Chapter I. Preliminaries: Set theory and categories

Algebra: Chapter 0, by Paolo Aluffi

I.1. Naive set theory

Problem 2

Prove that if $\sim$ is an equivalence relation on a set S, then the corresponding family $\mathcal{P}_\sim$ is indeed a partition of S: that is, its elements are nonempty, disjoint, and their union is in S.

Proof. Here $\mathcal{P}_\sim$ is the set of equivalence classes w.r.t $\sim$ over S:

$$\mathcal{P}_\sim := \{[a]_\sim \mid a \in S\}.$$

Consider any $[a]_\sim \in \mathcal{P}_\sim$. Then $[a]_\sim$ is nonempty, because $a \in [a]_\sim$ by reflexivity. Now consider any other $[b]_\sim \in \mathcal{P}_\sim$ where $a \not\sim b$. By transitivity, for any element $b_n \in [b]_\sim$ we know that $a \not\sim b_n$ and so $[a]_\sim$ and $[b]_\sim$ must be disjoint. Finally, consider any $c \in S$. Then we must have that $c \in [c]_\sim \in \mathcal{P}_\sim$, and so we can write:

$$S = \bigcup_{[c]_\sim \in \mathcal{P}_\sim} [c]_\sim.$$

So each element of the family $\mathcal{P}_\sim$ is nonempty, disjoint, and their union equals S, as desired. $\square$

I.2. Functions between sets

I.3. Categories

I.4. Morphisms

I.5. Universal properties