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Statistical Methods and Data Analysis (EN.625.603)
Problem Set 8

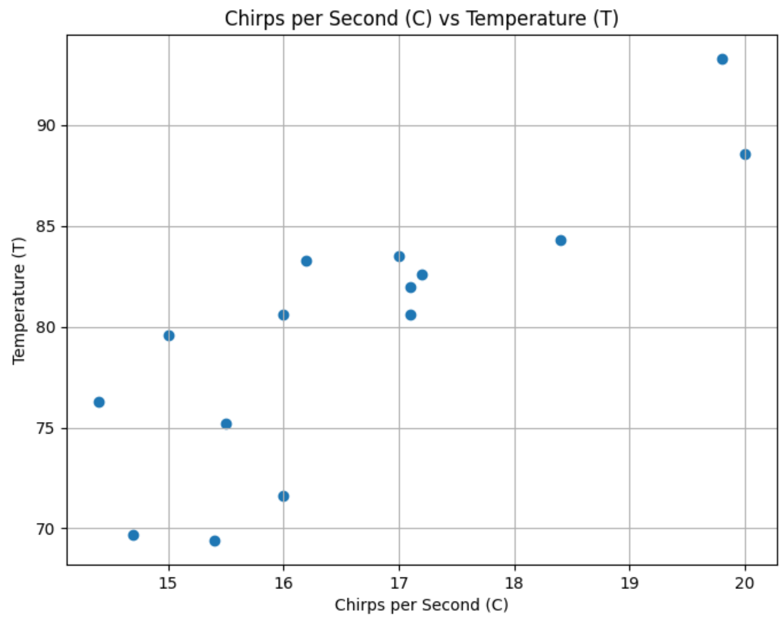
Question 11.2.1

Observation No.	Chirps per Second (x)	Temperature (y)
1	20.0	88.6
2	16.0	71.6
3	19.8	93.3
4	18.4	84.3
5	17.1	80.6
6	15.5	75.2
7	14.7	69.7
8	17.1	82.0
9	15.4	69.4
10	16.2	83.3
11	15.0	79.6
12	17.2	82.6
13	16.0	80.6
14	17.0	83.5
15	14.4	76.3

$$\sum_{i=1}^{15} x_1 = 249.8 \qquad \sum_{i=1}^{15} x_i^2 = 4200.56$$
$$\sum_{i=1}^{15} y_i = 1200.6 \qquad \sum_{i=1}^{15} x_i y_i = 20127.47$$

Solution
Using **Theorem 11.2.1**, we can calculate the slope as follows:

$$\begin{aligned}
b &= \frac{n\sum_1^{15} x_i y_i - \sum_1^{15} x_i \sum_1^{15} y_i}{n\sum_1^{15} x_i^2 - (\sum_1^{15} x_i)^2} \\
&= \frac{15(20127.47) - (249.8)(1200.6)}{15(4200.56) - (249.8)^2} = 3.291 \\
a &= \frac{\sum_1^{15} y_i}{n} - b\frac{\sum_1^{15} x_i}{n} \\
&= \frac{1200.6}{15} - (3.291)\frac{249.8}{15} = 25.234 \\
y(18) &= 25.234 + (3.291)(18) = 84.472
\end{aligned}$$



Question 11.2.2

Age, x	Proof, y
0	104.6
0.5	104.1
1	104.4
2	105
3	106
4	106.8
5	107.7
6	108.7
7	110.6
8	112.1

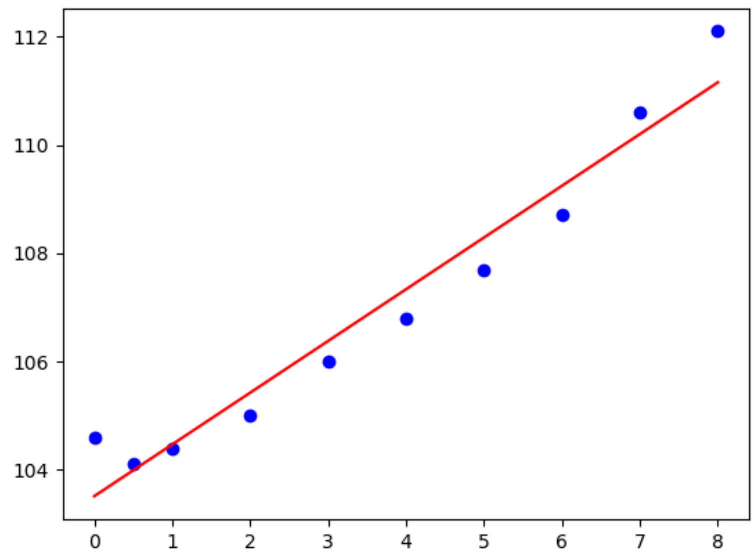
$$\sum_{i=1}^{10} x_i = 36.5 \qquad \sum_{i=1}^{10} x_i^2 = 204.25$$

$$\sum_{i=1}^{10} y_i = 1070 \qquad \sum_{i=1}^{10} x_i y_i = 3973.35$$

Solution

Using **Theorem 11.2.1**, we can calculate the slope as follows:

$$\begin{aligned}
b &= \frac{n\sum_1^{10} x_i y_i - \sum_1^{10} x_i \sum_1^{10} y_i}{n\sum_1^{10} x_i^2 - (\sum_1^{10} x_i)^2} \\
&= \frac{10(3973.35) - (36.5)(1070)}{10(204.25) - (36.5)^2} \\
&= 0.955 \\
a &= \frac{\sum_1^{10} y_i}{n} - b\frac{\sum_1^{10} x_i}{n} \\
&= \frac{1070}{10} - (0.955)\frac{36.5}{10} \\
&= 103.514 \\
y &= 0.955x + 103.514
\end{aligned}$$



Question 11.2.3

Temperature, x	Parts Dissolved, y
0	66.7
4	71.0
10	76.3
15	80.6
21	85.7
29	92.9
36	99.4
51	113.6
68	125.1

$$\sum_{i=1}^{9} x_i = 234 \qquad \sum_{i=1}^{9} x_i^2 = 10144$$

$$\sum_{i=1}^{9} y_i = 811.3 \qquad \sum_{i=1}^{9} x_i y_i = 24628.6$$

Solution
Using **Theorem 11.2.1**, we can calculate the slope as follows:

$$\begin{aligned}
b &= \frac{n\sum_1^9 x_i y_i - \sum_1^9 x_i \sum_1^9 y_i}{n\sum_1^9 x_i^2 - (\sum_1^9 x_i)^2} \\
&= \frac{9(24628.6) - (234)(811.3)}{9(10144) - (234)^2} \\
&= 0.871 \\
a &= \frac{\sum_1^9 y_i}{n} - b\frac{\sum_1^9 x_i}{n} \\
&= \frac{811.3}{9} - (0.871)\frac{234}{9} \\
&= 67.498 \\
y &= 0.871x + 67.498
\end{aligned}$$

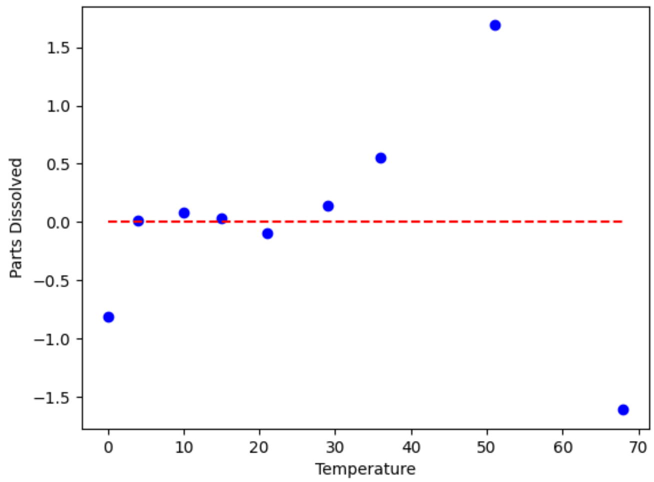
Using Definition 11.2.1, we can calculate the residuals as follows:

$$\begin{aligned}
r_i &= y_i - \hat{y}_i \\
&= y_i - (0.871)x_i - 67.498 \\
r_1 &= 66.7 - (0.871)(0) - 67.498 = -0.798 \\
r_2 &= 71.0 - (0.871)(4) - 67.498 = 0.018 \\
r_3 &= 76.3 - (0.871)(10) - 67.498 = 0.092 \\
r_4 &= 80.6 - (0.871)(15) - 67.498 = 0.037 \\
r_5 &= 85.7 - (0.871)(21) - 67.498 = -0.089 \\
r_6 &= 92.9 - (0.871)(29) - 67.498 = 0.143 \\
r_7 &= 99.4 - (0.871)(36) - 67.498 = 0.546 \\
r_8 &= 113.6 - (0.871)(51) - 67.498 = 1.681 \\
r_9 &= 125.1 - (0.871)(68) - 67.498 = -1.626
\end{aligned}$$

Question 11.2.7

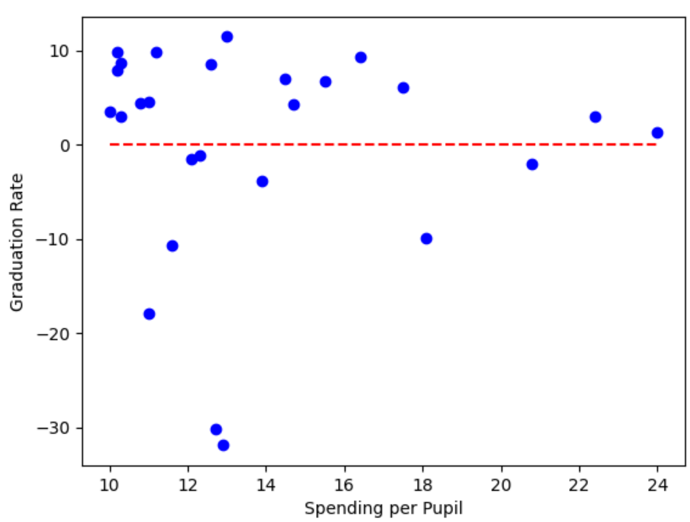
$$\sum_{i=1}^{26} x_i = 360 \qquad \sum_{i=1}^{26} x_i^2 = 5365.08$$

$$\sum_{i=1}^{10} y_i = 2256.6 \qquad \sum_{i=1}^{26} x_i y_i = 31402$$



Solution
From **Theorem 11.2.1**, we can calculate the slope as follows:

$$
\begin{aligned}
b &= \frac{n\sum_1^{26} x_i y_i - \sum_1^{26} x_i \sum_1^{26} y_i}{n\sum_1^{26} x_i^2 - (\sum_1^{26} x_i)^2} \\
&= \frac{26(31402) - (360)(2256.6)}{26(5365.08) - (360)^2} \\
&= 0.412 \\
a &= \frac{\sum_1^{26} y_i}{n} - b\frac{\sum_1^{26} x_i}{n} \\
&= \frac{2256.6}{26} - (0.412)\frac{360}{26} \\
&= 81.088 \\
y &= 0.412x + 81.088
\end{aligned}
$$



Question 11.2.13
Prove that a least squares straight line must necessarily pass through the point $(\bar{x}, \bar{y})$.
Solution
Proof as follows:

$$\begin{aligned}
\bar{y} &= \frac{\sum_{i=1}^{n} y_i}{n} \\
&= \frac{\sum_{i=1}^{n}(ax_i + b)}{n} \\
&= \frac{a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} 1}{n} \\
&= \frac{a\sum_{i=1}^{n} x_i + bn}{n} \\
&= a\bar{x} + b
\end{aligned}$$

Question 11.2.14
In some regression situation, there are a priori reasons for assuming that the xy-relationship being approximated passes through the origin. If so, the equation to be fit to the $(x_i, y_i)'s$ has the form $y = bx$. Use the least squares criterio to show that the "best" slope in that case is given by

$$b = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$

Solution
Proof as follows:

$$\begin{aligned}
\sum_{i=1}^{n} r_i^2 &= \sum_{i=1}^{n}(y_i - bx_i)^2 \\
&= \sum_{i=1}^{n}(y_i^2 - 2bx_i y_i + b^2 x_i^2) \\
&= \sum_{i=1}^{n} y_i^2 - 2b\sum_{i=1}^{n} x_i y_i + b^2\sum_{i=1}^{n} x_i^2 \\
\frac{d}{db}\sum_{i=1}^{n} r_i^2 &= -2\sum_{i=1}^{n} x_i y_i + 2b\sum_{i=1}^{n} x_i^2 \\
\frac{d}{db}\sum_{i=1}^{n} r_i^2 &= 0 \\
-2\sum_{i=1}^{n} x_i y_i + 2b\sum_{i=1}^{n} x_i^2 &= 0 \\
2b\sum_{i=1}^{n} x_i^2 &= 2\sum_{i=1}^{n} x_i y_i \\
b &= \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}
\end{aligned}$$

Question 11.3.2
The best straight line through the Massachusetts funding/graduation rate described in Question 11.2.7 has the equation $y = 81.088 + 0.412x$, where $s = 11.78848$.

(a) Construct a 95% confident interval for β_1.

(b) What does your answer to part (a) imply about the outcome of testing $H_0 : \beta_1 = 0$ versus $H_1 : \beta_1 \neq 0$? at the $\alpha = 0.05$ level of significance?

(c) Graph teh data and superimpose the regressionline. How would you summarize these data, and their implicatios, to a meeting of the state School Board?

Solution

(a) From Question 11.2.7,

$$\sum_{i=1}^{26} x_i = 360 \qquad \sum_{i=1}^{26} x_i^2 = 5365.08$$

$$\sum_{i=1}^{10} y_i = 2256.6 \qquad \sum_{i=1}^{26} x_i y_i = 31402$$

$$\begin{aligned}\hat{\beta}_0 &= 81.088\\ \hat{\beta}_1 &= 0.412\\ y &= 0.412x + 81.088\end{aligned}$$

First, we can find the degrees of freedom and t random variable as follows

$$\begin{aligned} df &= n-2\\ &= 24\\ t_{\alpha/2,n-2} &= t_{0.025,24}\\ &= 2.064\end{aligned}$$

Using **Theorem 11.3.6**, we can calculate the 95% confidence interval as follows:

$$\begin{aligned} &= \hat{\beta}_1 \pm t_{\alpha/2,n-2}\frac{s}{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2}}\\ &= 0.412 \pm 2.064\frac{11.78848}{\sqrt{380.46}}\\ &= (-0.835, 1.659)\end{aligned}$$

(b) Since 0 is in the confidence interval, we fail to reject the null hypothesis.

(c) As we shown in the graph above in the solution to question 11.2.7, there is no strong linear relationship between the funding and graduation rate. The regression line is not a good fit for the data.

Question 11.3.14

Construct a 90% confidence interval for σ^2 in the cigarette-consumption/CHD mortality data given in Case Study 11.3.1.

Solution

From Case Study 11.3.1, we have

$$\begin{aligned}\beta_0 &= 15.771\\ \beta_1 &= 0.0601\\ s^2 &= 2181.588\end{aligned}$$

First we can find the degrees of freedom and χ^2 random variable as follows

$$\begin{aligned} df &= n - 2 \\ &= 19 \\ \chi^2_{\alpha/2,n-2} &= \chi^2_{0.05,19} \\ &= 10.117 \\ \chi^2_{1-\alpha/2,n-2} &= \chi^2_{0.95,19} \\ &= 30.144 \end{aligned}$$

Using **drawing interferences about** σ^2, we can calculate the 90% confidence interval as follows

$$\begin{aligned} &= \left(\frac{(n-2)s^2}{\chi^2_{1-\alpha/2,n-2}}, \frac{(n-2)s^2}{\chi^2_{\alpha/2,n-2}} \right) \\ &= (1375.072, 4097.081) \end{aligned}$$

Question 11.3.16

$$\begin{aligned} y &= -0.104 + 0.988x \\ \beta_0 &= -0.104 \\ \beta_1 &= 0.988 \end{aligned}$$

Solution

(a) For $x = 14$,

$$\begin{aligned} \hat{y} &= -0.104 + 0.988(14) \\ &= 13.728 \end{aligned}$$

We can calculate degress of freedom and t random variable as follows

$$\begin{aligned} df &= 18 - 2 \\ &= 16 \\ t_{\alpha/2,n-2} &= t_{0.025,16} \\ &= 2.1199 \end{aligned}$$

Applying **Theorem 11.3.8**, we can calculate the 95% confidence interval as follows

$$\begin{aligned} w &= t_{\alpha/2,n-2} \sqrt{ s^2 \left(\frac{1}{n} + \frac{(x - \bar{x})^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \right) } \\ w &= 0.109 \\ CI_{95\%} &= = 13.728 \pm 0.109 \\ &= (13.619, 13.837) \end{aligned}$$

(b) Using the result from (a), we can construct the $CI_{95\%}$ prediction interval as follows

$$\begin{aligned} w &= t_{\alpha/2,n-2} \sqrt{ s^2 \left(1 + \frac{1}{n} + \frac{(x - \bar{x})^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \right) } \\ w &= 0.442 \\ CI_{95\%} &= = 13.728 \pm 0.442 \\ &= (13.286, 14.170) \end{aligned}$$

Question 11.3.17 Construct a 95% confidence interval $E(Y|2.750)$ using the connecting rod data given in Case Study 11.2.1.

$$\sum_{i=1}^{25} x_i = 66.075 \qquad \sum_{i=1}^{25} x_i^2 = 174.672925$$
$$\sum_{i=1}^{25} y_i = 50.12 \qquad \sum_{i=1}^{25} y_i^2 = 100.49865$$
$$\sum_{i=1}^{25} x_i y_i = 132.490725$$
$$b = 0.642 \qquad a = 0.308$$

Solution
For $x = 2.750$,

$$\begin{aligned}\hat{y} &= 0.308 + 0.642(2.750)\\ &= 2.0735\end{aligned}$$

We can calculate the degrees of freedom and t random variable as follows

$$\begin{aligned} df &= 25 - 2\\ &= 23\\ t_{\alpha/2,n-2} &= t_{0.025,23}\\ &= 2.069\end{aligned}$$

We can calculate sample variance as follows

$$\begin{aligned} s^2 &= \frac{1}{n-2}\left[\sum_{i=1}^{n} y_i^2 - b\sum_{i=1}^{n} x_i y_i\right]\\ &= 0.0001149\end{aligned}$$

We can calculate the 95% confidence interval as follows

$$\begin{aligned} w &= t_{\alpha/2,n-2}\sqrt{s^2\left(\frac{1}{n} + \frac{(x-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right)}\\ w &= 0.013\\ CI_{95\%} &= = 2.074 \pm 0.013\\ &= (2.061, 2.087)\end{aligned}$$

Question 11.4.12

$$\sum_{i=1}^{30} x_i = 1300.69 \qquad \sum_{i=1}^{30} y_i = 323$$
$$\sum_{i=1}^{30} x_i^2 = 86754.6939 \qquad \sum_{i=1}^{30} y_i^2 = 11881$$
$$\sum_{i=1}^{30} x_i y_i = 7807.36$$

Solution

Using **sample correlation coefficient** formula, we can calculate the sample correlation coefficient as follows

$$
\begin{aligned}
R &= \frac{n\sum_{i=1}^{n} X_iY_i - \sum_{i=1}^{n} X_i \sum_{i=1}^{n} Y_i}{\sqrt{n\sum_{i=1}^{n} X_i^2 - \left(\sum_{i=1}^{n} X_i\right)^2}\sqrt{n\sum_{i=1}^{n} Y_i^2 - \left(\sum_{i=1}^{n} Y_i\right)^2}} \\
&= \frac{30(7807.36) - (1300.69)(323)}{\sqrt{30(86754.6939) - (1300.69)^2}\sqrt{30(11881) - (323)^2}} \\
&= -0.388
\end{aligned}
$$

It suggests that increasing bonus is associated with decreasing performance.