t = \exp\left\{\kappa w_t - \frac{1}{2}\kappa^2 t\right\}$. *Then* ξ_t *is a martingale and the probability measure* Q *with* $\frac{dQ}{dP} = \xi_T$ *is the unique equivalent martingale measure.*

Proof. This has two parts. Given κ and ξ_t above it says that: (1) ξ_t is a martingale; and (2) Q is the unique equivalent martingale measure.

(1) Claim: ξ_t is a martingale.

For any $s \geq t$,

$$\begin{aligned}
E\left[\xi_s | \mathcal{F}_t\right] &= E\left[\exp\left\{\kappa w_s - \frac{1}{2}\kappa^2 s\right\}\middle| \mathcal{F}_t\right] \\
&= E\left[\exp\left\{\kappa w_t - \frac{1}{2}\kappa^2 t + \kappa\left(w_s - w_t\right) - \frac{\kappa^2}{2}\left(s-t\right)\right\}\middle| \mathcal{F}_t\right] \\
&= \xi_t E\left[\exp\left\{\kappa\left(w_s - w_t\right) - \frac{\kappa^2}{2}\left(s-t\right)\right\}\middle| \mathcal{F}_t\right]
\end{aligned}$$

By independent of Brownian increments, we can disregard the time-t conditioning. Call $K = \kappa\left(w_s - w_t\right) - \frac{\kappa^2}{2}\left(s-t\right)$. Then by normality of Brownian increments, we have

$$K \sim N\left(-\frac{\kappa^2}{2}\left(s-t\right), \kappa^2\left(s-t\right)\right)$$

so that we can write

$$\begin{aligned}
E\left[\xi_s | \mathcal{F}_t\right] &= \xi_t \exp\left\{E\left(K\right) + \frac{1}{2}\mathrm{Var}\left(K\right)\right\} \\
&= \xi_t \exp\left\{-\frac{\kappa^2}{2}\left(s-t\right) + \frac{\kappa^2}{2}\left(s-t\right)\right\} \\
&= \xi_t
\end{aligned}$$

This show the first claim.

(2) Claim: Q defined as in the theorem with $\frac{dQ}{dP} = \xi_T$ is an equivalent martingale measure.

The ξ_T satisfies $E\left[\xi_T\right] = \xi_0 = 1$ and $\xi_T > 0$ a.s., so it is a Radon-Nikodym derivative (with appropriately defined Q). For EMM[9], we have strictly positive ξ_T, so $Q \sim P$, and $\frac{dQ}{dP} \in$

[9]Recall EMM Q requires

(i) $Q \sim P$

(ii) $\frac{1}{B_T}\frac{dQ}{dP} \in L^2\left(P\right)$

(iii) G^* is a martingale under Q

$L^2(P)$ because we know it is log normally distributed (and hence finite second moments). It remains to show that G^* is a martingale under Q.

Take

$$dG_t^* = S_t^* (\mu + \delta - r) dt + S_t^* \sigma dw_t$$

Girsanov's theorem defines a new Brownian motion under Q, and tells you how to compute it.

It says that if we have an Ito process $dX_t = \mu_t dt + \sigma_t dw_t$ and some probability measure $Q \sim P$, then there exists some $\phi \in \mathcal{L}^2$ with

$$\frac{dQ}{dP} = \exp\left\{\int_0^T \phi_t dw_t - \frac{1}{2}\int_0^T |\phi_t|^2 dt\right\}$$

and we can write the same Ito process with BM under Q

$$dX_t = (\mu_t + \sigma_t \phi_t) dt + \sigma_t dw_t^*$$

where $w_t^* = w_t - \int_0^t \phi_s ds$ is the Q Brownian motion.

But notice if we set $\phi_t = \kappa$ constant, then $\frac{dQ}{dP} = \xi_T$ exactly! So we have our ϕ_t to formulate our Ito process G_t^* and BM under Q:

$$\begin{aligned} dG_t^* &= S_t^* (\mu + \delta - r) dt + S_t^* \sigma dw_t \\ &= S_t^* (\mu + \delta - r) dt + S_t^* \sigma (dw_t^* + \kappa dt) \\ &= S_t^* (\mu + \delta - r + \sigma\kappa) dt + S_t^* \sigma dw_t^* \\ &= S_t^* \sigma dw_t^* \end{aligned}$$

Thus G^* is the process

$$G_t^* = G_0^* + \int_0^t S_s^* \sigma dw_s^*$$

Lemma 107. *An Ito process with zero drift is a local martingale.*

G^* is then a Q-local martingale. If $\sigma S_t^* \in \mathcal{H}^2$ or

$$E^Q\left[\int_0^T |S_t^* \sigma|^2 dt\right] < \infty$$

, then G^* is a proper L^2 martingale under Q. I claim this is the case:

$$\begin{aligned} dS_t^* &= S_t^* (\mu - r) dt + S_t^* \sigma dw_t \\ &= S_t^* (\mu - r) dt + S_t^* \sigma (dw_t^* + \kappa dt) \\ &= S_t^* (\mu - r + \kappa\sigma) dt + S_t^* \sigma dw_t^* \\ &= S_t^* (-\delta) dt + S_t^* \sigma dw_t^* \end{aligned}$$

This is sort of like a geometric Brownian motion with mean $-\delta$ and we know how to solve that one:

$$\begin{aligned} S_t^* &= S_0^* \exp\left\{\left(-\delta - \frac{\sigma^2}{2}\right) t + \sigma dw_t^*\right\} \\ &= S_0^* e^{-\left(\delta + \frac{\sigma^2}{2}\right) t + \sigma w_t^*} \end{aligned}$$

which is log-normally distributed under Q. Now in an application of Fubini's theorem,

$$E^Q\left[\int_0^T |S_T^* \sigma|^2 dt\right] = \int_0^T E^Q\left[|S_t^* \sigma|^2\right] dt < \infty$$

Why? $S_t^* \sigma$ is log-normal, which means $(S_t^* \sigma)^2$ is also log-normal. Now we move on to our final claim..

Claim: Q is the unique EMM.

Let $\hat{Q}$ be an EMM in this environment. To show uniqueness, we then want to show that $\hat{Q} = Q$. Since $\hat{Q}$ is EMM, then we have $\hat{Q} \sim P$. Then by Girsanov's theorem, $\exists \phi \in \mathcal{L}^2$ such that

$$\frac{dQ}{dP} = \exp\left\{\int_0^T \phi_t dw_t - \frac{1}{2}\int_0^T |\phi_t|^2 \, dt\right\}$$

and with $d\hat{w}_t = dw_t - \phi_t dt$, we have

$$dG_t^* = S_t^* \left(\mu + \delta - r + \sigma\phi_t\right) dt + S_t^* \sigma d\hat{w}_t$$

where $\hat{w}_t$ is a Q Brownian motion.

What must be then is that G_t^* is a martingale under $\hat{Q}$ (as before), so its drift must be zero:

$$S_t^* \left(\mu + \delta - r + \sigma\phi_t\right) = 0, \text{ a.e.}$$

Since we know $S_t^* > 0$, it must be that

$$\phi_t = -\frac{\mu + \delta - r}{\sigma} = \kappa$$

If $\phi_t = \kappa$, then this means that the Radon-Nikodym derivative is the same, which in turn means that

$$\hat{Q} = Q$$

□

Remark 108. The fact that we can assign a unique price to everything is the same as being dynamically complete.

The next question is how we should define trading strategies. Consider (α, θ) tight strategies, with no jumps (which makes predictable processes same as progressively measurable). The trading strategy (α, θ) is tight if

$$\begin{aligned}
\alpha_t B_t + \theta_t S_t &= \alpha_0 B_0 + \theta_0 S_0 + \int_0^t \alpha_s dB_s + \int_0^t \theta_s dG_s \\
&= \alpha_0 B_0 + \theta_0 S_0 + \int_0^t \alpha_s B_s r ds + \int_0^t \theta_s \left[S_s \left(\mu + \delta\right) ds + S_s \sigma dw_s\right]
\end{aligned}$$

To characterize (α, θ), consider each of the integrals on the right. How can we assure that the (α, θ) are well-defined? The first integral $\int_0^t \alpha_s B_s r ds$ is a path-by-path Lebesgue integral, as α could potentially be random. The same goes for $\int_0^t \theta_s S_s \left(\mu + \delta\right) ds$, as $\theta \cdot S$ is random. Lastly, $\int_0^t \theta_s S_s \sigma dw_s$ is a stochastic integral. This yield the following necessary conditions for well-definedness of the trading strategies:

(i) $\int_0^T |\alpha_t B_t| \, dt < \infty$

(ii) $\int_0^T |\theta_t S_t| \, dt < \infty$

(iii) $\int_0^T |\theta_t S_t|^2 < \infty$ (or $\theta_t S_t \in \mathcal{L}^2$)

Thus we have the following (preliminary) space of trading strategies:

$$\hat{\Theta} = \left\{(\alpha, \theta) : \begin{array}{c} \int_0^T |\alpha_t B_t| \, dt < \infty, \int_0^T |\theta_t S_t|^2 < \infty \\ \text{and } (\alpha, \theta) \text{ is tight} \end{array}\right\}$$

What are some equivalenet ways to characterize Θ? In the notes we have

$$\hat{\Theta} = \left\{(\alpha, \theta) : \begin{array}{c} \int_0^T |\alpha_t| \, dt < \infty, \int_0^T |\theta_t S_t|^2 < \infty \\ \text{and } (\alpha, \theta) \text{ is tight} \end{array}\right\}$$

This is enough because $B_t = B_0 e^{rt}$ for all $t \in [0, T]$ is always bounded above and below away from zero. I argue that the more compact way to denote the same space is

$$\hat{\Theta} = \left\{ (\alpha, \theta) : \int_0^T |\theta_t S_t|^2 < \infty \text{ and } (\alpha, \theta) \text{ is tight} \right\},$$

or in other words, the condition that $\int_0^T |\alpha_t B_t| \, dt < \infty$ or $\int_0^T |\alpha_t| \, dt < \infty$ is redundant.

Why is this true? Let

$$W_t = \alpha_t B_t + \theta_t S_t = W_0 + \int_0^t \alpha_s dB_s + \int_0^t \theta_s dG_s$$

Then,

$$\alpha_t B_t = W_t - \theta_t S_t$$

By triangle inequality,

$$\begin{aligned} \int_0^T |\alpha_t B_t| \, dt &\leq \int_0^T |W_t| \, dt + \int_0^T |\theta_t S_t| \, dt \\ &\leq \int_0^T |W_t| \, dt + \left(\int_0^T |\theta_t S_t|^2 \, dt \right)^{1/2} \end{aligned}$$

and the second inequality follows by Jensen's Inequality[10]. The above is finite because each integral is finite. We have $\int_0^T |W_t| \, dt < \infty$ because W_t has continuous sample paths over $[0, T]$. W_t has continuous sample paths because it is $W_t = W_0 + \int_0^t \alpha_s dB_s + \int_0^t \theta_s dG_s$, which is a constant + integrals (which is continuous). $\left(\int_0^T |\theta_t S_t|^2 \, dt \right)^{1/2} < \infty$ is given by the assumptions of $\hat{\Theta}$.

In a sense, we have that $\hat{\Theta}$ is large enough to dynamically complete the market. We show this in the next proposition that this is the case. But as we will see later, it is also "too large", because it will include some arbitrage opportunities.

Proposition 109. *Any consumption plan $x \in \mathcal{C}$ is financed by a trading strategy $(\alpha, \theta) \in \hat{\Theta}$ where*

$$\hat{\Theta} = \left\{ (\alpha, \theta) : \int_0^T |\theta_t S_t|^2 < \infty \textit{ and } (\alpha, \theta) \textit{ is tight} \right\}$$

Proof. As before, let $x \in \mathcal{C} = L^2(P)$. Our goal is to show that there is some $(\alpha, \theta) \in \hat{\Theta}$ that finances this arbitrary consumption plan x, which holds $\alpha_T B_T + \theta_T^\top S_T = x$. Let's begin by defining a process

$$M_t = E^Q \left[\left. \frac{x}{B_T} \right| \mathcal{F}_t \right]$$

This process is well-defined because $x \in L^2(P)$, so it is squared-integrable, which means it is integrable in Q by Holder's inequality[11]. It is also a "conditional expectation" process, so it's a martingale.

[10] The version I use is that for ϕ a convex function, and some non-negative Lebesgue integrable f,

$$\phi \left(\int_a^b f(x) \, dx \right) \leq \int_a^b \phi(f(x)) \, dx$$

applued to $\phi(x) = x^2$, so that

$$\left(\int_0^T |\theta_t S_t| \, dt \right)^2 \leq \int_0^T |\theta_t S_t|^2 \, dt$$

where the result follows by taking the square root of both sides.

[11] Why? Apply Holder's Inequality to x with $p = 2$ and ξ_T with $q = 2$:

$$E^Q[|x|] = E^P \left[|x| \frac{dQ}{dP} \right] = E^P[|x| \, \xi_T] \leq \left(E\left[|X|^2\right] \right)^{1/2} \left(E\left[|\xi_T|^2\right] \right)^{1/2}$$

where X is squared integrable by assumption, and ξ_T is log-normal (and hence finite second moments).

Given, that it's a martingale, we can apply the Martingale Representation Theorem, so $\exists \phi \in \mathcal{L}^2$ such that

$$M_t = M_0 + \int_0^t \phi_s dw_s^*$$

But to apply the MRT under Q, we need to check that the martingale is adapted to a Brownian motion under Q. Check the filtration in this economy: $w_t^* = w_t - \kappa t$, so $F^{w^*} = F^w$, so we're good to go.

Now let's further define the trading strategies

$$\theta_t = \frac{\phi_t}{\sigma S_t^*}$$
$$\alpha_t = M_t - \theta_t S_t^*$$

Then, simply rearranging the definition of α_t, the value of the portfolio becomes

$$\alpha_t + \theta_t S_t^* = M_t = M_0 + \int_0^t \phi_s dw_s^*$$

Check that this (α, θ) finances x (by construction) because by multiplying both sides by B_T in the above at $t = T$,

$$\alpha_T B_T + \theta_T S_T = M_T B_T = x$$

Now if we're at $t = 0$, this would mean $M_0 = \alpha_0 + \theta_0 S_0^*$, so making this substitution, and that $\phi_t = \sigma S_t^* \theta_t$,

$$\begin{aligned} M_t &= \alpha_0 + \theta_0 S_0^* + \int_0^t \sigma \theta_s S_s^* dw_s^* \\ &= \alpha_0 + \theta_0 S_0^* + \int_0^t \theta_s dG_s^* \end{aligned}$$

[12]Notice that the last expression is simply the value of the portfolio in bond numeraire! As we already shown,

$$\alpha_t + \theta_t S_t^* = \alpha_0 + \theta_0 S_0^* + \int_0^t \theta_s dG_s^*$$

implies the strategy is tight, as tightness is invariant to changes in numeraire. So this trading strategy is tight.

So far we have shown that (1) (α, θ) finances x; and (2) (α, θ) is tight. It remains to show square-integrability of $\theta \cdot S$.

$$\int_0^T |\theta_t S_t|^2 \, dt = \int_0^T \left| \frac{\phi_t B_t}{\sigma} \right|^2 dt \le \left(\frac{B_0 e^{|r|T}}{\sigma} \right)^2 \int_0^T |\phi_t|^2 \, dt$$

where each term in the last expression is finite. Thus, $(\alpha, \theta) \in \hat{\Theta}$. □

As mentioned, however, this space of trading strategies $\hat{\Theta}$ is too large, as it contains arbitrage opportunities! An example of these arbitrage opportunities will be the type of "doubling strategy" that we have shown before.

[12]because

$$\begin{aligned} \int_0^t \sigma \theta_s S_s^* dw_s^* &= \int_0^t \sigma \theta_s S_s^* dw_s - \kappa \int_0^t \sigma \theta_s S_s^* ds \\ &= \int_0^t \sigma \theta_s S_s^* dw_s + \int_0^t (\mu + \delta - r) \theta_s S_s^* ds \\ &= \int_0^t \theta_s \left[S_s^* (\mu + \delta - r) ds + \sigma S_s^* dw_s \right] \\ &= \int_0^t \theta_s dG_s^* \end{aligned}$$

4.3 Doubling Strategies

There is a type of doubling strategy that is in $\hat{\Theta}$

$$\hat{\Theta} = \left\{ (\alpha, \theta) : \int_0^T |\theta_t S_t|^2 < \infty \text{ and } (\alpha, \theta) \text{ is tight} \right\}$$

that also constitutes an arbitrage opportunity.

Let $\phi_t = \frac{1}{\sqrt{T-t}}$ and $M_t = \int_0^t \phi_s dw_s^*$ for $t \in [0, T)$. Is the stochastic integral well-defined? It is, if the integrand is square-integrable, i.e. $\int_0^t |\phi_s|^2 ds < \infty$ for all $t \in [0, T)$. Note,

$$\int_0^t |\phi_s|^2 ds = \int_0^t \frac{1}{T-s} ds = \log\left(\frac{T}{T-t}\right) < \infty$$

which shows M_t is well-defined on the half-closed interval $[0, T)$.

Now let take some positive number $L > 0$, and define the stopping time

$$\tau = \inf\{t : M_t > L\}$$

So τ is the first time this M_t process exceeds L. It can be shown that $\tau < T$ a.s., because

$$E^Q\left[M_t^2\right] = E^Q\left[\int_0^t |\phi_s|^2 ds\right] = \log\left(\frac{T}{T-t}\right) \to \infty \text{ as } t \to T$$

where the first equality follows from the expectation of products property of stochastic integrals[13]. In other words, the variance is exploding to infinity as $t \to T$, so it will "hit" any finite value before T.

So far M_t is just a process, and we still need to identify a trading strategy in $\hat{\Theta}$ to conclude the existence of an arbitrage opportunity. Let

$$\begin{aligned}
\theta_t &= \frac{\phi_t}{\sigma S_t^*} \mathbb{I}\{0 \leq t \leq \tau\} \\
\alpha_t &= M_{t \wedge \tau} - \theta_t S_t^*
\end{aligned}$$

Let's look at α_t more closely:

$$\begin{aligned}
\alpha_t &= M_{t \wedge \tau} - \theta_t S_t^* \\
&= \int_0^{t \wedge \tau} \phi_s dw_s^* - \theta_t S_t^* \\
&= \int_0^t \sigma S_t^* \theta_t dw_s^* - \theta_t S_t^* \\
&= \int_0^t \theta_s dG_s^* - \theta_t S_t^*
\end{aligned}$$

where we used the fact that $\sigma S_t^* dw_t^* = dG_t^*$ in the Black-Scholes economy. Note that the bounds of integration changed because once $\tau < t$, the indicator function in θ_t will kick in and make the integrand zero.

Then, we know that by rearranging the expression for α_t in the first and the last lines,

$$\alpha_t + \theta_t S_t^* = M_{t \wedge \tau} = \int_0^t \theta_s dG_s^*$$

For $t = 0$, we know that $\alpha_0 + \theta_0 S_0^* = M_{0 \wedge \tau} = M_0 = 0$, we can just write

$$\alpha_t + \theta_t^* S_t^* = \alpha_0 + \theta_0 S_0^* + \int_0^t \theta_s dG_s^*$$

[13] For any $\theta_1, \theta_2 \in \mathcal{H}^2$

$$E\left[\left(\int_0^t \theta_1(s) dw_s\right)\left(\int_0^t \theta_2(s) dw_s\right)\right] = E\left[\int_0^t \theta_1(s)\theta_2(s) ds\right]$$

which shows that (α, θ) is tight. Furthermore,

$$\begin{aligned}
\int_0^T |\theta_t S_t|^2 \, dt &= \int_0^T \left| \frac{\phi_t B_t}{\sigma} \mathbb{I}\{0 \le t \le \tau\} \right|^2 dt = \int_0^\tau \left| \frac{\phi_t B_t}{\sigma} \right|^2 dt \\
&\le \left(\frac{B_0 e^{|r|T}}{\sigma} \right)^2 \int_0^T |\phi_t|^2 \, dt \\
&= \left(\frac{B_0 e^{|r|T}}{\sigma} \right)^2 \log \left(\frac{T}{T-\tau} \right) < \infty \text{ a.s.}
\end{aligned}$$

The last line follows as we have that $\tau < T$ almost surely. This is a random object that is finite with probability 1.

This concludes our proof that $(\alpha, \theta) \in \hat{\Theta}$. Is it an arbitrage opportunity? Recall that an arbitrage opportunity has zero cost, and has a terminal value greater than zero:

$$\begin{aligned}
\alpha_0 + \theta_0 S_0^* &= 0 \\
\alpha_T + \theta_T S_T^* &= M_{T \wedge \tau} = M_\tau \ge L
\end{aligned}$$

In fact, $M_\tau = L$ because time is continuous. So we have $\alpha_T + \theta_T S_T^* = L > 0$. Thus, this trading strategy constitutes an arbitrage.

Remark 110. What's happening here?

- Position gets larger and larger as $t \to T$. The agent is simultaneously borrowing to finance his purchases.
- As soon as you make \$$L$ at τ, you stop immediately.
- Note the value of strategy is

  $$\alpha_t + \theta_t S_t^* = \alpha_0 + \theta_0 S_0^* + \int_0^t \theta_s dG_s^*$$

  This is an example of a stochastic integral that is NOT a martingale:

  $$\begin{aligned}
  \alpha_0 + \theta_0 S_0^* &= 0 \\
  \alpha_T + \theta_T S_T^* &= L > 0
  \end{aligned}$$

  Thus

  $$\alpha_0 + \theta_0 S_0^* \ne E^Q \left[\alpha_T + \theta_T S_T^* \right]$$

- This trading strategy also requires one to borrow unboundedly large amounts of money (potentially). One way to rule out doubling strategies, then, is to rule out this possibility.

What is the best way to proceed? We would like to have no arbitrage, and thus rule out doubling strategies. One hint is to think of Q-martingales! First note if (α, θ) is tight, then

$$\begin{aligned}
\alpha_t + \theta_t S_t^* &= \alpha_0 + \theta_0 S_0^* + \int_0^t \theta_s S_s^* \sigma dw_s^* \\
&= \alpha_0 + \theta_0 S_0^* + \int_0^t \theta_s dG_s^*
\end{aligned}$$

Now suppose the stochastic integral were a Q-martingale, so that process $Z_t = (\alpha_t + \theta_t S_t^*) - (\alpha_0 + \theta_0 S_0^*)$ is a Q-martingale. Then, taking time-0 expectation of Z_T,

$$\alpha_0 + \theta_0 S_0^* = E^Q \left[\alpha_T + \theta_T S_T^* \right]$$

But to have arbitrage, we need

$$\begin{aligned}
\alpha_T + \theta_T S_T^* &> 0 \\
\alpha_0 + \theta_0 S_0^* &\le 0
\end{aligned}$$

which is impossible if the stochastic integral were a Q-martingale!

So this means that if we somehow make the stochastic integral $\int_0^t \theta_s dG_s^*$ a Q-martingale, this will effectively rule out aribtrage, including the type of doubling strategies that we have seen.

Therefore, we revise the space of trading strategies as follows:

$$\Theta = \left\{ (\alpha, \theta) : E^P \left[\int_0^T |\theta_t S_t|^2 \, dt \right] < \infty \text{ and } (\alpha, \theta) \text{ is tight} \right\}$$

Now we have a proposition that says if a trading strategy is in this space, then this gives us the property that $Z_t = (\alpha_t + \theta_t S_t^*) - (\alpha_0 + \theta_0 S_0^*)$ is a Q-martingale (which in turn rules out arbitrage).

Proposition 111. *Let $(\alpha, \theta) \in \theta$. Then, $W^* = \alpha + \theta S^*$ is a $L^p(Q)$ martingale for all $p \in (1,2)$.*

Proof. Fix some $p \in (1,2)$. Then, the Martingale Representation Theorem says that

$$\alpha_t + \theta_t S_t^* = \alpha_0 + \theta_0 S_0^* + \int_0^t \theta_s S_s^* \sigma dw_s^*$$

is a $L^p(Q)$ martingale if and only if

$$E^Q \left[\left(\int_0^T |\theta_s S_s^*|^2 \, ds \right)^{p/2} \right] < \infty$$

Making the change of measure

$$E^Q \left[\left(\int_0^T |\theta_s S_s^*|^2 \, ds \right)^{p/2} \right] = E^P \left[\xi_T \left(\int_0^T |\theta_s S_s^*|^2 \, ds \right)^{p/2} \right]$$

Apply Holder's inequality on ξ_T with $\hat{q} = \frac{\hat{p}}{\hat{p}-1}$ and $\left(\int_0^T |\theta_s S_s^*|^2 \, ds \right)^{p/2}$ with $\hat{p} = \frac{2}{p} \in (1,2)$. Then, we have $\frac{1}{\hat{p}} + \frac{1}{\hat{q}} = 1$ by construction, and that

$$E^P \left[\xi_T \left(\int_0^T |\theta_s S_s^*|^2 \, ds \right)^{p/2} \right] \leq E^P \left[(\xi_T)^{\hat{q}} \right]^{1/\hat{q}} E^P \left[\left(\int_0^T |\theta_s S_s^*|^2 \, ds \right)^{\frac{p}{2} \cdot \hat{p}} \right]^{1/\hat{p}}$$

where $\frac{p}{2} \cdot \hat{p} = 1$. Now $E^P \left[(\xi_T)^{\hat{q}} \right]^{1/\hat{q}} < \infty$ because ξ_T is log-normal, and the second term is finite because we have $E^P \left[\int_0^T |\theta_t S_t|^2 \, dt \right] < \infty$ by definition of Θ. $\square$

Remark 112. A few remarks are in order:

- This is an extremely powerful result. It says that, no matter which trading strategy that we pick $(\alpha, \theta) \in \Theta$, wealth in units of bond, will always be a martingale. This is a result that we will exploit over and over again.
- Note we effectively made the space of trading strategies Θ smaller, so that we can rule out arbitrage. But how do we know we didn't make it too small? In particular, how do we know the market is still dynamically complete? This is what the next lemma explores.

Lemma 113. *Any $x \in L^p(P)$ where $p \in (2, \infty]$ is financed by a trading strategy $(\alpha, \theta) \in \Theta$, where*

$$\Theta = \left\{ (\alpha, \theta) : E^P \left[\int_0^T |\theta_t S_t|^2 \, dt \right] < \infty \textit{ and } (\alpha, \theta) \textit{ is tight} \right\}$$

Proof. Given as an exercise in homework. $\square$

Lemma 114. *The marketed space $\mathcal{M}$, which is the set of all consumption financed by trading strategies in Θ, is closed in $L^2(P)$.*

The proof for this is long and technical, so we skip it. But notice what it says. The limit of any sequence of marketed consumption $\{c^k\}$ is also a marketed consumption plan.

Theorem 115. *Any $x \in \mathcal{C} = L^2(P)$ is financed by a trading strategy $(\alpha, \theta) \in \Theta$.*

Proof. Fix $x \in L^2(P)$. Then, because $L^\infty(P)$ is dense in $L^2(P)$, there exists $\{x_n\} \subset L^\infty(P)$ such that

$$\{x_n\} \overset{L^2(P)}{\to} x$$

By Lemma 8, each $\{x_n\} \subset \mathcal{M}$, because any $x \in L^\infty(P)$ is financed by some trading strategy $(\alpha, \theta) \in \theta$. Now, since $\{x_n\} \to x$, and $\mathcal{M}$ is closed by Lemma 9, we have $x \in \mathcal{M}$. $\square$

Remark 116. Let's review what we have done so far.

(i) We have defined the specific application of Black-Scholes economy, and have shown that it is viable by constructing a unique equivalent martingale measure.

(ii) Then we began exploring the space of trading strategies that we should use, and said that requiring $\int_0^T |\theta_t S_t|^2 < \infty$ gave us dynamic completeness.

(iii) But this space was too large, in the sense that it allowed doubling strategies, and showed the more restrictive condition $E^P\left[\int_0^T |\theta_t S_t|^2 \, dt\right] < \infty$ was more appropriate because it kept dynamic completeness while ruling out arbitrage.

4.4 European Contingent Claims

We did a lot of work so far in carefully defining the space of trading strategies, showing results about the Q-martingale property of wealth, etc.

This was building up to the pricing of contingent claims.

Definition 117. A European contingent claim is a claim to a payoff $x \in \mathcal{C}$, where the term European means the cash-flow is at a certain time (at maturity T).

A European contingent claim is path independent if there exists a smooth functional ϕ such that $x = \phi(S_T)$. That is, it only depends (smoothly) on the terminal value of the stock.

What are some examples of path-dependent European contingent claims? For instance, if you take the average price over time, or the min / max price over time, it will give rise to path-dependency.

Definition 118. Let $x = \phi(S_T)$ be a path-independent contingent claim. We define

$$S_t^x = \alpha_t B_t + \theta_t S_t$$

where $(\alpha, \theta) \in \Theta$ finances x. That is, S_t^x is the time-t value of the trading strategy that replicates x.

Our key object of interest is this S_t^x, and how we can price this object.

Proposition 119. *Given path-independence,*

$$S_t^x = B_t E^Q\left[B_T^{-1} x \middle| \mathcal{F}_t\right]$$

Proof. Suppose $(\alpha, \theta) \in \Theta$ finances x. Then by definition of "finances x",

$$\alpha_T B_T + \theta_T S_T = x$$

Furthermore, by definition of S_t^x,

$$\begin{aligned} S_t^x &= (\alpha_t B_t + \theta_t S_t) = B_t (\alpha_t + \theta_t S_t^*) \\ &= B_t E^Q \left[\alpha_T + \theta_T S_T^* \middle| \mathcal{F}_t \right] \\ &= B_t E^Q \left[\frac{x}{B_T} \middle| \mathcal{F}_t \right] \end{aligned}$$

where the second line follows because $\alpha_t + \theta_t S_t^*$ is a $Q-$martingale, and the last line follows from rearranging the $\alpha_T B_T + \theta_T S_T = x$ condition. □

With path-independence, we can make a little more progress:

$$S_t^x = B_t E^Q \left[\frac{\phi(S_T)}{B_T} \middle| \mathcal{F}_t \right] = e^{-r(T-t)} E^Q \left[\phi(S_T) \middle| \mathcal{F}_t \right]$$

To get a sharp characterization of this object, we need to take $E^Q[\cdot]$. First, what are the dynamics of stock under Q?

$$\begin{aligned} dS_t &= S_t \mu dt + S_t \sigma dw_t \\ &= S_t (\mu + \sigma\kappa) dt + S_t \sigma dw_t^* \\ &= S_t (r - \delta) dt + S_t \sigma dw_t^* \end{aligned}$$

where we used that $dw_t = dw_t^* + \kappa dt$. This stochastic differential equation has the form of a geometric Brownian motion, and we can solve it:

$$S_t = S_0 \exp\left\{ \left(r - \delta - \frac{\sigma^2}{2} \right) t + \sigma w_t^* \right\}$$

At $t = T$, we have $S_T = S_0 \exp\left\{ \left(r - \delta - \frac{\sigma^2}{2} \right) T + \sigma w_T^* \right\}$. We can divide S_T by S_t to get

$$\begin{aligned} S_T &= S_t \exp\left\{ \left(r - \delta - \frac{\sigma^2}{2} \right) (T-t) + \sigma (w_T^* - w_t^*) \right\} \\ &= S_t \exp\left\{ \left(r - \delta - \frac{\sigma^2}{2} \right) (T-t) + \sigma z \sqrt{T-t} \right\} \end{aligned}$$

for $z \sim N(0,1)$. Note $Z \equiv \frac{w_T^* - w_t^*}{\sqrt{T-t}}$ is independent of $\mathcal{F}_t$, and the reason we took ratios is to take advantage of the independence property of Brownian increments. We also want to express S_T in terms of "state" variable S_t.

Going back to expectation, this gives us the formula:

$$\boxed{S_t^x = e^{-r(T-t)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \phi \left(S_t \exp\left\{ \left(r - \delta - \frac{\sigma^2}{2} \right) (T-t) + \sigma y \sqrt{T-t} \right\} \right) dy = F(S_t, t)}$$

given a smooth functional $\phi(\cdot)$.

Let's go through an example to see this put to work.

Example 120. ("European Call Option")

Let K be the strike price, $x = (S_T - K)^+$ be the contingent claim. Using the previous formula gives us

$$S_t^x = e^{-r(T-t)} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \left(S_t \exp\left\{ \left(r - \delta - \frac{\sigma^2}{2} \right) (T-t) + \sigma y \sqrt{T-t} \right\} - K \right)^+ dy$$

Let's evaluate this integral. From here on, it's just going to be a bunch of algebra. To evaluate $(\cdot)^{+}$, let y^{*} be the point at which the max operator kicks in,

$$S_t \exp\left\{\left(r-\delta-\frac{\sigma^2}{2}\right)(T-t)+\sigma y^{*}\sqrt{T-t}\right\}=K$$

which implies

$$y^{*}=\frac{\log\left(K/S_t\right)-\left(r-\delta-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

Then,

$$\begin{aligned}
S_t^x &= e^{-r(T-t)}\int_{y^*}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}\left(S_t\exp\left\{\left(r-\delta-\frac{\sigma^2}{2}\right)(T-t)+\sigma y\sqrt{T-t}\right\}-K\right)dy\\
&= S_t e^{-r(T-t)}\int_{y^*}^{+\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}y^2+\left(r-\delta-\frac{\sigma^2}{2}\right)(T-t)+\sigma y\sqrt{T-t}\right\}dy\\
&- Ke^{-r(T-t)}\int_{y^*}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}dy\\
&= S_t e^{-\delta(T-t)}\int_{y^*}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(y-\sigma\sqrt{T-t}\right)^2}dy-Ke^{-r(T-t)}\int_{y^*}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}dy
\end{aligned}$$

[14]Define new variables of integration $u := -\left(y-\sigma\left(T-t\right)\right)$ and $v := -y$, so that

$$S_t^x = S_t e^{-\delta(T-t)}\int_{-\infty}^{-\left(y^*-\sigma\sqrt{T-t}\right)}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}du-Ke^{-r(T-t)}\int_{-\infty}^{-y^*}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}v^2}dv$$

which gives us the Black-Scholes formula:

$$\boxed{S_t^x = S_t e^{-\delta(T-t)}\Phi\left(-\left(y^*-\sigma\sqrt{T-t}\right)\right)-Ke^{-r(T-t)}\Phi\left(-y^*\right)=F\left(S_t,t\right)}$$

where $\Phi(\cdot)$ is the CDF of standard normal.

Replicating Trading Strategies. Suppose now we ask a different question: How can we find the (α,θ) that finances the contingent claim? Furthermore, how can we find the contribution of stock and? We can do this by an application of Ito's Lemma under Q. Recall that stock dynamics under Q follow $dS_t = S_t\left(r-\delta\right)dt + S_t\sigma dw_t^*$.

Take the differential of

$$\begin{aligned}
d\left(B_t^{-1}F\left(S_t,t\right)\right) &= B_t^{-1}\left(DF\left(S_t,t\right)dt+F_s\left(\cdot\right)S_t\sigma dw_t^*\right)-B_t^{-1}F\left(\cdot\right)rdt\\
&= B_t^{-1}\left(\underbrace{DF\left(\cdot\right)-rF\left(\cdot\right)}_{=0}\right)dt+B_t^{-1}F_s\left(\cdot\right)S_t\sigma dw_t^*
\end{aligned}$$

where the drift term is zero by the martingale property. Note $DF\left(\right)=F_t+F_sS\left(r-\delta\right)+\frac{1}{2}F_{ss}S^2\sigma^2$.

Here, $DF(\cdot)$ is called the ***Dinkyn Operator*** and returns the drift term of the function. It is used fairly often in stochastic differential equations.

Earlier, we wrote $S_t^x = B_tE^Q\left[\left.B_T^{-1}\phi\left(S_T\right)\right|\mathcal{F}_t\right]$, so in turn we can write

$$\frac{F\left(S_t,t\right)}{B_t}=E^Q\left[\left.\frac{\phi\left(S_T\right)}{B_T}\right|\mathcal{F}_t\right]$$

[14]Note

$$\begin{aligned}
\exp\left\{-\frac{1}{2}y^2+\left(r-\delta-\frac{\sigma^2}{2}\right)(T-t)+\sigma y\sqrt{T-t}\right\} &= e^{(r-\delta)(T-t)}\exp\left\{-\frac{1}{2}\left(y^2-2y\sigma\sqrt{T-t}+\sigma^2\left(T-t\right)\right)\right\}\\
&= e^{(r-\delta)(T-t)}\exp\left\{-\frac{1}{2}\left(y-\sigma\sqrt{T-t}\right)^2\right\}
\end{aligned}$$

which is a Q-martingale, as it is a "conditional expectation" process. So the process $\frac{F(S_t,t)}{B_t}$ should have zero drift, as written above. Imposing zero drift,

$$\begin{aligned}\frac{F(S_t,t)}{B_t} &= \frac{F(S_0,0)}{B_0} + \int_0^t F_S(S_s,s)\,S_s^*\sigma dw_s^* \\ &= \frac{F(S_0,0)}{B_0} + \int_0^t F_S(S_s,s)\,dG_s^*\end{aligned}$$

where $S_s^*\sigma dw_s^* = dG_S^*$. Note that $\frac{F(S_t,t)}{B_t}$ is the time-t value of the portfolio position in bond numeraire, so whatever (α,θ) trading strategy that replicates this has the property that

$$\alpha_t + \theta_t S_t^* = \alpha_0 + \theta_0 S_0^* + \int_0^t F_S(S_s,s)\,dG_s^*$$

On the other hand, it is a property of tight trading strategies that

$$\alpha_t + \theta_t S_t^* = \alpha_0 + \theta_0 S_0^* + \int_0^t \theta_s dG_s^*$$

Next step is to match integrals, to say that

$$\int_0^t F_S(S_s,s)\,dG_s^* = \int_0^t \theta_s dG_s^*$$

so as to identify the stock position

$$\theta_t = F_S(S_t,t)$$

Furthermore, for the bond position,

$$F(S_t,t) = \alpha_t B_t + \theta_t S_t \implies \alpha_t = \frac{F(S_t,t) - F_S(S_t,t)\,S_t}{B_t}$$

We can use this to back out shares in stock as

$$S_t^x = \underbrace{S_t e^{-\delta(T-t)}\Phi\left(-\left(y^* - \sigma\sqrt{T-t}\right)\right)}_{=\theta_t} + \left[\underbrace{-Ke^{-r(T-t)}\Phi(-y^*)}_{=\alpha_t B_t}\right]$$

The PDE Approach. We also know that $F(S_t,t)$ must satisfy a 2nd order partial differential equation and a terminal condition:

$$\begin{aligned}0 &= F_S S(r-\delta) + F_t + \frac{1}{2}F_{SS}S^2\sigma^2 - rF \\ F(S_T,T) &= \phi(S_T)\end{aligned}$$

This is known as a ***Cauchy Problem***, and the ***Feynman-Kac formula*** tells us the solution is $F(S_t,t)$.

Black and Scholes did not know about equivalent martingale measures at the time, so they did something different instead. Try to identify gaps in their argument. It went as follows.

> Let's say the price of a call is $F(S_t,t)$. Suppose it's twice continuously differentiable, and apply Ito's Lemma under P measure:
>
> $$dF(S_t,t) = \left[F_S S_t\mu + F_t + \frac{1}{2}F_{SS}S_t\sigma^2\right]dt + F_S S_t\sigma dw_t$$
>
> Now consider the portfolio where you buy a call and short F_S units of stocks.:
>
> $$V(t) = F(S_t,t) - F_S(S_t,t)\,S_t$$

Change in the portfolio value must equal the change in value of call minus change in stock shares:

$$\begin{aligned} dV(t) &= dF(S_t, t) - F_S(S_t, t)\, dS_t \\ &= \left[F_t + \frac{1}{2} F_{SS} S_t \sigma^2\right] dt \end{aligned}$$

Return over the next instant has no Brownian motion term, so it must be equal to the riskless rate! Return is then $V(t)\, rdt$,so

$$dV(t) = [F - F_S S_t]\, rdt$$

The two square bracket terms must match, leading us to the conditions

$$\begin{aligned} 0 &= F_t + \frac{1}{2} F_{SS} S \sigma^2 - rF + F_S S r \\ F(S_T, T) &= (S_T - K)^+ \end{aligned}$$

for the case of no dividends.

Remark 121. Black and Scholes gap in argument is getting to the $dV(t)$ term, as that is only true if the trading strategy is self-financing.

- Black and Scholes got to the PDE fairly quickly, but had issues solving the PDE.
- This shows the advantage of the Martingale Approach: It takes us directly to the integral.
- It also makes it easier to solve consumption problems.

4.5 The Individual's Consumption Problem

4.5.1 The Stochastic Control Approach: An Introduction

Our focus in this section will be solving a dynamic optimization problem of the ***Bolza Type*:**

$$\begin{aligned} &\max_{c \in \mathcal{C}} E\left[\int_0^T u(c_t, X_t^c, t)\, dt + V(X_T^c)\right] \\ \text{s.t. } &X_t^c = X_0 + \int_0^t \mu(c_s, X_s^c, s)\, ds + \int_0^t \sigma(c_s, X_s^c, s)\, dw_s \end{aligned}$$

- Now we turn to solving the individual consumption problem in continuous-time. Merton was the first to import "stochastic control" to finance − which is the theory of maximizing an objective that depends on a stochastic process.
- Here, a control c is a multidimensional process that can affect $\mu(\cdot)$ and $\sigma(\cdot)$. Note here both u and V are given functions, and so are μ and σ. The agent "controls" c.
- Control space is
$$\mathcal{C} = \{c : c_t \in K \subset \mathbb{R}^m, \forall t \in [0, T]\}$$
Our goal is then to characterize the optimal control c^*.

First step is to show that you can do no better than to choose a Markov control,

$$c_t = c(X_t^c, t)$$

and given Brownian processes, you can restrict yourself to these processes without loss of generality.

Substituting back to μ and σ, then we obtain that the state follows a diffusion:

$$X_t^c = X_0 + \int_0^t \mu\left(c\left(X_t^c, t\right), X_s^c, s\right) ds + \int_0^t \sigma\left(c\left(X_t^c, t\right), X_s^c, s\right) dw_s$$

Diffusions are Markov!

Define *reward function* J^c: this is the value of the objective starting at t and following an optimal control c afterwards:

$$J^c\left(X_t, t\right) = E\left[\int_t^T u\left(c_s, X_s^c, s\right) ds + V\left(X_T^c\right) \middle| \mathcal{F}_t\right]$$

The fact that $J^c = J^c\left(X_t, t\right)$ is a consequence of the Markov property. The expectation of time t and onwards only depend on the X_t^c state.

Define ***value function*** J:

$$J\left(X_t, t\right) = \sup_{c \in \mathcal{C}} J^c\left(X_t, t\right)$$

Note if c^* is optimal, then $J\left(X_t, t\right) = J^{c^*}\left(X_t, t\right)$

4.5.2 Deriving the Hamilton-Jacobi-Bellman Equation.

Consider any $t < \tau \leq T$:

$$\begin{aligned} J\left(X_t^c, t\right) &= \max_{c \in \mathcal{C}} E\left[\int_t^\tau u\left(c_s, X_s^c, s\right) ds + \underbrace{\int_\tau^T u\left(c_s, X_s^c, s\right) ds + V\left(X_T^c\right)}_{\text{time}-\tau \text{ value}} \middle| \mathcal{F}_t\right] \\ &= \max_{c \in \mathcal{C}} E\left[\int_t^\tau u\left(c_s, X_s^c, s\right) ds + J\left(X_\tau^c, \tau\right) \middle| X_t^c = X\right] \end{aligned}$$

where we replaced the sup with the max, assuming the solution exists. By Markov, we switch the filtration $\mathcal{F}_t$ for $X_t^c = X$. This is the *principle of optimality* at work: the value at time t is equal to the maximized value of flow over (t, τ) and the continuation value at τ.

By Ito's Lemma on $J\left(X_t^c, \tau\right)$, we have

$$J\left(X_\tau^c, \tau\right) = J\left(X_t^c, t\right) + \int_t^\tau DJ\left(\cdot\right) ds + \int_t^\tau J_x\left(\cdot\right)^\top \sigma\left(\cdot\right) dw_s$$

It turns out $\int_t^\tau J_x\left(\cdot\right)^\top \sigma\left(\cdot\right) dw_s$ is a P-local martingale. If it is an actual martingale, (not just a local-MG),

$$E\left[J\left(X_\tau^c, \tau\right) \middle| \mathcal{F}_t\right] = J\left(X_t^c, t\right) + E\left[\int_t^\tau DJ\left(\cdot\right) ds \middle| \mathcal{F}_t\right]$$

because we would then have $E\left[\int_t^\tau J_x\left(\cdot\right)^\top \sigma\left(\cdot\right) dw_s \middle| \mathcal{F}_t\right] = \int_t^t J_x\left(\cdot\right)^\top \sigma\left(\cdot\right) dw_s = 0$ as time t expectation of a future τ value is equal to its time t value (which happens to be zero).

Substitute this into our earlier expression of $J\left(X_t^c, t\right)$:

$$\begin{aligned} J\left(X_t^c, t\right) &= \max_{c \in \mathcal{C}} E\left[\int_t^\tau \left(u\left(c_s, X_s^c, s\right) + DJ\left(\cdot\right)\right) ds \middle| X_t^c = X\right] + J\left(X_t^c, t\right) \\ \implies 0 &= \max_{c \in \mathcal{C}} E\left[\int_t^\tau \left(u\left(c_s, X_s^c, s\right) + DJ\left(\cdot\right)\right) ds \middle| X_t^c = X\right] \end{aligned}$$

Divide by the time interval, and take limits:

$$0 = \lim_{\tau \downarrow t} \left\{\max_{c \in \mathcal{C}} E\left[\frac{1}{\tau - t} \int_t^\tau \left(u\left(c_s, X_s^c, s\right) + DJ\left(\cdot\right)\right) ds \middle| X_t^c = X\right]\right\}$$

If we can exchange the limit with the expectation,

$$0 = \max_{c\in\mathcal{C}} E\left[\lim_{\tau\downarrow t}\left\{\frac{1}{\tau - t}\int_t^\tau \left(u\left(c_s, X_s^c, s\right) + DJ\left(\cdot\right)\right) ds\right\}\middle| X_t^c = X\right]$$

Note that the term in the $\{\cdot\}$ is now the average value of the integral over $s \in [t,\tau)$. As $\tau \downarrow t$, this converges to time-t value, and we can ignore the expectation for the time t value.

This gets us to the Hamilton-Jacobi-Bellman Equation:

$$\begin{aligned} 0 &= \max_{c\in\mathcal{C}}\left\{u\left(c_t, X, t\right) + DJ\left(X,t\right)\right\} \\ J\left(X,T\right) &= V\left(X\right) \end{aligned}$$

where $DJ = J_x^\top \mu + J_t + \frac{1}{2}\text{tr}\left(J_{xx}\sigma\sigma^\top\right)$, a second-order differential equation.

Remark 122. The solution works if our steps so far (and the assumptions that we made along the way) are warranted. This includes the assumption that the stochastic integral is a real martingale, and that we could exchange limits with the expectations.

- *Verification Theorems* guarantee that c is the optimal control, and that J is the value function. It is what guarantees that the solution to the HJB equation is the value function.

4.5.3 Investment Problem

The problem we are interested in has utility

$$U\left(W\right) = E\left[\frac{W^{1-b}}{1-b}\right]$$

with $b > 0$, $y_0 > 0$.

The problem is

$$\max_{(\alpha,\theta)\in\Theta} E\left[\frac{\left(\alpha_T B_T + \theta_T S_T\right)^{1-b}}{1-b}\right]$$

$$\begin{aligned} \alpha_0 B_0 + \theta_0 S_0 &\leq y_0 \\ \alpha_T B_T + \theta_T S_T &\in \mathcal{C}_+ \end{aligned}$$

where $\mathcal{C}_+$ is the positive orthant of the consumption space. Here we are considering tight trading strategies, so we really just have to check that the initial cost of the trading strategy is something the agent can afford.

To analyze this problem using the machinery developed in the previous section, we would like it to be in the same framework as before. In other words, can we appropriately define the objects $u\left(\cdot\right)$, $V\left(\cdot\right)$, state and control variables?

Well to start, note that the terminal value $V\left(X_T^c\right)$ in the objective can only depend on the value of the state variable – so at the very least, the state needs to include wealth. Let's try taking the trading strategy (α,θ) as the control variable. Thus we have the controls c and state X_t where

$$\begin{aligned} c &= (\alpha,\theta) \\ X_t = W_t &= \alpha_t B_t + \theta_t S_t \end{aligned}$$

where given a tight strategy (α,θ) we have

$$W_t = W_0 + \int_0^t \alpha_s B_s r ds + \int_0^t \theta_s S_s\left(\mu+\delta\right) ds + \int_0^t \theta_s S_s \sigma dw_s$$

Does this satisfy our framework? No! Recall the Bolza type problem has

$$\max_{c\in\mathcal{C}} E\left[\int_0^T u\left(c_t, X_t^c, t\right) dt + V\left(X_T^c\right)\right]$$

$$\text{s.t. } X_t^c = X_0 + \int_0^t a ds + \int_0^t \sigma\left(c_s, X_s^c, s\right) dw_s$$

which means in particular, the drift term and volatility terms have the following form:

$$\mu\left(c_t, X_t^c, t\right) \quad \text{and} \quad \sigma\left(c_t, X_t^c, t\right)$$

But in our case, the process for wealth features a drift term and a volatility term that both depend on the stock price S_t! So this doesn't fit.

At this point we have two ways to proceed: (1) We can include S_t as a state variable, so that $X_t^c = (W_t, S_t)$, a 2-dimensional state process; or (2) we can change the controls to be relative *weights on risk asset* instead:

$$X_t^c = \frac{\theta_t S_t}{W_t}$$

Note that to follow the second route, we need $W_t \neq 0$ for all t, as we can't divide by zero. For CRRA utility, the marginal utility at zero is infinite, so such an agent would avoid it to the extent that he can afford it: thus $\alpha_T B_T + \theta_T S_T > 0$ almost surely[15]. Now, see that

$$\alpha_t + \theta_t S_t^* = E^Q\left[\frac{\alpha_T B_T + \theta_T S_T}{B_T}\middle|\mathcal{F}_t\right] > 0, \quad \text{a.s.}$$

because $\frac{\alpha_T B_T + \theta_T S_T}{B_T} > 0$ almost surely, so is the expectation. This implies that for all t, $B_t\left(\alpha_t + \theta_t S_t^*\right) > 0$ almost surely.

Reformulating with weights. Okay now let's reformulate the problem with risky weights $x_t = \frac{\theta_t S_t}{W_t}$. This means

$$\begin{aligned}
\alpha_t B_t &= \left(1 - x_t\right) W_t \\
\theta_t S_t &= x_t W_t
\end{aligned}$$

and the wealth process now becomes

$$\begin{aligned}
W_t &= W_0 + \int_0^t W_s\left(1 - x_s\right) r ds + \int_0^t W_s x_s\left(\mu + \delta\right) ds + \int_0^t W_s x_s \sigma dW_s \\
&= W_0 + \int_0^t \underbrace{W_s\left[r + x\left(\mu + \delta - r\right)\right]}_{\equiv \mu\left(X_t^c, c, s\right)} ds + \int_0^t \underbrace{W_s x_s \sigma d}_{\equiv \sigma\left(X_t^c, c, s\right)} W_s
\end{aligned}$$

where now the functional dependence is as we want it to be! Thus the framework applies.

Let's go through some technical details. So far the optimization problem becomes

$$\max_x E\left[\frac{W_T^{1-b}}{1-b}\right]$$

subject to

$$\begin{aligned}
W_t &= W_0 + \int_0^t W_s\left[r + x\left(\mu + \delta - r\right)\right] ds + \int_0^t W_s x_s \sigma dw_s \\
W_0 &= y_0
\end{aligned}$$

where y_0 is the initial endowment[16]. In addition, we had $\alpha_T B_T + \theta_T S_T \in \mathcal{C}_+$, but as $\alpha_T B_T + \theta_T S_T > 0$ almost surely, we only just require that $\alpha_T B_T + \theta_T S_T \in \mathcal{C}$, where $\mathcal{C} = L^2$, it suffices to require $E\left(W_T^2\right) < \infty$.

[15] As a side comment, notice that this is true even if S_t were a Brownian motion instead of a geometric Brownian motion, because the agent could place all his money in a bond instead. So here we really just need CRRA. Note this is not true for exponential utility.

[16] To justify the switch from inequality to equal sign in the second line, note that by monotone preferences our initial

$$\alpha_0 B_0 + \theta_0 S_0 = W_0 \leq y_0$$

must hold with equality.

How about $(\alpha, \theta) \in \Theta$? Earlier we had

$$\Theta = \left\{ (\alpha, \theta) : E\left[\int_0^T |\theta_t S_t|^2 \, dt \right] < \infty \right\}$$

which in terms of the weights x_t becomes the condition on our set of control x_t that

$$E\left[\int_0^T |x_t W_t|^2 \, dt \right] < \infty$$

Final version of the optimization problem.

$$\max_x E\left(\frac{W_T^{1-b}}{1-b} \right)$$

subject to

$$W_t = W_0 + \int_0^t W_s \left[r + x\left(\mu + \delta - r\right) \right] ds + \int_0^t W_s x_s \sigma dw_s$$

$$E\left(W_T^2\right) < \infty$$
$$E\left[\int_0^T |x_t W_t|^2 \, dt \right] < \infty$$

When we are actually solving this, we will assume the last conditions don't bind, and proceed.

Optimal control solution. What is the value function $J(w, t)$? Take the HJB equation

$$\begin{aligned} 0 &= \max_x \mathcal{D}J(W, t) \\ &= \max_x \left\{ J_W W \left(r + x\left(\mu + \delta - r\right) \right) + J_t + \frac{1}{2} J_{WW} W^2 x^2 \sigma^2 \right\} \end{aligned}$$

Take the first order condition to arrive at the optimal x^*

$$0 = J_W W \left(\mu + \delta - r\right) + J_{WW} W^2 \sigma^2 x$$

which implies

$$x^* = \frac{J_W}{W J_{WW}} \frac{\kappa}{\sigma}$$

Substitute this expression for x^* back into the HJB equation and solve the PDE:

$$\begin{aligned} 0 &= J_W W \left(r + x\left(\mu + \delta - r\right) \right) + J_t + \frac{1}{2} J_{WW} W^2 x^2 \sigma^2 \\ &= J_W W \left(r + \left(\frac{J_W}{W J_{WW}} \frac{\kappa}{\sigma} \right) \left(\mu + \delta - r\right) \right) + J_t + \frac{1}{2} J_{WW} W^2 \left(\frac{J_W}{W J_{WW}} \frac{\kappa}{\sigma} \right)^2 \sigma^2 \\ &= J_W W r - \frac{1}{2} \frac{J_W^2}{J_{WW}} \kappa^2 + J_t \end{aligned}$$

To obtain an explicit expression for the value function $J(W, t)$, solve the following PDE with terminal condition:

$$\begin{cases} 0 &= J_W W r - \frac{1}{2} \frac{J_W^2}{J_{WW}} \kappa^2 + J_t \\ J(W, T) &= \frac{W^{1-b}}{1-b} \end{cases}$$

Guess and verify is the way to go here. What's a good guess? Well, we know that at time T, it must somehow become a power function (due to the terminal condition). A good guess in that (1) it's sensible; and (2) makes progress for us in terms of pinning down J is that it is separable in W and t:

$$J(W,t) = g(t)\frac{W^{1-b}}{1-b}, \quad g(T) = 1$$

So let's try this. Plugging in, we get

$$0 = \frac{W^{1-b}}{1-b}\left[g'(t) + (1-b)\left(r + \frac{\kappa^2}{2b}\right)g(t)\right]$$

Since this holds for all W and t, we need the expression inside the square brackets to be zero. Define the constant $\gamma := (1-b)\left(r + \frac{\kappa^2}{2b}\right)$, and we get ourselves a first order ODE:

$$g'(t) + \gamma g(t) = 0$$

This is what I meant by making progress earlier – this is still a differential equation, but as far as differential equations go, this is about as simple as it gets. Solving for g,

$$g(t) = e^{\gamma(T-t)}, \quad g(T) = 1$$

Now we have our solutions:

$$J(W,t) = e^{\gamma(T-t)}\frac{W^{1-b}}{1-b}$$
$$x^* = -\frac{\kappa}{\sigma} = \frac{\mu+\delta-r}{b\sigma^2}$$

where $\gamma = (1-b)\left(r + \frac{\kappa^2}{2b}\right)$. Once you have J you can get x^* via our earlier FOC $x^* = \frac{J_W}{WJ_{WW}}\frac{\kappa}{\sigma}$.

Remark 123. What does this mean? An optimal rule is to invest in a constant fraction into the stock:

$$x_t^* = \frac{\mu+\delta-r}{b\sigma^2} = f\left(\overset{+}{\mu}, \overset{+}{\delta}, \overset{-}{r}, \overset{-}{b}, \overset{-}{\sigma}\right)$$

where the "+" means x^* is increasing and "−" means decreasing.

Also check that $E\left(W_T^2\right) < \infty$ as W_t now has some constant drift term and a constant volatility term. Thus W_t is a geometric Brownian motion (hence the lognormal distribution gives us finite second moments). Similarly we have $E\left[\int_0^T |x_t W_t|^2\, dt\right] < \infty$.

Remark 124. Comments:

- This is the simplest type of optimization we can do.
- Check verification theorems at the appendix in lecture notes.
- Now we look at an alternative way to solve this.

4.5.4 The Martingale Approach

Since markets are complete, we can finance any contingent claim at time T. As an alternative approach, let's ignore trading strategies altogether, and cut directly to terminal wealth. You'll see that in fact this is a simpler way to do things.

We can do this as long as he can afford it:

$$\max_{W_T} E\left[\frac{W_T^{1-b}}{1-b}\right]$$

$$W_0 = B_0 E^Q \left[B_T^{-1} W_T \right] \leq y_0$$

The issue? The objective features expectations under P where as the constraint has expectation under Q. Let's fix this via change of measure:

$$B_0 E^Q \left[B_T^{-1} W_T \right] = E^P \left[\frac{B_0}{B_T} \xi_T W_T \right] = E^P \left[\pi W_T \right]$$

where $\pi = \frac{B_0}{B_T} \xi_T$ is the unique state price density in the Black-Scholes economy. Thus the constraint simply becomes

$$E \left[\pi W_T \right] \leq y_0$$

Thus we have the following.

Final optimization problem under the martingale approach.

$$\max_{W_T \in \mathcal{C}} E \left[\frac{W_T^{1-b}}{1-b} \right]$$

subject to

$$E \left[\pi W_T \right] \leq y_0$$

Solve this with the usual Lagrangian method:

$$\begin{aligned} &\max_{W_T \in \mathcal{C}} E \left[\frac{W_T^{1-b}}{1-b} \right] + \psi \left[y_0 - E \left[\pi W_T \right] \right] \\ =& \max_{W_T \in \mathcal{C}} E \left[\frac{W_T^{1-b}}{1-b} + \psi \left[y_0 - \pi W_T \right] \right] \end{aligned}$$

Notice what we are doing here: we are maximizing an expectation by choosing a random variable. And we do this by maximizing state-by-state (ω-by-ω): FOC $W_T^{-b} - \psi \pi = 0$ gives us

$$W_T = (\psi \pi)^{-1/b}$$

Or in other words, the marginal utility of wealth is proportional to the state price density π.

Trading strategy and multiplier ψ. The next steps are to understand: (1) What (α, θ) finances this? (2) What is the Lagrange multiplier ψ?

Well, first the portfolio value in the bond numeraire is a Q-martingale, so

$$\begin{aligned} B_t^{-1} W_t &= E^Q \left[B_T^{-1} W \middle| \mathcal{F}_t \right] \\ &= E^Q \left[B_T^{-1} (\psi \pi)^{-1/b} \middle| \mathcal{F}_t \right] \end{aligned}$$

where I used the optimality condition. Since we have

$$\pi = \frac{B_0}{B_T} \xi_T = \exp \left\{ - \left(r + \frac{\kappa^2}{2} \right) T + \kappa w_T \right\}$$

we get

$$\begin{aligned} &E^Q \left[B_T^{-1} \psi^{-1/b} \exp \left\{ \frac{1}{b} \left(r + \frac{\kappa^2}{2} \right) T - \frac{\kappa}{b} w_T \right\} \middle| \mathcal{F}_t \right] \\ =& B_T^{-1} \psi^{-1/b} \exp \left\{ \frac{1}{b} \left(r + \frac{\kappa^2}{2} \right) T - \frac{\kappa^2}{b} T \right\} E^Q \left[\exp \left\{ -\frac{1}{b} \kappa \left(w_T - \kappa T \right) \right\} \middle| \mathcal{F}_t \right] \end{aligned}$$

where I added and subtracted $\frac{1}{b}\kappa^2 T$. Since Brownian motion under Q is $w_T^* = w_T - \kappa T$,

$$
\begin{aligned}
&= B_T^{-1}\psi^{-1/b}\exp\left\{\frac{1}{b}\left(r-\frac{\kappa^2}{2}\right)T - \frac{\kappa}{b}w_t^*\right\} E^Q\left[\exp\left\{-\frac{1}{b}\kappa\left(w_T^* - w_t^*\right)\right\}\middle|\mathcal{F}_t\right] \\
&= B_T^{-1}\psi^{-1/b}\exp\left\{\frac{1}{b}\left(r-\frac{\kappa^2}{2}\right)T - \frac{\kappa}{b}w_t^* + \frac{1}{2}\frac{\kappa^2}{b^2}\left(T-t\right)\right\} \\
&= B_T^{-1}\psi^{-1/b}\exp\left\{\frac{1}{b}\left(r+\frac{1-b}{2b}\kappa^2\right)T - \frac{1}{b}\kappa w_t^* - \frac{1}{2}\frac{\kappa^2}{b^2}t\right\}
\end{aligned}
$$

where I use the moment generating function property that $E\left[e^{tX}\right] = e^{E[X]t+\frac{t^2}{2}\mathrm{Var}(X)}$ for any t.

All this algebra gives us an expression for $B_t^{-1}W_t$:

$$
B_t^{-1}W_t = B_T^{-1}\psi^{-1/b}\exp\left\{\frac{1}{b}\left(r+\frac{1-b}{2b}\kappa^2\right)T - \frac{1}{b}\kappa w_t^* - \frac{1}{2}\frac{\kappa^2}{b^2}t\right\}
$$

Because it's a martingale, the optimal wealth W satisfies

$$
B_0^{-1}W_0 = E^Q\left[B_T^{-1}W_T\right] = B_T^{-1}\psi^{-1/b}\exp\left(\frac{1}{b}\left(r+\frac{1-b}{2b}\kappa^2\right)T\right) = B_0^{-1}y_0
$$

by setting $t=0$ and using our constraint $W_0 = y_0$.

Note that this tells us what ψ should be! Rearrange the last equality (which has just ψ and other parameters) to get $\psi = B_T^{-1}\psi^{-1/b}\exp\left\{\frac{1}{b}\left(r+\frac{1-b}{2b}\kappa^2\right)T - \frac{1}{b}\kappa w_t^* - \frac{1}{2}\frac{\kappa^2}{b^2}t\right\}$. Substitute this back into the expression for $B_t^{-1}W_t$ to obtain that

$$
B_t^{-1}W_t = B_0^{-1}y_0\exp\left\{-\frac{\kappa}{b}w_t^* - \frac{1}{2}\frac{\kappa^2}{b^2}t\right\}
$$

To figure out the process, apply Ito's Lemma:

$$
\begin{aligned}
d\left(B_t^{-1}W_t\right) &= -B_t^{-1}W_t\frac{\kappa}{b}dw_t^* \\
&= -\frac{\kappa}{b\sigma}\frac{W_t}{S_t}\frac{S_t}{B_t}\sigma dw_t^* \\
&= -\frac{\kappa}{b\sigma}\frac{W_t}{S_t}dG_t^*
\end{aligned}
$$

Recall that a tight (α,θ) requires $d\left(B_t^{-1}W_t\right) = \theta_t dG_t^*$. So writing out the coefficients,

$$
\begin{aligned}
\theta_t &= -\frac{\kappa}{b\sigma}\frac{W_t}{S_t} \\
\alpha_t &= \frac{W_t - \theta_t S_t}{B_t}
\end{aligned}
$$

To compare this to our earlier solution (solved via HJB equation), the portion in stocks is

$$
\frac{\theta_t S_t}{W_t} = -\frac{\kappa}{b\sigma} = \frac{\mu+\delta-r}{b\sigma^2}
$$

Remark 125. Comments:

- Here we look at it as a Lagrangian problem
- Time-0 wealth gives you the Lagrangian multiplier ψ
- There are no differential equation s to solve in this case.

For the next section, we move back to a more general setting.

4.6 Appendix: Stochastic Control

5 The Standard Economy

Here we look at a more general setting than the Black-Scholes economy. We allow asset prices to be general Ito processes, driven by multi-dimensional Brownian motions. We revise our set of admissible trading strategies, and we now allow for intertemporal consumption c_t (not just terminal consumption W_T). However, we still refrain from incorporating jump processes.

We will learn how to price European contingent claims in this more general setting; pricing of forwards and futures; and the pricing of American contingent claims.

5.1 The Environment

The setup is as follows:

- Time $t \in [0, T]$ where $T < \infty$
- $(\Omega, \mathcal{F}, F, P)$ where $F = F^w$ for w a d-dimensional Brownian motion, $w = (w_1, \cdots, w_d)^\top$. We also impose that there is no residual uncertainty at the terminal period, $\mathcal{F} = \mathcal{F}_T$.
- The consumption space:
  - Previously we had L^2 consumption space, with preferences over just the terminal wealth.
  - Now we allow for intertemporal consumption ("c_t") and terminal wealth ("W"):

$$\mathcal{C} = \left\{ (c, W) : \int_0^T |c_t| \, dt < \infty, \text{ a.s.} \quad \text{and} \quad W < \infty, \text{ a.s.} \right\}$$

- The market:
  - Assume agents can trade $n + 1$ long-lived securities. Long-lived means tradable for all $t \in [0, T]$.
  - An example of a non- long-lived security would be a zero coupon bond with maturity before T.
- The bond:
  - This is again basically a money market account that follows

$$dB_t = B_t r_t dt$$

which implies

$$B_t = B_0 \exp\left(\int_0^t r_s ds \right)$$

where $\int_0^T |r_t| \, dt < \infty$ a.s. and r_t is potentially stochastic.

- Stocks:
  - The other n securities are "stocks" with

$$S_t = S_0 + \int_0^t I_S(s) \mu(s) \, ds + \int_0^t I_S(s) \sigma(s) \, dw(s)$$

where the dimensions are

$$\underset{(n\times 1)}{S_t}, \quad \underset{(n\times n)}{I_S} = \begin{bmatrix} S_1 & & & 0 \\ & S_2 & & \\ & & \ddots & \\ 0 & & & S_n \end{bmatrix}, \quad \underset{(n\times 1)}{\mu(s)}, \quad \underset{(n\times d)}{\sigma(s)}, \quad \underset{(d\times 1)}{dw(s)}$$

We write $\sigma^k(t)$ for the k^{th} row of the matrix $\sigma(t)$ so that

$$\sigma(t) = \begin{bmatrix} \sigma^1(t) \\ \sigma^2(t) \\ \vdots \\ \sigma^n(t) \end{bmatrix}$$

.

- Solving for each of S_t^k for $k = 1, \cdots, n$,

$$S_t^k = S_0^k \exp\left(\int_0^t \left(\mu_s^k - \frac{1}{2}\left|\sigma^k(s)\right|^2\right) ds + \int_0^t \sigma^k(s)^\top dw(s)\right)$$

where $\sigma^k(s)^\top$ is $(1 \times d)$.

- This generalizes the Brownian motion processes in the Black-Scholes economy that we had before, where we had regularity conditions

$$\int_0^T |\mu_t|\, dt < \infty, \quad a.s.$$
$$\int_0^T |\sigma_t|^2\, dt < \infty, \quad a.s.$$

- Note you can model any correlation structure via the $\sigma(\cdot)$ process for the stocks.
- We further assume that $n \leq d$ where n is the number of stocks and d is the number of processes (to "complete the markets" we only need $n = d$), so that

$$\text{rank}\left(\underset{(n\times d)}{\sigma_t}\right) = n, \quad a.e.$$

Note ant $m \times n$ matrix has rank at most $\min\{m, n\}$. This assumption gets rid of redundancy.

- Dividends:
  - Stocks pay dividends continuously at a (potentially stochastic) rate δ, so that

$$\underset{(n\times 1)}{D_t} = \int_0^t I_s(s)\,\delta(s)\, ds$$

where we require $\int_0^T |\delta_t|\, dt < \infty$ a.s.

This completes the description of the "Standard Economy" environment that we will work with. Note that we are in a partial equilibrium setting, not general equilibrium. We will need to make additional assumptions to have the Equivalent Martingale Measure ("EMM"), which in turn implies viability.

Let κ_t be the *relative risk premium* process

$$\underset{(d\times 1)}{\kappa_t} = -\underset{(d\times n)}{\sigma_t^\top}\left(\underbrace{\sigma_t\sigma_t^\top}_{(n\times n)}\right)^{-1}\left(\underset{(n\times 1)}{\mu_t + \sigma_t - r_t\bar{\mathbf{1}}}\right)$$

where $\bar{\mathbf{1}}$ **is just the** $(n \times 1)$ ones, i.e.

$$\underset{(n\times 1)}{\bar{\mathbf{1}}} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

We will require that

$$\int_0^T |\kappa_t|^2 \, dt < \infty, \quad a.s.$$

and that the local martingale

$$\xi_t = \exp\left(\int_0^t \kappa_s^\top dw_s - \frac{1}{2} \int_0^t |\kappa_s|^2 \, ds \right)$$

is in fact a martingale[17].

It then follows that (you should check this) we then have

$$d\xi_t = \xi_t \kappa_t^\top dw_t \implies \xi_t = 1 + \int_0^t \xi_s \kappa_s^\top dw_s$$

Remark 126. Note what we did above is that we took processes μ_t, σ_t and r_t that we defined previously, and defined a new object κ_t. We then required some regularity conditions on the new object κ_t and ξ_t (which is defined in terms of κ_t). This is not 100% desirable, and less preferred to say, imposing conditions on μ_t, σ_t and r_t outright that make this happen.

But to get a sense of what is required on the process κ_t for ξ_t to be a martingale, we introduce the ***Novikov condition.*** **Formally, a sufficient, but not necessary, condition for ξ_t to be a martingale is that**

$$E\left[\exp\left(\frac{1}{2} \int_0^T |\kappa_t|^2 \, dt \right) \right] < \infty$$

which is known as the Novikov condition. Note that in the case $n = 1$, then κ_t simply reduces to the case of our Black-Scholes economy.

Remark 127. In order to exploit the connection with PDEs, it is often useful to specialize the setup to the case in which the market coefficients r, μ, σ and δ are *deterministic* functions of the objects (S, Y, t) where Y is an $m-$dimensional vector of state variables with

$$Y(t) = Y(0) + \int_0^t \gamma(Y(s), s) \, ds + \int_0^t \beta(Y(s), s) \, dw(s)$$

Under appropriate conditions (e.g. Lipschitz and growth conditions on r, μ, σ, γ and β), it follows that the 3 stochastic differential equations for (B, S, Y) where $Y(t)$ is given above, and

$$\begin{aligned}
B(t) &= B(0) \exp\left(\int_0^t r_s ds \right) \\
S(t) &= S(0) + \int_0^t I_S(s) \mu(s) \, ds + \int_0^t I_S(s) \sigma(s) \, dw(s)
\end{aligned}$$

as previously given, have a unique solution which is a Markov process. We refer to this as the Markovian model.

5.2 Equivalent Martingale Measures

Now we derive the Equivalent Martingale Measures in this new environment, the Standard Economy.

Let's normalize everything in terms of the bond process B_t, by defining

$$\begin{aligned}
S_t^* &= B_t^{-1} S_t \\
D_t^* &= \int_0^t B_s^{-1} dD_s \\
G_t^* &= S_t^* + D_t^*
\end{aligned}$$

[17] Recall that a stochastic integral is always a local martingale. Additional assumptions make it a martingale.

as we did in the economy. S_t^* is the prices of stocks scaled by bond price, D_t^* is the *cumulative* dividend process scaled by bond prices, while G_t^* is the sum of the two.

To deal with a more general economy than the Black-Scholes economy, we also have to relax our notion of what an equivalent martingale measure is in this more general setting. The following definition clarifies this point.

Definition 128. A probability measure Q on $(\Omega, \mathcal{F})$ is an *Equivalent Martingale Measure* if

(i) $Q \sim P$

(ii) G^* is a $Q-$local martingale.

Couple of comments are due. Note that previously, in the Black-Scholes economy, we had $dG_t^* = S_t^* (\mu + \delta - r)\, dt + S_t^* \sigma dw_t$. We also required G^* to be a Q-martingale, instead of a local martingale.

Why the local-martingale condition? Here the process for G_t^* is

$$dG_t^* = B_t^{-1} I_S(t) \left(\mu_t + \delta_t - r_t \bar{\mathbf{1}}\right) dt + B_t^{-1} I_S(t)\, \sigma(T)\, dw_t$$

What happens if we stick to our old requirement of G^* being a Q-martingale?

Note that by choosing our Q-measure, we can only affect the drift (and the volatility). The best we can do is remove the drift. But, still with $\sigma(\cdot)$ stochastic and not unbounded, we end up with a local martingale, since $\sigma(\cdot)$ is stochastic.

Doing this, we get that

$$G_t^* = G_0^* + \int_0^t \underbrace{B_s^{-1} I_S(s)\, \sigma(s)}_{(*)} dw_s^*$$

In general, we cannot force the $(*)$ term to be such that G_t^* is a real martingale. That is too restrictive, and would rule out too much.

But G_t^* being a local-martingale still gets us what we want! So this is how we proceed.

The following proposition helps us characterize equivalent martingale measures.

Proposition 129. *A probability measure $\hat{Q} \sim P$ is an EMM if and only if*

$$\frac{d\hat{Q}}{dP} = \exp\left(\int_0^T \hat{\kappa}_t^\top dw_t - \frac{1}{2}\int_0^T \left|\hat{\kappa}_t\right|^2 dt\right)$$

for some d-dimensional stochastic process $\hat{\kappa}_t$ such that (1) $\int_0^T |\hat{\kappa}_t|\, dt < \infty$ a.s. and (2) $\mu_t + \delta_t - r_t \bar{\mathbf{1}} + \sigma_t \hat{\kappa}_t = 0$ a.e.

Proof. As long as $\hat{Q} \sim P$, by Girsanov theorem we have that $\exists \hat{\kappa} \in \mathcal{L}^2$ such that

$$\frac{d\hat{Q}}{dP} = \exp\left(\int_0^T \hat{\kappa}_t^\top dw_t - \frac{1}{2}\int_0^T \left|\hat{\kappa}_t\right|^2 dt\right)$$

and

$$\hat{w}_t = w_t - \int_0^t \hat{\kappa}_s ds$$

is a $\hat{Q}$-Brownian motion[18]. Moreover,

$$dG_t^* = B_t^{-1} I_S(t) \left(\underbrace{\mu_t + \delta_t - r_t \bar{\mathbf{1}} + \sigma_t \hat{\kappa}_t}_{=0}\right) dt + B_t^{-1} I_S(t)\, \sigma_t d\hat{w}_t$$

A local martingale cannot have positive or negative drift - it must vanish. If the drift is equal to zero, it is a local martingale. □

[18]Note intuitively, drift under $\hat{Q}$ is the drift under P plus some adjustment. The volatility is the same.

Note that if we define

$$\kappa_t = -\sigma_t^\top \left(\sigma_t \sigma_t^\top\right)^{-1} \left(\mu_t + \delta_t - r_t \bar{\mathbf{1}}\right)$$

then condition (2) above is satisfied.

Remark 130. Say $\hat{\kappa}_t$ satisfies

$$\hat{\kappa}_t = \kappa_t + \nu_t$$

where ν_t is a $(d \times 1)$ stochastic process with $\sigma_t \nu_t = 0$ a.e. (that is, σ is orthogonal to ν). Then the probability measure Q with

$$\frac{dQ}{dP} = \exp\left(\int_0^T \kappa_t^\top dw_t - \frac{1}{2}\int_0^T \left|\hat{\kappa}_t\right|^2 dt\right)$$

is an equivalent martingale measure.

Moreover, if $n = d$, then Q is the ***unique*** equivalent martingale measure. Why? If we consider the object

$$\underset{(n \times n)}{\sigma} \nu$$

then only $\nu = 0$, the zero vector, is orthogonal to σ. This implies $\hat{\kappa}_t = \kappa_t$.

We then have the following corollary as an immediate consequence. This is a restatement of the above.

Corollary 131. *The probability measure Q with*

$$Q(A) = \int_A \xi(T)\, dP$$

is an EMM, that we will now refer to as the standard equivalent martingale measure. If $n = d$, this is the unique equivalent martingale measure.

Remark 132. If we have a unique EMM, then the economy is viable, and prices are uniquely defined. This strongly suggests that if we set the space of trading strategies Θ correctly, then we can get completeness.

For what follows, let Q be the EMM with

$$\frac{dQ}{dP} = \exp\left(\int_0^T \kappa_t^\top dw_t - \frac{1}{2}\int_0^T \left|\kappa_t\right|^2 dt\right)$$

$$w_t^* = w_t - \int_0^t \kappa_s ds$$

so that Ito's Lemma gives us

$$dG_t^* = B_t^{-1} I_S(t)\, \sigma_t dw_t^*$$

or in integral form

$$G^*(t) = S^*(0) + \int_0^t B(s)^{-1} I_S(s)\, \sigma(s)\, dw^*(s)$$

G^* must be a local martingale under Q, and $w^*$is a Brownian motion under Q.

5.3 Trading Strategies

We now make a minor change in notation: a trading strategy (α, θ) now denote amount of dollars invested, *not shares* as we have previously. The convention we use is that α is (1×1) scalar and θ is $(n \times 1)$.

Definition 133. A **trading strategy** (α, θ) is a $(n + 1)$-dimensional adapted process.

A trading strategy (α, θ) ***finances*** a consumption plan $(c, W) \in \mathcal{C}$ if

- $\alpha_T + \theta_T \bar{\mathbf{1}} = W$ so that it finances the terminal wealth W
- $\forall t \in [0, T]$ we have

$$\alpha_t + \theta_t \bar{\mathbf{1}} = \alpha_0 + \theta_0^\top \bar{\mathbf{1}} + \underbrace{\int_0^t B_s^{-1} \alpha_s dB_s + \int_0^t \left(I_S (s)^{-1} \theta_s \right)^\top dG_s}_{\text{gains from trading}}$$
$$- \underbrace{\int_0^t c_s ds}_{\text{consumption}} - \underbrace{C_t}_{\text{free disposal}} \tag{5.1}$$

for some nondecreasing process C with $C(0) = 0$.

If the above holds with $C = 0$, we say that (α, θ) is *tight*. We interpret $C(t)$ as the cumulative wealth withdrawn from the portfolio by time t.

Note here that $B_s^{-1} \alpha_s$ refers to the number of shares in the bond. We also require

$$\int_0^T \left| \theta(t)^\top \sigma(t) \right|^2 dt < \infty$$

Lemma 134. *In terms of the Brownian motion under Q, dw_t^*, the dynamic budget constraint becomes*

$$\alpha_t + \theta_t \bar{\mathbf{1}} = \alpha_0 + \theta_0^\top \bar{\mathbf{1}} + \int_0^t \left(\alpha_s + \theta_s^\top \bar{\mathbf{1}} \right) r_s ds + \int_0^t \theta_s^\top \sigma_s dw_s^* - \int_0^t c_s ds - C_t$$

Proof. The algebraic steps are as follows. Recall that

$$\begin{aligned} dB_t &= B_t r_t dt \\ dG_t &= I_S(t) (\mu_t + \delta_t) dt + I_S(t) \sigma_t dw_t \end{aligned}$$

so that the condition in the second bullet point above becomes (after substitution)

$$\begin{aligned} \alpha_t + \theta_t \bar{\mathbf{1}} &= \alpha_0 + \theta_0^\top \bar{\mathbf{1}} + \int_0^t \alpha_s r_s ds + \int_0^t \theta_s^\top (\mu_s + \delta_s) ds + \int_0^t \theta_s^\top \sigma_s dw_s - \int_0^t c_s ds - C_t \\ &= \alpha_0 + \theta_0^\top \bar{\mathbf{1}} + \int_0^t \left(\alpha_s + \theta_s^\top \bar{\mathbf{1}} \right) r_s ds + \int_0^t \theta_s^\top (\mu_s + \delta_s - r_s \bar{\mathbf{1}}) ds \\ &+ \int_0^t \theta_s^\top \sigma_s dw_s - \int_0^t c_s ds - C_t \end{aligned}$$

Now since $dw_t^* = dw_t - \kappa_t dt$ and $\mu_t + \delta_t - r_t \bar{\mathbf{1}} + \sigma_t \kappa_t = 0$, we can write in terms of dw_s^* so that[19]

$$\alpha_t + \theta_t \bar{\mathbf{1}} = \alpha_0 + \theta_0^\top \bar{\mathbf{1}} + \int_0^t \left(\alpha_s + \theta_s^\top \bar{\mathbf{1}} \right) r_s ds + \int_0^t \theta_s^\top \sigma_s dw_s^* - \int_0^t c_s ds - C_t$$

□

Remark 135. The dynamic budget constraint can also be stated in discounted terms (i.e. in bond numeraire) as

$$\boxed{\frac{\alpha_t + \theta_t^\top \bar{\mathbf{1}}}{B_t} = \frac{\alpha_0 + \theta_0 \bar{\mathbf{1}}}{B_0} + \int_0^t B_s^{-1} \theta_s^\top \sigma_s dw_s^* - \int_0^t B_s^{-1} c_s ds - C_t^*} \tag{5.2}$$

[19] Here I used

$$\begin{aligned} \int_0^t \theta_s^\top \sigma_s dw_s &= \int_0^t \theta_s^\top \sigma_s (dw_s^* + \kappa_s ds) \\ &= \int_0^t \theta_s^\top \sigma_s dw_s^* - \int_0^t \theta_s^\top (\mu_t + \delta_t - r_t \bar{\mathbf{1}}) ds \end{aligned}$$

where $C^*(t) = \int_0^t B(s)^{-1} dC(s)$. This shows that a trading strategy financing a contingent claim (c, W) is completely specified by the initial investment $\alpha(0) + \theta(0)^\top \bar{\mathbf{1}}$, its stock component θ, and the withdrawal process C [20].

Remark 136. Consider the stochastic integral in 5.2, $\int_0^t B_s^{-1}\theta_s^\top \sigma_s dw_s^*$. How can we guarantee that this is well -defined? For one, we can restrict the space of trading strategies Θ as

$$\Theta = \left\{ (\alpha, \theta) : \int_0^T \left| B_t^{-1}\theta_t^\top \sigma_t \right|^2 dt < \infty, \quad a.s. \right\}$$

which is a path-by-path ("ω-by-ω") condition. Since B_t is continuous and each continuous path of B_t is over a compact interval $[0, T]$, this means B_t is bounded (hence B_t^{-1} is bounded). So we can write the above as

$$\Theta = \left\{ (\alpha, \theta) : \int_0^T \left| \theta_t^\top \sigma_t \right|^2 dt < \infty, \quad a.s. \right\}$$

Recall that in the Black-Scholes economy, we defined Θ to be the set of admissible self-financing strategies, where admissible meant $E\left(\int_0^T |\theta(t) S(t)|^2 dt\right) < \infty$, but was able to still complete the market because we had $\mathcal{C} = L^2(P)$. But if the space of consumption $\mathcal{C}$ was larger, this would no longer be true. Here, if we define trading strategies as $E(\cdot) < \infty$ above, this would narrow down the set of trading strategies Θ too much, and we would get market incompleteness. So we try something that's more economically meaningful.

We also need to rule out doubling strategies. Doubling strategies require unbounded short positions without posting collateral. One way to rule them out is to simply prevent borrowing in unboundedly large amounts.

Definition 137. A trading strategy (α, θ) is *admissible* if the discounted portfolio value process $B^{-1}\left(\alpha + \theta^\top \bar{\mathbf{1}}\right)$ is bounded below, i.e.

$$\exists K > 0 \quad s.t. \quad \frac{\alpha_t + \theta_t \bar{\mathbf{1}}}{B_t} \geq -K, \quad \forall t \in [0, T]$$

To be clear, this is an "ex-ante" K, chosen a priori.

Why do we take this route? To explain, let's recap what we did previously. As we already know, the set of trading strategies we developed so far is too large, in the sense that it allows for arbitrage opportunities in the form of doubling strategies. In our section on the Black-Scholes economy, we saw that the possibility of doubling strategies depends on being able to (i) borrow arbitrarily large amounts and to (ii) invest arbitrarily large amounts in the stocks. We then ruled out the doubling strategies by restricting the stock portfolio θ to be *martingale generating* (i.e. to be such that the stochastic integral $\int_0^t B_s^{-1}\theta_s^\top \sigma_s dw_s^*$ is a martingale under Q.

However, this condition is difficult to motivate economically and not very convenient to use outside the constant-coefficients Black-Scholes economy. Therefore, we henceforth follow a different route and rule out doubling strategies by assuming that investors do not have unlimited borrowing power and hence that the portfolio value cannot become arbitrarily negative.e

Two important questions we need to address are:

- Does this rule out any conceivable arbitrage? (not just doubling strategies)
- Do we still have market completeness?

We address the first question by developing a (revised) notion of arbitrage.

[20] Intuitively, the $\int_0^t \left(\alpha_s + \theta_s^\top \bar{\mathbf{1}}\right) r_s ds$ term disappears because $r_t = 0$ for the bond numeraire.

Definition 138. A trading strategy (α, θ) is an *arbitrage opportunity* (or a *free lunch*) if it finances a nonnegative consumption process $(c, W) \in \mathcal{C}_+$ with

$$P\left(\left\{\int_0^T c_t dt + W > 0\right\}\right) > 0$$

and $\alpha_0 + \theta_0^\top \bar{\mathbf{1}} \leq 0$.

We can see that a lower bound on the discounted portfolio value rules out the doubling strategies. We verify next that this lower bound in fact rules out all free lunches.

To do this, first let $\mathcal{C}_b$ denote the subset of consumption plans $(c, W) \in \mathcal{C}$ defined below.

Definition 139. $\mathcal{C}_b = \left\{(c, W) \in \mathcal{C} : \int_0^T c_t^- dt + W^- \text{ is bounded}\right\}$ where $x^- = \max\{0, -x\}$ is the "negative part".

Proposition 140. *If (α, θ) finances a consumption plan $(c, W) \in \mathcal{C}_b$, then the stochastic process*

$$\frac{\alpha_t + \theta_t^\top \bar{\mathbf{1}}}{B_t} + \int_0^t B_s^{-1} c_s ds + C_t^*$$

is a Q-local martingale bounded below (hence a $Q-$supermartingale).

As result, we have

$$\alpha_t + \theta_t^\top \bar{\mathbf{1}} \geq B_t E^Q\left[\int_t^T B_s^{-1} c_s ds + B_T^{-1} W \middle| \mathcal{F}_t\right], \quad \forall t \in [0, T] \tag{5.3}$$

where equality holds in 5.3 if and only if (α, θ) is tight and martingale generating.

Proof. Suppose $(\alpha, \theta) \in \Theta$ finances $(c, W) \in \mathcal{C}_b$. Then, simply by rearranging 5.2,

$$\underbrace{\frac{\alpha_t + \theta_t^\top \bar{\mathbf{1}}}{B_t}}_{\substack{\text{bndd below}\\ \text{by } \geq -K}} + \underbrace{\int_0^t B_s^{-1} c_s ds}_{\substack{\text{bndd below}\\ \text{by } \mathcal{C}_b}} + \underbrace{C_t^*}_{\substack{C_0 = 0\\ \text{nondec}}} = \underbrace{\frac{\alpha_0 + \theta_0 \bar{\mathbf{1}}}{B_0}}_{\text{constant}} + \underbrace{\int_0^t B_s^{-1} \theta_s^\top \sigma_s dw_s^*}_{\text{stochastic integral}}$$

The right hand side is a constant + a stochastic integral. Hence, left hand side is a Q-local martingale. Shown by Fatou's Lemma, we also have LHS is a supermartingale because it is bounded below. Thus, the left hand side is a Q-supermartingale as required. □

Remark 141. $\mathcal{C}_+ \subset \mathcal{C}_b$

Corollary 142. *There are no free lunches in Θ.*

Proof. (Corollary) Suppose $(\alpha, \theta) \in \Theta$ finances a consumption plan $(c, W) \in \mathcal{C}_+$. By the proposition stating the Q-supermartingale property, we have

$$\begin{aligned}
\frac{\alpha_0 + \theta_0^\top \bar{\mathbf{1}}}{B_0} &\geq E^Q\left[\frac{\alpha_T + \theta_T^\top \bar{\mathbf{1}}}{B_T} + \int_0^T B_s^{-1} c_s ds + C_T^*\right] \\
&= E^Q\left[\underbrace{B_T^{-1} W + \int_0^T B_s^{-1} c_s ds}_{\geq 0 \text{ since } (c,W) \in \mathcal{C}_+} + C_T^*\right] \geq 0
\end{aligned}$$

This says that initial wealth in the bond numeraire is bounded below by the Q-expectation of the sum of discounted terminal wealth, discounted consumption and the discounted withdrawal. $C_T^* \geq 0$ because $C_0 = 0$ and it is nondecreasing. □

Hence we settle on our final version of the space of trading strategies Θ which is

$$\boxed{\Theta := \left\{(\alpha,\theta) : \begin{array}{c} \int_0^T \left|\theta_t^\top \sigma_t\right|^2 dt < \infty, \quad \text{and} \\ \exists K > 0 \quad s.t. \quad \frac{\alpha_t + \theta_t \bar{\mathbf{1}}}{B_t} \geq -K, \quad \forall t \in [0,T] \end{array}\right\}}$$

Remark 143. Let's apply the proposition. Since we know that the stochastic process $\frac{\alpha_t + \theta_t^\top \bar{\mathbf{1}}}{B_t} + \int_0^t B_s^{-1} c_s ds + C_t^*$ is a Q-supermartingale, we know that its time-t value is bounded below by its Q-expectation (conditional on time-t information) of its time-T value.

That is, we have

$$\frac{\alpha_t + \theta_t \bar{\mathbf{1}}}{B_t} + \int_0^t B_s^{-1} c_s ds + C_t^* \geq E^Q \left[\frac{\alpha_T + \theta_T \bar{\mathbf{1}}}{B_T} + \int_0^T B_s^{-1} c_s ds + C_T^* \middle| \mathcal{F}_t \right]$$

Using the fact that C_t^* is nondecreasing (so $C_T^* - C_t^* \geq 0$), we can derive

$$\frac{\alpha_t + \theta_t \bar{\mathbf{1}}}{B_t} \geq E^Q \left[B_T^{-1} W_T + \int_t^T B_s^{-1} c_s ds \middle| \mathcal{F}_t \right] \tag{5.4}$$

In other words, if (α,θ) finances (c,W), then portfolio value is bounded below by the Q-expectation of terminal wealth plus future consumption. *This will be important for pricing in the general economy.*

An interesting additional question is, "*When will we have equality in 5.4 instead of an inequality?*" The answer is that we need two conditions to be satisfied:

(i) The stochastic process $\frac{\alpha_t + \theta_t^\top \bar{\mathbf{1}}}{B_t} + \int_0^t B_s^{-1} c_s ds + C_t^*$ is a $Q-$martingale (instead of a supermartingale); and

(ii) The trading strategy (α,θ) is tight, so that $C_T^* - C_t^* = 0$ $\forall t$.

In that case, we will have

$$\frac{\alpha_t + \theta_t \bar{\mathbf{1}}}{B_t} = E^Q \left[B_T^{-1} W_T + \int_t^T B_s^{-1} c_s ds \middle| \mathcal{F}_t \right]$$

Remark 144. Even if an admissible trading strategy cannot turn zero wealth in to a strictly positive consumption plan, it is still possible for an admissible trading strategy to turn positive wealth into a zero consumption plan by running the doubling strategy in reverse. These ***suicidal strategies*** however pose no problem, and will be ruled out by optimality conditions.

Now we move onto market completeness.

5.4 Market Completeness

We thus turn to the issue of market completeness. To define market completeness in this setting, we need to set the space of consumption plans first. The space of $Q-$integrable contingent claims (in the bond numeraire), $\mathcal{C}_b^1$, will be our focus.

Definition 145. Let

$$\mathcal{C}_b^1 = \left\{ (c,W) \in \mathcal{C}_b : E^Q \left[\int_0^T B_t^{-1} c_t dt + B_T^{-1} W_T \right] < \infty \right\}$$

Recall that $\mathcal{C}_b$ denotes the subset of consumption plans $(c,W) \in \mathcal{C}$ such that the random variable $\int_0^T B_t^{-1} c_t^- dt + B_T^{-1} W_T^-$ is essentially bounded (x^- denotes the negative part of the real number x).

Note we really only look at $\mathcal{C}_+ \subset \mathcal{C}_b$. The object $\int_0^T B_t^{-1} c_t dt + B_T^{-1} W_T$ inside the expectation is the cost of financing a consumption plan. There's no need to consider arbitrarily (infinitely) large sets.

Here is the definition of market completeness.

Definition 146. The market is ***dynamically complete*** if every contingent claim (i.e. consumption plan) $(c, W) \in \mathcal{C}_b^1$ is financed by a tight, martingale-generating trading strategy $(\alpha, \theta) \in \Theta$, where martingale generating means

$$\frac{\alpha_t + \theta_t^\top \mathbf{\bar{1}}}{B_t} + \int_0^t B_s^{-1} c_s ds + C_t^*$$

is a Q-martingale.

The following theorem characterizes the notion of dynamic completeness.

Theorem 147. *The market is dynamically complete if and only if $n = d$, i.e. if and only if the number of stocks traded equals the dimension of the underlying Brownian motion.*

Its proof relies on the following result, which we will use repeatedly.

Lemma 148. *(Representation of Q-martingales) Let X be a Q-martingale adapted to $F = F^w$. Then, $\exists \phi \in \mathcal{L}^2$ [21] such that*

$$X_t = X_0 + \int_0^t \phi_s^\top dw_s^*, \quad \forall t \in [0, T]$$

Proof. Let X be a Q-martingale. Then, ξX is a P-martingale. Hence, $\exists \hat{\phi} \in \mathcal{L}^2$ such that

$$\xi_t X_t = \xi_0 X_0 + \int_0^t \hat{\phi}_s^\top dw_s$$

Recall that

$$d\xi_t = \xi_t \kappa_t^\top dw_t$$

By the multidimensional Ito's lemma,

$$d\xi_t^{-1} = \xi_t^{-1} \left(-\kappa_t^\top dw_t + |\kappa_t|^2 dt \right)$$

We then in turn express X_t in terms of $d\left(\xi_t^{-1}\right)$ and $d\left(\xi_t X_t\right)$ processes using the Integration by Parts formula as

$$\begin{aligned}
dX_t &= d\left(\xi_t^{-1} \xi_t X_t\right) \\
&= \xi_t^{-1} d\left(\xi_t X_t\right) + \xi_t X_t d\left(\xi_t^{-1}\right) + d\left[\xi^{-1}, \xi X\right]_t \\
&= \xi_t^{-1} \hat{\phi}_t^\top dw_t + \xi_t X_t \xi_t^{-1} \left(-\kappa_t^\top dw_t + |\kappa_t|^2 dt \right) - \xi_t^{-1} \hat{\phi}_t{}^\top \kappa_t dt \\
&= \left(\xi_t^{-1} \hat{\phi}_t - X_t \kappa_t \right)^\top (dw_t - \kappa_t dt)
\end{aligned}$$

Define $\phi_t = \xi_t^{-1} \hat{\phi}_t - X_t \kappa_t$ and recall that $dw_t^* = dw_t - \kappa_t dt$. Then, we get the result that

$$dX_t = \phi_t^\top dw_t^*$$

and we have shown the existence of ϕ_t as required.

To complete the proof, note that $\phi_t \in \mathcal{L}^2$ because:

- $\hat{\phi}_t \in \mathcal{L}^2$, ξ_t^{-1} has continuous sample paths, and is hence bounded over the compact interval $[0, T]$.
- $\kappa_t \in \mathcal{L}^2$, X_t has continuous sample paths because its a stochastic integral
- difference of two $\mathcal{L}^2$ processes is $\mathcal{L}^2$

[21] To clarify, ϕ is an adapted d-dimensional process with

$$\int_0^T |\phi(t)|^2 dt < \infty, \quad a.s.$$

$\square$

Now we prove the theorem that the market is complete if and only if $n = d$.

Proof. Let us prove the theorem in two parts.

($\Longleftarrow$) First, suppose $n = d$ and let $(c, W) \in \mathcal{C}_b^1$. Let

$$X_t = E^Q \left[\int_0^T B_t^{-1} c_t dt + B_T^{-1} W \middle| \mathcal{F}_t \right]$$

Then, X_t is a Q-martingale, and by Lemma (148), $\exists \phi \in \mathcal{L}^2$ such that

$$X_t = X_0 + \int_0^t \phi_s^\top dw_s^*$$

Let us now define the trading strategies (α, θ) as follows:

$$\begin{aligned} \theta_t &= \left(B_t \phi_t^\top \sigma_t^{-1}\right)^\top \\ \alpha_t &= B_t \left(X_t - \int_0^t B_s^{-1} c_s ds \right) - \theta_t^\top \bar{\mathbf{1}} \end{aligned}$$

Our goal is to show that this particular (α, θ) is a tight, martingale-generating trading strategy that finances (c, W).

To see that it finances (c, W), note

$$\begin{aligned} \frac{\alpha_t + \theta_t^\top \bar{\mathbf{1}}}{B_t} &= B_t^{-1} \left[B_t \left(X_t - \int_0^t B_s^{-1} c_s ds \right) - \theta_t^\top \bar{\mathbf{1}} + \theta_t^\top \bar{\mathbf{1}} \right] \\ &= X_t - \int_0^t B_s^{-1} c_s ds \end{aligned}$$

where using the Q-martingale representation, we have

$$\frac{\alpha_t + \theta_t^\top \bar{\mathbf{1}}}{B_t} = \left(\frac{\alpha_0 + \theta_0^\top \bar{\mathbf{1}}}{B_0} + \int_0^t B_s^{-1} \theta_s^\top \sigma_s dw_s^* \right) - \int_0^t B_s^{-1} c_s ds$$

and

$$\frac{\alpha_T + \theta_T^\top \bar{\mathbf{1}}}{B_T} = X_T - \int_0^T B_s^{-1} c_s ds = \frac{W}{B_T}$$

Hence (α, θ) finances (c, W) as required.

Is $(\alpha, \theta) \in \Theta$? Let's check the two conditions:

- $\int_0^T \left|\theta_t^\top \sigma_t\right|^2 dt = \int_0^T \left|B_t \phi_t\right|^2 dt < \infty$
- Check that $\frac{\alpha_t + \theta_t^\top \bar{\mathbf{1}}}{B_t} = X_t - \int_0^t B_s^{-1} c_s ds = E^Q \left[\int_t^T B_s^{-1} c_s ds + B_T^{-1} W \middle| \mathcal{F}_t \right]$. The question is how negative can this expectation term go. The "negative part" $c_t^-, W^- \geq 0$, so that

$$E^Q \left[\int_t^T B_s^{-1} c_s ds + B_T^{-1} W \middle| \mathcal{F}_t \right] \geq -E^Q \left[\int_t^T B_s^{-1} c_s^- ds + B_T^{-1} W^- \middle| \mathcal{F}_t \right]$$

  But recall we restricted consumption plans as $(c, W) \in \mathcal{C}_b^1$ where $\int_0^T B_s^{-1} c_s^- ds + B_T^{-1} W^-$ is bounded. So we can further expand the region of integraion and bound that term below by

$$\begin{aligned} \frac{\alpha_t + \theta_t^\top \bar{\mathbf{1}}}{B_t} &\geq -E^Q \left[\int_0^T B_s^{-1} c_s^- ds + B_T^{-1} W^- \middle| \mathcal{F}_t \right] \\ &\geq -K \end{aligned}$$

  for some $K > 0$.

Hence we have $(\alpha, \theta) \in \Theta$.

Is the trading strategy martingale generating? The answer is yes, as $\theta_t = \left(B_t \phi_t^\top \sigma_t^{-1}\right)^\top$ implies

$$\int_0^t B_s^{-1} \theta_t^\top \sigma_s dw_s^* = \int_0^t \phi_s^\top dw_s^*$$

by the definition of X_t (i.e. since it's a conditional expectation process).

($\Longrightarrow$) We show the second part of the proof (i.e. "if market is complete then $n = d$") via contradiction. Suppose the market is dynamically complete and $n < d$. Recall we can find some non-trivial ϕ_t such that

$$\underset{(n\times d)}{\sigma} \underset{(d\times 1)}{\phi_t} = 0$$

where ϕ_t is orthogonal to σ. In other words, $\exists \phi_t \in \mathcal{L}^2$ such that $\sigma\phi = 0$ a.e. Since we can scale ϕ by any normalization, without loss of generality, we could say $\phi \neq 0$ and that ϕ is bounded.

Next approach is then to take some (c, W) and find some (α, θ) financing it (via dynamic completeness). Let's write $X = (0, B_T M_T)$ (so with $c_t = 0$), where we define M_t to be

$$M_t = \exp\left(\int_0^t \phi_s^\top dw_s^* - \frac{1}{2}\int_0^t |\phi_s|^2 \, ds\right)$$

this implies we can write

$$dM_t = M_t \phi_t^\top dw_t^*$$

which is a local martingale because it's a stochastic integral. Now because $M_t > 0$, its negative part is equal to zero, and hence $X \in \mathcal{C}_b$. On the other hand, we stated that ϕ is bounded, so Novikov's condition is satisfied[22], which implies that the $\exp(\cdot)$ process given by M_t is a $Q-$martingale.

Then using this, and the fact that $c_t = 0$ by assumption,

$$E^Q\left[\int_0^T B_t^{-1} c_t dt + B_T^{-1} W\right] = E^Q\left[\frac{B_T M_T}{B_T}\right] = M_0 < \infty$$

as $W = B_T M_T$ and M_T is a Q-martingale. Thus, $X = (0, B_T M_T) \in \mathcal{C}_b^1$, and is a consumption process we should be able to finance using some $(\alpha, \theta) \in \Theta$, given dynamic market completeness.

Hence, by assumption of dynamic market completeness, $\exists (\alpha, \theta) \in \Theta$ that is tight, MG-generating, and finances X. Consider such a trading strategy. By definition of (α, θ) financing X, we have

$$\frac{\alpha_t + \theta_t^\top \bar{\mathbf{1}}}{B_t} = \frac{\alpha_0 + \theta_0 \bar{\mathbf{1}}}{B_0} + \underbrace{\int_0^t B_s^{-1} \theta_s^\top \sigma_s dw_s^*}_{\text{martingale}} - \underbrace{\int_0^t B_s^{-1} c_s ds}_{=0} - \underbrace{C_t^*}_{=0}$$

where $\int_0^t B_s^{-1} c_s ds = 0$ by assumption (i.e. $c_t = 0$) and $C_t^* = 0$ because we have a tight strategy. This shows that $\frac{\alpha_t + \theta_t^\top \bar{\mathbf{1}}}{B_t}$ is a martingale, so that we can write

$$\begin{aligned} \frac{\alpha_t + \theta_t^\top \bar{\mathbf{1}}}{B_t} &= E^Q\left[\left.\frac{\alpha_T + \theta_T^\top \bar{\mathbf{1}}}{B_T}\right| \mathcal{F}_t\right] = E^Q\left[M_T | \mathcal{F}_t\right] = M_t \\ &= M_0 + \int_0^t M_s \phi_s^\top dw_s^* \end{aligned}$$

[22]Novikov's condition states that given some stochastic process X_t, if

$$E\left[\exp\left(\frac{1}{2}\int_0^T |X_t|^2 \, dt\right)\right] < \infty$$

is satisfied, then the following process is a martingale:

$$\exp\left(\int_0^t X_s dw_s - \frac{1}{2}\int_0^t |X_s|^2 \, ds\right), \quad \forall t \in [0, T]$$

where we used the fact that $\alpha_T + \theta_T^\top \bar{\mathbf{1}} = W = B_T M_T$ and that M_t is a Q-martingale via $dM_t = M_t \phi_t^\top dw_t^*$. Note $1 = M_0$ via the definition. We can also conclude $\frac{\alpha_0 + \theta_0 \bar{\mathbf{1}}}{B_0} = 1$ by substituting $t = 0$ in the last expression above. Also, as stated earlier, because the trading strategy is tight, we can also write

$$\frac{\alpha_t + \theta_t^\top \bar{\mathbf{1}}}{B_t} = 1 + \int_0^t B_s^{-1} \theta_s^\top \sigma_s dw_s^*$$

Combining, we get that

$$M_0 + \int_0^t M_s \phi_s^\top dw_s^* = \frac{\alpha_t + \theta_t^\top \bar{\mathbf{1}}}{B_t} = 1 + \int_0^t B_s^{-1} \theta_s^\top \sigma_s dw_s^*, \quad \forall t \in [0, T]$$

Because this is a statement that holds for all $t \in [0, T]$, the integrand must be the same, and we write

$$\phi_t = \frac{B_t^{-1} \sigma_t^\top \theta_t}{M_t}$$

which implies that

$$|\phi_t|^2 = \phi_t^\top \phi_t = \frac{\left(B_t^{-1} \sigma_t^\top \theta_t\right)^\top}{M_t} \phi_t = \frac{B_t^{-1} \theta_t^\top (\sigma_t \phi_t)}{M_t} = 0$$

where the last equality follows because $\sigma_t \phi_t = 0$ as we initially presumed. However, this is a contradiction because we assumed $\phi_t \neq 0$, and thus shows if market is complete then $n = d$. □

5.5 Pricing European Contingent Claims

Given our discussion on market completeness, we now let $n = d$ and $(c, W) \in \mathcal{C}_b^1$. This implies the market is complete, and that any consumption plan can be financed.

Recall that in previous sections, where we restricted ourselves to martingale-generating trading strategies, we showed that the price process for any marketed contingent claim was *uniquely defined*, corresponded to the value process for any tight trading strategy financing the contingent claim, and could be obtained by taking the conditional expectations under any equivalent martingale measure. But it turns out this is no longer true if we allow trading strategies that are not martingale generating.

In particular, if we want to price claims in this new environment, we need to ask the following: *If two trading strategies finance the same consumption claim, do they have the same process?* In other words, can we uniquely pin down (α, θ) where

$$S_t^x = \alpha_t + \theta_t^\top \bar{\mathbf{1}} \text{ such that } (\alpha, \theta) \in \Theta \text{ finances } (c, W)$$

for some value of x process S_t^x? That is, do they have the same cost at t? Note that if the answer is no, we may have a problem in terms of pinning down trading strategies.

And indeed, it seems that by no arbitrage, it should be the same. But no - we bounded strategies for Θ. The idea is that if (α, θ) is any admissible tight trading strategy financing a contingent claim $x = (c, W) \in \mathcal{C}$, then the trading strategy (α', θ') obtained by adding a *suicidal strategy* to (α, θ) is also an admissible tight trading stategy financing (c, W) but

$$\alpha_0' + \theta_0'^\top \bar{\mathbf{1}} > \alpha_0 + \theta_0^\top \bar{\mathbf{1}}$$

so initial wealth is higher.

The following remark explains more about sucidal trading strategies.

Remark 149. (Suicidal Trading Strategies) This is also known as a reverse doubling strategy. To illustrate by a simple analogy, suppose we bet money on a series of coin flips. Start by betting \$1 that the first flip will show heads. If the first flip turns out to be tails, you lose your money, and you stop. If its heads and you win, you bet on heads again, repeating the process until you lose.

The outcome is that you lose your \$1 with probability 1. Now imagine you do htis in continuous-tim with stocks. Unfortunately, this is not ruled out by constraints, because we start out with strictly positive wealth, and it stays positive.

The takeaway here is that you can always "add" sucidal trading strategies by "wasting" wealth. By monotonic preferences, however, it should be clear which one investors prefer (i.e. the one with the lowest cost). Thus we have the following definition.

Definition 150. Let $x = (c, W) \in \mathcal{C}$ be a contingent claim. Then, the price of x at a time t, which we denote by S_t^x, is given by

$$S_t^x = \min\left\{\alpha_t + \theta_t^\top \bar{\mathbf{1}} : (\alpha, \theta) \in \Theta \text{ finances } x\right\}$$

provided that the above minimum exists.

Recall our discussion on the stochastic process $\frac{\alpha_t + \theta_t^\top \bar{\mathbf{1}}}{B_t} + \int_0^t B_s^{-1} c_s ds + C_t^*$, that because it is a local martingale bounded below, it is a Q-supermartingale. That condition translates to placing the lower bound $E^Q\left[\frac{\alpha_T + \theta_T^\top \bar{\mathbf{1}}}{B_T} + \int_0^T B_s^{-1} c_s ds + C_T^* \middle| \mathcal{F}_t\right]$ on the said process. Rearranging, we get that

$$\begin{aligned}
\frac{\alpha_t + \theta_t^\top \bar{\mathbf{1}}}{B_t} &\geq E^Q\left[B_T^{-1} W + \int_t^T B_s^{-1} c_s ds + (C_T^* - C_t) \middle| \mathcal{F}_t\right] \\
&\geq E^Q\left[B_T^{-1} W + \int_t^T B_s^{-1} c_s ds \middle| \mathcal{F}_t\right]
\end{aligned}$$

If $\int_0^t B_s^{-1} \theta_s^\top \sigma_s dw_s^*$ is a martingale (hence "martingale-generating"), then we have equality in the first line. If the trading strategy is tight, then we have equality in the second line.

The trading strategy with the cheapest cost, is martingale-generating and tight. Any other strategy is more expensive. Thus we have the following theorem.

Theorem 151. *Suppose that the market is dynamically complete, and let $x = (c, W) \in \mathcal{C}_b^1$ be a contingent claim. Then,*

$$\begin{aligned}
S_t^x &= \min\left\{\alpha_t + \theta_t^\top \bar{\mathbf{1}} : (\alpha, \theta) \in \Theta \text{ finances } x\right\} \\
&= B_t E^Q\left[B_T^{-1} W + \int_t^T B_s^{-1} c_s ds \middle| \mathcal{F}_t\right]
\end{aligned}$$

where c is dividend and W is terminal wealth.

Next step is to obtain a *partial differential equations (PDE)* characterization of S_t^x.

In principle, S_t^x may potentially depend on the entire history, in which case we cannot use PDE methods. So let's assume we are in a Markovian setting.

Let Y_t be a vector of *additional state variables* with some finite number of dimensions. We make the assumption that μ, δ, σ, and r are deterministic functions of (S_t, Y_t) where Y_t is a diffusion. That is,

$$dY_t = \gamma(t, Y_t)\, dt + \beta(t, Y_t)\, dw_t$$

for some drift γ and volatility β. Thus, (S_t, Y_t) is a diffusion (certainly true for Y_t and true for S_t by earlier assumptions).

As we have previous in an earlier section, we introduce the notino of path-independency.

Definition 152. A contingent claim $(c, W) \in \mathcal{C}$ is *path-independent* if

$$\begin{aligned}
c_t &= \phi_1(t, S_t) \\
W &= \phi_2(S_T)
\end{aligned}$$

for some continuous functions $\phi_1 : [0, T] \times \mathbb{R}_+^n \to \mathbb{R}$ and $\phi_2 : \mathbb{R}_+^n \to \mathbb{R}$.

If $x = (c, W)$ is path-independent, then

$$\begin{aligned}
S_t^x &= B_t E^Q\left[\int_t^T B_s^{-1} c_s ds + B_T^{-1} W \middle| \mathcal{F}_t\right] \\
&= E^Q\left[\left(\int_t^T e^{-\int_t^s r_u du} \phi_1(s, S_s)\, ds\right) + \left(e^{-\int_t^T r_u du} \phi_2(S_T)\right) \middle| \mathcal{F}_t\right]
\end{aligned}$$

where we used $B_t = B_0 \exp\left(\int_0^t r_s ds\right)$. The fact that we have a diffusion implies the environment is Markov, which means that the expectation term $E^Q[\cdot|\mathcal{F}_t]$ is a function of the current process. Thus we have that

$$S_t^x = F(t, S_t, Y_t)$$

Now that we have a characterization of S_t^x as a function, we can approach using PDE methods! The steps are, generally speaking, as follows:

(i) Find a martingale related to $F(\cdot)$.

(ii) Compute the drift using Ito's Lemma.

(iii) Set it equal to zero.

To take the first step, we ask: *Is $F(t, S_t, Y_t)$ itself a martingale?* We have to be a little careful here, as we have something like $M_t = E^Q[X|\mathcal{F}_t]$, whereas we should be taking the expectation of the same random variable for it to be a martingale[23].

One way to get around this issue is instead of $S_t^x = F(t, S_t, Y_t)$, we consider

$$\frac{F(t, S_t, Y_t)}{B_t} + \int_0^t B_s^{-1} \phi_1(s, S_s)\, ds = E^Q\left[\int_0^T B_s^{-1} \underbrace{\phi_1(s, S_s)}_{=c_s} ds + B_T^{-1} \underbrace{\phi_2(S_T)}_{=W} \middle| \mathcal{F}_t\right]$$

which gets rid of the time-t dependence on the right hand side.

Apply Ito's Lemma under Q (because its a martingale under Q) to the left hand side object. This part is going to be a bit long, but the idea is that the martingale process should have zero drift. So when we apply Ito's lemma, setting the drift part equal to zero will give us the PDE we want.

$$\begin{aligned}
&\frac{F(t, S_t, Y_t)}{B_t} + \int_0^t B_s^{-1} \phi_1(s, S_s)\, ds \\
=&\frac{F(0, S_0, Y_0)}{B_0} + \int_0^t \left[B_s^{-1}\left(F_S(\cdot)^\top I_S(s)\right)(r(\cdot)\mathbf{\bar{1}} - \delta(\cdot)) + F_Y(\cdot)^\top(\gamma(\cdot) + \beta(\cdot)\kappa(\cdot)) + F_t(\cdot)\cdots\right]
\end{aligned}$$

where F_S is a vector of partials; $\kappa = -\sigma^\top\left(\sigma\sigma^\top\right)^{-1}(\mu + \delta - r\bar{1})$; $I_S(\mu + \sigma\kappa) = I_S(r\mathbf{\bar{1}} - \delta)$[24].

Let's collect the drift and volatility terms as

$$\begin{aligned}
&\frac{F(t, S_t, Y_t)}{B_t} + \int_0^t B_s^{-1} \phi_1(s, S_s)\, ds \\
=&\frac{F(0, S_0, Y_0)}{B_0} \\
&+\int_0^t B_s^{-1} \left\{\begin{array}{c} \left(F_S^\top I_S(s)(r\mathbf{\bar{1}} - \delta) + F_Y^\top(\gamma + \beta\kappa) + F_t\right) \\ +\frac{1}{2}\mathrm{tr}\left(F_{SS} I_S \sigma\sigma^\top I_S\right) + \frac{1}{2}\mathrm{tr}\left(F_{YY}\beta\beta^\top\right) \\ +\mathrm{tr}\left(F_{SY} I_S \sigma\beta^\top\right) - rF + \phi_1 \end{array}\right\} ds \\
&+\int_0^t \left[F_S^\top I_S \sigma + F_Y^\top \beta\right] dw_s^*
\end{aligned}$$

---

[23]The key problem here is that when we write

$$S_t^x = E^Q\left[\left(\int_t^T e^{-\int_t^s r_u du}\phi_1(s, S_s)\, ds\right) + \left(e^{-\int_t^T r_u du}\phi_2(S_T)\right)\middle| \mathcal{F}_t\right],$$

on the right hand side, there is time-t dependence *inside the expectation term* $E^Q[\cdot|\mathcal{F}_t]$ via the bounds of integraion.

[24]Also remember that we specified the processes as

$$\begin{aligned}
S_t &= S_0 + \int_0^t I_S(s)\mu_s ds + \int_0^t I_S(s)\sigma_s ds \\
Y_t &= Y_0 + \int_0^t \gamma(s, Y_s)\, ds + \int_0^t \beta(s, Y_S)\, dw_s
\end{aligned}$$

This looks really long, but the point is that the term inside $\{\cdot\}$ should be equal to zero.

A more compact way to write it is that, if F is sufficiently smooth for an application of Ito's lemma, then it must satisfy the PDE with boundary condition[25]

$$\boxed{\begin{gathered} 0 = \mathcal{D}F + \phi_1(t, S) - rF \\ F(T, S_T, Y_T) = \phi_2(S_T) \end{gathered}}$$

5.6 Forwards and Futures

Here we apply our knowledge from the previous section to specific applications. Assume now $n = d = 1$, so that we consider derivatives with a single underlying stock, driven by a single Brownian motion. Recall that given a consumption plan $(c, W) \in \mathcal{C}_b^1$, we have that the price process should be

$$S_t^x = B_t E^Q \left[\int_t^T B_s^{-1} c_s ds + B_T^{-1} W \middle| \mathcal{F}_t \right]$$

Let's first consider forwards.

Example 153. (Forward Contract) Consider a forward contract to buy a given asset (stock) at time T for a delivery price F. This gives no dividends and offers a single cash flow on the delivery date, which is

$$x = (c, W) = (0, S_T - F)$$

Applying our previous formula, the value at time t of this contract is

$$\begin{aligned} S_t^x &= B_t E^Q \left[B_T^{-1} (S_T - F) \middle| \mathcal{F}_t \right] \\ &= B_t \left[E^Q \left(B_T^{-1} S_T \middle| \mathcal{F}_t \right) - F E^Q \left(B_T^{-1} \middle| \mathcal{F}_t \right) \right] \\ &= 0 \end{aligned}$$

The last line follows from the idea that the forward price F_t is set at time t so that $S_t^x = 0$ for a newly written contract.

Thus, the forward price at time t for delivery at time T is given by

$$\boxed{F_t = \frac{E^Q \left(B_T^{-1} S_T \middle| \mathcal{F}_t \right)}{E^Q \left(B_T^{-1} \middle| \mathcal{F}_t \right)}}$$

This generalizes the simple cost-of-carry formula obtained in the previous section.

Now we analyze futures. The difference with forwards are that: (1) they are marked to market on a daily basis; and (2) the could be settled day-to-day. The cash flow of a futures contract is the closing settlement today minus the closing settlement yesterday. If this change is positive, the difference is debited; if it's negative, it's credited. It is not absolutely continuous with respect to cash flow .

Definition 154. A *continuously settled futures contract* is a claim to a cumulative cash flow

$$D_t = \int_0^t df_s$$

where f is the futures price of the underlying stock.

---

[25] Here $\mathcal{D}F$ is the Dynkin operator with

$$\mathcal{D}F = \left(F_S^\top I_S(s) (r\bar{\mathbf{1}} - \delta) + F_Y^\top (\gamma + \beta\kappa) + F_t \right) + \frac{1}{2} \text{tr} \left(F_{SS} I_S \sigma \sigma^\top I_S \right) + \frac{1}{2} \text{tr} \left(F_{YY} \beta \beta^\top \right) + \text{tr} \left(F_{SY} I_S \sigma \beta^\top \right)$$

Let V_t^f be the value of a futures contract,

$$V_t^f = B_t E^Q \left(\int_t^T B_s^{-1} df_s \middle| \mathcal{F}_t \right)$$

Since there is no initial cost to set up futures, value must adjust to become a fair trade for both parties. Note that the futures price f is set so that $V^f \equiv 0$ and $f_T = S_T$ (i.e. at the terminal price, it becomes the spot price). It turns out these conditions uniquely determine the futures price process, as stated in the next proposition.

Proposition 155. *(Pricing a Futures Contract) Suppose that* $E^Q\left[S_T^2\right] < \infty$ *and that the bond price process* B_t *is bounded above and below away from zero. Then,* $\exists$*a unique Ito process* f *such that*

$$\begin{aligned} V^f &\equiv 0 \\ f_T &= S_T \end{aligned}$$

and given by

$$\boxed{f_t = E^Q\left[S_T \middle| \mathcal{F}_t\right]}$$

Proof. Let $f_t = E^Q\left[S_T \middle| \mathcal{F}_t\right]$. Then, f is a $L^2(Q)$-martingale and hence there exists some ϕ_t such that

$$f_t = f_0 + \int_0^t \phi_s^\top dw_s^*$$

with $E^Q\left[\int_0^T |\phi_t|^2 dt\right] < \infty$ (or equivalently, $\phi_t \in \mathcal{H}_2$).

Now, let

$$D_t^* = \int_0^t B_s^{-1} df_s = \int_0^t B_s^{-1} \phi_s^\top dw_s^*$$

Because B_s is bounded from zero, B_s^{-1} is bounded above, and with $\phi_t \in \mathcal{H}_2$, this implies D_t^* is a Q-martingale. Using that, we can write

$$\begin{aligned} 0 &= E^Q\left[D_T^* - D_t^* \middle| \mathcal{F}_t\right] = E^Q\left[\int_t^T B_s^{-1} df_s \middle| \mathcal{F}_t\right] \\ &= B_t^{-1} V_t^f \end{aligned}$$

where the last line follows from the definition of V_t^f. This shows that $V^f \equiv 0$ and that also $f_T = S_T$.

So far, we have shown that our candidate process for f_t satisfies the two conditions we would like it to. It remains to show uniqueness: now suppose $\hat{f}$ is an Ito process such that $V^{\hat{f}} \equiv 0$ and $\hat{f}_T = S_T$. For uniqueness, it suffices to show that this implies $\hat{f} = f$. Since an Ito process under P is an Ito process under Q, we can write

$$d\hat{f}_t = \hat{\mu}_t dt + \hat{\sigma}_t dw_t^*$$

Furthermore, we can write the cumulative dividend process for $\hat{f}$ as well,

$$\hat{D}_t^* = \int_0^t B_s^{-1} d\hat{f}_s = \int_0^t B_s^{-1} \hat{\mu}_t dt + \int_0^t B_s^{-1} \hat{\sigma}_t dw_t^*$$

Since for all $t \in [0, T]$,

$$0 = V_t^{\hat{f}} = B_t E^Q\left[\int_t^T B_s^{-1} d\hat{f}_s \middle| \mathcal{F}_t\right] = B_t E^Q\left[\hat{D}_T^* - \hat{D}_t^* \middle| \mathcal{F}_t\right]$$

This tells us that the expected increment at any future $T-t$ is zero, and thus implies that D^* is a Q-martingale, which in turn means that $\hat{\mu} \equiv 0$. Hence,

$$\hat{f}_t = \hat{f}_0 + \int_0^t \hat{\sigma}_s dw_s^*$$

a stochastic integral. Applying this at time T,

$$E^Q\left[\left(\hat{f}_T - \hat{f}_0\right)^2\right] = E^Q\left[\left(S_T - \hat{f}_0\right)^2\right] < \infty$$

where the first equality follows by the terminal condition $\hat{f}_T = S_T$. Note $\hat{f}_0$ is a constant. This means that

$$E^Q\left[\left(\hat{f}_T - \hat{f}_0\right)^2\right] = E^Q\left[\left(\int_0^T \hat{\sigma}_t dw_t^*\right)^2\right] = E^Q\left[\int_0^T \left|\hat{\sigma}_t\right|^2 dt\right] < \infty$$

by Ito isometry, so we conclude that $\hat{\sigma}_t \in \mathcal{H}_2$. Because a square-integrable local martingale is a (real) $Q-$martingale,

$$\hat{f}_t = E^Q\left[S_T \middle| \mathcal{F}_t\right] = f_t$$

and both are the same processes. □

Remark 156. (Futures vs. Forwards prices) How can we compare futures and forward prices? Note since

$$\operatorname{Cov}_t^Q(X,Y) = E_t^Q(XY) - E_t^Q(X)\, E_t^Q(Y)$$

We can start by writing the forward prices as

$$F_t = \frac{E^Q\left(B_T^{-1} S_T \middle| \mathcal{F}_t\right)}{E^Q\left(B_T^{-1} \middle| \mathcal{F}_t\right)} = \frac{E^Q\left(B_T^{-1} \middle| \mathcal{F}_t\right) E^Q\left(S_T \middle| \mathcal{F}_t\right) + \operatorname{Cov}_t^Q\left(B_T^{-1}, S_T\right)}{E^Q\left(B_T^{-1} \middle| \mathcal{F}_t\right)}$$

so that we have

$$\boxed{F_t = f_t + \frac{\operatorname{Cov}_t^Q\left(B_T^{-1}, S_T\right)}{E^Q\left(B_T^{-1} \middle| \mathcal{F}_t\right)}}$$

What does this mean? It means that the futures price of an asset coincides with its forward price if and only if B_T^{-1} and S_T are uncorrelated under Q. In particular, if interest rates are non-stochastic, then $F_t = f_t = E^Q\left(S_T \middle| \mathcal{F}_t\right)$ because $\operatorname{Cov}_t^Q(\cdot) = 0$.

What about the signs on the covariance term? Suppose $\operatorname{Cov}_t^Q\left(B_T^{-1}, S_T\right) > 0$. What happens?

- Gains from long futures positions when interest rates are low.
- Losses from long futures positions when interest rates are high.

Note also that f_t is more valuable, so ex-ante, $F_t > f_t$ to be fair pricing. Only need monotonicity.. marignal utility doesn't matter to draw this conclusion.

Remember that for forwards, all gains and losses are realized at the end, whereas futures are settled daily. Payoffs can be summarized as

$$\begin{aligned} \text{Futures:} \quad & f_T - f_T = S_t - f_t \\ \text{Forwads:} \quad & S_T - F_t \end{aligned}$$

5.7 Pricing American Contingent Claims

6 Optimal Consumption wiht Complete Markets

6.1 The Investor's Problem

6.1.1 The Basic Setting

6.1.2 The Martingale Approach

6.1.3 The Stochastic Control Approach

6.2 The Stochastic Control Approach

6.2.1 Charaterization of Optimal Policies

6.2.2 Closed Form solutions

6.3 The Martingale Approach

6.3.1 Characterization of Optimal Policies

6.3.2 Closed Form Solutions

6.4 Extensions

6.4.1 Binding Non-Negativity Constraint on Optimal Consumption

6.4.2 Stochastic Endowment

Part IV

Dynamic Equilibrium

7 General Equilibrium with Complete Markets

We study the general equilibrium problem for a pure-exchange economy with complete markets. In particular, we aim to derive a charaterization for the equilibrium interest rate process and stock risk premia. Up to this point, we took both prices as given (exogenously). Our goal is to endogenize them here.

After going over the details of the economy, we establish some results regarding what the equilibrium quantities are going to look like. Then, we use that to get an expression for the equilibrium interest rate and the risk premia. We cover both the cases of general uncertainty (consumption-based CAPM) and Markovian uncertainty (intertemporal CAPM).

7.1 Setup

7.1.1 The Economy

Here we first go over the setup of the economy that we consider in this section.

- Finite horizon $t \in [0, T]$
- Uncertainty represented by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where
  - Ω the sample space represent the set of states of the world
  - $\mathcal{F} = \mathcal{F}_T$[26] a sigma-field representing the set of events to which the agents can assign probability
  - $\mathbb{F} = \{\mathcal{F}_t : t \in [0, T]\}$ is the filtration generated by a $d-$dimensional Brownian motion, represents the way that agents are going to be learning over time; and
  - P is the probability measure that represents the common beliefs held by all the agents in the economy
- Single consumption good (the numeraire) in consumption space where total intertemporal consumption is finite with probability one, i.e.[27]

$$\mathcal{C} = \left\{ c : c \text{ is a progressive SP with } \int_0^T |c_t|\, dt < \infty \quad \text{a.s.} \right\}$$

  - Note agents only have preferences over intertemporal consumption (i.e. no bequest function). The reason is that we are looking at an economy that ends at time $t = T$. As we are not considering a structure like overlapping generations, there is really no rationale for a bequest function.
- Finite number of agents, indexed by $i \in \mathcal{I} = \{1, \cdots, I\}$.

[26] Sigma field $\mathcal{F} = \mathcal{F}_T$ so that the events that agents can assign probability to are the events that agents eventually learn at the terminal date.

[27] Recall our notation for alternative consumption spaces

$$\mathcal{C}_b = \left\{ c \in \mathcal{C} : \exists K \geq 0 \text{ s.t. } \int_0^T B_t^{-1} c_t^- \, ds \leq K \quad \text{a.s.} \right\}$$

where b is as in discounted total consumption is "bounded below" (which is the same as the negative part being bounded above). Also,

$$\mathcal{C}_b^1 = \left\{ c \in \mathcal{C}_b : E^Q \left[\int_0^T B_s^{-1} c_s ds \right] < \infty \right\}$$

where 1 is as in finite first moment (expectation) under the EMM.

- Preferences for each agent $i \in \mathcal{I}$ represented by a time-additive expected utility for intertemporal consumption:

$$U_i(c) = E^P\left[\int_0^T u_i(c_i, t)\, dt\right]$$

where $u_i \in \mathcal{U}$ with $\mathcal{U}$ denoting the set of functions that satisfy the four conditions in the footnotes[28][29].

- Each agent $i \in \mathcal{I}$ is endowed with a strictly positive CP $\tilde{c}_i \in \mathcal{C}$ such that the aggregate endowment $\tilde{c} = \sum_{i=1}^{I} \tilde{c}_i$ is bounded above and below away from zero and sastifsfies the SDE (i.e. an Ito process)

$$\tilde{c}(t) = \tilde{c}(0) + \int_0^t \tilde{c}(s)\mu(s)\, ds + \int_0^t \tilde{c}(s)\tilde{\sigma}(s)^\top\, dw(s)$$

for some stochastic processes $\tilde{\mu}$ and $\tilde{\sigma}$ with $\tilde{\sigma}$ satisfying the Novikov condition[30].

- Agent can trade in $n+1$ long lived (available for trading for all t) securities, where $n = d$. Bonds with intermediate maturities (which we need to consider when studying the term structure in general equilibrium) are not long lived for example.

- All securities are in zero net supply. This is without loss of generality, given that our agents receive an arbitrary random stream of endowment.

  – Why? Suppose we would like to have some securities in positive net supply. In the beginning we then must endow some agents with some of the securities, so that the total endowment of securities is equal to the aggregate supply of the securities. But then endowing the agents with stocks is the same as endowing the dividend stream that the stock is a claim to, so we can just call this the stochastic endowment that they receive, $\tilde{c}$. Once we do this, we can go back to our zero net supply assumption: the equilibrium CP and prices are going to be exactly the same in the two economies.

- The first security is a money market account ("bond") that pays the instantaneous interest rate r (determined endogenously) and has price process

$$B(t) = 1 + \int_0^t B(s) r(s)\, ds = \exp\left(\int_0^t r(s)\, ds\right)$$

with the normalizing assumption $B(0) = 1$.

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[28] The four conditions are:

(i) (Differentiability)$u_i \in C^{3,1}$

(ii) (Strictly increasing and concave) $u_i(\cdot, t)$ is strictly increasing and concave for all $t \in [0, T]$

(iii) (Inada) marginal utility $u_{ic}(\cdot, t)$ satisfies the Inada conditions for all $t \in [0, T]$:

$$\lim_{c\uparrow\infty} u_{ic}(c, t) = 0 \quad \text{and} \quad \lim_{c\downarrow 0} u_{ic}(c, t) = +\infty$$

(iv) (Positive marginal util.) for all $c > 0$, $\inf_{t\in[0,T]} u_{ic}(c, t) > 0$

[29] The conditions here are a little stronger than in the previous section. Previously we assumed continuously differentiable in consumption and continuous in time. $C^{1,0}$. The reason is that in order to characterize the equilibrium interest rate and the equilibrium risk premum, we need to apply Ito's lemma to the marginal utilities.

Assumption 4 is pretty mild. If the utility function is separable in c and t, for example as $u(c, t) = e^{-\rho t} f(c)$, then this assumption is automatically satisfied.

[30] Recall that this condition is

$$E^P\left[\exp\left(\frac{1}{2}\int_0^T |\tilde{\sigma}(t)|^2\, dt\right)\right] < \infty$$

and implies that the process given by $\exp\left(\int_0^t \tilde{\sigma}(s)\, dw(s) - \frac{1}{2}\int_0^t |\tilde{\sigma}(s)|^2\, ds\right)$ is a martingale.

- The remaining n securities ("stocks") are claims to the exogenous cumulative dividend process $D = (D_1, \cdots, D_n)$ where $D(t) = \int_0^t \delta(s)\, ds$ for some stochastic process $\delta \geq 0$ and $E^P[D(T)] < \infty$ (total dividend has finite expectation). .
  - In a pure-exchange economy, the dividend process for each stock is exogenous. We model stocks as claims to these dividend processes with prices process S to be determined endogenously in equilibrium
- The set Θ of admissible trading strategies is as defined in Sections 5 and 6.
  - Thus rules out free lunches such as doubling strategies via non-negativity constraints on discounted wealth, which amounts to an upper bound on how much the agents can borrow.

Given this environment, the optimization problem for each agent $i \in \mathcal{I}$ is then

$$\max_{(c,\alpha,\theta)\in\mathcal{C}_+\times\Theta} U_i(c) \tag{7.1}$$

$$\text{s.t. } (\alpha,\theta) \text{ finances } c - \tilde{c}_i \text{ and } \alpha(0) + \theta(0)^\top \bar{1} \leq 0$$

requiring nonpositive time zero cost for the trading strategy that finances the net consumption plan.

7.1.2 The Equilibrium Concept

Recall Theorem 4 in Section 3 ("Harrison and Kreps"). Say the economy is admissible in the sense of HK, at least one agent that can find an optimal consumption plan, and the filtration is the one generated by BM. Then, the discounted gain process for the stocks $G = S + D$ (in equilibrium) must be an Ito process

$$G(t) = G(0) + \int_0^t I_S(s)\,\mu(s)\,ds + \int_0^t I_S(s)\,\sigma(s)\,dw(s)$$

for some stochastic processes μ and σ (to be determined in equlibrium). Note if we are in equilibrium, then each agent is following an optimal consumption plan. Also, because the money market account is an Ito process (following from the bond process $B(t)$ given above), then G^*, the gain process in bond units, is also an Ito process. Thus the same must be true in the original numeraire.

Here's the equilibrium concept that we will use in this setting.

Definition 157. (CME) A ***complete-market equilibrium*** is the collection of prices and optimal policies, i.e.

$$(r, S, \{(c_i^*, \alpha_i^*, \theta_i^*) : i \in \mathcal{I}\}),$$

such that:

(i) (*optimality*) for each $i \in \mathcal{I}$, $(c_i^*, \alpha_i^*, \theta_i^*)$ solves agent i's problem in equation (7.1);

(ii) (*market clearing*) $\sum_{i=1}^I (c_i^*, \alpha_i^*, \theta_i^*) = (\tilde{c}, 0, 0)$

(iii) (*market completeness*) σ has full rank a.e.

Given that we have as many stocks as Brownian motions, σ will be both the volatility of the gains process and the stock process. This is because $G = S + D$ and $D(t) = \int_0^t \delta(s)\, ds$ so there is no volatility part. Thus σ having full rank a.e. is a necessary and sufficient condition for the market to be dynamically complete.

Furthermore, just for convenience, we let $\mu(t)$ to be the drift of the gains process (not the stock). We'll just have to be careful how other things change accordingly, for example κ.

7.2 Characterization and Existence of Equilibrium

Next thing we do is try to solve for an equilibrium in general without specifying utility functions and endowment processes.

7.2.1 Necessary Conditions for an Equilibrium

The object we need to pin down to get the tuple is the equilibrium state-price density π. Given the partial equilibrium problem we studied in section 6, we know that the optimal consumption c_i^* should be the inverse marginal utility of $\psi_i \pi_t$. We also know that if we get π, we can get (r, S) given the exogenous dividend process.

Suppose that a CME (complete market equilibrium) exists. We don't know this yet, but if we did, then the stochastic process

$$\pi(t) = \exp\left(\int_0^t \kappa(s)^\top dw(s) - \int_0^t \left(r(s) + \frac{1}{2}|\kappa(s)|^2\right) ds\right)$$

where

$$\kappa(t) = -\sigma(t)^{-1}\left(\mu(t) - r(t)\bar{1}\right)$$

is the unique state price density.

We now conjecture and verify later, under the stated assumptions of utility functions and endowments, the equilibrium SPD is bounded above and below away from zero. Then, because for all $i \in \mathcal{I}$, $0 < \tilde{c}_i < \tilde{c}$ and $\tilde{c}$ is bounded (as $\tilde{c}_i > 0$ and sum to $\tilde{c}$ which is bounded), then it follows that

$$0 < E^P\left[\int_0^T \pi(t)\tilde{c}_i(t)\,dt\right] < \infty \tag{7.2}$$

as the integrand is bounded.

Now let's consider the marginal utility for an individual agent. Felicity function is strictly increasing and strictly concave, and $u_i \in C^{3,1}$. So the marginal utility exists, is strictly positive, and is strictly decreasing. Inada condition gives that it takes all values between $+\infty$ and zero.

For each $i \in \mathcal{I}$, let $f_i(\cdot, t) = u_c^{-1}(\cdot, t)$ denote the inverse of $u_{ic}(\cdot, t)$. Then, $f_i(\cdot, t)$ is a continuous and strictly decreasing map from $\mathbb{R}_{++}$onto itself. This implies that the map

$$\psi \mapsto E^P\left[\int_0^T \pi(t) f_i(\psi\pi_t, t)\,dt\right]$$

is also a continuous and strictly decreasing map from $\mathbb{R}_{++}$ onto iteself.

Recall that we conjectured that π_t is bounded above. And because f_i is continuous, that's going to imply f_i is a continuous function of a bounded quantity, and hence will be bounded. $\pi(t)$ is bounded so $\pi(t) f_i(\psi\pi_t, t)$ is bounded. So as ψ varies from 0 to ∞, $f_i(\cdot, t)$ does also, and so the $E^P[\cdot]$ takes all possible values between 0 and infinity.

Hence, because of equation (7.2) and the intermediate value theorem, there exists a $\psi_i^* \in \mathbb{R}_{++}$ such that

$$E^P\left[\int_0^T \pi(t) f_i(\psi_i^*, \pi_t, t)\,dt\right] = E^P\left[\int_0^T \pi(t)\tilde{c}_i(t)\,dt\right] \tag{7.3}$$

This in turn implies that if we take that particular value of ψ_i^*, defined as above, we know from our discussion on optimal consumption (via the martingale approach) in section 6 that for each $i \in \mathcal{I}$,

$$c_i^*(t) = f_i(\psi_i^*\pi_t, t)$$

solves agent i's problem in equation (7.1). This is because equation (7.3) just established that ψ_i^* satisfies the budget constraint with equality. This established that if a CME exists, and our conjecture that in equilibrium π_t is bounded away from zero and above is true, then the optimal consumption plan for each agent must have the form given here.

In addition, this means that by market clearing, if a CME exists then

$$\sum_{i \in \mathcal{I}} f_i\left(\psi_i^* \pi_t, t\right) = \tilde{c}\left(t\right)$$

for all $t \in [0, T]$. Note that $f_i\left(\cdot, t\right)$ and $\tilde{c}\left(t\right)$ are both exogenous, whereas π_t and ψ_i^* are endogenous. Our plan is to then try to *invert this* equation, write something for ψ_i^*, so that we can write π_t in terms of things that we know.

Given an aribtrary vector $\Psi = (\psi_1, \cdots, \psi_I) \in \mathbb{R}_{++}^I$, define the functional $f\left(x, t; \Psi\right)$ on $\mathbb{R}_{++} \times [0, T]$ by

$$f\left(x, t; \Psi\right) = \sum_{i=1}^{I} f_i\left(\psi_i x, t\right) \tag{7.4}$$

Then, for all $t \in [0, T]$, each $f_i\left(\psi_i x, t\right)$ is a strictly decreasing map from $\mathbb{R}_{++}$onto itself, and thus so is $f\left(\cdot, t; \Psi\right)$. As a result, $f\left(\cdot, t, ; \Psi\right)$ has an inverse $f^{-1}\left(\cdot, t; \Psi\right)$ that is also a strictly decreasing map from $\mathbb{R}_{++}$ onto itself.

Why define f? This is because the equation that we want to invert can just be written as

$$f\left(\pi_t, t; \Psi^*\right) = \sum_{i \in \mathcal{I}} f_i\left(\psi_i^* \pi_t, t\right) = \tilde{c}\left(t\right)$$

where $\Psi^* = (\psi_1^*, \cdots, \psi_I^*) \in \mathbb{R}_{++}^I$, it follows that if an equilibrium exists, then

$$\pi\left(t\right) = f^{-1}\left(\tilde{c}_t, t; \Psi^*\right) \tag{7.5}$$

for all $t \in [0, T]$.

Equation (7.5) provides an explicit expression for the equilibrium SPD int terms of the exogenous aggregate endowment process $\tilde{c}$ and the endogenous vector Ψ^* of Lagrangeian multipliers. If we characterize Ψ^*, then we have a full charaterization of the SPD.

But before we do that, let's be more explicit about what f^{-1} is. This will get an alternative charaterization of f^{-1} that we are more familiar with.

7.2.2 The Representative Agent

The following proposition gives an alternative characterization of the function f^{-1} in equation (7.5) in terms of a *representative agent*'s utility function.

Proposition 158. *Given an arbitrary vector $\Lambda = (\lambda_1, \cdots, \lambda_I) \in \mathbb{R}_{++}^I$, define the functional $u\left(c, t; \Lambda\right)$ on $\mathbb{R}_{++} \times [0, T]$ by*

$$u\left(c, t; \Lambda\right) = \max_{\substack{(c_1, \cdots, c_I) \in \mathbb{R}_+^I \\ c_1 + \cdots + c_I = c}} \sum_{i=1}^{I} \lambda_i u_i\left(c_i, t\right) \tag{7.6}$$

Then $u\left(\cdot, \cdot; \Lambda\right) \in \mathcal{U}$. Moreover,

$$u_c\left(c, t; \Lambda\right) = f^{-1}\left(c, t; \Psi\right), \tag{7.7}$$

where $\Psi = (1/\lambda_1, \cdots, 1/\lambda_I)$.

If you take the weights of the function f to be Lagrangian multipliers, then the weights in the utility function are going to be the reciprocals of the Lagrangean multilpiers corresponding to the optimal allocation of each agent.

With equation (7.7), the more familar expression emerges: the state price density in equation (7.5) is the marginal utility of the representative agent evaluated at the aggregate endowment.

Proof. Take some $\Lambda = (\lambda_1, \cdots, \lambda_I) \in \mathbb{R}^I_{++}$ and let $\Psi = (1/\lambda_1, \cdots, 1/\lambda_I)$. We begin by showing that

$$c_i^* = f_i\left(\psi_i f^{-1}(c, t; \Psi), t\right) \tag{7.8}$$

for all $i \in \mathcal{I}$ achieves the maximum in equation (7.6). We do this by plugging in this value of c_i^* in various places, as below.

First, let's show feasibility. Note that from the definition of f in equation (7.4) that

$$\begin{aligned}
\sum_{i=1}^I c_i^* &= \sum_{i=1}^I f_i\left(\psi_i f^{-1}(c, t; \Psi), t\right) \\
&= f\left(f^{-1}(c, t; \Psi), t; \Psi\right) \\
&= c
\end{aligned}$$

so that $(c_1^*, \cdots, c_I^*)$ is feasible, i.e. it holds that $c_1^* + \cdots + c_I^* = c$.

Second, since we defined $f_i(\cdot, t) = u_{ci}^{-1}(\cdot, t)$, we have

$$\begin{aligned}
\lambda_i u_{ic}(c_i^*, t) &= \lambda_i u_{ic}\left(f_i\left(\psi_i f^{-1}(c, t; \Psi), t\right), t\right) \\
&= \lambda_i \psi_i f^{-1}(c, t; \Psi) \\
&= f^{-1}(c, t; \Psi)
\end{aligned}$$

where we used that $\psi_i = 1/\lambda_i$. We will use this property directly below.

Now consider any other feasible allocation, i.e. any other $(c_1, \cdots, c_I) \in \mathbb{R}^I_{++}$ with $\sum_{i=1}^I c_i = c$, we have that

$$\begin{aligned}
\sum_{i=1}^I \lambda_i u_i(c_i, t) &\leq \sum_{i=1}^I \lambda_i \left[u_i(c_i^*, t) + u_{ic}(c_i^*, t)(c_i - c_i^*)\right] \\
&= \sum_{i=1}^I \lambda_i u_i(c_i^*, t) + f^{-1}(c, t; \Psi) \underbrace{\sum_{i=1}^I (c_i - c_i^*)}_{=0} \\
&= \sum_{i=1}^I \lambda_i u_i(c_i^*, t)
\end{aligned}$$

where the first inequality follows from the concavity of $u_i(\cdot, t)$ for all $i \in \mathcal{I}$[31], and the equalities from above.

This thus far shows the optimality of the allocations in equation (7.8). Therefore, we can rewrite $u(c, t; \Lambda)$ by eliminating the max operator using the allocation above,

$$\begin{aligned}
u(c, t; \Lambda) &= \sum_{i=1}^I \lambda_i u_i(c_i^*, t) \\
&= \sum_{i=1}^I \lambda_i u_i\left(f_i\left(\psi_i f^{-1}(c, t; \Psi), t\right), t\right)
\end{aligned}$$

Now we proceed to establish equation (7.7).

[31] Try drawing a picture! For some concave $f(\cdot)$, we have

$$f(y) \leq f(x) + f'(x)(y - x)$$

where the right hand side is like a linear approximation of f at point x (taking the slope at x), extrapolated to the point y.

Another way to view this is as a first order Taylor approximation of $u_i(c_i, t)$ that can be made exact by adding inside the square brackets

$$\frac{1}{2} u_{icc}(\tilde{c}_i, t)(c_i - c_i^*)^2$$

with $\tilde{c}_i \in [c_i, c_i^*]$. Since $u_{icc}(\tilde{c}_i, t) < 0$, dropping this term gives the inequality above.

Use chain rule to differentiate with respect to c. Note we can now do this because the right hand side is also a function of c.

$$\begin{aligned}
u_c(c,t;\Lambda) &= \sum_{i=1}^{I} \lambda_i u_{ic}\left(\underbrace{f_i\left(\psi_i f^{-1}(c,t;\Psi),t\right)}_{=c_i^*},t\right) \times \frac{\partial}{\partial c} f_i\left(\psi_i f^{-1}(c,t;\Psi),t\right) \\
&= \sum_{i=1}^{I} f^{-1}(c,t;\Psi) \times \frac{\partial}{\partial c} f_i\left(\psi_i f^{-1}(c,t;\Psi),t\right) \\
&= f^{-1}(c,t;\Psi) \times \frac{\partial}{\partial c} \sum_{i=1}^{I} f_i\left(\psi_i f^{-1}(c,t;\Psi),t\right) \\
&= f^{-1}(c,t;\Psi)
\end{aligned}$$

where again in the second equality we used the property that $\lambda_i u_{ic}(c_i^*,t) = f^{-1}(c,t;\Psi)$; in the third equality, the property that the sum of the derivatives is equal to the derivative of the sum; and in the last,

$$\sum_{i=1}^{I} f_i\left(\psi_i f^{-1}(c,t;\Psi),t\right) = \sum_{i=1}^{I} c_i^* = c$$

a restatement of the feasibility condition. This proves equation (7.7).

Finally, we show that $u \in \mathcal{U}$, which is tedious but straight forward.

First, $u(\cdot,\cdot;\Lambda) \in C^{3,1}$ follows from the fact that for all $i \in \mathcal{I}$, $u_i \in C^{3,1}$ so $u_{ic}, f_i, f(\cdot,\cdot,;\Psi), f^{-1}(\cdot,\cdot;\Psi)$ and $u_c(\cdot,\cdot;\Psi)$ are all of the class $C^{2,1}$. Second, that $u(\cdot,t;\Lambda)$ is increasing, strictly concave and satisfies the Inada conditions follows from the fact that $u_c(\cdot,t;\Lambda) = f^{-1}(\cdot,t;\Psi)$ is a strictly decreasing map from $\mathbb{R}_{++}$ onto itself. Third, the fact that

$$\inf_{t\in[0,T]} u_c(c,t;\Lambda) > 0, \quad \text{for all } c > 0$$

follows from the previously stated fact that for any $i \in \mathcal{I}$,

$$u_c(c,t;\Lambda) = f^{-1}(c,t;\Psi) = \lambda_i u_{ic}(c_i^*,t)$$

where from equation (7.8), c_i^* is a monotonically increasing function of c with

$$\lim_{c\uparrow+\infty} c_i^*(c) = +\infty \quad \text{and} \quad \lim_{c\downarrow 0} c_i^*(c) = 0.$$

This concludes the proof. □

Given the proposition above, we can rewrite the equilibrium SPD as

$$\pi(t) = u_c(\tilde{c}_t,t;\Lambda^*) \tag{7.9}$$

where $\Lambda^* = (1/\psi_1^*,\cdots,1/\psi_I^*)$. With this expression, we can show our previous claim that the SPD is bounded above and below away from zero.

How? Well since the aggregate endowment is assumed to be bounded above and below (away from zero), there exists constants $\underline{c}$ and $\bar{c}$ such that for all $t \in [0,T]$

$$0 < \underline{c} < \tilde{c}(t) \leq \bar{c} < \infty$$

Now given our new expression for $\pi(t)$ and that $u \in \mathcal{U}$, we have

$$0 < \inf_{t\in[0,T]} u_c(\bar{c},t;\Lambda^*) \leq \pi(T) \leq \sup_{t\in[0,T]} u_c(\underline{c},t;\Lambda^*) < +\infty$$

where in the first inequality we used the property of $\mathcal{U}$; the last equality follows from taking a supremum of a continuous function over a bounded set. This proves that the equilibrium SPD is bounded above and below away from zero.

7.2.3 Key Results

Equation (7.9) provides an explicit expression for the equilibrium SPD in terms of the unknown vector of representative agent's weights $\Lambda^* = (1/\psi_1^*, \cdots, 1/\psi_I^*)$.

Furthermore, it follows from equations (7.3) and (7.9) that if an equilibrium exists, then Λ^* must solve the system of I non-linear equations

$$E^P\left[\int_0^T u_c(\tilde{c}_t, t; \Lambda)\left(f_i\left(\frac{u_c(\tilde{c}_t, t; \Lambda)}{\lambda_i}, t\right) - \tilde{c}_i(t)\right)dt\right] = 0 \quad \forall i \in \mathcal{I} \tag{7.10}$$

The two equations (7.9) and (7.10) together pin down the equilibrium SPD. Although this is I equations and I unknowns, the solution is not necessarily unique. Notice, for example, that a solution to equation (7.10) is only unique up to a positive linear transformation; that is, if Λ^* solves it, then so does $\Lambda^{**} = \eta\Lambda^*$ for any $\eta > 0$. Why? This is a consequence of the fact that

$$u(c, t; \eta\Lambda) = \max_{\substack{(c_1, \cdots, c_I) \in \mathbb{R}_+^I \\ c_1 + \cdots + c_I = c}} \sum_{i=1}^I \eta\lambda_i u_i(c_i, t) = \eta u(c, t; \Lambda)$$

and hence

$$u_c(c, t; \eta\Lambda) = \eta u_c(c, t; \Lambda)$$

Substituting $\eta\Lambda^*$ into equation (7.10), one can see that this is indeed the case.

Nevertheless, this non-uniqueness to equation (7.10) is not a problem - that is, it does not interfere with the identification of Λ^* or of the equilibrium SPD. To see why, consider (7.9) at $t = 0$. We know that $\pi(0) = 1$. Now, if the Λ^* is the "right" solution (i.e. the reciprocal of the Lagrangean multipliers), then it must also satisfy

$$\pi(0) = 1 = u_c(\tilde{c}_0, 0; \Lambda^*)$$

If, however, we were to take a scalar multiple of the weights $\eta\Lambda^*$, then we would get

$$u_c(\tilde{c}_0, 0; \eta\Lambda^*) = \eta u_c(\tilde{c}_0, 0; \Lambda^*) = \eta \tag{7.11}$$

so we would know how to scale by η as needed. Furthermore,

$$u_c(\tilde{c}_t, t; \eta\Lambda^*) = \eta u_c(\tilde{c}_t, t; \Lambda^*) = u_c(\tilde{c}_0, 0; \eta\Lambda^*)\, u_c(\tilde{c}_t, t; \Lambda^*)$$

Thus we can reformulate as follows. Take an arbitrary solution Λ^{**} to equation (7.10). We can either retrieve the weights $\Lambda^* = (1/\psi_1^*, \cdots, 1/\psi_I^*)$ by rescaling Λ^{**} by $1/\eta$, where η is determined by equation (7.11). Or we can simply define the SPD more generally as

$$\pi(t) = \frac{u_c(\tilde{c}_t, t; \Lambda^{**})}{u_c(\tilde{c}_0, 0; \Lambda^{**})} \tag{7.12}$$

Of course if we have Λ^* exactly as the reciprocal of Lagrangean multipliers, then the denominator is one and we're good to go. Thus writing this new version instead of $\pi(t) = u_c(\tilde{c}_t, t; \Lambda^*)$ let's us account for the non-uniqueness of the solution to the system of equations when we solve for Λ.

Now we establish the key characterization of equilibrium in this section in the following theorem.

Theorem 159. *(Characterization of Equilibrium) Suppose $\Lambda^{**} \in \mathbb{R}_{++}^I$ solves equation (7.10). Let π be the stochastic process in equation (7.12) and let*

$$\begin{aligned}
c_i^*(t) &= f_i\left(\frac{u_c(\tilde{c}_t, t; \Lambda^{**})}{\lambda_i^{**}}, t\right) \\
r(t) &= -\frac{1}{\pi(t)}\,\mathit{drift}[\pi(t)] \\
S(t) &= \frac{1}{\pi(t)}E^P\left[\int_t^T \pi(s)\,\delta(s)\,ds \,\middle|\, \mathcal{F}_t\right] \\
\sigma(t) &= I_S(t)^{-1}\,\mathit{diff}[S(t)]
\end{aligned}$$

If σ has full rank a.e., then there exist trading strategies $\{(\alpha_i^*, \theta_i^*) : i \in \mathcal{I}\}$ such that $(r, S, \{(c_i^*, \alpha_i^*, \theta_i^*) : i \in \mathcal{I}\})$ is a CME and π is the equilibrium SPD.

Note that if $X$is an Ito process then the notation

$$\text{drift}\left[X\left(t\right)\right] = a\left(t\right) \quad \text{and} \quad \text{diff}\left[X\left(t\right)\right] = b\left(t\right)$$

means that

$$X\left(t\right) = X\left(0\right) + \int_0^t a\left(s\right) ds + \int_0^t b\left(s\right) dw\left(s\right).$$

This theorem chracterizes all the equilibrium objects – the only thing that it leaves open is whether there exists such a vector Λ^{**} that solves the system of equations, and then whether σ has full rank a.e. Unfortunately, there is no easy sufficient condition to guarantee if σ has full rank a.e. The way we usually proceed is we solve for $\Lambda^{**}, S\left(t\right), \sigma\left(t\right)$ and check whether that is the case. Generically, it happens to be the case, and you can conclude that you have a complete market equilibrium.

Now let's prove the above theorem.

Proof. (Theorem 159) Suppose Λ^{**} solves the system of equations in equation (7.10) and π to be the SP in equation (7.12).

The first thing we need to show is when we write $r\left(t\right)$, we are not dividing by zero, i.e. $\pi\left(t\right) > 0$. Since $\pi\left(t\right) = \frac{u_c(\tilde{c}_t, t; \Lambda^{**})}{u_c(\tilde{c}_0, 0; \Lambda^{**})}$ and $u \in \mathcal{U}$, by Ito's lemma, π is a smooth transformation of an Ito process, and hence it is an Ito process. Since u is monotonically strictly increasing, $u_c > 0$, and the interest rate process $r\left(t\right)$ is well-defined. Similarly, $S\left(t\right)$ is well-defined because $\pi > 0$, π is bounded, and δ total dividend has finite expectation under P $E^P\left[\int_0^T \delta\left(t\right) dt\right] < \infty$. We also know $S\left(t\right) > 0$.

In addition, note that

$$\begin{aligned}
\pi_t S_t &= E^P\left[\left.\int_t^T \pi\left(s\right)\delta\left(s\right) ds\right| \mathcal{F}_t\right] \\
&= E^P\left[\left.\int_0^T \pi\left(s\right)\delta\left(s\right) ds\right| \mathcal{F}_t\right] - \int_0^t \pi\left(s\right)\delta\left(s\right) ds
\end{aligned} \tag{7.13}$$

so that by Martingale Representation Theorem, the conditional expectation is a stochastic integral. Thus, $\pi_t S_t$ is equal to a stochastic integral minus a time integral, thus an Ito process. Since we already established π_t is an Ito process, then S_t is the ratio of two Ito processes. By the multidimensional Ito's lemma, the ratio of two Ito processes is an Ito process. Thus its diffusion is well-defined, and $S\left(t\right) > 0$ so $I_S\left(t\right)$ is also invertible because it is a diagonal matrix with positive entries on the diagonal. This shows the quantities stated in the theorem are well-defined.

Next, we will confirm that π as defined in equation (7.12) is the unique SPD for the economy. Recall that should have the representation

$$\pi\left(t\right) = \exp\left(\int_0^t \kappa\left(s\right)^\top dw\left(s\right) - \int_0^t \left(r\left(s\right) + \frac{1}{2}\left|\kappa\left(s\right)\right|^2\right) ds\right)$$

and this is what we check.

Let $G = S + D$ and let $\mu\left(t\right) = I_S\left(t\right)^{-1} \text{drift}\left[G_t\right]$, so that

$$G_t = G_0 + \int_0^t I_S\left(s\right)\mu_s ds + \int_0^t I_S\left(s\right)\sigma_s dw_s$$

as the cumulative dividend process does not contribute to the volatility of G. In addition, define

$$\kappa_t = \frac{1}{\pi_t}\text{diff}\left[\pi_t\right] = \frac{1}{\pi_t}\frac{u_{cc}\left(\tilde{c}_t, t; \Lambda^{**}\right)}{u_c\left(\tilde{c}_0, 0; \Lambda^{**}\right)}\tilde{c}_t\tilde{\sigma}_t \tag{7.14}$$

where the second inequality follows from the endowment dynamics and equation (7.12). Then,

$$\pi_t = 1 - \int_0^t \pi_s r_s ds + \int_0^t \pi_s \kappa_s^\top dw_s$$

This is practically by construction, as $r_t = -\frac{1}{\pi_t}\text{drift}\left[\pi_t\right]$ and $\kappa_t = \frac{1}{\pi_t}\text{diff}\left[\pi_t\right]$.

We need to show that π_t constructed as above is indeed the equilibrium state price density, or that our κ_t we specified in the beginning, i.e.

$$\kappa_t = -\sigma_t^{-1}\left(\mu_t - r_t \bar{1}\right) \tag{7.15}$$

is consistent with our κ_t above in terms of the diffusion of π_t.

To do this, consider the stochastic process

$$M_t = \pi_t \left(G_t - \int_0^t \delta_s ds\right) + \int_0^t \pi_s \delta_s dt = E^P\left[\int_0^T \pi_s \delta_s ds \middle| \mathcal{F}_t\right]$$

which really is just equation (7.13) rearranged. This means that M_t is a P-martingale, so that

$$\begin{aligned}
0 &= \text{drift} M_t \\
&= \pi_t \left(I_S(t)\mu_t - \delta_t\right) - \left(G_t - \int_0^t \delta_s ds\right)\pi_t r_t + \pi_t I_S(t)\sigma_t \kappa_t + \pi_t \delta_t \\
&= \pi_t I_S(t)\left(\mu_t - r_t \bar{1} + \sigma_t \kappa_t\right)
\end{aligned}$$

via Ito's product rule, processes for G_t and π_t. Hence, κ_t is consistent with both expressions equations (7.15) and (7.14) and π_t is the unique SPD at the equilibrium prices.

Having established this, we now show that the consumption plans stated in the theorem are indeed the optimal consumption plans for each agent $i \in \mathcal{I}$. What do we need to show? That the $c_i^*(t)$ satisfies the first order conditions: Given

$$c_i^*(t) = f_i\left(\frac{u_c\left(\tilde{c}_t, t; \Lambda^{**}\right)}{\lambda_i^{**}}, t\right),$$

we can apply $u_{ic}(\cdot, t)$ to both sides to obtain

$$u_{ic}\left(c_i^*(t), t\right) = \frac{u_c\left(\tilde{c}_t, t; \Lambda^{**}\right)}{\lambda_i^{**}} = \frac{u_c\left(\tilde{c}_t, t; \Lambda^{*}\right)}{\lambda_i^{*}} = \psi_i^* \pi_t$$

then scaling the numerator and the denominator by the same positive constant. Hence c_i^* is optimal for agent i.

Let's now show market clearing. Letting $\Psi^{**} = \left(1/\lambda_1^{**}, \cdots, 1/\lambda_I^{**}\right)$ we have

$$\begin{aligned}
\sum_{i=1}^I c_{it}^* &= \sum_{i=1}^I f_i\left(\frac{u_c\left(\tilde{c}_t, t; \Lambda^{**}\right)}{\lambda_i^{**}}, t\right) \\
&= f\left(u_c\left(\tilde{c}_t, t; \Lambda^{**}\right), t; \Psi^{**}\right) \\
&= f\left(f^{-1}\left(\tilde{c}_t, t; \Psi^{**}\right), t; \Psi^{**}\right) \\
&= \tilde{c}_t
\end{aligned}$$

where second equality follows from equations (7.4), and the third from (7.7). Hence the market for the consumption good clears.

Finally, let's confirm the statement on trading strategies. The first thing is to require them to be MG-generating, and check the existence of EMM Q in equilibrium. Let

$$\xi_t = 1 + \int_0^t \xi_s \kappa_s^\top dw_s$$

We want ξ_T to be an admissible Radon Nikodym derivative (i.e. $E\xi_T = 1$). Also, ξ_t needs to be not just a local martingale, but a real martingale under P. This is going to follow from κ_t satisfying the Novikov condition,

$$E^P\left[\exp\left(\frac{1}{2}\int_0^t |\kappa_t|^2\, dt\right)\right] < \infty$$

which holds because $\kappa_t = \frac{u_{cc}(\tilde{c}_t,t,\Lambda^{**})}{u_c(\tilde{c}_t,t,\Lambda^{**})}\tilde{c}_t\tilde{\sigma}_t$ where $\tilde{\sigma}_t$ satisfies the Novikov condition and is bounded. Thus ξ is a P-MG and the probability measure Q with $\frac{dQ}{dP} = \xi_T$ is the unique equilibrium EMM.

The net consumption $c_i^* - \tilde{c}_i \in \mathcal{C}_b^1$ because c_i^* is bndded below $\tilde{c}_i$ is bndd. Also, c_i^* has finite expectation under Q (as it's in $\mathcal{C}_b^1$ satisfies the budget constraint). $\tilde{c}_i$ is bndd so finite expectation under Q. The market is complete because $n = d$ and σ is full rank a.e. Hence, there exists a tight, MG generating $(\alpha_i^*, \theta_i^*) \in \Theta$ financing the net CP $c_i^* - \tilde{c}_i$ and (α_i^*, θ_i^*) is optimal for agent i.

It remains to check the market for securities clears. Note the trading strategy that consists of the sum of individual strategies, $\left(\sum_{i=1}^I \alpha_i^*, \sum_{i=1}^I \theta_i^*\right)$, is a tight, martingale generating TS that finances the aggregate net consumption

$$\sum_{i=1}^I (c_i^* - \tilde{c}_i) = 0$$

This in turn implies

$$\left(\sum_{i=1}^I \alpha_i^*, \sum_{i=1}^I \theta_i^*\right) = (0,0)$$

as it is the cheapest trading strategy that finances the zero consumption plan. It cannot be negative because then there will be an arbitrage opportunity; it cannot be positive because then it wouldn't be the cheapest one. This concludes the proof. $\square$

The two assumptions of the above theorem is that (1) the system of equations on 7.10 has a solution; and (2) σ has full rank almost everywhere. Provided that those two assumptions are met, then the theorem gives us a complete characterization of an equilibrium.

Now, the following theorem says that our initial assumptions on the utility functions imply that the system of equations in 7.10 has solution.

Theorem 160. *There is a $\Lambda^{**} \in \mathbb{R}_{++}^I$ that solves the system of equations 7.10.*

Lemma 161. *(Knaster-Kuratowski-Mazurkiewicz) Let $\Delta \subset \mathbb{R}^n$ be a $(n-1)$ dimensional simplex with vertices $\{e_1, \cdots, e_n\}$, i.e.*

$$\Delta = \left\{\sum_{i=1}^n \lambda_i e_i : \lambda_i \geq 0 \text{ for all } i \in \{1,\cdots,n\} \text{ and } \sum_{i=1}^n \lambda_i = 1\right\}$$

and let $\{C_1, \cdots, C_n\}$ be a ***KKM covering*** *of Δ, i.e. a collection of closed subsets of Δ with the property that*

$$\forall \mathcal{J} \subseteq \{1,\cdots,n\} : \left\{\sum_{i\in\mathcal{J}} \lambda_i e_i : \lambda_i \geq 0 \text{ for all } i \in \mathcal{J} \text{ and } \sum_{i\in\mathcal{J}} \lambda_i = 1\right\} \subseteq \bigcup_{i\in\mathcal{J}} C_i.$$

Then

$$\bigcap_{i\in\mathcal{J}} C_i \neq \emptyset.$$

We will not prove this lemma, but will simply apply it to prove the theorem above. Say you have some $n-1$ dimensional simplex Δ in $\mathbb{R}^n$ with n vertices. A KKM covering is defined a set $C_1, \cdots, C_n$ of closed sets such that for any subset of vertices $\mathcal{J}$, the convex hull of the vertices corresponding to $\mathcal{J}$ is covered by $\bigcup_{i\in\mathcal{J}} C_i$. The KKM lemma, in turn, states that a KKM covering has a non-empty intersection. Remember from topology that a cover of a set X is a collection of sets whose union includes X as a subset, i.e. $C = \{U_\alpha : \alpha \in A\}$, where A is some index set, is a cover of X if $X \subseteq \bigcup_{\alpha \in A} U_\alpha$.

We now move to the proof of the above theorem.

Proof. Let $\{e_1, \cdots, e_n\}$ denote the standard basis of $\mathbb{R}^I$ and let $\Delta \subset \mathbb{R}^I_+$ be the $(I-1)$ dimensional simplex with vertices $\{e_1, \cdots, e_I\}$, i.e. the unit simplex

$$\Delta = \left\{ (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^I_+ : \sum_{i=1}^{n} \lambda_i = 1 \right\}$$

Since a solution Λ^{**} to equation 7.10 is only unique up to a positive linear transformation, we can WLOG look for a solution in Δ, i.e. focus our attention on a solution that satisfies $\sum_{i=1}^{I} \lambda_i^{**} = 1$.

We had previous defined the representative agent's utility function $u(c, t; \Lambda)$ for $\Lambda \in \mathbb{R}^I_{++}$ in equation (7.6), but we could technically define $u(c, t; \Lambda)$ exactly the same way even if $\lambda_i = 0$ for some $i \in \mathcal{I}$, as long as $\Lambda \neq 0$. In thaat case, the solution to the allocation problem in equation (7.6) would be

$$\hat{c}_{it} = \begin{cases} f_i\left(\frac{u_c(c,t;\Lambda)}{\lambda_i}, t\right) & \text{if } \lambda_i > 0 \\ 0 & \text{if } \lambda_i = 0 \end{cases}$$

and $u_c(c, t; \cdot)$ is continuous on Δ, since $f_i\left(\frac{u_c(c,t;\Lambda)}{\lambda_i}, t\right)$ goes to zero as λ_i goes to zero.

Now for some agent $i \in \mathcal{I}$ and $\Lambda \in \Delta$, let

$$F_i(\Lambda) = E^P\left[\int_0^T u_c(\tilde{c}_t, t; \Lambda)(\hat{c}_{it} - \tilde{c}_{it})\, dt\right]$$

Then, this new object F_i is also well-defined and continuous on Δ. If we let

$$C_i = \{\Lambda \in \Delta : F_i(\Lambda) \geq 0\}$$

it then follows from the continuity of F_i that C_i is closed.

Furthermore, observe that

$$F_i(\Lambda) < 0 \quad \text{if } \lambda_i = 0$$

because then $\hat{c}_{it} - \tilde{c}_{it} < 0$ and $u_c > 0$. Also,

$$\sum_{i \in \mathcal{I}} F_i(\Lambda) = 0 \quad \text{for all } \Lambda \in \Delta$$

as $\sum_{i \in \mathcal{I}} (\hat{c}_{it} - \tilde{c}_{it}) = 0$ for all $t \in [0, T]$.

Suppose now we take some subset of the agents $\mathcal{J} \subseteq \mathcal{I}$ and take a $\tilde{\Lambda}$ where

$$\begin{aligned} \tilde{\Lambda} \in & \left\{ \sum_{i \in \mathcal{J}} \lambda_i e_i : \lambda_i \geq 0 \text{ for all } i \in \mathcal{J} \text{ and } \sum_{i \in \mathcal{J}} \lambda_i = 1 \right\} \\ & = \{(\lambda_1, \cdots, \lambda_I) \in \Delta : \lambda_i = 0 \text{ for all } i \in (\mathcal{I} \backslash \mathcal{J})\} \end{aligned}$$

Then,

$$\sum_{i \in (\mathcal{I} \backslash \mathcal{J})} F_i\left(\tilde{\Lambda}\right) \leq 0$$

with the inequality being strict if $\mathcal{J}$ is a proper subset of $\mathcal{I}$. This follows as $\lambda_i = 0$ for all $i \in (\mathcal{I} \backslash \mathcal{J})$ but we previously noted $F_i(\Lambda) < 0$ if $\lambda_i = 0$.

Since the entire sum over $\mathcal{I}$ has to be zero, i.e. $\sum_{i \in \mathcal{I}} F_i\left(\tilde{\Lambda}\right) = 0$ but taking a partial sum over $(\mathcal{I} \backslash \mathcal{J})$ gives a nonpositive number, i.e. $\sum_{i \in (\mathcal{I} \backslash \mathcal{J})} F_i\left(\tilde{\Lambda}\right) \leq 0$, this then must mean that

$$\sum_{i \in \mathcal{J}} F_i\left(\tilde{\Lambda}\right) \geq 0$$

This implies that $\tilde{\Lambda} \in \bigcup_{i \in \mathcal{J}} C_i$. Why? Suppose $\tilde{\Lambda} \notin \bigcup_{i \in \mathcal{J}} C_i$, so that this $\tilde{\Lambda}$ that we took is not an element of any C_i for $i \in \mathcal{J}$. By definition of C_i, that would mean we have $F_i\left(\tilde{\Lambda}\right) < 0$ for all $i \in \mathcal{J}$, and in that case, it would not be possible for the sum $\sum_{i \in \mathcal{J}} F_i\left(\tilde{\Lambda}\right)$ to be nonnegative.

Thus, we now have that the collection $\{C_1, \cdots, C_I\}$ we constructed is a KKM covering of our choice of Δ, the unit simplex.

Applying the KKM lemma, it then follows that the intersection of any of the covering sets is nonempty. Trivially, take $\mathcal{J} = \mathcal{I}$, so then we can conclude there exists a $\Lambda^{**} \in \bigcap_{i \in \mathcal{I}} C_i$; that is, there exists a Λ^{**} such that

$$F_i\left(\Lambda^{**}\right) \geq 0 \text{ for all } i \in \mathcal{I}.$$

As we previously concluded that $\sum_{i \in \mathcal{I}} F_i(\Lambda) = 0$ for all $\Lambda \in \Delta$, this then means that for all $i \in \mathcal{I}$, $F_i\left(\Lambda^{**}\right) = 0$ and hence $\lambda_i^{**} > 0$ (if not and $\lambda_i^{**} = 0$ we would get $F_i\left(\Lambda^{**}\right) < 0$ instead).

It remains to map this back to our system of equations. From the way we defined $F_i(\cdot)$ and $\hat{c}_{it}$, this is almost trivial: we therefore have

$$0 = F_i\left(\Lambda^{**}\right) = E^P\left[\int_0^T u_c\left(\tilde{c}_t, t; \Lambda^{**}\right)\left(f_i\left(\frac{u_c\left(\tilde{c}_t, t; \Lambda^{**}\right)}{\lambda_i^{**}}\right) - \tilde{c}_i(t)\right) dt\right]$$

for all $i \in \mathcal{I}$, so that our $\Lambda^{**} \in \mathbb{R}_{++}^I$ solves 7.10, as required. $\square$

We now go over a couple of examples. Before we dive in, however, let me outline the general approach that we will take. To characterize the equilibrium, the key object that we need an explicit expression of is the state price density. We can obtain this if we have the marginal utility of the representative agent as in (7.12), and that's exactly how far we'll go for now.

The steps are:

(i) Compute the individual inverse marginal utility function f_i's.

(ii) Sum them up to get $f(x, t; \Psi) = \sum_{i=1}^I f_i(\psi_i x, t)$.

(iii) Invert to get f^{-1}.

(iv) Use f_i and f^{-1} to obtain the optimal allocation c_i^*'s.

(v) Plug in to get $u(c, t; \Lambda)$ and $u_c(c, t; \Lambda)$.

(vi) Use $u_c(\tilde{c}_t, t; \Lambda)$ and f_i to solve for Λ^* via (7.10).

Let's look at some examples.

Example 162. (*Identical Logarithmic Utilities*) As a special case of our previous framework, let

$$u_i(c, t) = e^{-\rho t} \log(c) \text{ for all } i \in \mathcal{I}$$

so that all agents have identical logarithmic utility with exponential discounting. Then the inverse marginal utility function becomes

$$f_i(x, t) = \left(e^{\rho t} x\right)^{-1}$$

so that

$$f(x, t; \Psi) = \sum_{i=1}^I f_i(\psi_i x, t) = e^{-\rho t}\left(\sum_{i=1}^I \psi_i^{-1}\right) x^{-1}$$

and

$$f^{-1}(c, t; \Psi) = e^{-\rho t}\left(\sum_{i=1}^I \psi_i^{-1}\right) c^{-1}.$$

To characterize the equilibrium objects, we need to specify the equilibrium state price density , which is a function of the marginal utility of the representative agent. To compute this, we make a couple of preliminary calculations.

Using the first order conditions, the optimal allocation in (7.6) is

$$
\begin{aligned}
c_i^* &= f_i\left(\psi_i f^{-1}(c,t;\Psi),t\right) \\
&= \left(e^{\rho t}\psi_i e^{-\rho t}\left(\sum_{i=1}^{I}\psi_i^{-1}\right)c^{-1}\right)^{-1} \\
&= \frac{\psi_i^{-1}}{\left(\sum_{i=1}^{I}\psi_i^{-1}\right)}c \\
&= \frac{\lambda_i}{\sum_{j=1}^{I}\lambda_j}c
\end{aligned}
$$

for all $i \in \mathcal{I}$ where $\lambda_i = 1/\psi_i$. Each agent receives a share of the aggregate endowment that is proportional to his weight in the representative agent utility.

We then have the expression for the representative agent utility that is free of a max operator and is evaluated at the optimal allocation,

$$
u(c,t;\Lambda) = \sum_{i=1}^{I}\lambda_i u_i(c_i^*,t) = e^{-\rho t}\sum_{i=1}^{I}\lambda_i \log\left(\frac{\lambda_i}{\sum_{j=1}^{I}\lambda_j}c\right)
$$

To get the marginal utility, we can just differentiate with respect to c (or remember our previous discussion that $u_c(c,t;\Lambda) = f^{-1}(c,t;\Psi)$) to get

$$
u_c(c,t;\Lambda) = f^{-1}(c,t;\Psi) = e^{-\rho t}\left(\sum_{j=1}^{I}\lambda_j\right)c^{-1}
$$

so that the marginal utility of the representative agent is proportional to the inverse of the aggregate consumption. Once we get the weights λ_i's, we can get fully identify $u_c(c,t;\Lambda)$.

The equilibrium weights Λ^* must solve the following:

$$
\begin{aligned}
0 &= E^P\left[\int_0^T u_c(\tilde{c}_t,t;\Lambda)\left(f_i\left(\frac{u_c(\tilde{c}_t,t;\Lambda)}{\lambda_i},t\right)-\tilde{c}_i(t)\right)dt\right] \\
&= E^P\left[\int_0^T e^{-\rho t}\left(\sum_{j=1}^{I}\lambda_j\right)\tilde{c}(t)^{-1}\left(\frac{\lambda_i}{\sum_{j=1}^{I}\lambda_j}\tilde{c}(t)-\tilde{c}_i(t)\right)dt\right] \\
&= \left(\sum_{j=1}^{I}\lambda_j\right)E^P\left[\int_0^T e^{-\rho t}\left(\frac{\lambda_i}{\sum_{j=1}^{I}\lambda_j}-\frac{\tilde{c}_i(t)}{\tilde{c}(t)}\right)dt\right]
\end{aligned}
$$

for all $i \in \mathcal{I}$. Hence,

$$
\lambda_i^* \propto \frac{E^P\left[\int_0^T e^{-\rho t}\frac{\tilde{c}_i(t)}{\tilde{c}(t)}dt\right]}{\int_0^T e^{-\rho t}dt}.
$$

Recall that we can only identify the weights of the representative utility up to a scalar multiple, so we only need to solve the weights up to a proportionality constant. To see what this means, remember that the marginal utility of the representative agent is proportional to the inverse of the aggregate endowment $\tilde{c}^{-1}$, as in $u_c(\tilde{c},t;\Lambda) = e^{-\rho t}\left(\sum_{j=1}^{I}\lambda_j\right)\tilde{c}^{-1}$, and thus proportional to the state price density. So then the numerator $E^P\left[\int_0^T e^{-\rho t}\frac{\tilde{c}_i(t)}{\tilde{c}(t)}dt\right]$ is proportional to the value of the stream of individual endowment, and so is the individual equilibrium weight λ_i^*.

Applying the theorem then gives us a full characterization of the equilibrium. Once we write out the interest rates and the risk premium more explicitly, we will return to this example.

Example 163. (*Idential Power Utilities*) As another example, assume

$$u_i(c,t) = e^{-\rho t}\frac{c^{1-b}}{1-b} \text{ for all } i \in \mathcal{I}$$

where $b > 0$ and $b \neq 1$. Then, $f_i(x,t) = (e^{\rho t}x)^{-1/b}$ so that

$$f(x,t;\Psi) = \sum_{i=1}^{I} f_i(\psi_i x, t) = e^{-\frac{\rho}{b}t}\left(\sum_{i=1}^{I}\psi_i^{-\frac{1}{b}}\right)x^{-\frac{1}{b}}$$

and

$$f^{-1}(c,t;\Psi) = e^{-\rho t}\left(\sum_{i=1}^{I}\psi_i^{-1/b}\right)^{b} c^{-b}$$

By (7.8), once we have f_i and f^{-1}, we can identify the optimal allocation as

$$\begin{aligned}
c_i^* &= f_i\left(\psi_i f^{-1}(c,t;\Psi), t\right) \\
&= \left(e^{\rho t}\psi_i e^{-\rho t}\left(\sum_{i=1}^{I}\psi_i^{-\frac{1}{b}}\right)^{b} c^{-b}\right)^{-\frac{1}{b}} \\
&= \frac{\psi_i^{-1/b}}{\sum_{j=1}^{I}\psi_i^{-1/b}}c = \frac{\lambda_i^{1/b}}{\sum_{j=1}^{I}\lambda_j^{1/b}}c
\end{aligned}$$

for all $i \in \mathcal{I}$ as $\lambda_i = 1/\psi_i$.

Next step is to plug these optimal allocation values in to obtain an explicit expression for the representative utility

$$u(c,t;\Lambda) = \sum_{i=1}^{I}\lambda_i u_i(c_i^*, t) = e^{-\rho t}\left(\sum_{i=1}^{I}\lambda_i^{1/b}\right)^{b}\frac{c^{1-b}}{1-b}$$

and for the marginal utility of the representative agent

$$u_c(c,t;\Lambda) = f^{-1}(c,t;\Psi) = e^{-\rho t}\left(\sum_{i=1}^{I}\lambda_i^{1/b}\right)^{b} c^{-b}$$

Again, what's left are the equilibrium weights Λ^* that solve

$$\begin{aligned}
0 &= E^P\left[\int_0^T u_c(\tilde{c}_t,t;\Lambda)\left(f_i\left(\frac{u_c(\tilde{c}_t,t;\Lambda)}{\lambda_i},t\right) - \tilde{c}_i(t)\right)dt\right] \\
&= E^P\left[\int_0^T e^{-\rho t}\left(\sum_{j=1}^{I}\lambda_j^{1/b}\right)^{b}\tilde{c}(t)^{-b}\left(\frac{\lambda_i^{1/b}}{\sum_{j=1}^{I}\lambda_j^{1/b}}\tilde{c}(t) - \tilde{c}_i(t)\right)dt\right] \\
&= \left(\sum_{j=1}^{I}\lambda_j^{1/b}\right)^{b} E^P\left[\int_0^T e^{-\rho t}\tilde{c}(t)^{1-b}\left(\frac{\lambda_i^{1/b}}{\sum_{j=1}^{I}\lambda_j^{1/b}} - \frac{\tilde{c}_i(t)}{\tilde{c}(t)}\right)dt\right]
\end{aligned}$$

for all $i \in \mathcal{I}$. Hence,

$$\lambda_i^* \propto \left(\frac{E^P\left[\int_0^T e^{-\rho t}\tilde{c}(t)^{1-b}\frac{\tilde{c}_i(t)}{\tilde{c}(t)}dt\right]}{E^P\left[\int_0^T e^{-\rho t}\tilde{c}(t)^{1-b}dt\right]}\right)^{b}$$

Example 164. (*Heterogeneous Preferences - Logarithmic and Square Root Utilities*) As a third example, we consider a case in which agents have heterogeneous utilities. With $I = 2$, assume

$$u_1(c,t) = e^{-\rho t} \log c$$
$$u_2(c,t) = e^{-\rho t} \frac{c^{1-b}}{1-b}$$

where $b = 1/2$. Then, we have that $f_1(x,t) = (e^{\rho t} x)^{-1}$ and $f_2(x,t) = (e^{\rho t} x)^{-2}$, so that

$$f(x,t;\Psi) = \sum_{i=1}^{I} f_i(\psi_i x, t) = \left(e^{\rho t}\psi_1 x\right)^{-1} + \left(e^{\rho t}\psi_2 x\right)^{-2}$$

To invert this, recall the requirement that $f^{-1} > 0$, to arrive at

$$f^{-1}(c,t;\Psi) = e^{-\rho t} \frac{2\psi_1/\psi_2^2}{\sqrt{1 + 4(\psi_1/\psi_2)^2 c} - 1}$$

Then again using $c_i^* = f_i\left(\psi_i f^{-1}(c,t;\Psi), t\right)$, we pin down the optimal allocation as

$$c_1^* = \frac{\sqrt{1+4(\lambda_2/\lambda_1)^2 c} - 1}{2(\lambda_2/\lambda_1)^2}$$
$$c_2^* = \left(\frac{\sqrt{1+4(\lambda_2/\lambda_1)^2 c} - 1}{2(\lambda_2/\lambda_1)}\right)^2$$

using $\lambda_i = 1/\psi_i$.

Next step is the representative utility

$$\begin{aligned} u(c,t;\Lambda) &= \lambda_1 u_1(c_1^*, t) + \lambda_2 u_2(c_2^*, t) \\ &= e^{-\rho t}\left[\lambda_1 \log\left(\frac{\sqrt{1+4(\lambda_2/\lambda_1)^2 c} - 1}{2(\lambda_2/\lambda_1)^2}\right) + 2\lambda_2 \frac{\sqrt{1+4(\lambda_2/\lambda_1)^2 c} - 1}{2(\lambda_2/\lambda_1)}\right] \end{aligned}$$

and

$$u_c(c,t;\Lambda) = f^{-1}(c,t;\Psi) = e^{-\rho t} \frac{2\lambda_2^2/\lambda_1}{\sqrt{1+4(\lambda_2/\lambda_1)^2 c} - 1}$$

The equilibrium weights Λ^* solve

$$\begin{aligned} 0 &= E^P\left[\int_0^T u_c(\tilde{c}_t, t; \Lambda)\left(f_1\left(\frac{u_c(\tilde{c}_t,t;\Lambda)}{\lambda_1}, t\right) - \tilde{c}_1(t)\right) dt\right] \\ &= \lambda_1 E^P\left[\int_0^T e^{-\rho t}\left(1 - \frac{2(\lambda_2/\lambda_1)^2 \tilde{c}_1(t)}{\sqrt{1+4(\lambda_2/\lambda_1)^2 \tilde{c}(t)} - 1}\right) dt\right] \end{aligned}$$

and

$$0 = E^P\left[\int_0^T u_c(\tilde{c}_t, t; \Lambda)\left(f_2\left(\frac{u_c(\tilde{c}_t,t;\Lambda)}{\lambda_2}, t\right) - \tilde{c}_2(t)\right) dt\right]$$

This says that the value of net consumption for each agent must be equal to zero. However, the second equation is redundant by market clearing because

$$f_1\left(\frac{u_c(\tilde{c}_t,t;\Lambda)}{\lambda_1}, t\right) + f_2\left(\frac{u_c(\tilde{c}_t,t;\Lambda)}{\lambda_2}, t\right) = \tilde{c}(t) = \tilde{c}_1(t) + \tilde{c}_2(t)$$

reflecting the fact that in the case of two agents, we can only uniquely determine the ratio λ_2^*/λ_1^*.

7.3 Equilibrium Returns

Let's use the results we had in the theorem to get a more explicit representation of the equilibrium interest rate and risk premium.

7.3.1 Interest Rate

Let Λ^{**} solve the system of equations (7.10) and let π be the equilibrium SPD. We saw that the SPD is equal to the marginal rate of substitution for the representative agent via

$$\pi(t) = \frac{u_c(\tilde{c}_t, t; \Lambda^{**})}{u_c(\tilde{c}_0, 0; \Lambda^{**})}$$

Under the slightly stronger assumptions we made for the utility functions in this section, the marginal utility is twice continuously differentiable in the first argument and continuously differentiable in the second. We can then apply Ito's lemma to obtain

$$\pi(t) = 1 + \int_0^t \frac{\mathcal{D}u_c(\tilde{c}_s, s; \Lambda^{**})}{u_c(\tilde{c}_0, 0; \Lambda^{**})} ds + \int_0^t \frac{u_{cc}(\tilde{c}_s, s; \Lambda^{**})}{u_c(\tilde{c}_0, 0; \Lambda^{**})} \tilde{c}(s)\, \tilde{\sigma}(s)^\top dw(s) \tag{7.16}$$

where

$$\mathcal{D}u_c(c, t; \Lambda) = u_{cc}(c, t; \Lambda)\, c\tilde{\mu}(t) + \frac{1}{2} u_{ccc}(c, t; \Lambda)\, c^2 |\tilde{\sigma}(t)|^2 + u_{ct}(c, t; \Lambda)$$

Let's go back to the expression we have for the equilibrium interest rate in our theorem, which was $r(t) = -\frac{1}{\pi(t)} \text{drift}[\pi(t)]$. We can use this, equation (7.16), and $\pi(t) = \frac{u_c(\tilde{c}_t, t; \Lambda^{**})}{u_c(\tilde{c}_0, 0; \Lambda^{**})}$ to obtain an expression for the equilibrium interest rate.

$$\begin{aligned} r(t) &= -\frac{\mathcal{D}u_c(\tilde{c}_t, t; \Lambda^{**})}{u_c(\tilde{c}_t, t; \Lambda^{**})} \\ &= -\frac{u_{ct}(\tilde{c}_t, t; \Lambda^{**})}{u_c(\tilde{c}_t, t; \Lambda^{**})} + a(\tilde{c}_t, t; \Lambda^{**})\, \tilde{\mu}(t) - \frac{1}{2} a(\tilde{c}_t, t; \Lambda^{**})\, p(\tilde{c}_t, t; \Lambda^{**})\, |\tilde{\sigma}(t)|^2 \end{aligned}$$

where we can respectively define the representative agent's Arrow-Pratt coefficient of relative risk aversion and the Kimball coefficient of relative prudence

$$a(c, t; \Lambda) = -\frac{c u_{cc}(c, t; \Lambda)}{u_c(c, t; \Lambda)} \text{ and } p(c, t; \Lambda) = -\frac{c u_{ccc}(c, t; \Lambda)}{u_{cc}(c, t; \Lambda)}$$

to simplify the expression.

This expression holds with full generality in our pure-exchange economy. What does this expression say? The equilibrium interest rate gives the equilibrium compensation that agents require in order to save; that is, in order to postpone their consumption.

The first term says that the interest rate must compensate the agent for how his marginal utility decays through time. For example, if the agent has exponential discounting via a $e^{-\rho t}$ term and utility that is (multiplicatively) separable in time and consumption, then the first term just becomes the coefficient of time preference, i.e. $-\frac{u_{ct}(c, t; \Lambda^{**})}{u_c(\tilde{c}_0, 0; \Lambda^{**})} = \rho$. In general, if the representative agent discounts the future more heavily, then the agent prefers to consume more today and demands a higher interest rate. This is true even for an economy with no uncertainty.

The second term says that higher the expected growth of aggregate consumption, the higher the equilibrium interest rate. Note $\tilde{\mu}(t)$ is the drift of the aggregate endowment, specifying how fast the economy is expected to grow. If the agent internalizes the fact that the economy is expected to grow more, then he expects to consume more tomorrow because we saw in the expression for c_i^* that the indiviual consumption is increasing in aggregate consumption.

Now as long as marginal utility is decreasing in consumption, this means that the agent's marginal utility is lower tomorrow relative to today. As agents get a lower amount of utility tomorrow for a given amount of consumption tomorrow, this implies the amount of consumption they would require tomorrow to give up a unit of consumption today is larger. Hence the equilibrium interest rate must increase.

Second term also includes a relative risk aversion $a(\tilde{c}_t, t; \Lambda^{**})$ term, which is also (in the case of expected utility) the inverse of the elasticity of intertemporal substitution. This merely says that the effect of a larger consumption tomorrow on the equilibrium interest rate is larger the more inelastic intertemporal substitution is.

The third term on the right hand side, $-\frac{1}{2}a(\tilde{c}_t, t; \Lambda^{**})\, p(\tilde{c}_t, t; \Lambda^{**})\, |\tilde{\sigma}(t)|^2$, includes the volatility of the aggregate endowment $\tilde{\sigma}(t)$. As agents are risk-averse, they try to allocate consumption through time and states to achieve a consumption profile that is as smooth as possible (i.e. affected as little as possible by the various realizations of the economy). Thus, if agents face a lot of uncertainty regarding aggregate consumption tomorrow (and hence individual optimal consumption c_i^* which is increasing in $\tilde{c}$), then agents will be inclined to save to protect themselves against large negative shocks to their consumption tomorrow. This gives the *prudence motive for saving*, which is related to the coefficient of relative prudence as noted by Kimball. So the larger this precautionary savings motive, the lower the equilibrium interest rate the agent demands.

7.3.2 The Consumption-based CAPM

Recall we had $\kappa(t) = -\sigma(t)^{-1}[\mu(t) - r(t)\bar{1}]$, equations (7.16), (7.14), and (7.12) to see that

$$\begin{aligned}
\mu(t) - r(t)\bar{1} &= -\sigma(t)\kappa(t) = -\sigma(t)\left[\frac{1}{\pi(t)}\mathrm{diff}\pi(t)\right] \\
&= a(\tilde{c}_t, t; \Lambda^{**})\,\sigma(t)\,\tilde{\sigma}(t) \\
&= a(\tilde{c}_t, t; \Lambda^{**})\frac{d}{dt}[\log(S), \log(\tilde{c})](t)
\end{aligned}$$

where the last equality follows from the fact that

$$\begin{aligned}
d\log(S_t) &= (\cdot)\,dt + \sigma(t)\,dw(t) \\
d\log(\tilde{c}_t) &= (\cdot)\,dt + \tilde{\sigma}(t)^\top dw(t)
\end{aligned}$$

As the covariation process between two Ito processes is just the time integral of the product of their diffusions,

$$[\log(S), \log(\tilde{c})](t) = \int_0^t \sigma(s)\,\tilde{\sigma}(s)\,ds.$$

Taking the time derivative gets us to the third line.

The expression

$$\mu(t) - r(t)\bar{1} = a(\tilde{c}_t, t; \Lambda^{**})\frac{d}{dt}[\log(S), \log(\tilde{c})](t)$$

is known as the *Consumption-based Capital Asset Pricing Model* (C-CAPM).

What does this expression say? The risk premium $\mu(t) - r(t)\bar{1}$ is equal to the relative risk aversion coefficient of the representative agent times the covariation term we have above. Sometimes this is written sloppily as the covariation between the change in log stock price and the change in log consumption, but the proper way to write it in continuous time is given above.

The expression suggests that if the stock price tends to go up (i.e. provide positive return) when aggregate consumption is large (i.e. when marginal utility of consumption is low), then all else equal the investor would tend to demand a larger risk premium (a larger return) in order to hold the stock. This provides a general charaterization of the equilibrium risk premium in the standard economy.

Now let's return to the previous examples.

Example 165. (Identical Logarithmic or Power Utilities) In the log utility case $u_i(c,t) = e^{-\rho t}\log(c)$, we use the marginal utility of the representative agents to compute

$$a(c, t; \Lambda) = 1 \text{ and } p(c, t; \Lambda) = 2$$

so that

$$
\begin{aligned}
r(t) &= \rho + \tilde{\mu}(t) - |\tilde{\sigma}(t)|^2 \\
\mu(t) - r(t)\bar{1} &= \sigma(t)\tilde{\sigma}(t)
\end{aligned}
$$

Similarly, in the power utility case $u_i(c,t) = e^{-\rho t}\frac{c^{1-b}}{1-b}$ where $b>0, b\neq 1$, we have

$$
a(c,t;\Lambda) = b \text{ and } p(c,t;\Lambda) = b+1
$$

so that

$$
\begin{aligned}
r(t) &= \rho + b\tilde{\mu}(t) - \frac{1}{2}b(b+1)|\tilde{\sigma}(t)|^2 \\
\mu(t) - r(t)\bar{1} &= b\sigma(t)\tilde{\sigma}(t)
\end{aligned}
$$

Example 166. (Heterogeneous Preferences - Log and Power Utilities) Continuing in the previous example with $I=2$, consider $u_1(c,t) = e^{-\rho t}\log(c)$ and $u_2(c,t) = e^{-\rho t}\frac{c^{1-b}}{1-b}$ with $b = 1/2$. By differentiating the marginal utility function, we get that the relative risk aversion coefficient and the relative prudence coefficients are

$$
\begin{aligned}
a(c,t;\Lambda) &= \frac{1}{2}\left(1 + \frac{1}{\sqrt{1+4(\lambda_2/\lambda_1)^2 c}}\right) \in \left(\frac{1}{2},1\right) \\
p(c,t;\Lambda) &= \left(\frac{1+6(\lambda_2/\lambda_1)^2 c}{1+4(\lambda_2/\lambda_1)^2 c} + \frac{1}{\sqrt{1+4(\lambda_2/\lambda_1)^2 c}}\right) \in \left(\frac{3}{2},2\right)
\end{aligned}
$$

Note both will now vary stochastically depending on the aggregate consumption $\tilde{c}$. In particular, $a(\cdot)$ varies between the relative risk aversion of the second agent $\frac{1}{2}$ and that of the first agent 1. Similarly, the prudence coefficient also varies between $\frac{3}{2}$ for the second agent and 2 for the first agent.

7.3.3 The Intertemporal CAPM

The characterizations so far were at the generality of the standard economy; that is, we allowed for any arbitrary, general form of uncertainty adapted to the filtration generated by the Brownian motions.

Now, we specialize to the simpler uncertainty, the Markovian case. The exogenous quantities, instead of arbitrary processes adapted to our filtration, are deterministic functions of a vector of state variables that follow a diffusion.

As now we are looking at the Markovian setup of the equilibrium version of the standard economy, what we take to be deterministic functions of the state variables are just the exogenous quantities. These are the dividend rate δ and the aggregate endowment $\tilde{c}$, so that we write

$$
\delta(Y,t) \text{ and } \tilde{c}(Y,t)
$$

where Y is a $m-$dimensional vector of state variables that follow

$$
Y(t) = Y(0) + \int_0^t \gamma(Y_s,s)\,ds + \int_0^t \beta(Y_s,s)\,dw(s).
$$

Under these additional assumptions, it follows from our expression of $S(t)$ and the Markov property of diffusions that

$$
S(t) = E^P\left[\int_t^T \frac{u_c(\tilde{c}(Y_s,s),s;\Lambda^{**})}{u_c(\tilde{c}(Y_t,t),t;\Lambda^{**})}\delta(Y_s,s)\,ds \middle| \mathcal{F}_t\right] = S(Y_t,t)
$$

for some function $S(\cdot,\cdot)$. Notice that everything inside the expectation that is stochastic depends only on the future value of the vector of state variables.

Let's see what this implies for the equilibrium risk premium.

$$\begin{aligned}
I_S(t)\left[\mu(t) - r(t)\bar{1}\right] &= a\left(\tilde{c}(Y_t,t),t;\Lambda^{**}\right) I_S(t)\,\sigma(t)\,\tilde{\sigma}(t) \\
&= a\left(\tilde{c}(Y_t,t),t;\Lambda^{**}\right) S_Y(Y_t,t)\,\beta(Y_t,t)\left[\tilde{c}_Y(Y_t,t)^\top \beta(Y_t,t)\right]^\top \\
&= S_Y(Y_t,t)\,\phi(Y_t,t;\Lambda^{**})
\end{aligned}$$

where $S_Y(Y,t)$ denotes the $n \times m$ matrix of partial derivatives of $S(Y,t)$ above, and

$$\phi(Y,t;\Lambda) = a\left(\tilde{c}(Y,t),t;\Lambda\right)\beta(Y,t)\,\beta(Y,t)^\top\,\tilde{c}_Y(Y,t)$$

is a m-dimensional vector of risk premia associated with the state variables. Note to get to the second line, we applied Ito's lemma to $S(Y,t)$ and $\tilde{c}(Y,t)$ respectively to substitute above

$$\begin{aligned}
I_S(t)\,\sigma(t) &= \text{diff}\left[S(Y,t)\right] \\
\tilde{c}(Y,t)\,\tilde{\sigma}(t) &= \text{diff}\left[\tilde{c}(Y,t)\right]
\end{aligned}$$

The expression

$$I_S(t)\left[\mu(t) - r(t)\bar{1}\right] = S_Y(Y_t,t)\,\phi(Y_t,t;\Lambda^{**})$$

is known as the *Intertemporal Capital Asset Pricing Model* (I-CAPM). This was derived by Merton (1976), before Breeden (1979)'s consumption CAPM.

What does this say? First note that the left hand side is the dollar risk-premium for the stocks, as $\mu(t) - r(t)\bar{1}$ is the risk premium as percentage of stock prices. If we are in a Markovian economy, then the dollar risk-premium for the stocks is equal to the vector of sensitivities of the stock prices to the each of the m state variables $S_Y(Y,t)$ multiplied by a m dimensional vector ϕ, the components of which you can interpret as the risk premium associated with each of the m state variables in Y_t.

Essentially, in a Markovian standard economy, the risk premia are going to satisfy a linear factor structure in equilibrium, with the state variables serving as the factors that affect dividends $\delta(t)$ and the aggregate endowment $\tilde{c}(t)$. The risk premium for each stock is going to be proportional to the sensitivity of each stock to the factors multiplied by the equilibrium risk premia for each factor $\phi(Y,t;\Lambda)$. Note that nothing in $\phi(\cdot)$ is asset-specific – the only thing that is asset-specific on the right hand side is the sensitivities $S_Y(Y_t,t)$.

Part V

Models of the Term Structure

8 The Term Structure of Interest Rates

Let's look at how to model the term structure of interest rates. There are mainly two ways to model the evolution of interest rates of different tenors through time: (1) First is the *general equilibrium approach*, where we construct an economy where agents can trade bonds of different maturities. The equilibrium prices of these different bonds are going to be determining the equilibrium interest rates, and hence by market clearing, the evolution of the term structure of interest rates. (2) Second is the *no-arbitrage approah*. This is a reduced form approach where we model the interest rates of different tenors in a way that would be consistent with some preferences and some aggregate endowments (and thus some general equilibrium), but without specifying what those preferences and aggregate endowments are. In essence, we just make sure that the evolution of term structure does not allow arbitrage opportunities. This in fact would allow the evolution of interests to be supported as a general equilibrium for some preferences.

We will look at examples of both.

8.1 The One-Factor Cox-Ingersoll-Ross Model

This is the first and possibly the most famous continuous-time general equilibrium term structure model.

8.1.1 The Economy

First, we will describe the economy that we will be constructing for the purposes of studying term structure of interest rates.

- $t \in [0, T]$ continuous-time, finite horizon.
- $(\Omega, \mathcal{F}, \mathbb{F}, P)$ a filtered probability space where.
  - $\mathbb{F} = \{\mathcal{F}_t : t \in [0, T]\}$ is the filtration generated by a d-dimensional Brownian motion w.
  - $\mathcal{F} = \mathcal{F}_T$ (i.e. all the uncertainty is resolved at the end)

A bit different from previous sections in that we won't consider a pure-exchange economy. The aggregate consumption is equal to aggregate endowment (which is exogenous) there. Instead, the aggregate consumption process is determined by production choices of agents in the economy. Hence, this is a *production economy* as opposed to a pure exchange economy.

The consumption good is something like corn, where we can eat it (consume) or plant it (produce more).

- Single good (numeraire) which can be allocated either to consumption of production.
- $\mathcal{C} = \left\{ c : c \text{ is a progressive SP with } \int_0^T |c_t|\, dt < \infty \text{ a.s.} \right\}$ the consumption space. (total consumption is finite w.p. 1)
- Finite number of agents with identical preferences and endowments.
- $U(c) = E\left[\int_0^T e^{-\rho t} \log c_t dt\right]$ utility function of each agent.
- $e_0 > 0$ endowment of the consumption good for each agent at time 0.

Production opportunity.

- Production opportunities consists of a set of $n = d$ linear processes. Consumption good is the only input (i.e. no labor or capital input).

- Each agent has free access to all of the production processes.
- The returns to the production technology is linear implying constant returns to scale.
- Investing amount $\theta(t) \in \mathbb{R}^n$ of the consumption good in the n production processes at time t generates the instantaneous return

$$d\theta(t) = I_\theta(t)\,\mu(Y_t)\,dt + I_\theta(t)\,\sigma(Y_t)\,dw(t)$$

  where $I_\theta(t) = \operatorname{diag}(\theta_t)$ and Y is a one-dimensional state variable.

  - Note this is CRS. Output is proportional to input (imagine multipying θ by some amount).
  - Specifically, $\mu(Y) = \bar{\mu}Y$ for some vector $\bar{\mu} \in \mathbb{R}^n$ and $\sigma(Y) = \bar{\sigma}\sqrt{Y}$ for some matrix $\bar{\sigma} \in \mathbb{R}^{n\times n}$ having full rank (thus Ω is invertible).
  - In addition, letting

$$\Omega = \bar{\sigma}\bar{\sigma}^\top \tag{8.1}$$

    and the scalar

$$\eta = \frac{\bar{1}^\top\Omega^{-1}\bar{\mu} - 1}{\bar{1}^\top\Omega^{-1}\bar{1}}, \tag{8.2}$$

    we have $\eta > 0$ and the n-dimensional vector

$$\Omega^{-1}\left(\bar{\mu} - \eta\bar{1}\right) \geq 0. \tag{8.3}$$

- Finally, the state variable Y follows the *square-root process*

$$Y(t) = Y(0) + \int_0^t \gamma\left(\bar{Y} - Y(s)\right)ds + \int_0^t \sqrt{Y(s)}\bar{\beta}^\top dw(s) \tag{8.4}$$

  for some $\gamma, Y(0), \bar{Y} \in \mathbb{R}_{++}$ and $\bar{\beta} \in \mathbb{R}^n \backslash \{0\}$ ($\bar{\beta}$ is not the zero vector as that would eliminate uncertainty from the CIR economy).

Financial securities

- Agents can invest in financial securities in addition to the production processes.
- A money market account that pays the instantaneous interest rate r (endogenous) and has price process:

$$B(t) = 1 + \int_0^t B(s)\,r(s)\,ds = \exp\left(\int_0^t r(s)\,ds\right)$$

- A set of default-free zero-coupon bonds (ZCB) with unit face value and arbitrary maturity $\tau \in (0,T]$ (and thus an uncountable number of ZCBs traded). We will denote the price at time t of the ZCB with maturity τ by $P(t,\tau)$.
- All securities are in net zero supply.

8.1.2 The Equilibrium

Since agents are identical and the securities are in zero net supply, no agent holds securities in equilibrium. If any agent would like to hold positive or negative holdings of a security, then so would others, implying the market would not clear.

Letting (α, θ) denote, respectively, the amount of the consumption good invested by each agent in the money market account and in the n technologies, we have the following definition of an equilibrium.

Definition 167. An equilibrium is a tuple $(r, \{P(\cdot,\tau) : \tau \in (0,T]\}, c^*, \theta^*)$ such that[32]:

[32] Note there is no i subscript as all agents are identical. All agents consume the same c^* and allocate the same amount of the consumption good in to the vector of production opportunities θ^*.

(i) $(c^*, 0, \theta^*)$ solves

$$\max_{(c,\alpha,\theta)} E\left[\int_0^T e^{-\rho t}\log(c_t)\,dt\right] \tag{8.5}$$

$$\text{s.t.} \quad \alpha_t + \theta_t^\top \bar{1} = e_0 + \int_0^t \left(\alpha_s r_s + \theta_s^\top \mu(Y_s) - c_s\right) ds + \int_0^t \theta_s^\top \sigma(Y_s)\,dw(s),$$
$$c \geq 0, \theta \geq 0, \alpha + \theta^\top \bar{1} \geq 0$$

(ii) The agents have no incentive to deviate from the consumption-investment choice above if given the opportunity to also invest in the ZCBs.

Note where the right hand side of the budget constraint represents the total amount of consumption good in the economy. $\alpha+\theta^\top\bar{1} \geq 0$ is imposed to rule out arbitrage opportunities such as doubling strategies. The optimization problem guarantees that the consumption and allocation to production opportunities are optimally chosen, and that the investment in the money market account is zero.

However, notice that this did not include the opportunity to trade in any of the zero coupon bonds! This is covered in the second part of the definition. It says that, if in addition to the production technologies and the money market account, the agents are given access to the ZCBs, they will continue to invest nothing in ZCBs.

The following theorem provides an explicit characterization of the equilibrium.

Theorem 168. *The tuple* $(r, \{P(\cdot,\tau) : \tau \in (0,T]\}, c^*, \theta^*)$ *with*

$$r(t) = \eta Y(t) \tag{8.6}$$

$$P(t,\tau) = \frac{1}{\pi(t)} E\left[\pi(\tau) \mid \mathcal{F}_t\right] \tag{8.7}$$

$$c^*(t) = \frac{\rho}{1-e^{-\rho(T-t)}} W(t)$$

$$\theta^*(t) = \Omega^{-1}(\bar{\mu} - \eta\bar{1}) W(t)$$

where Ω *and* η *are constants given in equations (8.1)-(8.2),*

$$\kappa(t) = -\bar{\sigma}^{-1}(\bar{\mu} - \eta\bar{1})\sqrt{Y(t)}$$

$$\pi(t) = \exp\left(\int_0^t \kappa(s)^\top dw(s) - \int_0^t \left(r(s) + \frac{1}{2}|\kappa(s)|^2\right) ds\right) \tag{8.8}$$

$$W(t) = \frac{e^{-\rho t} - e^{-\rho T}}{1-e^{-\rho T}} \frac{e_0}{\pi(t)}$$

is the unique equilibrium of the one-factor CIR economy.

Consumption is a deterministic proportion of wealth, similarly for the investment in the production technology.

Proof. We start by solving the optimization problem given in equation (8.5). To do this, note that apart from the non-negativity constraint on θ, the problem is identical to the optimal consumption problem for an agent who can invest in the money market account and n stocks with drift $\mu(Y_t)$ and volatility $\sigma(Y_t)$.

Note that this is general. If we have a production economy where the production opportunities have constant returns to scale, so that the returns are linear in the amount allocated to the production technology, then it is isomorphic to an economy with the type of securities we have been discussing so far (apart from the non-negativity constraint).

The equilibrium problem is actually easier if we have linear production technologies rather than stocks. The reason for this is that, if these were stocks, we would have to determine the drift and volatility ourselves; if we call it linear production technologies, we can take them as given (and not solve for them endogenously).

Either way, we have seen this problem before. First let's conjecture that the non-negativity constraint on θ is not binding (to be verified later). If so, it follows from our previous section that the optimal policies $(c^*, \alpha^*, \theta^*)$ in equation (8.5) are

$$c^*(t) = \frac{e^{-\rho t}}{\psi \pi(t)} = \frac{\rho}{1 - e^{-\rho(T-t)}} W(t) \tag{8.9}$$

$$\theta^*(t) = W(t) \left[\sigma(Y_t) \sigma(Y_t)^\top\right]^{-1} \left(\mu(Y_t) - r(t) \bar{1}\right) \tag{8.10}$$

where consumption c^* is a deterministic fraction of wealth and investment in production θ^* is the investment in the growth-optimal portfolio. Furthermore,

$$\begin{aligned}
\alpha^*(t) &= W(t) - \bar{1}^\top \theta^*(t) \\
&= W(t) \left[1 - \left[\sigma(Y_t) \sigma(Y_t)^\top\right]^{-1} \left(\mu(Y_t) - r(t) \bar{1}\right)\right],
\end{aligned}$$

where

$$W(t) = \frac{e^{-\rho t} - e^{-\rho T}}{\rho \psi \pi(t)} = \frac{e^{-\rho t} - e^{-\rho T}}{1 - e^{-\rho T}} \frac{e_0}{\pi(t)}$$

and π is the SPD in equation (8.8)[33].

This is true for an arbitrary interest rate. Notice our expression for α^* is a linear expression of the interest rate $r(t)$.

We now solve for the level of $r(t)$ at which $\alpha^*(t) = 0$. After carefully rearranging the matrix equation we have above in the expression for $\alpha^*(t)$, this becomes

$$\begin{aligned}
r(t) &= \frac{\bar{1}^\top \left(\sigma(Y_t) \sigma(Y_t)^\top\right)^{-1} \mu(Y_t) - 1}{\bar{1}^\top \left(\sigma(Y_t) \sigma(Y_t)^\top\right)^{-1} \bar{1}} \\
&= \frac{\bar{1}^\top (\Omega Y_t)^{-1} \bar{\mu} Y_t - 1}{\bar{1}^\top (\Omega Y_t)^{-1} \bar{1}} \\
&= \eta Y(t)
\end{aligned}$$

which conveniently turns out to be linear in $Y(t)$. This is where our previous assumptions on $\mu(Y_t)$ and $\sigma(Y_t)$ were helpful. Defining the matrix and scalar constants Ω and η just makes it easier to write things.

Moreover, substituting $r(t) = \eta Y(t)$ in equation (8.10) gives

$$\theta^*(t) = W(t) \Omega^{-1} (\bar{\mu} - \eta \bar{1}) \geq 0$$

where the inequality follows from (8.3). Hence, $(c^*, 0, \theta^*)$ solves the optimization problem in equation (8.5) when the interest rate is as given in equation (8.6).

Now it remains to show that if we define the prices of ZCBs as in (8.7), then agents would not want to deviate from optimal consumption allocation c^*. It that case, they then have no reason to deviate from the trading strategies that finances c^*.

The expression in equation (8.7) says that we are pricing the ZCBs using the state price densities $\pi(t)$. This is defined as the usual way, where we take $\kappa(t)$ to be $-\sigma^{-1}(\mu(Y) - r\bar{1}) = -\bar{\sigma}^{-1}(\bar{\mu} - \eta\bar{1})\sqrt{Y(t)}$. The $\pi(t)$ reflects our usual definition of the state price density if we were to treat technologies as stocks, and define the SPD accordingly. Note here that the SPD is implied entirely by the returns to the technologies.

Now it also turns out the same SPD $\pi(t)$ prices the ZCBs. This implies that if we look at the augmented economy, in which the agents can not only invest in the money market

[33] For the logarithmic utility case, we determined that $c^*(t) = \frac{\eta_1 e^{-\rho t}}{\psi \pi(t)}$ via the first order condition and $W(t) = \frac{e^{-\rho t}}{\psi \pi(t)} h(t)$ by evaluating the portfolio value process where $h(t) = \eta_1 \frac{1 - e^{-\rho(T-t)}}{\rho} + \eta_2 e^{-\rho(T-t)}$. Thus,

$$\frac{c^*(t)}{W(t)} = \frac{\eta_1}{h(t)}.$$

account and the production technologies, but also the ZCBs, $\pi(t)$ as defined in equation (8.8) is still the state price density that correctly prices returns to technology and ZCB. If the state price density is the same, this implies the optimal consumption c^* (equal to the inverse marginal utility evaluated at Lagrangean multiplier times the SPD) is unchanged. And if consumption policy c^* is unchanged once we introduce the ZCBs (because SPD unchanged), the agents have no reason to invest in the ZCBs.

Uniqueness of the equilibrium follows from the fact that $r(t) = \eta Y(t)$ is the only interest rate at which $\alpha^*(t) = 0$ and the equilibrium SPD is uniquely determined by the interest rate. □

From the fact that the state variable Y_t follows a square-root process as in (8.4), our expression for equilibrium interest rate equation (8.6), and that $\eta > 0$, we conclude that the equilibrium interest also follows the square-root process

$$\begin{aligned} r(t) &= \eta Y(0) + \eta \int_0^t \gamma \left(\bar{Y} - Y(s)\right) ds + \eta \int_0^t \sqrt{Y(s)} \bar{\beta}^\top dw(s) \\ &= r(0) + \int_0^t \gamma (\bar{r} - r(s)) ds + \int_0^t \sqrt{r(s)} \sigma_r^\top dw(s) \end{aligned} \tag{8.11}$$

where $r(0) = \eta Y(0) > 0$, $\bar{r} = \eta \bar{Y}$, and $\sigma_r = \sqrt{\eta} \bar{\beta} \neq 0$.

A few comments about this expression. The drift term is proportional to $r(t)$ while the volatility term is proportional to $\sqrt{r(t)}$. Also, this equation tells us that the interest rate is mean-reverting, being pull back toward the value $\bar{r}$ at a speed determined by the parameter $\gamma > 0$. Furthermore, as the diffusion coefficient vanishes as $r(t) = 0$, mean reversion ensures that the interest rate cannot become negative. That is, if $r(t)$ ever equals zero, then the change over the next instant is strictly positive (and volatility is zero). As $r(t)$ has continuous sample paths and starts positive, this implies $r(t)$ will never become negative.

In fact, if either the speed of mean reversion γ or the level of reversion $\bar{r}$ is large enough, then the interest rate remains strictly positive. Specifically, uf $\gamma \bar{r} \geq |\sigma_r|^2 / 2$, the mean reversion is strong enough that the interest rate does not reach zero either (i.e. stays strictly positive).

As for the non-defaultable ZCBs, equation (8.7) tells us that its price at time t with maturity $\tau \in [0, T]$ is given by

$$P(t, \tau) = \frac{1}{\pi(t)} E\left[\pi(\tau) | \mathcal{F}_t\right] = B(t) E^Q \left[B(\tau)^{-1} \middle| \mathcal{F}_t \right] \tag{8.12}$$

where $\pi(t) = \frac{B(0)}{B(t)} \xi_t$ and Q is the probability measure with

$$\frac{dQ}{dP} = \exp\left(\int_0^T \kappa(t)^\top dw(t) - \frac{1}{2} \int_0^t |\kappa(t)|^2 dt \right) \tag{8.13}$$

Note that Q is the unique equivalent martingale measure in this case.

Let's try to further characterize the price of these zero coupon bonds more sharply. Our strategy, similar to our previous approaches, is to write $r(t)$ as a diffusion process under Q, write $P(t, \tau)$ as a function of $r(t)$ (as it's written in terms of $B(t)$ and thus a time integral of $r(t)$), and identify a $Q-$martingale to set the drift equal to zero.

To do this, we need an expression of $r(t)$ under the Q measure. An application of Girsanov's theorem to equation (8.11), we obtain

$$\begin{aligned} dr(t) &= \left[\gamma(\bar{r} - r(t)) + \sqrt{r(t)} \sigma_r^\top \kappa(t) \right] dt + \sqrt{r(t)} \sigma_r^\top dw^*(t) \\ &= \left[\gamma(\bar{r} - r(t)) - \lambda r(t) \right] dt + \sqrt{r(t)} \sigma_r^\top dw^*(t) \end{aligned} \tag{8.14}$$

where $\lambda = \sigma_r^\top \left(\sqrt{\eta} \bar{\sigma} \right)^{-1} (\bar{\mu} - \eta \bar{1})$ and $dw^*(t) = dw(t) - \kappa(t) dt$ is a Brownian motion under Q. Notice that r follows a diffusion process under Q and thus has the Markov property.

Returning to our expression for the price of ZCBs,

$$\begin{aligned}
P(t,\tau) &= B(t)\, E^Q\left[B(\tau)^{-1} \middle| \mathcal{F}_t \right] \\
&= E^Q\left[e^{-\int_t^\tau r(s)ds} \middle| \mathcal{F}_t \right] \\
&= F(r_t, t, \tau)
\end{aligned}$$

for some function F. As the price of the ZCB is a conditional $Q-$expectation of something that depends entirely on the future path of the interest rate (and as $r(t)$ follows a diffusion under Q), it follows that the price $P(t,\tau)$ is just a function of interest rate $r(t)$, time t, and maturity τ.

To construct a martingale, consider the stochastic process

$$M(t) = e^{-\int_0^t r(s)ds} F(r_t, t, \tau) = E^Q\left[e^{-\int_0^\tau r(s)ds} \middle| \mathcal{F}_t \right].$$

This is a conditional expectation process and the term inside the expectation operator does not include t, so it is a Q-martingale (thus has zero drift). It follows from Ito's lemma that

$$\begin{aligned}
0 &= \text{drift}^Q(M_t) \\
&= e^{-\int_0^t r(s)ds}\left[F_r(\gamma(\bar{r}-r) - \lambda r) + F_t + \frac{1}{2}F_{rr} r |\sigma_r|^2 - rF \right]
\end{aligned} \tag{8.15}$$

This implies that

$$\begin{aligned}
\text{drift}^P(P(t,\tau)) - r_t P(t,\tau) &= F_r(r,t,\tau)\,\gamma(\bar{r}-r_t) + F_t(r,t,\tau) \\
&\quad + \frac{1}{2}F_{rr}(r,t,\tau)\, r\, |\sigma_r|^2 - rF(r,t,\tau) \\
&= F_r(r,t,\tau)\,\lambda r
\end{aligned}$$

The first equality follows from applying Ito's lemma under the physical measure to $P(t,\tau)$. The second follows from rearranging equation 8.15.

Writing it out this way is useful because the expression is readily interpretable. In particular, it gives a way to give meaning to the λr term that comes from switching measures:

$$\underbrace{\text{drift}^P(P(t,\tau)) - r_t P(t,\tau)}_{\text{dollar risk premium}} = \underbrace{F_r(r,t,\tau)}_{\text{sensitivity to } r} \times \underbrace{\lambda r}_{\substack{\text{ICAPM} \\ \text{risk premium}}}$$

What does this say? $\text{drift}^P(P(t,\tau))$ is the expected dollar return over the next instant of holding a zero coupon bond. $r_t P(t,\tau)$ is what you would get by putting the same money in the riskfree interest rate. The left hand side is then the dollar risk premium. On the right hand side, the first term is the sensitivity of the price of ZCB with respect to the interest rate (i.e. the unique state variable in the CIR economy), $F_r(r,t,\tau)$. This is then multiplied by λr which is a term that's independent of the particular ZCB. Thus λr is the ICAPM risk premium associated with interest rate risk.

Recall that in the ICAPM, we had written[34]

$$\begin{array}{c}\text{risk premium} \\ \text{(in dollars)}\end{array} = (\text{price sensitivity to state variables}) \times (\text{factor risk premium})$$

The ICAPM must hold in the Cox-Ingersoll-Ross economy because this is the Markovian version of the general economy that we considered, with the interest rate $r(t)$ as the single state variable.

[34] The precise expression being

$$I_S(t)\left[\mu(t) - r(t)\right] = S_Y(Y,t)\,\phi(Y,t;\Lambda^{**})$$

In addition, the second order linear PDE we are aiming to solve via equation 8.15 is

$$0 = F_r\left(\gamma\left(\bar{r} - r\right) - \lambda r\right) + F_t + \frac{1}{2}F_{rr}r\left|\sigma_r\right|^2 - rF$$

with terminal condition

$$F\left(r, \tau, \tau\right) = 1$$

as zero coupon pulls to par (i.e. price becomes face value of one).

The solution to this PDE can be obtained explicitly[35] as

$$P\left(t, \tau\right) = F\left(r, t, \tau\right) = A\left(\tau - t\right)e^{-B(\tau - t)r} \tag{8.16}$$

where

$$\begin{aligned} A\left(s\right) &= \left(\frac{2ae^{bs/2}}{2a + b\left(e^{as} - 1\right)}\right)^{\frac{2\gamma\bar{r}}{\left|\sigma_r\right|^2}} \\ B\left(s\right) &= \left(\frac{2\left(e^{as} - 1\right)}{2a + b\left(e^{as} - 1\right)}\right) \\ a &= \sqrt{\left(\gamma + \lambda\right)^2 + 2\left|\sigma_r\right|^2} \\ b &= a + \gamma + \lambda \end{aligned}$$

This is the main result in the Cox, Ingersoll, Ross paper.

Because we derived what the equilibrium prices of zero coupon bonds of all maturities must be in the CIR economy, we can use those prices to determine the interest rates of arbitrary tenor. It follows from equation (8.16) that the continuously-compounded interest rate[36] at time t for maturity τ is

$$R\left(t, \tau\right) = -\frac{\log P\left(t, \tau\right)}{\tau - t} = \frac{B\left(\tau - t\right)r_t - \log A\left(\tau - t\right)}{\tau - t} \tag{8.17}$$

Notice that this object is stochastic and depends on the instantaneous interest rate r_t; the interest rate, however, determines the entire term structure. Thus, the CIR model is a one factor model of the term structure of interest rates. In other words, interest rates of all tenors are just deterministic functions of a single state variable.

In particular, we get the following two interesting limits:

$$\begin{aligned} \lim_{\tau \downarrow t} R\left(t, \tau\right) &= r_t \\ \lim_{\tau \uparrow +\infty} R\left(t, \tau\right) &= \frac{2\gamma\bar{r}}{a + \gamma + \lambda} \end{aligned}$$

So what happens in this model is that the term structure is going to be fluctuating through time as a function of the level of the instantaneous interest rate. However, the asymptotic end is going to remain fixed - it will not vary with the interest rate. Note that equations (8.11) and (8.17) determine the evolution of the entire term structure of interest rates over time.

A counterfactual feature of this model is that this is a single factor term structure model; a single stochastic factor determines the entire term structure. Because interest rates are functions of a single stochastic factor (Y or equivalently r), applying Ito's lemma tells us that changes in interest rates of all tenors are locally perfectly correlated. And this is counterfactual.

[35]Because the process for the interest rate is affine, the price of ZCB is an exponential affine function of the interest rates.

[36]Note that for a zero coupon bond with \$1 principal and maturity T, the continuouly compounded rate $R\left(t, T\right)$ at time t over the time interval $[t, T]$ is defined by

$$P\left(t, T\right) = e^{-R(t,T)(T-t)}$$

or

$$R\left(t, T\right) = -\frac{\log P\left(t, T\right)}{T - t}$$

8.2 No-Arbitrage Models

Now we consider no-arbitrage models. The classical no-arbitrage approach was used a lot in the 80s and 90s. We also look at a newer approach known as the Heath-Jarrow-Morton approach.

Although general equilibrium models are conceptually straightforward, they become complicated very quickly once you move beyond the most restrictive assumptions that underlie the CIR model. Therefore, we may need to give enough flexibility to the models to (1) move past the very simple term structure dynamics of the CIR model; and (2) better match the various observed empirical properties of the term structure.

The approach that has followed in the literature is largely not an equilibrium approach, but a no-arbitrage one. Remember that when we model the evolution of the term structure in continuous-time, we are specifying the evolution of an uncountable number of prices of various tenors. Thus, you have to be careful in how you are specifying the dynamics to not allow arbitrage opportunities across bonds of different maturities.

8.2.1 The Classical No-Arbitrage Approach

Let's see how the no-arbitrage approach has typically been formulated.

The classical no-arbitrage model, as introduced by Vasicek (1977), is based on the following three assumptions.

(i) Prices of default-free ZCBs with arbitrary maturity τ satisfy $P(t,\tau) = F(Y,t,\tau)$ for some function F and a given $m-$dimensional set of state variables Y whose evolution is given by the diffusion

$$dY_t = \gamma(Y_t,t)\,dt + \beta(Y_t,t)\,dw_t$$

where γ and β are given functions.

(ii) The instantaneous interest rate r is given by $r_t = r(Y_t,t)$ where r is a given function[37].

(iii) The vector of ICAPM risk premia associated with Y is given by $\phi(Y_t,t)$, where ϕ is a given function.

It then follows from the ICAPM that

$$F_Y^\top \gamma + F_t + \frac{1}{2}\mathrm{tr}\left(F_{YY}\beta\beta^\top\right) - rF = F_Y^\top \phi \tag{8.18}$$

Hence, prices of zero coupon bonds can be determined by solving the PDE in equation 8.18 subject to the terminal condition $F(r,\tau,\tau) = 1$.

Let's just recap the approach. You assume: the vector of relevant state variables Y, its dynamics dY, instantaneous interest rate $r(\cdot)$, and ICAPM risk premia $\phi(\cdot)$; you then try to deduce what the price of ZCB $P(\cdot,\cdot)$ should be by solving for F by using the ICAPM.

And again, the ICAPM says that the dollar risk premium on traded assets in a Markovian setting follows a linear factor structure; the sensitivities multiplied by the factor risk premium.

On the other hand, you can equivalently state the PDE in equation 8.18 by requiring that the prices of zero coupon bonds in units of the money market account must be martingales under the equivalent martingale measure. If you do that, you get something that's equivalent to the PDE based on ICAPM

Here's the alternative formulation:

(i) Prices of default-free ZCBs with arbitrary maturity τ satisfy $P(t,\tau) = F(Y,t,\tau)$ for some function F and a given $m-$dimensional set of state variables Y whose evolution is given by the diffusion

$$dY_t = \gamma^*(Y_t,t)\,dt + \beta(Y_t,t)\,dw_t^*$$

where γ^* and β are given functions, and w^* is a BM under the EMM Q.

[37] Note the instantaneous interest rate r is just the yield to maturity of a zero coupon bond with infinitesimal maturity.

(ii) The instantaneous interest rate r is given by $r_t = r(Y_t, t)$ where r is a given function.

It then follws from the fact that $M_t = e^{-\int_0^t r(s)ds} F(r, t, \tau)$ is a Q-martingale that

$$F_Y^\top \gamma^* + F_t + \frac{1}{2} \operatorname{tr}\left(F_{YY} \beta \beta^\top\right) - rF = 0 \tag{8.19}$$

If you compare the two equations (8.18) and (8.19), you see that the two formulations are equivalent as long as

$$\begin{aligned} F_Y(Y, t, \tau)^\top \phi(Y, t) &= -F_Y(Y, t, \tau)^\top \left[\gamma^*(Y, t) - \gamma(Y, t)\right] \\ &= -F_Y(Y, t, \tau)^\top \beta(Y, t)\, \kappa(t) \end{aligned} \tag{8.20}$$

where the second equality follows from Girsanov's theorem. The validity of this expression can be verified using the equilibrium expressions for κ and ϕ from Section 7, which were

$$\kappa_t = \frac{1}{\pi_t} \operatorname{diff}[\pi_t] = -\sigma_t^{-1}\left(\mu_t - r_t \bar{1}\right)$$
$$\phi(Y, t) = a(\tilde{c}, t)\, \beta(Y, t)\, \beta(Y, t)^\top \tilde{c}_Y(Y, t)$$

Example 169. (Classical One-Factor No-Arbitrage Models) With that, we can look at the history of early development of continuous-time term structure models.

- Vasicek (1977): $m = 1$, $r_t = Y_t$, $\phi(r, t) = \lambda$ and

  $$dr_t = \gamma(\bar{r} - r_t)\, dt + \sigma dw_t$$

  so that the interest rate $r(t)$ follows a simple OU process; mean-reverting with constant volatility. The interest rate at all future dates conditionally normally distributed. Interest rates can take arbitrary negative values (i.e. unbounded below). ICAPM risk premium is a constant.

  - Following models tried to preclude interest rates of negative values, however.

- Dothan (1978): $m = 1$, $r_t = Y_t$, $\phi(r_t, t) = \lambda r_t$ and

  $$dr_t = r_t \sigma dw_t$$

  - Interest rate $r(t)$ follows geometric Brownian motion without drift (hence no longer takes negative values)
  - Single state variable taken as r_t
  - ICAPM risk premium proportional to level of interest rate

- Cox-Ingersoll-Ross (1985): $m = 1$, $r_t = Y_t$, $\phi(r, t) = \lambda r_t$ and

  $$dr_t = \gamma(\bar{r} - r_t)\, dt + \sqrt{r_t}\sigma dw_t$$

  - Could just as well have been derived via no-arbitrage approach

- Hull-White (1990): $m = 1$, $r_t = Y_t$, $\phi(r_t, t) = \lambda(t)$ and

  $$dr_t = \left[\eta(t) + \gamma(\bar{r} - r_t)\right] dt + \sigma dw_t$$

  for some functions λ and η .

  - $r(t)$ follows generalized version of the dynamics of Vasicek, just with time dependent parameter $\eta(t)$ in the drift.
  - If $\lambda(t) = \lambda$ constant and $\eta(t) = 0$ this reduces to the Vasicek model.
  - When $\gamma = 0$ this is the continuous-time version of the model of Ho-Lee (1986) , which is the first model that had a time dependent drift term $\eta(t)$.

– When $\gamma > 0$ this model is also known as the *extended Vasicek model.*

A remark on the time dependent $\eta(t)$ term in the Hull-White (1990) and the Ho-Lee (1986) models. Often term structure models are used to price fixed income derivatives, and in those cases, the $\eta(t)$ term is key.

Why? Suppose you calibrate your term structure model. You pick $r_0, \gamma, \bar{r}, \sigma$, but once you selected those parameters, the model pins down the entire term structure of interest rates. Since you pick only a few parameters, you don't have nearly enough degrees of freedom to match the implied term structure of the calibrated model to what is actually observed.

To see this, note that the price of contingent claims that we compute are the cheapest cost of setting up a portfolio which replicates the payoffs to the contingent claim that you are trying to price. If you do that with fixed income derivatives, the replicating portfolio that you are going to use dynamically trades not stocks, but simple bonds (e.g. ZCBs with different tenors).

And here's the problem; if the model that you are using to price fixed income derivatives does not correctly obtain the current prices of the ZCBs in the replicating portfolio because it doesn't exactly match the initial term structure of interest rates (which reflects the current prices of the ZCBs of different tenors), then there is no way that it would price the fixed income derivative correctly, as it does so by replicating (via dynamic trading) a portfolio ZCBs. It would be like using Black-Scholes to price a call option, but instead of just inputting the stock price directly, you try to infer the stock price as an equilibrium object.

So having the time-dependent parameter in the dynamics of interest rates that you can calibrate any way that you want, gives the flexbility that you need to ensure that the model matches the initial term structure of interest rates. This is key if you want to use the model to price more complicated fixed income derivatives.

Example 170. (Calibration of the Hull-White Model) The inclusion of the time-dependent parameter η in the Hull-White (HW) model allows it to be calibrated so as to match the initial term strucutre of interest rates, which is a key requirement to use the model to price interest rate derivatives.

To see this in action, note that in the HW model, the PDE solution to equation 8.18 is

$$P(t,\tau) = F(r,t,\tau) = A(t,\tau)\, e^{-B(t,\tau) r_t},$$

where

$$\begin{aligned} A(t,\tau) &= \exp\left(-\int_t^\tau \zeta(s)\, ds + B(t,\tau)\,\zeta(t) + \frac{1}{2} v(t,\tau)\right) \\ B(t,\tau) &= \frac{1}{\gamma}\left(1 - e^{-\gamma(\tau - t)}\right) \end{aligned}$$

$$\begin{aligned} v(t,\tau) &= \frac{|\sigma|^2}{\gamma}\left(\frac{\tau - t - B(t,\tau)}{\gamma} - \frac{B(t,\tau)^2}{2}\right) \\ \zeta(t) &= e^{-\gamma t}\left(r_0 + \int_0^t e^{\gamma s}\left(\eta(s) - \lambda(s) + \gamma \bar{r}\right) ds\right) \end{aligned}$$

This gives the prices of the zero coupon bonds.

Now, we would like to pick the time dependent parameter η so that the prices of zero coupon bonds at different tenors at $t = 0$ matches the prices you observe today at time zero (i.e. the initial term structure). In the Hull-White model, this amounts to

$$P^{mkt}(0,\tau) = A(0,\tau)\, e^{-B(0,\tau) r_0} \tag{8.21}$$

for all $\tau \in (0, T]$ where $P^{mkt}(0,\tau)$ denotes the market price at time 0 of a default free ZCB with maturity τ. Note that for $\tau = 0$ the above condition holds.

Taking the log of both sides and taking the derivative with respect to τ,

$$-f^{mkt}(0,\tau) = \frac{\partial}{\partial\tau}\log\left(A(0,\tau)\right) - \frac{\partial}{\partial\tau}B(0,\tau)\,r_0 \tag{8.22}$$
$$= -\zeta(\tau) + \frac{\sigma^2}{2\gamma^2}\left(1-e^{-\gamma\tau}\right)^2$$

where
$$f^{mkt}(0,\tau) = -\frac{\partial}{\partial\tau}\log P^{mkt}(0,\tau) \tag{8.23}$$

is the market instantaneous forward rate at time 0 for date $\tau > t$[38]. Note that the $\eta(t)$ that we are trying to calibrate is inside the $\zeta(t)$ term in the instantaneous forward rate. Of course, as $\tau \downarrow 0$, the forward rate becomes the instantaneous interest rate $r(t)$. The same thing happens on the right hand side; $-\zeta(\tau)$ becomes the instantaneous interest rate, and the second term becomes zero.

For equation (8.22) to hold, it is enough that the derivatives of both sides match. Again taking the derivative,

$$-\frac{\partial}{\partial\tau}f^{mkt}(0,\tau) = -\frac{\partial}{\partial\tau}\zeta(\tau) + \frac{\sigma^2}{2\gamma^2}\frac{\partial}{\partial\tau}\left(1-e^{-\gamma\tau}\right)^2$$
$$= \gamma\zeta(\tau) - \eta(\tau) + \lambda(\tau) - \gamma\bar{r} + \frac{\sigma^2}{\gamma}e^{-\gamma\tau}\left(1-e^{-\gamma\tau}\right)$$
$$= \gamma f^{mkt}(0,\tau) - \eta(\tau) + \lambda(\tau) - \gamma\bar{r} + \frac{\sigma^2}{2\gamma}\left(1-e^{-2\gamma\tau}\right), \tag{8.24}$$

which we obtain after a moderate amount of algebra. The last equality follows from equation (8.22). Further rearranging (8.24) gives

$$\eta(\tau) = \lambda(\tau) - \gamma\bar{r} + \frac{\partial}{\partial\tau}f^{mkt}(0,\tau) + \gamma f^{mkt}(0,\tau) + \frac{\sigma^2}{2\gamma}\left(1-e^{-2\gamma\tau}\right) \tag{8.25}$$

This gives you the value of $\eta(\tau)$ in terms of the observable current curve at $t=0$ using forward rates, and the derivative of that curve. You can use these to match the time-dependent parameter $\eta(\cdot)$. With these specification, the initial term structure as implied by the model matches the observed initial term structure of interest rates by construction.

If you take these $\eta(t)$ and substitute them into the assumed dynamics for the interest rate $r(t)$, you obtain

$$dr_t = \left[\eta(t) + \gamma(\bar{r} - r_t)\right]dt + \sigma dw_t$$
$$= \left[\lambda(t) + \frac{\partial}{\partial\tau}f^{mkt}(0,t) + \gamma f^{mkt}(0,t) + \frac{\sigma^2}{2\gamma}\left(1-e^{-2\gamma t}\right) - \gamma r_t\right]dt + \sigma dw_t$$
$$= \left[\frac{\partial}{\partial\tau}f^{mkt}(0,t) + \gamma f^{mkt}(0,t) + \frac{\sigma^2}{2\gamma}\left(1-e^{-2\gamma t}\right) - \gamma r_t\right]dt + \sigma dw_t^* \tag{8.26}$$

where the last equality follows from equation (8.20) and the fact that $F_r(r,t,\tau) \neq 0$.

The second line gives the dynamics under P and the third under Q. Moreover, given that the model assumes a particular form for the ICAPM risk premium $\phi(r,t) = \lambda(t)$, this also pins down the dynamics under the EMM as we saw in the CIR case.

What you should take away from this discussion of the classical no-arbitrage models is that they build not only on assumptions regarding the evolution of instantaneous interest rates $dr(t)$, but also on assumptions regarding the ICAPM risk premia $\phi(r,t)$ which are unobservable. Hence, there really is no way to directly validate or invalidate those assumptions.

[38] The continuously compounded $\tau_1 \times \tau_2$ forward rate at time t (where $t < \tau_1 < \tau_2$) is defined by $f(t,\tau_1,\tau_2) = \frac{1}{\tau_2-\tau_1}\log\left(P(t,\tau_1)/P(t,\tau_2)\right)$, so that $f(t,\tau) = \lim_{\epsilon\downarrow 0} f(t,\tau,\tau+\epsilon)$. This is the usual forward rate as the tenor of the forward becomes infinitesimally small.

The other problem with this approach is that to the extent you allow for time dependent parameters $\eta(t)$ to match the initial term structure (which is critical if you want to model fixed income derivatives), the calibration of those parameters is far from straightforward. This was seen in the Hull-White model as an example, but as the model becomes more sophisticated, the calibration exercise becomes even more complicated.

8.2.2 The Heath-Jarrow-Morton Approach

The next approach we introduce is by Heath-Jarrow-Morton (1992), which is now the de facto standard among no arbitrage models of term structure. The key insight is that it is easier and more convenient to model the evolution of instantaneous forward rates of different tenors, instead of the spot rate r.

There are several advantages here. First, it takes the initial term strucutre (i.e. the forward rate curve) as an input and hence the calibration is straightforward. It does not go through the complexity associated with introducing a time-dependent parameter. As you can compute spot rates from forward rates, matching the initial term structure of forward rates implies you match the initial term structure of spot rates as well. Second, while the classical approach requires making assumptions regarding non-observable quantities such as the ICAPM risk premia, the Heath-Jarrow-Morton only requires assumptions regarding directly observable quantities, i.e. volatilities of forward rates of different tenors. That is, everything is specified once you specify what you believe to be plausible parameters for the volatilities of forward rates.

The HJM approach is based on the following result.

Theorem 171. *Suppose that the instantaneous forward rates for all dates $\tau \in [0, T]$ are Ito processes, i.e.*

$$f(t,\tau) = f(0,\tau) + \int_0^t \alpha(s,\tau)\,ds + \int_0^t \beta(s,\tau)^\top dw_s^*, \tag{8.27}$$

where $\{f(0,\tau) : \tau \in [0,T]\}$ is the initial forward curve, w^* is a d-dimensional BM under the EMM Q and, for each $\tau \in [0,T]$, $\alpha(\cdot,\tau)$ and $\beta(\cdot,\tau)$ are bounded SPs. Then,

$$\alpha(t,\tau) = \beta(t,\tau)^\top \int_t^\tau \beta(t,s)\,ds \tag{8.28}$$

Equation 8.28, which is known as the *forward rate drift restriction*, shows that the drifts under Q of instantaneous forward rates for all dates τ are uniquely determined by the term structure of forward rate volatilities. Note $\alpha(t,\tau)$ and $\beta(t,\tau)$ are stochastic processes. Since things are written in terms of Brownian motion under Q, $\alpha(t,\tau)$ is a drift under Q. Thus it is non-observable, but it turns out that this is not really a problem as we will see in a bit. While the paper is more general (they assume integrability), we assume that $\alpha(t,\tau)$ and $\beta(t,\tau)$ are bounded for technical convenience.

The key result is that the drift under Q given by $\alpha(t,\tau)$ is entirely pinned down by the diffusion under Q (which is the same as diffusion under P by Girsanov), $\beta(t,\tau)$. Thus we only need to specify the diffusion of forward rates of different tenors, which is empirically observable. The result is given by no-arbitrage considerations. We'll see an example on how to apply this later.

You can think of the instantaneous forward rate $f(t,\tau)$ to be the interest rate you negotiate today at time t for a forwad loan, which rather than being initiated right now, is initiated at a future date τ, and is repaid one instant later.

Note if $\tau \to t$, then the forward rate converges to the instantaneous spot rate r. Thus, $f(t,t) = r(t)$, and the above equation gives that

$$r(t) = f(0,t) + \int_0^t \alpha(s,t)\,ds + \int_0^t \beta(s,t)^\top dw_s^* \tag{8.29}$$

The two equations 8.28 and 8.29 show that the evolution of the interest rate r under Q is also uniquely determined by the term strucutre of forward rate volatilities.

What's more is that, the next proposition actually shows that the evolution under Q of the price of the default-free ZCBs is similarly uniquely determined by the term structure of forward rate volatilities.

Proposition 172. *Under the assumptions of Theorem 171, prices of default-free ZCBs with arbitrary tenors $\tau \in [0, T]$ satisfy*

$$P(t,\tau) = P(0,\tau) + \int_0^t P(s,\tau)\, r(s)\, ds + \int_0^t P(s,\tau)\, \sigma_P(s,\tau)^\top dw_s^* \tag{8.30}$$

where

$$\sigma_P(t,\tau) = -\int_t^\tau \beta(t,s)\, ds \tag{8.31}$$

The dynamics of ZCBs under EMM is entirely determined by the specification of $\beta(\cdot)$. Having the dynamics of the instantaneous spot rate $r(t)$ and all of the the ZCBs $P(t,\tau)$ under the EMM is all you need to price more complicated fixed income derivatives.

Now we look at the proof for the theorem and the proposition above.

Proof. □

Example 173. (Exponential Forward Rate Volatility)

Suppose that $d = 1$ and

$$\beta(t,\tau) = \sigma e^{-\gamma(\tau - t)}$$

for some $\sigma, \gamma \in \mathbb{R}$ with $\gamma > 0$.

From equation (8.28), we can then write the drift term under Q as

$$\alpha(t,\tau) = \beta(t,\tau)^\top \int_t^\tau \beta(t,s)\, ds = \frac{\sigma^2}{\gamma} e^{-\gamma(\tau-t)} \left(1 - e^{-\gamma(\tau-t)}\right)$$

and from equation (8.29), the spot rate as

$$\begin{aligned} r(t) &= f(0,t) + \int_0^t \alpha(s,t)\, ds + \int_0^t \beta(s,t)\, dw_s^* \\ &= f(0,t) + \frac{\sigma^2}{2\gamma^2}\left(1 - e^{-\gamma t}\right)^2 + \sigma \int_0^t e^{-\gamma(t-s)} dw_s^* \end{aligned} \tag{8.32}$$

Now we apply Ito's lemma to equation (8.29). Be mindful as the integrand in the stochastic integral changes with t. You have to apply something like a stochastic integral version of Leibniz rule.

$$\begin{aligned} dr(t) &= \left[\frac{\partial}{\partial \tau} f(0,t) + \frac{\sigma^2}{\gamma} e^{-\gamma t}\left(1 - e^{-\gamma t}\right) - \gamma\sigma \int_0^t e^{-\gamma(t-s)} dw_s^*\right] dt + \sigma dw_t^* \\ &= \left[\frac{\partial}{\partial \tau} f(0,t) + \frac{\sigma^2}{\gamma} e^{-\gamma t}\left(1 - e^{-\gamma t}\right) - \gamma\left(r(t) - f(0,t) - \frac{\sigma^2}{2\gamma^2}\left(1 - e^{-\gamma t}\right)^2\right)\right] dt + \sigma dw_t^* \\ &= \left[\frac{\partial}{\partial \tau} f(0,t) + \gamma f(0,t) + \frac{\sigma^2}{2\gamma}\left(1 - e^{-2\gamma t}\right) - \gamma r(t)\right] dt + \sigma dw_t^* \end{aligned}$$

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Part VI
Portfolio Constraints

9 Optimal Consumption with Portfolio Constraints

Previously, we studied the martingale approach under the complete markets assumption, solving for individual optimal consumption and such. Here, we consider how to take the martingale approach when we have incomplete markets.

First let's recall the following: What was the essence of the martingale approach under complete markets? The idea is that we decompose the individual optimal consumption into a two step problem. Step one is to let the agent choose an optimal consumption plan. We can do this because under complete markets, every marketed consumption plan can be implemented at some cost (and we can compute this cost as well). Step two is then to retrieve the cheapest trading strategy that delivers this consumption plan.

Well, this approach seems to break down under incomplete markets because now you need to keep track of which consumption plans can be financed by a trading strategy. The alternative then seems to be to simultaneously choose consumption and trading strategy so as to enforce that the trading strategy finances the consumption plan, but of course this is different from the martingale approach. The advantage of martingale approach, which is that we can solve the first step of choosing optimal consumption very simply using Lagrange theory (and nothing more), is lost.

We now go over a simple example that illustrates how we can take the martingale approach under incomplete markets.

Example 174. (Event-Tree Economy with Incomplete Market) Let's return to the simplest possible discrete time setting for intuition. Consider an economy with just two dates $t = 0, T$ and d possible nodes. The uncertainty can then be represented by an event tree with $d + 1$ nodes, denoted by $\omega_0, \omega_1, \cdots, \omega_d$. We can then identify a stochastic process X with a vector $(X_0, X_1, \cdots, X_d) \in \mathbb{R}^{1+d}$, where X_i denotes the value of the SP at node ω_i.

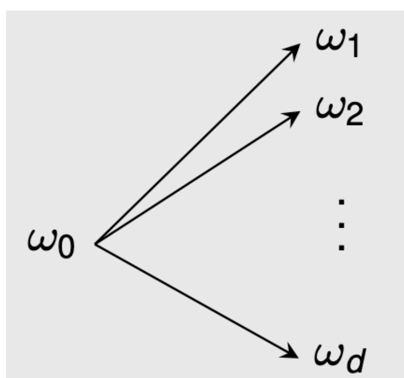

Figure 9.1: Event tree in discrete time

There are $n \leq d$ traded assets with price process $\left(S^1, \cdots, S^n\right)$ satisfying the following conditions:

(i) the price process for the first asset is strictly positive, i.e. $S^1 \in \mathbb{R}^{1+d}_{++}$

(ii) the $d \times n$ matrix

$$S_T = \begin{pmatrix} S_1^1 & \cdots & S_1^n \\ \vdots & \ddots & \vdots \\ S_d^1 & \cdots & S_d^n \end{pmatrix}$$

has rank n.

Each row of the S_T matrix corresponds to one of d states, whereas each column corresponds to one of n assets. A $d \times n$ matrix can have rank no larger than n. The full rank assumption states that there are no stocks that can be replicated by forming a portfolio of the remaining assets.

Now we define a number of familiar objects in this simplest possible setting. A *consumption plan* is a stochastic process $c \in \mathbb{R}_+^{1+d}$. A *trading strategy* is a vector $\theta \in \mathbb{R}^n$ denoting the number of shares (not dollar invested) of each asset purchased at $t = 0$. One says that a consumption plan c is *marketed* if there exists a trading strategy θ such that

$$c_T = S_T \theta \tag{9.1}$$

where $c_T = (c_1, \cdots, c_d)$. As before, you can think of a marketed consumption plan as one that can be replicated by the asset holdings. The market is complete if and only if each consumption plan is marketed (i.e. $n = d$) If the equation (9.1) holds, then the trading strategy θ is said to finance the CP c and the cost of the CP at $t = 0$ is $c_0 + S_0\theta$, where $S_0 = \left(S_0^1, \cdots, S_0^n\right)$.

An *endowment* is a SP $\tilde{c} \in \mathbb{R}_+^{1+d}$. A CP c is *budget feasible* for a given endowment $\tilde{c}$ if the net consumption $c - \tilde{c}$ is marketed and the cost of $c - \tilde{c}$ at $t = 0$ is 0 (i.e. if $c_0 - \tilde{c}_0 + S_0\theta = 0$ where θ finances the net consumption $c - \tilde{c}$). Thus c is budget feasible if and only if there exists a TS θ such that

$$c - \tilde{c} = X\theta$$

where

$$X = \begin{pmatrix} -S_0 \\ S_T \end{pmatrix} = \begin{pmatrix} -S_0^1 & \cdots & -S_0^n \\ S_1^1 & \cdots & S_1^n \\ \vdots & \ddots & \vdots \\ S_d^1 & \cdots & S_d^n \end{pmatrix}$$

is the $(d+1) \times n$ *payoff matrix.* Note this equation just requires that (i) in the initial period $c_0 - \tilde{c}_0 = -S_0\theta$; and (ii) in each state ω_i we have the net consumption plan financed by the trading strategy $c_i - \tilde{c}_i = S_i\theta$ for $i = 1, \cdots, d$. In matrix notation, $c_T = S_T\theta$.

Now let's introduce an agent that has time-additive expected utility function:

$$U(c) = \sum_{i=0}^{d} p_i u(c_i)$$

where

- $p_i > 0$ denotes probability of node $\omega_i$for $i = 1, \cdots, d$ and $p_0 = 1$;
- u is strictly increasing, strictly concave, and continuously differentiable on $\mathbb{R}_{++}$;
- u satisfies the Inada conditions

$$\lim_{c \downarrow 0} u'(c) = +\infty \quad \text{and} \quad \lim_{c \uparrow +\infty} u'(c) = 0$$

Since we are interested in the effect of incomplete markets, we are going to focus on the simple case where the non-negativity constraint on consumption is not binding. Note that time discounting can be introduced at little cost, but we keep it simple here.

The agent's optimal consumption problem is then

$$\begin{aligned} &\max_{(c,\theta) \in \mathbb{R}_+^{1+d} \times \mathbb{R}^n} U(c) \\ &\text{s.t. } c - \tilde{c} \leq X\theta \end{aligned} \tag{9.2}$$

Note we wrote the budget feasibility as an inequality (rather than as an equality as we did before), but with monotonicity, this makes little difference.

To take the martingale approach, we want to write this problem as an optimization problem purely over consumption without the trading strategy. To do this, we introduce the notion of state price process.

A *state price process* (SPP) is a stochastic process $\pi \in \mathbb{R}_{++}^{1+d}$ such that $\pi_0 = 1$ and $\pi^\top X = 0$. As before, we denote a SP by a strictly positive vector; π_0 corresponds to the price of consumption good at date 0, and because time-0 consumption good is the numeraire, it must equal 1. The second condition requires that a state price π must correctly "price" the traded assets. This means the price at date 0 of each of the n assets is equal to the possible payoffs at the terminal date T multiplied by the state price, i.e.

$$S_0^j = \sum_{i=1}^{d} \pi_i S_i^j \quad \text{for each asset } j = 1, \cdots, n$$

as $\pi_0 = 1$.

Note that under complete markets, we will have a unique state price process. But under incomplete markets, this is not the case. Instead, we will see that there are an infinite number of SPPs that are consistent with the prices given.

Let Π denote the set of all SPPs and $\bar{\Pi}$ denote the closure of Π where

$$\bar{\Pi} = \left\{\pi \in \mathbb{R}_{+}^{1+d} : \pi_0 = 1 \text{ and } \pi^\top X = 0\right\}$$

Note that since π has strictly positive components, the elements of the closure of Π has non-negative components. We refer to a stochastic process $\pi \in \bar{\Pi}$ as a *generalized state price process*. The only difference here is that the components can take values of zero.

We assume that there are no arbitrage opportunities, so that Π (and hence $\bar{\Pi}$) is nonempty. Given no arbitrage, Π (and hence $\bar{\Pi}$) is a singleton if $n = d$ (complete market) and otherwise constains an infinite number of elements. Moreover, consider the case $n < d$, where there are an infinite number of state price processes. If we add more assets to this economy, the number of potential SPPs would decrease, all the way to a singleton set of Π once we add $d - n$ assets. However, the SPPs still correctly price the initial n assets in the economy. Furthermore, each $\pi \in \Pi$ equals the unique state price process that would prevail in a fictitious completion of the original market.

Now we proceed to rewrite the optimization problem without the trading strategies.

By Farkas' lemma (i.e. theorems of alternatives), given an arbitrary net trade $c - \tilde{c} \in \mathbb{R}^{1+d}$, one of the following are true:

- either $\exists \theta \in \mathbb{R}^n$ such that $c - \tilde{c} \leq X\theta$,
- or $\exists \pi \in \bar{\Pi}$ such that $\pi^\top (c - \tilde{c}) > 0$.

Here's how we will use this result. Since Farka's lemma says one of the two above must be true, if we can rule out one of them via

$$\pi^\top (c - \tilde{c}) \leq 0 \text{ for all } \pi \in \bar{\Pi}$$

then it must be the case that there is a trading strategy θ that finances the net consumption plan. Hence, we can write

$$\begin{aligned} &\max_{c \in \mathbb{R}_+^{1+d}} U(c) \\ \text{s.t.} \quad &\sup_{\pi \in \bar{\Pi}} \pi^\top (c - \tilde{c}) \leq 0 \end{aligned} \tag{9.3}$$

The two optimization problems given in (9.2) and (9.3) are entirely equivalent. Once again, we achieved that the alternative formulation does not include the trading strategy; we don't need to keep track of the TSs to see whether or not the net trade is marketed or not. It just says that the net trade is marketed if it satisfies a budget constraint with respect to each and every one of the SPPs $\pi \in \bar{\Pi}$. We will see that this result carries over to the continuous-time economy.

The problem now is that the constraint is formulated in a rather complicated way, including a supremum. On the other hand, if we write $\pi^\top (c - \tilde{c}) \leq 0$ for all $\pi \in \bar{\Pi}$, then this has an infinite number of constraints. Note if we were to have complete markets, we would have a single constraint that corresponds to the martingale approach we had before.

We will proceed by using some methods that apply specifically to our discrete time, finite dimensional setting. However, the end results that we prove carry over to the continuous-time, infinite-dimensional setting.

Let's use some geometry. Because $\bar{\Pi}$ is a bounded polyhedral set[39], there exists a finite set of (what we call) extreme points $\{\pi^1, \cdots, \pi^k\} \subset \bar{\Pi}$ such that the bounded polyhedral set $\bar{\Pi}$ is a convex hull of those points, i.e.

$$\bar{\Pi} = \left\{ \sum_{j=1}^{k} \lambda^j \pi^j : \lambda^j \geq 0, \forall j \in \{1, \cdots, k\}, \sum_{j=1}^{k} \lambda^j = 1 \right\}.$$

Accordingly, we can equivalently write the investor's problem as

$$\begin{aligned} &\max_{c \in \mathbb{R}_+^{1+d}} U(c) \\ &\text{s.t. } \pi^\top (c - \tilde{c}) \leq 0 \text{ for } \pi \in \{\pi^1, \cdots, \pi^k\} \end{aligned} \tag{9.4}$$

and this is equivalent to the constraint in (9.3). We do not have to check for every $\pi \in \bar{\Pi}$, just that the constraint holds for every extreme point of the bounded polyhedral set. Since any generalized SPP π is a convex combination of those k extreme points, this guarantees that the constraint in (9.3) is also satisfied.

Now we use the Lagrangian for the optimization problem in (9.4):

$$\mathcal{L}(c, \psi^1, \cdots, \psi^k) = U(c) - \sum_{j=1}^{k} \psi^j (\pi^j)^\top (c - \tilde{c}),$$

where $(\psi^1, \cdots, \psi^k)$ are Lagrangian multipliers. The FOCs for an optimum are

$$\begin{aligned} p_i u'(c_i^*) - \sum_{j=1}^{k} \psi^j \pi_i^j = 0 \quad &\text{for } i = 0, 1, \cdots, d, \\ \psi^j \geq 0, (\pi^j)^\top (c^* - \tilde{c}) \leq 0, \psi^j (\pi^j)^\top (c^* - \tilde{c}) = 0 \quad &\text{for } j = 1, \cdots, k. \end{aligned} \tag{9.5}$$

Let $\psi = \sum_{j=1}^{k} \psi^j > 0$ and

$$\pi^* = \frac{\sum_{j=1}^{k} \psi^j \pi^j}{\sum_{j=1}^{k} \psi^j} \tag{9.6}$$

Note that this is just a convex combination of the extreme points π^j for $j = 1, \cdots, k$ as the weights are non-negative and add up to one. The FOCs in equation (9.5) imply that

$$p_i u'(c_i^*) = \psi \pi_i^* \quad \text{for } i = 0, 1, \cdots, d \tag{9.7}$$

and

$$(\pi^*)^\top (c^* - \tilde{c}) = 0 \tag{9.8}$$

as $\psi^j (\pi^j)^\top (c^* - \tilde{c}) = 0$ for each j, thus so must their sum. Thus, letting f denote the inverse of u' , the optimal CP is

$$c_i^* = f\left(\psi \frac{\pi_i^*}{p_i}\right) \quad \text{for } i = 0, 1, \cdots, d \tag{9.9}$$

where ψ solves the equation (9.8). This identifies the Lagrangean multipliers. Recall that the state price per unit of probability $\frac{\pi_i^*}{p_i}$ is what we previously called state price density.

[39] This is a subset of some Euclidean space $\mathbb{R}^n$ that is defined by a finite number of linear inequalities. This is true by how $\bar{\Pi}$ was defined. It is bounded because our asset prices have strictly positive components, and must satisfy $S_0^j = \sum_{i=1}^{d} \pi_i S_i^j$ for all j.

In addition, it follows from equations (9.6) and (9.7) that $\pi^* \in \Pi$. That is, not only is it a generalized state price process (with non-negative components), it has strictly positive components because the left hand side of (9.7) is strictly positive. So each π_i^* must be strictly positive.

Some comments on comparison with the complete markets case. Note that the FOCs look exactly the same, that the marginal utilities evaluated at the optimal policies must be proportional to the state price densities. However, in complete markets, there is only one state price density. So all you need to do is identify the Lagrangean multiplier and you're done. With incomplete markets, the problem here is that we do not know the π_i^*'s. Thus, equation (9.9) does not fully characterize the optimal consumption plans.

The optimal policy in an economy with an incomplete market coincides with the optimal policy in a fictitious economy with a complete market and SPP π^*. So the question is whether we will be able to identify these particular SPP that correspond to the fictitious completion of the incomplete markets. The answer turns out to be yes.

In light of equation (9.9), in order to identify the optimal CP c^* we only need to identify the SPP π^*, called the *minimax SPP*. This can be done as follows. For arbitrary $\pi \in \Pi$, consider the problem

$$\max_{c \in \mathbb{R}_+^{1+d}} U(c)$$

$$\text{s.t. } \pi^\top (c - \tilde{c}) \leq 0$$

and denote the value of this problem by $J(\pi)$.

Let V be the value of the investor's problem in equation (9.3). Then by examining the FOCs, we have that the optimal consumption policies under incomplete markets is the same as those under complete markets with a single budget constraint formulated via minimax SPP π^*. Furthermore, we saw that the incomplete markets case features more constraints (one for each $\pi \in \Pi$), and further constraining the program can only decrease the value.

Collecting these insights, we then have that

$$J(\pi^*) = V \leq J(\pi) \quad \text{for all } \pi \in \Pi,$$

so that

$$\pi^* = \arg\min_{\pi \in \Pi} J(\pi). \tag{9.10}$$

Thus, the minimax state price process solves the dual minimization problem in equation (9.10).

9.1 The Investor's Problem

What incomplete markets does is limit the trading opportunities of an agent; it is no longer going to be true that any consumption plan is traded. But that's very similar to what happens when agents are portfolio constrained. So we carry this analogy further, that incomplete markets is a special case of portfolio constraints.

Suppose we have incomplete markets, so $n < d$. Now throw in additional $d - n$ assets to dynamically complete the market and impose the condition that agents cannot trade those additional assets. From the point of view of agents, nothing has changed.

9.1.1 General Setting

Consider the standard economy with complete markets $n = d$, with the following additional assumptions:

- r is uniformly bounded;
- σ satisfies the non-degeneracy condition

  $$x^\top \sigma(t) \sigma(t)^\top x \geq \epsilon |x|^2, \text{a.s.}$$

  for all $x \in \mathbb{R}^n$, $t \in [0, T]$ and some $\epsilon > 0$.

In particular, the second condition says that there is no portfolio x of n assets that has negligible risk. This guarantees σ has full rank, invertible, and in fact stronger – σ^{-1} will have bounded matrix norm. Otherwise, the setting is similar to the Standard Economy in section 5. As in the section for general equilibrium, we also denote μ as the stock gain process (not just the price process).

We consider an investor with a time-additive expected utility function $U(c, W)$ over CPs $(c, W) \in \mathcal{C}_+$, i.e.

$$U(c, W) = E^P\left[\int_0^T \eta_1 u(c_t, t)\, dt + \eta_2 V(W)\right]$$

with the usual conditions:

(i) $\eta_1, \eta_2 \in \mathbb{R}_+$;

(ii) $u(\cdot, t)$ is differentiable, strictly increasing, strictly concave on $\mathbb{R}_{++}$ for all $t \in [0, T]$ and satisfies the Inada conditions $\lim_{c\uparrow+\infty} u_c(c, t) = 0$ and $\lim_{c\downarrow 0} u_c(c, t) = +\infty$;

(iii) $u(c, \cdot)$ is continuous on $[0, T]$ for all $c \in \mathbb{R}_{++}$;

(iv) V is differentiable, strictly increasing, strictly concave on $\mathbb{R}_{++}$ and satisfies the Inada conditions $\lim_{W\uparrow+\infty} V'(W) = 0$ and $\lim_{W\downarrow 0} V'(W) = +\infty$.

9.1.2 Portfolio Constraints

Now we lay out the parts of the framework that is different from previous sections, namely the portfolio constraints.

First, the investor is endowed with the amount $e_0 \geq 0$ at time 0 and with a bounded consumption process $\left(\tilde{c}, \tilde{W}\right) \in \mathcal{C}_+$ with either e_0 or $\left(\tilde{c}, \tilde{W}\right)$ being strictly greater than 0.

As before, the space of admissible trading strategies Θ is as defined in the section on Standard Economy; however, we assume that the trading strategies are subject to the additional constraint

$$(\alpha_t, \theta_t) \in A \text{ a.e.},$$

where $A \subset \mathbb{R}^{n+1}$ is a given nonempty closed convex cone[40] such that $(\alpha, 0) \in A$ for all $\alpha \geq 0$. Note (α_t, θ_t) is a random variable (despite the notation) and depends on the state of the world. In addition, if $A = \mathbb{R}^{n+1}$, then there is no constraint and we should be able to recover the case of optimal consumption under complete markets.

Let us define this refined set of trading strategies as

$$\Theta_A = \{(\alpha, \theta) \in \Theta : (\alpha_t, \theta_t) \in A \text{ a.e.}\}$$

and call it the set of *A-admissible trading strategies.*

Now let's look at some exampes.

Example 175. (Cone Constraints)

- *No Constraints (Complete Market)*:

$$A = \mathbb{R}^{n+1}$$

  We have as many risky assets as Brownian motions so this recovers the complete markets case.

- *Non-traded Assets (Incomplete Markets):*

$$A = \{(\alpha, \theta) \in \mathbb{R} \times \mathbb{R}^n : \theta_k = 0 \textit{ for } k = m + 1, \cdots, n\}$$

  for some $m \in \mathbb{N}$ with $0 \leq m < n$. This effectively bars the agent from trading the last $n - m$ stocks.

[40] Recall that a convex cone is a subset of a vector space that is closed under linear combinations with positive coefficients. That is, a subset C of a vector space V is a cone (or a linear cone) if

$$x \in C \text{ and } \alpha \in \mathbb{R}_+ \implies \alpha x \in C$$

- *Short-Sale Constraints*:

$$A = \{(\alpha, \theta) \in \mathbb{R} \times \mathbb{R}^n : \theta_k \geq 0 \text{ for } k = m+1, \cdots, n\}$$

for some $m \in \mathbb{N}$ with $0 \leq m < n$. This bars the agent from short-selling the last $n - m$ stocks.

- *Borrowing Constraint 1 (No Borrowing):*

$$A = \{(\alpha, \theta) \in \mathbb{R} \times \mathbb{R}^n : \alpha \geq 0\}$$

This is the most crude way of setting a borrowing constraint. The agent is only allowed to save.

- *Borrowing Constraint 2 (No Uncollateralized Borrowing):*

$$A = \left\{(\alpha, \theta) \in \mathbb{R} \times \mathbb{R}^n : \alpha \geq -(\mathbf{1} - \lambda)^\top \theta\right\}$$

for some $\lambda \in [0,1]^n$. The constraint on how much the agent can borrow (α going negative) is given by a $(1 - \lambda)$ fraction of the total value of his risky assets. This setting may arise when we assume that, what matters to lenders is the total wealth of the borrower, but he cannot pledge his future stochastic income. If $\lambda = \mathbf{0}$, then this corresponds to barring the agent's total portfolio value from going negative. If λ is strictly positive, then this corresponds to a haircut applied to the market value of his assets posted as collateral (e.g. as in repo markets).

- *Borrowing Constraint 3 (Margin Requirements):*

$$A = \left\{(\alpha, \theta) \in \mathbb{R} \times \mathbb{R}^n : \alpha \geq (\mathbf{1} + \lambda_-)^\top \theta^- - (\mathbf{1} - \lambda_+)^\top \theta^+\right\}$$

for some $\lambda_-, \lambda_+ \in [0,1]^n$. Note here θ^- and θ^+ are respectively the negative and positive part of θ. Here, λ_+ is the margin requirement for purchases so that the maximum amount that can be borrowed on margin is $(1 - \lambda_+)$. The λ_- is the margin applicable to short positions. These are typically the same as λ_+ but they don't have to be. The short sale proceeds cannot just be used, they must be posted as collateral (with additional fraction). So if $n = 1$ and $\theta < 0$, the investor must hold a strictly positive amount of bond $\alpha \geq (1 + \lambda_-)\,\theta^-$.

Note that the commonality among all these examples is that these are non-empty, closed cones (you can check). We also always require that taking a zero position in stocks and a non-negative bond position is allowed, as written above $(\alpha, 0) \in A$ for all $\alpha \geq 0$.

We could generalize by setting A to be arbitrary non-empty convex sets and this is done in Domenico's paper.

We can then write the investor's problem as

$$\begin{aligned} &\max_{(c,W,\alpha,\theta) \in \mathcal{C}_+ \times \Theta_A} U(c, W) \\ \text{s.t. } &(\alpha, \theta) \text{ finances } \left(c - \tilde{c}, W - \tilde{W}\right) \text{and } \alpha(0) + \mathbf{1}^\top \theta(0) \leq e_0 \end{aligned} \tag{9.11}$$

This would the usual formulation of the dynamic optimal consumption problem.

Our strategy is again to see if we can solve this problem in two steps: first only over consumption, and second over trading strategies, as had been the strategy in the martingale approach.

9.2 Duality

9.2.1 State Price Densities and Marketed Consumption Plans

First thing we need to do is start introducing state prices.

Definition 176. A *constrained state price density* is a SP π with the property that:

(i) $\pi(0) = 1$

(ii) If a CP $(c, W) \in \mathcal{C}_b$ is financed by a TS $(\alpha, \theta) \in \Theta_A$, then

$$\frac{1}{\pi(t)} E\left[\int_t^T \pi(s)\, c(s)\, ds + \pi(T)\, W \middle| \mathcal{F}_t\right] \leq \alpha(t) + \mathbf{1}^\top \theta(t)$$

for all $t \in [0, T]$.

The idea here is that the state price density π must correctly price any consumption plan that is financed by an A-admissible trading strategy. The LHS of the second condition is simply have we learned previously to price consumption plans. The condition then places an upper bound of the value of the CP; the issue is that we do not know that the TS (α, θ) is the cheapest one that finances that particular CP, so we have an inequality.

In order to characterize the set of constrained the state price densities, let

$$\tilde{A} = \left\{(\nu_0, \nu_-) \in \mathbb{R} \times \mathbb{R}^n : \alpha\nu_0 + \theta^\top \nu_- \geq 0 \text{ for all } (\alpha, \theta) \in A\right\} \tag{9.12}$$

denote the *dual cone* of A . Note $\tilde{A} \subset \mathbb{R}^{n+1}$ as well, and $\alpha\nu_0 + \theta^\top\nu_-$ is the inner product of (ν_0, ν_-) and (α, θ). Crucially, $\tilde{A}$ is the set of all points (ν_0, ν_-) such that the inner product is non-negative for *all points in A;* that is, no matter how you pick $(\alpha, \theta) \in A$.

Remark 177. We make two remarks related to this.

- The assumption of $(\alpha, 0) \in A$ for all $\alpha \geq 0$ implies that

$$\nu_0 \geq 0 \text{ for all } \nu \in \tilde{A} \tag{9.13}$$

- It is a standard result in convex analysis that

$$\tilde{\tilde{A}} = \left\{(\alpha, \theta) \in \mathbb{R} \times \mathbb{R}^n : \alpha\nu_0 + \theta^\top \nu_- \geq 0 \text{ for all } (\nu_0, \nu_-) \in \tilde{A}\right\} = A \tag{9.14}$$

Now let's go over some examples of dual cones $\tilde{A}$ for the sets of A that we covered previously.

Example 178. (Cone Constraints and Dual Cones)

- *No Constraints (Complete Market):*

$$\begin{aligned} A &= \mathbb{R}^{n+1} \\ \tilde{A} &= \{0\} \end{aligned}$$

- *Non-traded Assets (Incomplete Markets):*

$$\begin{aligned} A &= \{(\alpha, \theta) \in \mathbb{R} \times \mathbb{R}^n : \theta_k = 0 \text{ for } k = m+1, \cdots, n\} \\ \tilde{A} &= \{(\nu_0, \nu_-) \in \mathbb{R} \times \mathbb{R}^n : \nu_0 = 0 \text{ and } \nu_{-k} = 0 \text{ for } k = 1, \cdots, m\} \end{aligned}$$

for some $m \in \mathbb{N}$ with $0 \leq m < n$.

- *Short Sale Constraints*:

$$\begin{aligned} A &= \{(\alpha, \theta) \in \mathbb{R} \times \mathbb{R}^n : \theta_k \geq 0 \text{ for } k = m+1, \cdots, n\} \\ \tilde{A} &= \left\{(\nu_0, \nu_-) \in \mathbb{R} \times \mathbb{R}^n : \begin{array}{c} \nu_0 = 0, \nu_{-k} = 0 \text{ for } k = 1, \cdots, m \\ \text{and } \nu_{-k} \geq 0 \text{ for } k = m+1, \cdots, n \end{array}\right\} \end{aligned}$$

for some $m \in \mathbb{N}$ with $0 \leq m < n$.

- *Borrowing Constraint 1 (No Borrowing):*

$$A = \{(\alpha, \theta) \in \mathbb{R} \times \mathbb{R}^n : \alpha \geq 0\}$$
$$\tilde{A} = \{(\nu_0, \nu_-) \in \mathbb{R} \times \mathbb{R}^n : \nu_0 \geq 0 \textit{ and } \nu_- = 0\}$$

- *Borrowing Constraint 2 (No Uncollateralized Borrowing):*

$$A = \left\{(\alpha, \theta) \in \mathbb{R} \times \mathbb{R}^n : \alpha \geq -(\mathbf{1} - \lambda)^\top \theta\right\}$$
$$\tilde{A} = \{(\nu_0, \nu_-) \in \mathbb{R} \times \mathbb{R}^n : \nu_0 \geq 0 \textit{ and } \nu_- = \nu_0 (\mathbf{1} - \lambda)\}$$

  for some $\lambda \in [0, 1]^n$.

- *Borrowing Constraint 3 (Margin Requirements):*

$$A = \left\{(\alpha, \theta) \in \mathbb{R} \times \mathbb{R}^n : \alpha \geq (\mathbf{1} + \lambda_-)^\top \theta^- - (\mathbf{1} - \lambda_+)^\top \theta^+\right\}$$
$$\tilde{A} = \{(\nu_0, \nu_-) \in \mathbb{R} \times \mathbb{R}^n : \nu_0 \geq 0 \textit{ and } \nu_0 (\mathbf{1} - \lambda_+) \leq \nu_- \leq \nu_0 (\mathbf{1} + \lambda_-)\}$$

  for some $\lambda_-, \lambda_+ \in [0, 1]^n$.

Now we introduce the first key result in this setup.
Let

$$\mathcal{N} = \left\{\nu : \nu \text{ is a progressive } \tilde{A}\text{-valued SP with } E\left[\int_0^T |\nu(t)|^2 \, dt\right] < \infty\right\} \tag{9.15}$$

Let $\nu = (\nu_0, \nu_-) \in \mathcal{N}$. Note This is a $(n+1)$-dimensional SP that takes values in $\tilde{A}$ and satisfies a square integrability condition.

Now we define the objects that are used to, but doing that as if the interest rate is $r + \nu_0$ and drift is $\mu + \nu_-$. The reasons for this setup will become clear in a moment.

$$B_\nu(t) = B(0) \exp\left(\int_0^t (r(s) + \nu_0(s)) \, ds\right) \tag{9.16}$$

$$\begin{aligned} \kappa_\nu(t) &= -\sigma(t)^{-1} (\mu(t) + \nu_-(t) - (r(t) + \nu_0(t)) \mathbf{1}) \\ &= \kappa(t) - \sigma(t)^{-1} (\nu_-(t) - \nu_0(t) \mathbf{1}) \end{aligned} \tag{9.17}$$

$$\xi_\nu = \exp\left(-\frac{1}{2} \int_0^t |\kappa_\nu(s)|^2 \, ds + \int_0^t \kappa_\nu(s)^\top \, dw(s)\right) \tag{9.18}$$

$$\pi_\nu(t) = \frac{B_\nu(0)}{B_\nu(t)} \xi_\nu(t) \tag{9.19}$$

Finally, let

$$\mathcal{N}^* = \{\nu \in \mathcal{N} : \xi_v \text{ is a MG}\} \tag{9.20}$$

Given the way ξ_v is defined, it will always be a local martingale. $\mathcal{N}^*$ is the smaller set of ν processes such that ξ_v is not only a local martingale, but a P-MG.

Note that for each SP $v = (\nu_0, \nu_-) \in \mathcal{N}$, we have defined the collection of objects $\{B_\nu, \kappa_\nu, \xi_\nu, \pi_\nu\}$.

Now we introduce our key result, which is analogous to Farkas' lemma in the finite dimensional case, which was presented as a motivating example.

Theorem 179. *A CP $(c, W) \in \mathcal{C}_b$ is financed by a TS $(\alpha, \theta) \in \Theta_A$ if and only if*

$$\sup_{\nu \in \mathcal{N}^*} E\left[\int_0^T \pi_\nu(t) c(t) \, dt + \pi_\nu(T) W\right] < \infty$$

Moreover, if the above condition holds, then

$$\begin{aligned} &\min\left\{\alpha(t) + \mathbf{1}^\top \theta(t) : (\alpha, \theta) \in \Theta_A \textit{ finances } (c, W)\right\} \\ &= \sup_{\nu \in \mathcal{N}^*} \frac{1}{\pi_\nu(t)} E\left[\int_t^T \pi_\nu(s) c(s) \, ds + \pi_\nu(T) W \,\middle|\, \mathcal{F}_t\right] \end{aligned}$$

This characterizes, without explicitly writing the TS, what it means for (c, W) to be financed by some trading strategy in Θ_A. We want the expectation to be finite for every point $v \in \mathcal{N}^*$, and thus consequently for every π_ν defined by ν. If this is the case, then the minimal cost trading strategy financing the CP is given by that expectation.

This also provides a characterization of the set of SPD that is consistent with our constraints. It also shows that what was true in the finite case is true in our continuous-time analogue. If a CP satisfies the budget constraint with respect to every SPD that are consistent with the portfolio constraint, then indeed the CP is financed by an (constrained) admissible TS.

Corollary 180. *The SP π_ν is a constrained SPD for any $\nu \in \mathcal{N}^*$.*

This follows from the fact that if condition is satisfied at equality with the supremum (over $\nu \in \mathcal{N}^*$), then of course the condition is satisfied as an inequality over any other point in $\mathcal{N}^*$.

Proof. □

9.2.2 The Dual Problem

In light of Theorem 179, we can recast the investor's problem in equation (9.11)

$$\begin{aligned} &\max_{(c,W)\in\mathcal{C}_+} U(c,W) \\ \text{s.t. } &E\left[\int_0^T \pi_\nu(t)(c_t-\tilde{c}_t)\,dt + \pi_\nu(T)\left(W-\tilde{W}\right)\right] \le e_0 \quad \forall \nu \in \mathcal{N}^* \end{aligned} \tag{9.21}$$

Again, this allows us to formulate the problem without the TS. We can recover that later on. The problem now is that we have an infinite number of budget constraints, one for each $\nu \in \mathcal{N}^*$.

Recall what we did in the finite-dimensional economy. There we said that the solution to the above problem was equivalent to a problem subject to a single budget constraint using a minimax SPD. The minimax SPD was given its name because it solves a dual minimization problem where you first maximize the objective using a single budget constraint, and then minimize that value over the budget constraints (over the SPDs).

We proceed similarly here.

For any $\nu \in \mathcal{N}^*$, consider the problem

$$\begin{aligned} \max_{(c,W)\in\mathcal{C}_+} &E\left[\int_0^T \eta_1 u(c_t,t)\,dt + \eta_2 V(W)\right] \\ \text{s.t. } &E\left[\int_0^T \pi_\nu(t)(c_t-\tilde{c}_t)\,dt + \pi_\nu(T)\left(W-\tilde{W}\right)\right] \le e_0 \end{aligned} \tag{9.22}$$

Then formulate the Largrangian as

$$\begin{aligned} &L_\nu(c,W,\psi) \\ =&E\left[\int_0^T \eta_1 u(c_t,t)\,dt + \eta_2 V(W) - \psi\left(\int_0^T \pi_\nu(t)(c_t-\tilde{c}_t)\,dt + \pi_\nu(T)\left(W-\tilde{W}\right) - e_0\right)\right] \end{aligned}$$

The FOCs for an optimal CP (c_ν^*, W_ν^*) are

$$\begin{aligned} \eta_1 u_c(c_\nu^*(t),t) - \psi\pi_\nu(t) = 0 &\implies c_\nu^*(t) = f(\psi\pi_\nu(t),t) \\ \eta_2 V'(W_\nu^*) - \psi\pi_\nu(T) = 0 &\implies W_\nu^* = g(\psi\pi_\nu(T)) \end{aligned}$$

where the f and g are inverse marginal utilities as defined in previous sections.

Substitute the (c_ν^*, W_ν^*) back into the Lagrangian,

$$\begin{aligned} J(\psi,\nu) &= L_\nu(c_\nu^*, W_\nu^*, \psi) \\ &= E\left[\int_0^T \tilde{u}(\psi\pi_\nu, t)\,dt + \tilde{V}(\psi\pi_\nu(T)) + \psi\left(e_0 + \int_0^T \pi_\nu(t)\,\tilde{c}_t dt + \pi_\nu(T)\,\tilde{W}\right)\right] \end{aligned} \tag{9.23}$$

where the convex conjugates $\tilde{u}$ and $\tilde{V}$ are defined as

$$\tilde{u}(x,t) = \max_c\left[\eta_1 u(c,t) - xc\right] = \eta_1 u(f(x,t), t) - xf(x,t) \tag{9.24}$$

$$\tilde{V}(x) = \max_W\left[\eta_2 V(W) - xW\right] = \eta_2 V(g(x)) - xg(x) \tag{9.25}$$

purely for notational convenience.

Note we still haven't identified the Lagrangean multiplier ψ, but we know that it must satisfy the budget constraint as an equality. The maximized value of the Lagrangean L_ν is going to depend on c^* (and c^* depends on ψ), but the Envelope Theorem says that if we differentiate the maximal value with respect to ψ, then we can ignore the dependence of (c^*, W^*) over ψ.

$$\begin{aligned} \frac{\partial J(\psi,\nu)}{\partial\psi} &= \left.\frac{\partial L_\nu(c_\nu^*, W_\nu^*, \psi)}{\partial\psi}\right|_{(c,W)=(c_\nu^*, W_\nu^*)} \\ &= -E\left[\int_0^T \pi_\nu(t)\,\tilde{c}_t dt + \pi_\nu(T)\,\tilde{W} - e_0\right] \end{aligned} \tag{9.26}$$

Here's how we can use this condition to interpret what ψ does. If we are looking for a stationary point, i.e.$\frac{\partial J(\psi,\nu)}{\partial\psi} = 0$, then the Lagrangian multiplier ψ does this because it satisfies the budget constraint with equality.

And we can actually go even further. Because $\tilde{u}(\cdot,t)$ and $\tilde{V}(\cdot)$ are convex in y, ψ must minimize the maximized value of the Lagrangian $J(\psi,\nu)$ so that the value of the problem in equation (9.22) is

$$\min_{\psi>0} J(\psi,\nu)$$

Thus by analogy with the discrtete time case, this leads us to consider the dual problem

$$\min_{(\psi,\nu)\in(0,\infty)\times\mathcal{N}^*} J(\psi,\nu) \tag{9.27}$$

The discrete time finite dimensional case illustrates that we need to identify the minimax SPD, and this will be given by the constrained SPD π_ν such that ν solves the problem above.

The following theorem guides on how to approach the investor's problem with incomplete markets.

Theorem 181. *Suppose (ψ^*, ν^*) solves the dual minimization problem in equation (9.27). Then the consumption plan (c^*, W^*) with*

$$c^*(t) = f(\psi^*\pi_{\nu^*}(t), t) \tag{9.28}$$

$$W^* = g(\psi^*\pi_{\nu^*}(T)) \tag{9.29}$$

solves the investor's problem in equation (9.11) and the optimal TS (α^*, θ^*) satisfies

$$\alpha_t^* + \mathbf{1}^\top\theta_t^* = E\left[\int_t^T \frac{\pi_{\nu^*}(s)}{\pi_{\nu^*}(t)}(c_s^* - \tilde{c}_s)\,ds + \frac{\pi_{\nu^*}(T)}{\pi_{\nu^*}(t)}\left(W^* - \tilde{W}\right)\middle|\,\mathcal{F}_t\right]$$

The trading strategy is given by agent's financial wealth $\alpha_t^* + \mathbf{1}^\top\theta_t^*$, which is given by the the value of the agents net trade going forward, priced by π_{ν^*}. To identify (α^*, θ^*), we apply Ito's lemma on the wealth process and match the volatility terms to obtain θ^* (and then α^*).

Proof. (incomplete) □

9.3 Examples and Applications

Let's see how we can apply this framework. We will go over two examples: logarithmic utility case and the power utility case with deterministic price coefficients.

9.3.1 Closed Form Solutions

Example 182. (Logarithmic Utility)
Assume

$$\begin{aligned} u(c,t) &= e^{-\rho t}\log(c) \\ V(W) &= e^{-\rho T}\log(W) \end{aligned}$$

and $e_0 > 0$ and $\left(\tilde{c}, \tilde{W}\right) = 0$. Let's first compute the convex conjugates via (9.24)(9.25)

$$\begin{aligned} \tilde{u}(x,t) &= -\eta_1 e^{-\rho t}\left(\log(x) + \rho t - \log(\eta_1) + 1\right) \\ \tilde{V}(x) &= -\eta_2 e^{-\rho T}\left(\log(x) + \rho T - \log(\eta_2) + 1\right) \end{aligned}$$

and the maximal value via (9.23) as

$$\begin{aligned} J(\psi,\nu) &= E\left[\int_0^T \tilde{u}(\psi\pi_\nu(t), t)\,dt + \tilde{V}(\psi\pi_\nu(T)) + \psi e_0\right] \\ &= \eta_1\int_0^T e^{-\rho t}E\left[-\log\pi_\nu(t)\right]dt + \eta_2 e^{-\rho T}E\left[-\log\pi_\nu(T)\right] \\ &\quad - \left(\eta_1\frac{1-e^{-\rho T}}{\rho} + \eta_2 e^{-\rho T}\right)\log\psi + \psi e_0 + J_0 \end{aligned}$$

where $J_0 = -\eta_1\int_0^T e^{-\rho t}\left(\rho t - \log\eta_1 + 1\right)dt - \eta_2 e^{-\rho T}\left(\rho T - \log\eta_2 + 1\right)$.

By the fact that $\pi_\nu(t) = \frac{B_\nu(0)}{B_\nu(t)}\xi_\nu(t)$ and the definition of $\mathcal{N}^*$ in equations (9.15) and (9.20),

$$E\left[-\log\pi_\nu(t)\right] = E\left[\int_0^t\left(r(s) + \nu_0(s) + \frac{1}{2}\left|\kappa_\nu(s)\right|^2\right)ds - \cancelto{0}{\int_0^t \kappa_\nu(s)^\top dw(s)}\right]$$

Thus the solution of the dual problem can be obtained just by considering the terms with ψ

$$\begin{aligned} \psi^* &= \arg\min_{\psi>0}\left[-\left(\eta_1\frac{1-e^{-\rho T}}{\rho} + \eta_2 e^{-\rho T}\right)\log\psi + \psi e_0\right] \\ &= \left(\eta_1\frac{1-e^{-\rho T}}{\rho} + \eta_2 e^{-\rho T}\right)\frac{1}{e_0} \end{aligned}$$

and similarly for ν

$$\nu_t^* = \arg\min_{\nu\in\tilde{A}}\left[\nu_0 + \frac{1}{2}\left|\kappa_t - \sigma_t^{-1}\left(\nu_- - \nu_0\mathbf{1}\right)\right|^2\right]$$

Now we apply Theorem (181) to arrive at the optimal consumption plan (c^*, W^*), where

$$c_t^* = f\left(\psi^*\pi_{\nu^*}(t), t\right) = \frac{\eta_1 e^{-\rho t}}{\psi^*\pi_{\nu^*}(t)} \tag{9.30}$$

$$W^* = g\left(\psi^*\pi_{\nu^*}(T)\right) = \frac{\eta_2 e^{-\rho T}}{\psi^*\pi_{\nu^*}(T)}$$

Moreover, the portfolio value process for the optimal TS (α^*, θ^*) satisfies

$$\begin{aligned} W_t &= \frac{1}{\pi_{\nu^*}(t)}E\left[\int_t^T \pi_{\nu^*}(s)\,c_s^* ds + \pi_{\nu^*}(T)\,W^*\middle|\,\mathcal{F}_t\right] \\ &= \frac{1}{\psi^*\pi_{\nu^*}(t)}\left(\eta_1\frac{e^{-\rho t} - e^{-\rho T}}{\rho} + \eta_2 e^{-\rho T}\right) = \frac{e^{-\rho t}}{\psi\pi_\nu^*(t)}h(t) \end{aligned} \tag{9.31}$$

where as in the previous section $h(t) = \eta_1 \frac{1-e^{-\rho(T-t)}}{\rho} + \eta_2 e^{-\rho(T-t)}$.

Let's recover the trading strategies (α, θ). Apply Ito's lemma on (9.31) further shows that

$$\begin{aligned} \theta_t^* &= \left(\sigma_t \sigma_t^\top\right)^{-1} \left[\mu_t + \nu_-^*(t) - (r_t + \nu_0^*(t)) \mathbf{1}\right] W_t \\ \alpha_t^* &= W_t - \mathbf{1}^\top \theta_t^* \end{aligned}$$

which we get by matching volatility terms as in

$$\begin{aligned} dW(t) &= (\cdots)\, dt + \theta^*(t)\, \sigma(t)\, dw(t) \\ &= (\cdots)\, dt + W(t)\, \kappa_\nu(t)\, dw(t) \end{aligned}$$

where the first line is derived from tightness of TS and second line from applying Ito's lemma on (9.31). Expanding $\theta^*(t)\,\sigma(t) = W(t)\,\kappa_\nu(t)^\top$ gets us to the result.

What this tells us is that the optimal portfolio is the growth-optimal porfolio in a fictitious unconstrained economy with interest rate $r + \nu_0^*$ and expected stock returns $\mu + \nu_-^*$. We can also show the marginal propensity to consume by dividing the expression for optimal consumption by wealth:

$$\frac{c_t^*}{W_t} = \frac{\eta_1}{h(t)}$$

Hence, the marginal propensity to consume is unaffected by the portfolio constraint.

Let's consider two different portfolio constraints: No Borrowing and No Uncollateralized Borrowing.

First, in No Borrowing, we have that $\nu_-(t) = 0$ and $v_0 \geq 0$. So trying to pin down ν_0,

$$\nu_0^*(t) = \arg\min_{\nu_0 \geq 0} \left[\nu_0 + \frac{1}{2}\left|\kappa_t + \sigma_t^{-1}\nu_0 \mathbf{1}\right|^2\right] = \left(-\frac{1 + \left(\sigma_t^{-1}\mathbf{1}\right)^\top \kappa_t}{\left|\sigma_t^{-1}\mathbf{1}\right|^2}\right)^+$$

In this case, the optimal portfolio is the growth-optimal portfolio under a shadow (zero beta) interest rate $r + \nu_0^* \geq r$. This has the effect of discouraging borrowing and hence enforcing the no borrowing constraint.

Second, in No Uncollaterized Borrowing case, recall we had $\nu_0 \geq 0$ and $\nu_- = \nu_0(\mathbf{1} - \lambda)$. Thus again we just need to pin down ν_0^* via

$$\nu_0^*(t) = \arg\min_{\nu_0 \geq 0} \left[\nu_0 + \frac{1}{2}\left|\kappa_t + \sigma_t^{-1}\nu_0 \lambda\right|^2\right] = \left(-\frac{1 - \left(\sigma_t^{-1}\lambda\right)^\top \kappa_t}{\left|\sigma_t^{-1}\lambda\right|^2}\right)^+$$

In this case, the optimal portfolio is the growth-optimal portfolio under a shadow interest rate $r + \nu_0^* \geq r$ and a shadow stock risk premia $\mu - r\mathbf{1} - \nu_0^*\lambda$. The shadow risk premia discourage investment in stocks with large haircuts given by the vector λ, as it lowers the (fictitious) drift. All else equal, the agent prefers to invest in stocks with good collateral quality (i.e. those with lower haircuts) because it makes it easier for him to circumvent the added borrowing constraint.

In addition, if we were to consider the incomplete markets case with some non-traded assets, then the agent would simply invest in the growth optimal portfolio consisting of assets that he *can* trade.

Example 183. (Power Utility with Deterministic Price Coefficients)

(To be added)

9.3.2 Equilibrium Risk Premia

Here we consider what happens in general equilibrium. In particular, what happens to the C-CAPM when we have portfolio constraints?

Consider a pure-exchange economy with I agents similar to the one before, but with the difference that the TS of each agent $i = 1, \cdots, I$ is constrained to take values in a set A_i, where either $A_i = \mathbb{R}^{n+1}$ or $A_i = A \subset \mathbb{R}^{n+1}$. Thus, the constrained set of TS can vary among agents, but each are either unconstrained or constrained the same way.

Proposition 184. *Suppose that the equilibrium prices are Ito processes and that for each agent $i \in \{1, \cdots, I\}$ the dual problem has a solution. Then,*

$$\mu(t) - r(t)\mathbf{1} = a(\tilde{c}_t, t)\frac{d}{dt}[\log S, \log \tilde{c}](t) - \left(\nu_-^*(t) - \nu_0^*(t)\mathbf{1}\right)$$

for all $t \in [0, T]$, where

$$a(\tilde{c}_t, t) = \tilde{c}_t\left[\sum_{i=1}^{I}\left(-\frac{u_{ic}(c_i^*(t), t)}{u_{icc}(c_i^*(t), t)}\right)\right]^{-1}$$

is the aggregate relative risk aversion, $\tilde{c}$ is the aggregate consumption process and ν^* is an $\tilde{A}$-valued SP.

Previously, we knew from a result in Section 3 (by Huang) equilibrium prices had to be Ito processes. This is no longer the case with portfolio constraints. So first we presume that equilibrium prices are Ito processes. In addition, we ask that at those prices, the dual problem has a solution.

The first term is the usual CCAPM term, risk aversion multiplied by covariation of (log) stock price and (log) consumption. Note $-\frac{u_{ic}(c_i^*(t),t)}{u_{icc}\left(c_i^*(t),t\right)}$ is the absolute risk tolerance of agent i, and the reciprocal of aggregate risk tolerance is the aggregate risk aversion. This so far is identical to the case with complete markets and no constraints.

The second term arises due to the constraints. Notice that if $A = \mathbb{R}^{n+1}$ then $\tilde{A}$ only contains the zero vector, so that $\nu_- = \nu_0$, thereby retrieving the complete markets case (CCAPM). To obtain some intuition, consider the effect of various cone constraints on the CCAPM.

- Non-traded Assets: Only the first m assets are traded. Then, $\nu_0 = 0$ and $\nu_{-k} = 0$ for $k = 1, \cdots, m$ (the traded assets). Thus, CCAPM still holds for the assets that do get traded.

- Short Sale Constraints: Last $n - m$ assets cannot be shorted. Then $\nu_0 = 0$, $\nu_{-k} = 0$ for unconstrained assets, $\nu_{-k} \geq 0$ for those with short sale constraints. CCAPM still characterizes the risk premia for the unconstrained assets, whereas for the stocks that cannot be shorted CCAPM overestimates the risk premia.

- No Borrowing: Agent cannot short sell money market account, $\nu_0 \geq 0$ and $\nu_- = 0$. The risk premia is equal to CCAPM term plus some fixed non-negative term for all stocks. CCAPM underestimates the risk premia of all stocks by the same amount ν_0^*. Put differently, CCAPM holds provided risk premia are computed using a higher (shadow) interest rate.

- No Uncollateralized Borrowing: $\nu_0 \geq 0$ and $\nu_- = \nu_0(\mathbf{1} - \lambda)$. Thus

  $$\mu(t) - r(t)\mathbf{1} = a(\tilde{c}_t, t)\frac{d}{dt}[\log S, \log \tilde{c}](t) + \nu_0\lambda$$

  The amount by which CCAPM underestimate the the risk premia on different stocks is going to be different depending on the haircut on each. Higher haircuts λ commands a larger excess return above the CCAPM. That is, if investors face borrowing constraints with regards to assets of different collateral quality, they are going to demand a higher risk premium in equilibrium to hold assets of poor credit quality (i.e. larger haircuts).