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Statistical Methods and Data Analysis (EN.625.603)
Midterm Exam

Question 1
Suppose that a telephone number is 534-0826. If the first 3 digits of this number are written down in random order and then the last 4 digits of this number are written down in random order in an attempt to obtain the correct telephone number, what is the probability of each of the following events?

(a) All 7 digits are correctly placed.

(b) The first 3 digits are correctly placed and only 2 of the remaining digits are incorrectly placed.

Solution

(a) Probability the first 3 digits are correctly placed is

$$\frac{1}{3!} = \frac{1}{6}$$

Probability the last 4 digits are correctly placed is

$$\frac{1}{4!} = \frac{1}{24}$$

We observed that the order of the first 3 digits and the last 4 digits are independent. Thus the probability of all 7 digits are correctly placed is

$$\frac{1}{6} \times \frac{1}{24} = \mathbf{\frac{1}{144}}$$

(b) The number of ways for 2 digits to be incorrectly placed is

$$\binom{4}{2} = 6$$

The probability of 2 digits out of the last 4 digits are incorrectly placed is

$$\frac{6}{4!} = \frac{1}{4}$$

The probability of the first 3 digits are correctly placed and only 2 of the remaining digits are incorrectly placed is

$$\frac{1}{6} \times \frac{1}{4} = \mathbf{\frac{1}{24}}$$

Question 2

Given a collection of seven people, in which three are Data Science majors and four are ACM majors. If a committee of three people is picked at random, what is the probability that the committee contains one Data Science major and two ACM majors?

Solution

The number of ways to pick a committee of three people from seven people where one of them is data sience major and two of them are ACM majors is

$$\binom{3}{1} \times \binom{4}{2} = 3 \times 6 = 18$$

The number of ways to pick a committe of three people from seven people is

$$\binom{7}{3} = 35$$

The probability that the committee contains one Data Science major and two ACM majors is

$$\frac{18}{35} = \mathbf{\frac{18}{35}}$$

Question 3

Ninety percent of the disk drives manufactured by the COMDISK Company are known to function properly. For a collection of 18 disk drives, find the probability that at least 15 function properly.

Solution

Let X be the number of disk drives that function properly. X is a binomial random variable with $n = 18$ and $p = 0.9$. The probability that at least 15 function properly is

$$\begin{aligned}
P(X \geq 15) &= \sum_{k=15}^{18} \binom{18}{k} (0.9)^k (0.1)^{18-k} \\
&= \binom{18}{15}(0.9)^{15}(0.1)^3 + \binom{18}{16}(0.9)^{16}(0.1)^2 + \binom{18}{17}(0.9)^{17}(0.1)^1 + \binom{18}{18}(0.9)^{18}(0.1)^0 \\
&= 0.168 + 0.284 + 0.300 + 0.150 \\
&= \mathbf{0.902}
\end{aligned}$$

Question 4

A certain construction company buys 20%, 30%, and 50% of their nails from hardware suppliers A, B, and C, respectively. Suppose it is known that 0.5%, .02%, and .01% of the nails from A, B, and C respectively are defective. If a nail purchased by the construction company is defective, what is the probability that it came from supplier C?

Solution

Probability that a nail purchased by the construction company is defective is

$$\begin{aligned}
P(\text{defective}) &= P(\text{defective}|\text{A})P(\text{A}) + P(\text{defective}|\text{B})P(\text{B}) + P(\text{defective}|\text{C})P(\text{C}) \\
&= 0.5\% \times 20\% + 0.02\% \times 30\% + 0.01\% \times 50\% \\
&= 0.00111
\end{aligned}$$

Applying Bayes' theorem, the probability that a nail purchased by the construction company is from supplier C given that it is defective is

$$\begin{aligned} P(\text{C}|\text{defective}) &= \frac{P(\text{defective}|\text{C})P(\text{C})}{P(\text{defective})} \\ &= \frac{0.005\%}{0.00111} \\ &= \mathbf{0.045} \end{aligned}$$

Question 5
Given a coin for which the probability of head on any toss is 3/5. The coin is tossed three times. Determine the probability function (i.e., the random variables' values and associated probabilities) for:

(a) X, where X stands for the number of heads.

(b) Y, where Y stands for the absolute value of the number of heads minus the number of tails.

Solution

(a) The probability function for X is

$$\begin{aligned} P(X=0) &= \binom{3}{0}\left(\frac{3}{5}\right)^0\left(\frac{2}{5}\right)^3 = \mathbf{0.064} \\ P(X=1) &= \binom{3}{1}\left(\frac{3}{5}\right)^1\left(\frac{2}{5}\right)^2 = \mathbf{0.288} \\ P(X=2) &= \binom{3}{2}\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^1 = \mathbf{0.432} \\ P(X=3) &= \binom{3}{3}\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^0 = \mathbf{0.216} \end{aligned}$$

(b) The probability function for Y is

$$\begin{aligned} P(Y=0) &= P(X=0) + P(X=3) \\ &= 0.064 + 0.216 \\ &= \mathbf{0.280} \\ P(Y=1) &= P(X=1) + P(X=2) \\ &= 0.288 + 0.432 \\ &= \mathbf{0.720} \end{aligned}$$

Question 6
A coin is flipped, and a die is tossed simultaneously. Let X be the face of the coin (H = 0, T = 1). Let Y be the face of the die (1, 2, 3, 4, 5, 6).

(a) Calculate the joint probability distribution and show the values in a table.

(b) Let $Z = X - XY + 2Y$ represent the payoff associated with each outcome. Find the pdf for Z.

Solution

(a) The joint probability distribution is

$$
\begin{aligned}
P(X=0,Y=1) &= \frac{1}{2}\times\frac{1}{6} = \frac{1}{12}\\
P(X=0,Y=2) &= \frac{1}{2}\times\frac{1}{6} = \frac{1}{12}\\
P(X=0,Y=3) &= \frac{1}{2}\times\frac{1}{6} = \frac{1}{12}\\
P(X=0,Y=4) &= \frac{1}{2}\times\frac{1}{6} = \frac{1}{12}\\
P(X=0,Y=5) &= \frac{1}{2}\times\frac{1}{6} = \frac{1}{12}\\
P(X=0,Y=6) &= \frac{1}{2}\times\frac{1}{6} = \frac{1}{12}\\
P(X=1,Y=1) &= \frac{1}{2}\times\frac{1}{6} = \frac{1}{12}\\
P(X=1,Y=2) &= \frac{1}{2}\times\frac{1}{6} = \frac{1}{12}\\
P(X=1,Y=3) &= \frac{1}{2}\times\frac{1}{6} = \frac{1}{12}\\
P(X=1,Y=4) &= \frac{1}{2}\times\frac{1}{6} = \frac{1}{12}\\
P(X=1,Y=5) &= \frac{1}{2}\times\frac{1}{6} = \frac{1}{12}\\
P(X=1,Y=6) &= \frac{1}{2}\times\frac{1}{6} = \frac{1}{12}
\end{aligned}
$$

The table is

	Y= 1	Y= 2	Y= 3	Y= 4	Y= 5	Y= 6
$X=0$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$
$X=1$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

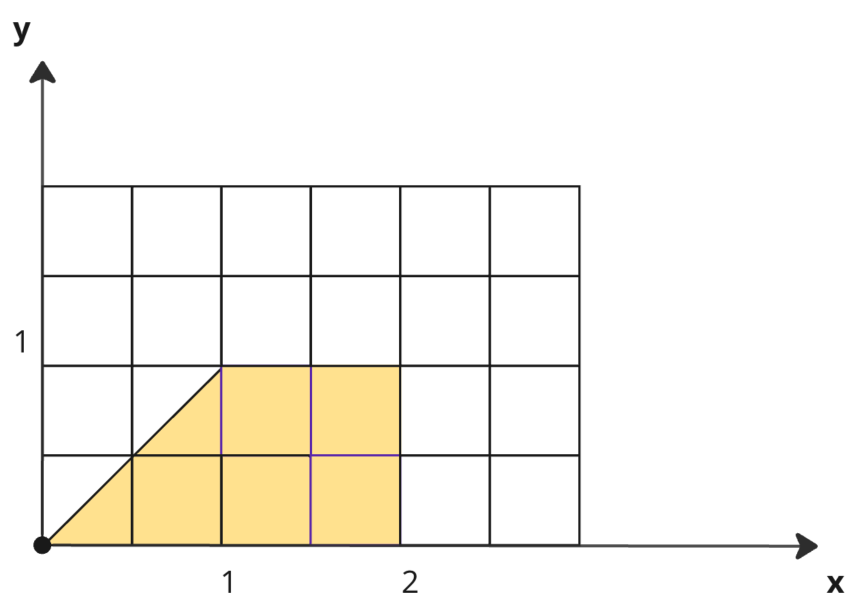
(b)

$$
\begin{aligned}
X=0,Y=1 &\implies Z = 0-0+2 = 2\\
X=0,Y=2 &\implies Z = 0-0+4 = 4\\
X=0,Y=3 &\implies Z = 0-0+6 = 6\\
X=0,Y=4 &\implies Z = 0-0+8 = 8\\
X=0,Y=5 &\implies Z = 0-0+10 = 10\\
X=0,Y=6 &\implies Z = 0-0+12 = 12\\
X=1,Y=1 &\implies Z = 1-1+2 = 2\\
X=1,Y=2 &\implies Z = 1-2+4 = 3\\
X=1,Y=3 &\implies Z = 1-3+6 = 4\\
X=1,Y=4 &\implies Z = 1-4+8 = 5\\
X=1,Y=5 &\implies Z = 1-5+10 = 6\\
X=1,Y=6 &\implies Z = 1-6+12 = 7
\end{aligned}
$$

The probability function for Z is

$$
\begin{aligned}
P(Z=2) &= P(X=0,Y=1)+P(X=1,Y=1)=\frac{1}{6}\\
P(Z=3) &= P(X=1,Y=2)=\frac{1}{12}\\
P(Z=4) &= P(X=0,Y=3)+P(X=1,Y=3)=\frac{1}{6}\\
P(Z=5) &= P(X=1,Y=4)=\frac{1}{12}\\
P(Z=6) &= P(X=0,Y=5)+P(X=1,Y=5)=\frac{1}{6}\\
P(Z=7) &= P(X=1,Y=6)=\frac{1}{12}\\
P(Z=8) &= P(X=0,Y=4)=\frac{1}{12}\\
P(Z=10) &= P(X=0,Y=5)=\frac{1}{12}\\
P(Z=12) &= P(X=0,Y=6)=\frac{1}{12}
\end{aligned}
$$

Question 7
Given the joint pdf of X and Y given by $f(x,y) = (\frac{1}{2})x^2y + (\frac{1}{3})y$ for $0 < x < 2;\ 0 < y < 1$ and $f(x,y) = 0$, elsewhere. Determine the possibility $P(Y \le X)$.
Solution
The area of $P(Y \le X)$ is *(image was created by using Miroboard)*



We calculate $P(Y \le X)$ as follows

$$
\begin{aligned}
P(Y \le X) &= \int_0^1 \int_y^2 f(x,y)\, dxdy \\
&= \int_0^1 \int_y^2 \left(\frac{1}{2}\right) x^2 y + \left(\frac{1}{3}\right) y \, dxdy \\
&= \int_0^1 \left(\frac{1}{6}\right) x^3 y + \left(\frac{1}{3}\right) xy \Big|_y^2 \, dy \\
&= \int_0^1 \left(\frac{4}{3}\right) y + \left(\frac{2}{3}\right) y - \left(\frac{1}{6}\right) y^4 - \frac{1}{3} y^2 \, dy \\
&= \left(\frac{2}{3}\right) y^2 + \left(\frac{1}{3}\right) y^2 - \left(\frac{1}{30}\right) y^5 - \left(\frac{1}{9}\right) y^3 \Big|_0^1 \, dy \\
&= \left(\frac{2}{3}\right) + \left(\frac{1}{3}\right) - \left(\frac{1}{30}\right) - \left(\frac{1}{9}\right) \\
&= \mathbf{\frac{77}{90}}
\end{aligned}
$$

Question 8

A box contains two red, three green, and five blue chips. Two chips are selected from the box. Let X_1 and X_2 denote the number of red and green chips obtained.

(a) Find the probabilities associated with all possible pairs of values (x_1, x_2).

(b) Determine the marginal probabilities associated with X_1, X_2.

(c) Determine the $f_{X_2}(x_2 = 1 | x_1 = 0)$.

Solution

(a) The probabilties associate with a pair of (x_1, x_2) can be calculated as follows

$$
P(X_1 = x_1, X_2 = x_2) = \frac{\binom{2}{x_1}\binom{3}{x_2}\binom{5}{5-x_1-x_2}}{\binom{10}{2}}
$$

The probabilities associated with all possible pairs of values (x_1, x_2) are

	$X_1 = 0$	$X_1 = 1$	$X_1 = 2$
$X_2 = 0$	$\frac{10}{45}$	$\frac{10}{45}$	$\frac{1}{45}$
$X_2 = 1$	$\frac{15}{45}$	$\frac{6}{45}$	0
$X_2 = 2$	$\frac{3}{45}$	0	0

(b) The marginal probabilities associated with X_1 and X_2 are

$$
\begin{aligned}
P(X_1 = 0) &= \frac{10}{45} + \frac{15}{45} + \frac{3}{45} = \mathbf{\frac{28}{45}} \\
P(X_1 = 1) &= \frac{10}{45} + \frac{6}{45} = \mathbf{\frac{16}{45}} \\
P(X_1 = 2) &= \mathbf{\frac{1}{45}} \\
P(X_2 = 0) &= \frac{10}{45} + \frac{10}{45} + \frac{1}{45} = \mathbf{\frac{21}{45}} \\
P(X_2 = 1) &= \frac{15}{45} + \frac{6}{45} = \mathbf{\frac{21}{45}} \\
P(X_2 = 2) &= \mathbf{\frac{3}{45}}
\end{aligned}
$$

(c) The conditional probability $f_{X_2}(x_2 = 1|x_1 = 0)$ is calculated as follows

$$
\begin{aligned}
f_{X_2}(x_2 = 1|x_1 = 0) &= \frac{P(X_1 = 0, X_2 = 1)}{P(X_1 = 0)} \\
&= \frac{\frac{15}{45}}{\frac{28}{45}} \\
&= \mathbf{\frac{15}{28}}
\end{aligned}
$$

Question 9
In a gambling game, five fair coins are tossed. For a bet of \$5, a gambler will win \$10 if three heads occur. Otherwise, the gambler loses the \$5 bet. What is the expected gain for a typical bet of \$5.
Solution
Probability of winning \$10 is

$$
\begin{aligned}
P(\text{win}) &= \binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 \\
&= \frac{5}{16}
\end{aligned}
$$

Probability of losing \$5 is

$$
\begin{aligned}
P(\text{lose}) &= 1 - P(\text{win}) \\
&= 1 - \frac{5}{16} \\
&= \frac{11}{16}
\end{aligned}
$$

The expected gain for a typical bet of \$5 is

$$
\begin{aligned}
E(\text{gain}) &= 10 \times P(\text{win}) - 5 \times P(\text{lose}) \\
&= 10 \times \frac{5}{16} - 5 \times \frac{11}{16} \\
&= \mathbf{-\frac{5}{16}}
\end{aligned}
$$

Question 10
In a certain country the heights for adult males are normally distributed with a mean of 68 inches and a standard deviation of 4 inches. Let X symbolize the height.

(a) Determine the probability $P(66 < X < 73)$.

(b) Determine the height which represents the 90^{th} percentile.

Solution

(a) The probability $P(66 < X < 73)$ is

$$\begin{aligned} P(66 < X < 73) &= P(\infty < X \leq 73) - P(\infty < X \leq 66) \\ &= \Phi\left(\frac{73-68}{4}\right) - \Phi\left(\frac{66-68}{4}\right) \\ &= \Phi(1.25) - \Phi(-0.5) \\ &= 0.8944 - 0.3085 \\ &= \mathbf{0.5859} \end{aligned}$$

(b) The height which represents the 90^{th} percentile is

$$\begin{aligned} \Phi(z) &= 0.9 \\ z &= \mathbf{1.28} \end{aligned}$$

The height which represents the 90^{th} percentile is

$$\begin{aligned} x &= 1.28 \times 4 + 68 \\ &= \mathbf{73.12} \end{aligned}$$

Question 11

Given that $f(x) = ke^{-x/3}$ for $x > 0$, and $f(x) = 0$ elsewhere.

(a) Determine k.

(b) Determine the CDF of X.

Solution

(a) The value of k is determined by

$$\begin{aligned} \int_0^\infty f(x)\, dx &= 1 \\ \int_0^\infty ke^{-x/3}\, dx &= 1 \\ k\int_0^\infty e^{-x/3}\, dx &= 1 \\ k\left(-3e^{-x/3}\right)\Big|_0^\infty &= 1 \\ k\left(-3e^{-\infty/3} + 3e^{-0/3}\right) &= 1 \\ k\left(-3 \times 0 + 3 \times 1\right) &= 1 \\ k &= \mathbf{\frac{1}{3}} \end{aligned}$$

(b) The CDF of X is

$$\begin{aligned} F(x) &= \int_0^x f(x)\, dx \\ &= \int_0^x \frac{1}{3} e^{-x/3}\, dx \\ &= \frac{1}{3} \int_0^x e^{-x/3}\, dx \\ &= \frac{1}{3} \left(-3e^{-x/3} \right) \Big|_0^x \\ &= \frac{1}{3} \left(-3e^{-x/3} + 3e^{-0/3} \right) \\ &= \mathbf{1 - e^{-x/3}} \end{aligned}$$

Bonus Question
Derive Bayes Theorem given in Theorem 2.4.2 on page 45.

$$P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^n P(B|A_i)P(A_i)}$$

Solution
Refer to the Definition 2.4.1 of conditional probability, we have the following

$$\begin{aligned} P(A_j|B) &= \frac{P(A_j \cap B)}{P(B)} \\ &= \frac{P(B|A_j)P(A_j)}{P(B)} \qquad (1) \end{aligned}$$

Refer to the Theorem 2.4.1 of the law of total probability, we have the following

$$P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i) \qquad (2)$$

From (1), (2), we derives Bayes Theorem in Theorem 2.4.2 as follows

$$P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^n P(B|A_i)P(A_i)}$$