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Statistical Methods and Data Analysis (EN.625.603)

Problem Set 6

Question 6.2.8b
Calculate the $P-$values for the hypothesis tests indicated in Question 6.2.1.
Do they agree with your decision on whether or not to reject H_0?
(b) $H_0 : \mu = 42.9$ versus $H_1 : \mu \neq 42.9; \bar{y} = 45.1, n = 16, \sigma = 3.2, \alpha = 0.01.$
Solution
Applying **Theorem 6.2.1(c)**, the test at the α level of significance

$$\text{Reject } H_0 \text{ if } |z| > z_{\alpha/2} = z_{0.005} = 2.575$$
$$\text{where } z = \frac{\bar{y} - \mu_0}{\sigma/\sqrt{n}} = \frac{45.1 - 42.9}{3.2/\sqrt{16}} = 2.75$$

Therefore, we reject H_0;
Using **Definition 6.2.4**, we calculate P-value as follows

$$P\text{-value} = 2 \times P(z > 2.75) = 2 \times (1 - P(z < 2.75)) = 2 \times (1 - 0.997) = \mathbf{0.006}$$

Since P-value $< \alpha$, we reject H_0 and **it agrees with the decision on rejecting H_0.**

Question 6.3.7
What α levels are possible with a decision rule of the form "Reject H_0 if $k \geq k^*$" when $H_0 : p = 0.5$ is to be tested against $H_1 : p > 0.5$ using a random sample of size $n = 7$?
Solution
For a test of size $n = 7$, $p > 0.5$ we have the following scenarios

- $k = 4$

$$\begin{aligned}\alpha &= P(\text{Reject } H_0 \mid H_0 \text{ is true}) \\ &= P(k \geq 4 \mid p = 0.5) \\ &= \sum_{k=4}^{7} \binom{7}{k} (0.5)^k (1 - 0.5)^{7-k} \\ &= 0.5\end{aligned}$$

- $k = 5$

$$\begin{aligned}\alpha &= P(\text{Reject } H_0 \mid H_0 \text{ is true}) \\ &= P(k \geq 5 \mid p = 0.5) \\ &= \sum_{k=5}^{7} \binom{7}{k} (0.5)^k (1 - 0.5)^{7-k} \\ &= 0.2265625\end{aligned}$$

- $k = 6$

$$\begin{aligned}\alpha &= P(\text{Reject } H_0 \mid H_0 \text{ is true}) \\ &= P(k \geq 6 \mid p = 0.5) \\ &= \sum_{k=6}^{7} \binom{7}{k} (0.5)^k (1-0.5)^{7-k} \\ &= 0.0625\end{aligned}$$

- $k = 7$

$$\begin{aligned}\alpha &= P(\text{Reject } H_0 \mid H_0 \text{ is true}) \\ &= P(k \geq 7 \mid p = 0.5) \\ &= \sum_{k=7}^{7} \binom{7}{k} (0.5)^k (1-0.5)^{7-k} \\ &= 0.0078125\end{aligned}$$

Thus, the possible α levels are $\mathbf{0.5, 0.2265625, 0.0625, 0.0078125}$.

Question 6.4.3
For the decision rule found in Question 6.2.2 to test $H_0 : \mu = 95$ versus $H_1 : \mu \neq 95$, at what $\alpha = 0.06$ level of significance, calculate $1 - \beta$ when $\mu = 90$.
Solution
From 6.2.2, we have $n = 22, \sigma = 15$, the decision rule for two-tailed is

$$\begin{aligned}\text{Reject } H_0 \text{ if } |z| > z_{\alpha/2} = z_{0.03} = 1.88 \\ \text{where } z = \frac{\bar{y} - \mu_0}{\sigma/\sqrt{n}} = \frac{\bar{y} - 95}{15/\sqrt{22}} \\ |\bar{y}| > 1.88 \times \frac{15}{\sqrt{22}} + 95 \\ \bar{y} > 101.01 \text{ or } \bar{y} < 88.99\end{aligned}$$

From **6.4**, we knnow that $1 - \beta = P(\text{Reject } H_0 \mid H_1 \text{ is true})$, thus, we can calculate $1 - \beta$ as follows

$$\begin{aligned}1 - \beta &= P(\text{Reject } H_0 \mid H_1 \text{ is true}) \\ &= P(\bar{y} > 101.01 \mid \mu = 90) + P(\bar{y} < 88.99 \mid \mu = 90) \\ &= P(z > \frac{101.01 - 90}{15/\sqrt{22}}) + P(z < \frac{88.99 - 90}{15/\sqrt{22}}) \\ &= P(z > 3.443) + P(z < -0.317) \\ &= 0.0003 + 0.3765 \\ &= \mathbf{0.3768}\end{aligned}$$

Question 6.4.4
Construct a power curve for the $\alpha = 0.05$ test of $H_0 : \mu = 60$ versus $H_1 : \mu \neq 60$ if the data consist of a random sample of size 16 from a normal distribution having $\sigma = 4$.

Solution

From **6.4**, *a graph of* $1-\beta$ *on the* $y-$*axis versus values of the parameter being tested on the* $x-$*axis is called a power curve.*

We have $\alpha = 0.05, n = 16, \sigma = 4$, the decision rule for two-tailed is

$$\begin{aligned}
\text{Reject } H_0 \text{ if } |z| > z_{\alpha/2} = z_{0.025} = 1.96 \\
\text{where } z = \frac{\bar{y}-\mu_0}{\sigma/\sqrt{n}} = \frac{\bar{y}-60}{4/\sqrt{16}} \\
|\bar{y}| > 1.96 \times \frac{4}{\sqrt{16}} + 60 \\
\bar{y} > 61.96 \text{ or } \bar{y} < 58.04
\end{aligned}$$

- For $\mu = 57$

$$\begin{aligned}
1-\beta &= P(\text{Reject } H_0 \mid H_1 \text{ is true}) \\
&= P(\bar{y} > 61.96 \mid \mu = 57) + P(\bar{y} < 58.04 \mid \mu = 57) \\
&= P(z > \frac{61.96-57}{4/\sqrt{16}}) + P(z < \frac{58.04-57}{4/\sqrt{16}}) \\
&= P(z > 4.96) + P(z < 1.04) \\
&\approx 0 + 0.851 = 0.851
\end{aligned}$$

- For $\mu = 58$

$$\begin{aligned}
1-\beta &= P(\text{Reject } H_0 \mid H_1 \text{ is true}) \\
&= P(\bar{y} > 61.96 \mid \mu = 58) + P(\bar{y} < 58.04 \mid \mu = 58) \\
&= P(z > \frac{61.96-58}{4/\sqrt{16}}) + P(z < \frac{58.04-58}{4/\sqrt{16}}) \\
&= P(z > 3.96) + P(z < 0.04) \\
&\approx 0 + 0.516 = 0.516
\end{aligned}$$

- For $\mu = 59$

$$\begin{aligned}
1-\beta &= P(\text{Reject } H_0 \mid H_1 \text{ is true}) \\
&= P(\bar{y} > 61.96 \mid \mu = 59) + P(\bar{y} < 58.04 \mid \mu = 59) \\
&= P(z > \frac{61.96-59}{4/\sqrt{16}}) + P(z < \frac{58.04-59}{4/\sqrt{16}}) \\
&= P(z > 2.96) + P(z < -0.96) \\
&\approx 0.0015 + 0.1685 = 0.17
\end{aligned}$$

- For $\mu = 60$

$$\begin{aligned}
1-\beta &= P(\text{Reject } H_0 \mid H_1 \text{ is true}) \\
&= P(\bar{y} > 61.96 \mid \mu = 60) + P(\bar{y} < 58.04 \mid \mu = 60) \\
&= P(z > \frac{61.96-60}{4/\sqrt{16}}) + P(z < \frac{58.04-60}{4/\sqrt{16}}) \\
&= P(z > 1.96) + P(z < -1.96) \\
&\approx 0.025 + 0.025 = 0.05
\end{aligned}$$

- For $\mu = 61$

$$\begin{aligned}1 - \beta &= P(\text{Reject } H_0 \mid H_1 \text{ is true}) \\ &= P(\bar{y} > 61.96 \mid \mu = 61) + P(\bar{y} < 58.04 \mid \mu = 61) \\ &= P(z > \frac{61.96 - 61}{4/\sqrt{16}}) + P(z < \frac{58.04 - 61}{4/\sqrt{16}}) \\ &= P(z > 0.96) + P(z < -2.96) \\ &\approx 0.1685 + 0.0015 = 0.17\end{aligned}$$

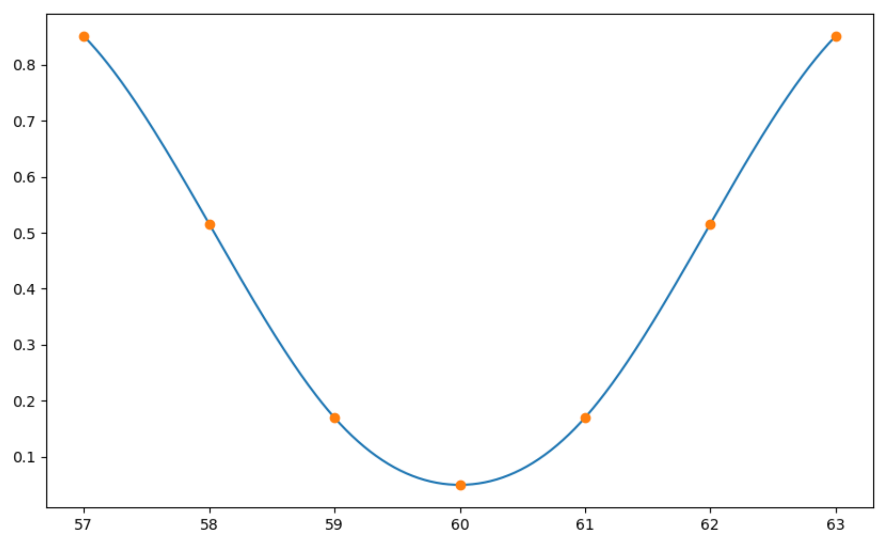
- For $\mu = 62$

$$\begin{aligned}1 - \beta &= P(\text{Reject } H_0 \mid H_1 \text{ is true}) \\ &= P(\bar{y} > 61.96 \mid \mu = 62) + P(\bar{y} < 58.04 \mid \mu = 62) \\ &= P(z > \frac{61.96 - 62}{4/\sqrt{16}}) + P(z < \frac{58.04 - 62}{4/\sqrt{16}}) \\ &= P(z > -0.04) + P(z < -3.96) \\ &\approx 0.516 + 0 = 0.516\end{aligned}$$

- For $\mu = 63$

$$\begin{aligned}1 - \beta &= P(\text{Reject } H_0 \mid H_1 \text{ is true}) \\ &= P(\bar{y} > 61.96 \mid \mu = 63) + P(\bar{y} < 58.04 \mid \mu = 63) \\ &= P(z > \frac{61.96 - 63}{4/\sqrt{16}}) + P(z < \frac{58.04 - 63}{4/\sqrt{16}}) \\ &= P(z > -1.04) + P(z < -4.96) \\ &\approx 0.851 + 0 = 0.851\end{aligned}$$

From the above results, using Python with matplotlib, we can construct the power curve as follows



Question 6.4.7

If $H_0 : \mu = 200$ is to be tested against $H_1 : \mu < 200$ at the $\alpha = 0.10$ level of significance based on a random sample of size n from a normal distribution where $\sigma = 15.0$, what is the smallest value for n that will make the power equal to at least 0.75 when $\mu = 197$?

Solution

From 6.4, the smallest sample size so that the power of a test to be at least 0.75 can be calculated as follows

$$\begin{aligned}
P(\text{Reject } H_0 \mid H_1 \text{ is true}) &\geq 0.75 \\
P(\bar{y} < 200 + z_{\alpha=0.10} \times \frac{\sigma}{\sqrt{n}} \mid \mu = 197) &\geq 0.75 \\
P(\bar{y} < 200 - 1.28 \times \frac{15}{\sqrt{n}} \mid \mu = 197) &\geq 0.75 \\
P(\frac{\bar{y} - 197}{15/\sqrt{n}} < \frac{3 - 19.2/\sqrt{n}}{15/\sqrt{n}}) &\geq 0.75 \\
P(z < \frac{3 - 19.2/\sqrt{n}}{15/\sqrt{n}}) &\geq 0.75 \\
\frac{3 - 19.2/\sqrt{n}}{15/\sqrt{n}} &\geq 0.675 \\
3 - 19.2/\sqrt{n} &\geq 10.125/\sqrt{n} \\
3 &\geq 29.325/\sqrt{n} \\
\boldsymbol{n} &\geq \mathbf{95.55}
\end{aligned}$$

Thus, the smallest value for n that will make the power equal to at least 0.75 when $\mu = 197$ is **96**.

Question 6.4.9

If $H_0 : \mu = 30$ is tested again $H_1 : \mu > 30$ using $n = 16$ observations (normally distributed) and if $1 - \beta = 0.85$ when $\mu = 34$, what does α equal? Assume $\sigma = 9$.

Solution

From **6.4**, we know that $1 - \beta = P(\text{Reject } H_0 \mid H_1 \text{ is true}) = 0.85$, thus, we can calculate $\bar{y}$ as follows

$$\begin{aligned}
1 - \beta &= P(\text{Reject } H_0 \mid H_1 \text{ is true}) \\
P(\bar{Y} > \bar{y} | \mu = 34) &= 0.85 \\
P(Z > \frac{\bar{y} - 34}{\sigma/\sqrt{n}}) &= 0.85 \\
\frac{\bar{y} - 34}{\sigma/\sqrt{n}} &\approx -1.035 \\
\frac{\bar{y} - 34}{9/\sqrt{16}} &\approx -1.035 \\
\bar{y} &\approx 31.67
\end{aligned}$$

From **6.2.1**, the test at the α level of significance can be calculated as follows

$$\begin{aligned}
\alpha &= P(\text{Reject } H_0 \mid H_0 \text{ is true}) \\
\alpha &= P(\bar{y} > 31.67 \mid \mu = 30) \\
\alpha &= P(z > \frac{31.67 - 30}{9/\sqrt{16}}) \\
\alpha &\approx P(z > 0.74) \\
\boldsymbol{\alpha} &\mathbf{= 0.23}
\end{aligned}$$

Question 7.3.2

Find the moment-generating function for a chi square random variable and use it to show that $E(\chi_n^2) = n$ and $V(\chi_n^2) = 2n$.

Solution

From **Theorem 7.3.1**, the moment-generating function for a chi square random variable is

$$\begin{aligned}
M(t) &= E(e^{tX}) \\
&= \int_{-\infty}^{\infty} e^{tx} \frac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} e^{-x/2} dx \\
&= \frac{1}{2^{n/2}\Gamma(n/2)} \int_{-\infty}^{\infty} x^{n/2-1} e^{-(1/2-t)x} dx
\end{aligned}$$

Let $u = -(1/2 - t)x$, then $du = -(1/2 - t)dx$, thus

$$\begin{aligned}
M(t) &= \frac{1}{2^{n/2}\Gamma(n/2)} \int_{-\infty}^{\infty} x^{n/2-1} e^{-(1/2-t)x} dx \\
M(t) &= \frac{1}{2^{n/2}\Gamma(n/2)} \int_{-\infty}^{\infty} \frac{u^{n/2-1}}{-(1/2-t)^{n/2-1}} e^{u} \frac{du}{-(1/2-t)} \\
M(t) &= \frac{1}{2^{n/2}\Gamma(n/2)} \frac{1}{-(1/2-t)^{n/2}} \int_{-\infty}^{\infty} u^{n/2-1} e^{u} du \\
M(t) &= \frac{1}{2^{n/2}\Gamma(n/2)} \frac{1}{(1/2-t)^{n/2}} \Gamma(n/2) \\
\boldsymbol{M(t)} &= \boldsymbol{\frac{1}{(1-2t)^{n/2}}}
\end{aligned}$$

We use the results above to show that $E(\chi_n^2) = n$ as follows

$$\begin{aligned}
E(\chi_n^2) &= M'(0) \\
E(\chi_n^2) &= \frac{d}{dt} \frac{1}{(1-2t)^{n/2}} \\
E(\chi_n^2) &= \frac{d}{dt} (1-2t)^{-n/2} \\
E(\chi_n^2) &= -\frac{n}{2} (1-2t)^{-n/2-1} (-2) \\
\boldsymbol{E(\chi_n^2)} &= \boldsymbol{n}
\end{aligned}$$

We use the results above to show that $V(\chi_n^2) = 2n$ as follows

$$
\begin{aligned}
V(\chi_n^2) &= M''(0) - (M'(0))^2 \\
V(\chi_n^2) &= \frac{d^2}{dt^2}(1-2t)^{-n/2} - n^2 \\
V(\chi_n^2) &= \frac{d}{dt}(-n/2)(1-2t)^{-n/2-1}(-2) - n^2 \\
V(\chi_n^2) &= -4(\frac{-n}{2} - 1)\frac{-n}{2} - n^2 \\
V(\chi_n^2) &= 4(\frac{n^2}{4} + \frac{n}{2}) - n^2 \\
\boldsymbol{V(\chi_n^2)} &\boldsymbol{= 2n}
\end{aligned}
$$

Question 7.4.5

Suppose a random sample of size $n = 11$ is drawn from a normal distribution with $\mu = 15.0$. For what value of k is the following true?

$$P\left(\left|\frac{\bar{Y} - 15.0}{S/\sqrt{11}}\right| \geq k\right) = 0.05$$

Solution

Applying **7.4.1**, we construct a confidence interval for μ and solve for k as follows

$$
\begin{aligned}
P\left(-t_{\alpha/2,n-1} \leq \frac{\bar{Y} - \mu}{S/\sqrt{n}} \leq t_{\alpha/2,n-1}\right) &= 1 - \alpha \\
P\left(-t_{0.025,10} \leq \frac{\bar{Y} - 15.0}{S/\sqrt{11}} \leq t_{0.025,10}\right) &= 0.95 \\
P\left(-2.228 \leq \frac{\bar{Y} - 15.0}{S/\sqrt{11}} \leq 2.228\right) &= 0.95 \\
P\left(\left|\frac{\bar{Y} - 15.0}{S/\sqrt{11}}\right| \geq 2.228\right) &= 0.05 \\
\boldsymbol{k} &\boldsymbol{= 2.228}
\end{aligned}
$$

Question 7.4.9

Solution

(a) First, we calculate $\bar{y}$ as follows

$$
\begin{aligned}
\bar{y} &= \frac{1}{12}(40 + 34 + 23 + 40 + 31 + 33 + 49 + 33 + 34 + 43 + 26 + 39) \\
&= 35.41
\end{aligned}
$$

Applying **5.4.1**, we calculate samlpe standard deviation S as follows

$$
\begin{aligned}
S &= \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2} \\
&= \sqrt{\frac{1}{12-1}\sum_{i=1}^{12}(y_i - 35.41)^2} \\
&= 7.23
\end{aligned}
$$

Applying **7.4.1**, we construct a 95% confidence interval as follows

$$
\begin{aligned}
P\big(-t_{\alpha/2,n-1} \le \frac{\bar{Y}-\mu}{S/\sqrt{n}} \le t_{\alpha/2,n-1}\big) &= 1-\alpha \\
P\big(-t_{0.025,11} \le \frac{\bar{Y}-35.41}{7.23/\sqrt{12}} \le t_{0.025,11}\big) &= 0.95 \\
P\big(-2.201 \le \frac{\bar{Y}-35.41}{7.23/\sqrt{12}} \le 2.201\big) &= 0.95 \\
P\big(\big|\frac{\bar{Y}-35.41}{7.23/\sqrt{12}}\big| \ge 2.201\big) &= 0.05 \\
\big|\frac{\bar{Y}-35.41}{7.23/\sqrt{12}}\big| &\ge 2.201 \\
\mathbf{30.81 \le \bar{Y} \le 40}
\end{aligned}
$$

Thus, for the 95% confidence interval above, scientiests do their best job between 30.81 to 40 years old.

(b) The graph of discovery by age over the year of discovery is as follows

From the graph above, the variability in the y_i's appears to be random with respect to time.

Question 7.4.19

Solution

First, we calculate $\bar{y}$ as follows

$$
\begin{aligned}
\bar{y} &= \frac{1}{15}(35+37+33+34+38+40+35+36+38+33+28+34+47+42+46) \\
&= 37.067
\end{aligned}
$$

Applying **5.4.1**, we calculate samlpe standard deviation S as follows

$$\begin{aligned} S &= \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2} \\ &= \sqrt{\frac{1}{15-1}\sum_{i=1}^{15}(y_i - 37.067)^2} \\ &= 5.049 \end{aligned}$$

We can construct hypothesis as follows

$$\begin{aligned} H_0 &: \mu = 40 \\ H_1 &: \mu < 40 \\ n &= 15 \\ \bar{y} &= 37.067 \\ S &= 5.049 \\ \alpha &= 0.05 \\ t_{\alpha,n-1} &= t_{0.05,14} = 1.7613 \end{aligned}$$

From **Theorem 7.4.2**, we can test the hypothesis above as follows

$$\begin{aligned} \text{Reject } H_0 \text{ if } t &< -t_{\alpha,n-1} \\ \text{where } t &= \frac{\bar{y}-\mu_0}{S/\sqrt{n}} = \frac{37.067-40}{5.049/\sqrt{15}} = -2.25 \\ t &< -t_{\alpha,n-1} \end{aligned}$$

Thus, we **reject H_0**.

Question 7.5.13
If a 90% confidence interval for σ^2 is reported to be (51.47, 261.90), what is the value of the sample standard deviation?
Solution
From **Theorem 7.5.1**, we know that

$$\begin{aligned} (51.47, 261.90) &= \left(\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}}, \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}\right) \\ (51.47, 261.90) &= \left(\frac{(n-1)S^2}{\chi^2_{0.05,n-1}}, \frac{(n-1)S^2}{\chi^2_{0.95,n-1}}\right) \\ \frac{\chi^2_{0.95,n-1}}{\chi^2_{0.05,n-1}} &= \frac{261.90}{51.47} \\ \frac{\chi^2_{0.95,n-1}}{\chi^2_{0.05,n-1}} &= 5.088 \end{aligned}$$

From **Table A3**, $n-1=9$ with $\frac{16.919}{3.325}=5.088$.
Thus, sample standard deviation S can be calculated as follows

$$\frac{(n-1)S^2}{\chi^2_{0.95,n-1}} = 51.47$$
$$S^2 = \frac{51.47 \times 16.919}{9}$$
$$S^2 = 96.76$$
$$\boldsymbol{S = 9.84}$$

Question 7.5.16
Standard deviation: $\sigma = 1.0$
Hypothesis: $H_0: \sigma^2 = 1$ versus $H_1: \sigma^2 > 1$
Level of significance: $\alpha = 0.05$
$\sum_{i=1}^{30} y_1 = 758.2$
$\sum_{i=1}^{30} y_1^2 = 19195.7938$
Solution
Applying **7.5**, we can calculate the sample variance S^2 as follows

$$\begin{aligned} S^2 &= \frac{30 \times 19195.7938 - 758.2^2}{30(30-1)} \\ &= 0.425 \end{aligned}$$

From **Theorem 7.5.2 (b)**, the critical value for the chi square ratio is

$$\chi^2_{1-\alpha,n-1} = \chi^2_{0.95,29} = 42.557$$

We can calculate the test statistic as follows

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2}$$
$$\chi^2 = \frac{29 \times 0.425}{1}$$
$$\chi^2 = 12.325$$
$$\chi^2 \le \chi^2_{1-\alpha,n-1} = 42.557$$

Thus, we **accept $\boldsymbol{H_0}$**