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**Matrix Theory (EN.625.609)**
Final Exam

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**Problem 1**
Let $A \in \mathbb{C}^{n\times n}$ be an invertible matrix.

(a) Prove that $A^*$ is invertible and $(A^*)^{-1} = (A^{-1})^*$.

(b) Define $B = A^{-1}A^*$. Prove that $B$ is unitary if and only if $A$ is normal.

**Solution:**

(a) *This problem was previously asked in* ***Problem 5(b), Module 3 Assignment.***

i. By property of identity matrix, $I = I^*$.

ii. By properties of invertible matrix, given that $A$ is invertible, there exists $A^{-1}$ such that $AA^{-1} = A^{-1}A = I$.

iii. By laws of Matrix Transpose $(AA^{-1})^* = (A^{-1})^*A^*$.

From (i), (ii), (iii)

$$\begin{aligned}(AA^{-1}) &= I\\ (AA^{-1})^* &= I^* = I\\ (A^{-1})^*A^* &= I\end{aligned}$$

Thus, by definition of invertible matrix, $A^*$ is invertible and $(A^*)^{-1} = (A^{-1})^*$.

(b) i. We show that $B$ is unitary implies $A$ is normal as follows:

$$\begin{aligned}B &= A^{-1}A^* && \text{from the definition of } B\\ BB^* = (A^{-1}A^*)(A^{-1}A^*)^* &= I && \text{definition of unitary matrix}\\ A^{-1}A^*A(A^{-1})^* &= I && \text{laws of Matrix Conjugate Transpose}\\ A^{-1}A^*A(A^*)^{-1} &= I && \text{result from (a)}\\ A^{-1}A^*A(A^*)^{-1}A^* &= A^* && \text{multiples both sides by } A^*\\ (A^{-1}A^*A) &= A^* && (A^*)^{-1}A^* = I\\ AA^{-1}A^*A &= AA^* && \text{multiples } A \text{ by both RHS \& LHS}\\ A^*A &= AA^* && \text{property of inverse matrix}\end{aligned}$$

Thus $A$ is normal by the definition of normal matrix.

ii. We show that $A$ is normal implies $B$ is unitary as follows:

(*) We show that $BB^* = I$ as follows

$$\begin{aligned}BB^* &= (A^{-1}A^*)(A^{-1}A^*)^* && \text{from the definition of } B\\ BB^* &= A^{-1}A^*A(A^{-1})^* && \text{laws of Matrix Conjugate Transpose}\\ BB^* &= A^{-1}AA^*(A^{-1})^* && \text{use the definition of normal matrix } A^*A = AA^*\\ BB^* &= I && (1)\end{aligned}$$

(*) We show that $B^*B = I$ as follows

$$\begin{aligned}
B^*B &= (A^{-1}A^*)^*(A^{-1}A^*) && \text{from the definition of } B \\
B^*B &= A(A^{-1})^*A^{-1}A^* && \text{laws of Matrix Conjugate Transpose} \\
(B^*B)^{-1} &= (A^*)^{-1}AA^*A^{-1} && \text{laws of Matrix Inverse \& result from (a)} \\
(B^*B)^{-1} &= (A^*)^{-1}A^*AA^{-1} && \text{use the definition of normal matrix } A^*A = AA^* \\
(B^*B)^{-1} &= I && \text{property of inverse matrix} \\
B^*B &= I^{-1} = I && (2)
\end{aligned}$$

(1), (2) implies that $B$ is an unitary matrix by definition.

(i), (ii) implies that $B$ is an unitary matrix if and only if $A$ is a normal matrix.

**Problem 2**

Consider the linear transformation $T : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^{4\times 1}$ defined by

$$T(a + bt + ct^2) = \begin{bmatrix} 2a + 10b + 8c \\ a - c \\ 3b + 3c \\ -b - c \end{bmatrix}$$

(a) Find an orthogonal basis $B$ for $R(T)$.

(b) Find the best approximation for $(1, 1, -1, 1)^T$ in $R(T)$.

(c) Find a least squares solution of $T(x) = (1, 1, -1, 1)^T$.

(d) Find an orthogonal basis $C$ for $\mathbb{R}^{4\times 1}$ such that $B$ is a subset of $C$.

**Solution:**

(a) First, we find the basis for $R(T)$ using the same procedure as in ***Problem 2(b), Module 3 Assignment*** as follows:
Consider the standard basis for $\mathcal{P}_2$. We can compute

$$T(1) = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad T(t) = \begin{bmatrix} 10 \\ 0 \\ 3 \\ -1 \end{bmatrix}, \qquad T(t^2) = \begin{bmatrix} 8 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

Thus, $R(T) = \text{span}((2, 1, 0, 0)^T, (10, 0, 3, -1)^T, (8, -1, 3, -1)^T)$.
Note that

$$\begin{bmatrix} 8 \\ -1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

so we can remove the $3^{rd}$ vector and the resulting set will still span $R(T)$. We claim this set is linearly independent - they have different indices for the zero entries. And as it spans $R(T)$ and is linearly independent, it is a basis of $R(T)$.

We now can applying the Gram-Schmidt orthogonalization as follows

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} 10 \\ 0 \\ 3 \\ -1 \end{bmatrix} - \frac{\langle (10,0,3,-1)^T, (2,1,0,0)^T \rangle}{\langle (2,1,0,0)^T, (2,1,0,0)^T \rangle}(2,1,0,0)^T$$

$$\mathbf{v}_2 = \begin{bmatrix} 2 \\ -4 \\ 3 \\ -1 \end{bmatrix}$$

Thus $B = \{(2,1,0,0)^T, (2,-4,3,-1)^T\}$ an orthogonal basis for $R(T)$.

(b) Using the result from (a), $U = \{(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0, 0)^T, (\frac{2}{\sqrt{30}}, \frac{-4}{\sqrt{30}}, \frac{3}{\sqrt{30}}, \frac{-1}{\sqrt{30}})^T\}$ is an orthonormal basis of $R(T)$. Let $\mathbf{v} = (1,1,-1,1)^T$, we can find $\text{proj}_U(\mathbf{v})$ using ***Definition 4.18*** as follows

$$\begin{aligned} \text{proj}_U(\mathbf{v}) =& \langle (1,1,-1,1)^T, (\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0, 0)^T \rangle (\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0, 0)^T + \\ & \langle (1,1,-1,1)^T, (\frac{2}{\sqrt{30}}, \frac{-4}{\sqrt{30}}, \frac{3}{\sqrt{30}}, \frac{-1}{\sqrt{30}})^T \rangle (\frac{2}{\sqrt{30}}, \frac{-4}{\sqrt{30}}, \frac{3}{\sqrt{30}}, \frac{-1}{\sqrt{30}})^T \\ \text{proj}_U(\mathbf{v}) =& (\frac{6}{5}, \frac{3}{5}, 0, 0)^T + (\frac{-2}{5}, \frac{4}{5}, \frac{-3}{5}, \frac{1}{5})^T \\ \text{proj}_U(\mathbf{v}) =& (\frac{4}{5}, \frac{7}{5}, \frac{-3}{5}, \frac{1}{5})^T \end{aligned}$$

Thus $(\frac{4}{5}, \frac{7}{5}, \frac{-3}{5}, \frac{1}{5})^T$ is the best approximation of $\mathbf{v}$ from $R(T)$ by ***Theorem 4.40***.

(c) Choose standard bases $P = \{e_1, e_2, e_3\}$ for $\mathcal{P}_2(\mathbb{R})$ and $E = \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$. Find the matrix representation of the linear transformation $T$ with respect to the ordered bases $B$ and $C$, denoted by $[T]_{E,P}$ using the same procedure as ***Example 2.39*** as follows

$$\begin{aligned} T(e_1) &= (2,1,0,0)^T = 2\tilde{e}_1 + \tilde{e}_2 \\ T(e_2) &= (10,0,3,-1)^T = 10\tilde{e}_1 + 3\tilde{e}_3 - \tilde{e}_4 \\ T(e_3) &= (8,-1,3,-1)^T = 8\tilde{e}_1 - \tilde{e}_2 + 3\tilde{e}_3 - \tilde{e}_4 \end{aligned}$$

Therefore

$$[T]_{E,P} = \begin{bmatrix} 2 & 10 & 8 \\ 1 & 0 & -1 \\ 0 & 3 & 3 \\ 0 & -1 & -1 \end{bmatrix}$$

From (a), we know that the columns of $[T]_{E,P}$ are not linearly independent and $T$ is not an injective transformation. Thus, the least squares solution is not unique by ***Theorm 4.45***..

Let $[A]_{E,P'} = \begin{bmatrix} 2 & 10 \\ 1 & 0 \\ 0 & 3 \\ 0 & -1 \end{bmatrix}$, where $P' = \{e_1, e_2\}$ is a standard base for $\mathcal{P}_1$ . Because the first two columns of $[T]_{E,P}$ spans $R(T)$, we can fix $c = 0$ and let $u' = a + bt$ and the following holds for all $a, b \in \mathbb{R}$

$$[A]_{E,P'}(a,b)^T = [T]_{E,P}(a,b,0)^T$$

Thus, the pair of $a, b$ which is the least squares solution of $A(x') = (1,1,-1,1)^T$ is also a least squares solution of $T(x) = (1,1,-1,1)^T$. Now we can use the same approach as ***Problem 4, Module 9*** to solve for the $QR$-factorization of $[A]_{E,P'}$.

i. First we find $Q^*$ by using the result from (b) as follows

$$Q = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{5}} & \frac{-4}{\sqrt{30}} \\ 0 & \frac{3}{\sqrt{30}} \\ 0 & \frac{-1}{\sqrt{30}} \end{bmatrix}$$

Since all entries of $Q$ are real, $Q^* = Q^T$. That is,

$$Q^* = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 & 0 \\ \frac{2}{\sqrt{30}} & \frac{-4}{\sqrt{30}} & \frac{3}{\sqrt{30}} & \frac{-1}{\sqrt{30}} \end{bmatrix}$$

ii. Second we find $R^{-1}$ by using the $\mathbf{a_1}, \mathbf{a_2}$ as columns of $[A]_{E,P'}$ and $\mathbf{q_1}, \mathbf{q_2}$ are columns of $Q$ as follows

$$\begin{aligned} R &= \begin{bmatrix} \langle \mathbf{a_1}, \mathbf{q_1} \rangle & \langle \mathbf{a_2}, \mathbf{q_1} \rangle \\ 0 & \langle \mathbf{a_2}, \mathbf{q_2} \rangle \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{5} & 4\sqrt{5} \\ 0 & \sqrt{30} \end{bmatrix} \end{aligned}$$

To compute $R^{-1}$ we augment $R$ with the $2 \times 2$ identity matrix and reduce to row reduced echelon form to get

$$\begin{bmatrix} \sqrt{5} & 4\sqrt{5} & 1 & 0 \\ 0 & \sqrt{30} & 0 & 1 \end{bmatrix} \xrightarrow{E_{1/\sqrt{5}}[1]} \begin{bmatrix} 1 & 4 & \frac{1}{\sqrt{5}} & 0 \\ 0 & \sqrt{30} & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{E_{1/\sqrt{30}}[2]} \begin{bmatrix} 1 & 4 & \frac{1}{\sqrt{5}} & 0 \\ 0 & 1 & 0 & 1/\sqrt{30} \end{bmatrix} \xrightarrow{E_{-4}[1,2]} \begin{bmatrix} 1 & 0 & \frac{1}{\sqrt{5}} & -4/\sqrt{30} \\ 0 & 1 & 0 & 1/\sqrt{30} \end{bmatrix}$$

iii. This gives us the least squares solution

$$(a,b)^T = R^{-1}Q^*(1,1,-1,1)^T$$

$$(a,b)^T = \begin{bmatrix} 1/\sqrt{5} & -4/\sqrt{30} \\ 0 & 1/\sqrt{30} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 & 0 \\ \frac{2}{\sqrt{30}} & \frac{-4}{\sqrt{30}} & \frac{3}{\sqrt{30}} & \frac{-1}{\sqrt{30}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$(a,b)^T = \begin{bmatrix} 2/15 & 11/15 & -2/5 & 2/15 \\ 1/15 & -2/15 & 1/10 & -1/30 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$(a,b)^T = \begin{bmatrix} 7/5 \\ -1/5 \end{bmatrix}$$

Thus $x = \frac{7}{5} - \frac{1}{5}t$ is a least square solutions of $T(x) = (1,1,-1,1)^T$

(d) From (a), the two vectors of $B = \{(2,1,0,0)^T, (2,-4,3,-1)^T\}$ will be the first two vectors $\mathbf{c}_1, \mathbf{c}_2$ of $C$ as they are orthogonal vectors and $B$ is a subset of $C$. Let $\mathbf{c}_3 = (a,b,c,d)^T$ where $a,b,c,d \in \mathbb{R}$ be the third vector of $C$. Because $C$ is an orthogonal basis, $\langle \mathbf{c}_1, \mathbf{c}_3 \rangle = 0$ and $\langle \mathbf{c}_2, \mathbf{c}_3 \rangle = 0$ by ***Definition 4.10***. Thus, any non-zero solution to the following system of equations can be $\mathbf{c}_3$.

$$\begin{cases} 2a + b = 0 \\ 2a - 4b + 3c - d = 0 \end{cases}$$

$$\begin{cases} 2a = -b \\ -5b + 3c - d = 0 \end{cases}$$

Set $a = b = 0$ to solve for the first equation of the system, then $c = 1, d = 3$ will make the system of equations above consistent, therefore, $\mathbf{c}_3 = (0,0,1,3)^T$ can be the third vector of $C$. Again set $\mathbf{c}_4 = (a,b,c,d)^T$ where $a,b,c,d \in \mathbb{R}$ and applying the same logic as above

$$\begin{cases} 2a + b = 0 \\ 2a - 4b + 3c - d = 0 \\ c + 3d = 0 \end{cases}$$

$$\begin{cases} 2a = -b \\ -5b - 10d = 0 \\ c = -3d \end{cases}$$

Set $b = 2, d = -1$ to solve for the second equation of the system, then $a = -1, c = 3$ will make the system of equations above consistent, therefore $\mathbf{c}_4 = (-1,2,3,-1)^T$ can be the forth vector of $C$.
Therefore, $C = \{(2,1,0,0)^T, (2,-4,3,-1)^T, (0,0,1,3)^T, (-1,2,3,-1)^T\}$.

**Problem 3**
Let $A \in \mathbb{C}^{3\times 3}$ such that $A^3$ is the zero matrix but $A^2$ is not the zero matrix. Find the Jordan matrix similar to $A$.
**Solution:**
By ***Theorem 6.19*** we know that there exists an invertible matrix $S \in \mathbb{C}^{3\times 3}$ such that $J = S^{-1}AS$ where $J \in \mathbb{C}^{3\times 3}$ is a Jordan matrix.

$$A^3 = (S^{-1}JS)^3 = S^{-1}J^3S = 0$$

Thus, $J^3 = 0$. Check all the possibilities of possible cycles of Jordan blocks which has length less than or equal to 3 as follows

$$J(\lambda,1) = \begin{bmatrix}\lambda\end{bmatrix}, (J(\lambda,1))^2 = \begin{bmatrix}\lambda^2\end{bmatrix}, (J(\lambda,1))^3 = \begin{bmatrix}\lambda^3\end{bmatrix} \qquad (1)$$

$$J(\lambda,2) = \begin{bmatrix}\lambda & 1 \\ 0 & \lambda\end{bmatrix}, (J(\lambda,2))^2 = \begin{bmatrix}\lambda^2 & 2\lambda \\ 0 & \lambda^2\end{bmatrix}, (J(\lambda,2))^3 = \begin{bmatrix}\lambda^3 & 3\lambda^2 \\ 0 & \lambda^3\end{bmatrix} \qquad (2)$$

$$J(\lambda,3) = \begin{bmatrix}\lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda\end{bmatrix}, (J(\lambda,3))^2 = \begin{bmatrix}\lambda^2 & 2\lambda & 1 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2\end{bmatrix}, (J(\lambda,3))^3 = \begin{bmatrix}\lambda^3 & 3\lambda^2 & 3\lambda \\ 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & \lambda^3\end{bmatrix} = 0 \qquad (3)$$

(1) implies that if $J(\lambda,1)$ exists in J, then

i If $\lambda = 0$; which then implies that the corresponding block in $J^2, J^3$ will both be zero.

ii If $\lambda \neq 0$; which then implies that the corresponding block in $J^2, J^3$ will both be non zero.

(2) implies that if $J(\lambda, 2)$ exists in J,

i If $\lambda = 0$; which then implies that the corresponding block in $J^2, J^3$ will both be zero.

ii If $\lambda \neq 0$; which then implies that the corresponding block in $J^2, J^3$ will both be non zero.

Thus, if only cycles of length 1 or 2 exist in $J$, then either

i $J^2 = 0, J^3 = 0$, which implies $A^2 = 0, A^3 = 0$.

ii $J^2 \neq 0, J^3 \neq 0$, which implies $A^2 \neq 0, A^3 \neq 0$.

Thus, the only possibility for $J$ is that it only has 1 cycle of length 3. In that case, from (3), $J^3 = 0$ if and only if $\lambda = 0$. Thus,

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

**Problem 4**

Let $\mathbf{w} \in \mathbb{R}^{n\times 1}$ with $\|\mathbf{w}\|_2 = 3$. Define $A = I - \mathbf{w}\mathbf{w}^T$ where $I$ is the $n \times n$ identity matrix.

(a) Show that $\mathbf{w}$ is an eigenvector of $A$ and find its corresponding eigenvalue.

(b) Let $\mathbf{v} \in \mathbb{R}^{n\times 1}$ such that $\mathbf{v}$ is orthogonal to $\mathbf{w}$. Show that $\mathbf{v}$ is an eigenvalue of $A$ and find its corresponding eigenvalue.

(c) Find $\sum$ such that $Q \sum P^*$ is a singular value decomposition of $A$.

(d) Prove that $A$ is diagonalizable.

(e) Is $A$ invertible?

**Solution:**

(a) Let $\mathbf{w} = (w_1, w_2, ...w_n)$, then the entry of $i^{th}$ row and $j^{th}$ column will be $w_i w_j$, thus

$$A = I - \mathbf{w}\mathbf{w}^T$$

$$A = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix} - \begin{bmatrix} w_1w_1 & w_1w_2 & \dots & w_1w_{n-1} & w_1w_n \\ w_2w_1 & \ddots & \ddots & \ddots & w_2w_n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ w_{n-1}w_1 & \ddots & \ddots & \ddots & w_{n-1}w_n \\ w_nw_1 & w_nw_2 & \dots & w_nw_{n-1} & w_nw_n \end{bmatrix}$$

$$A = \begin{bmatrix} 1 - w_1w_1 & -w_1w_2 & \dots & -w_1w_{n-1} & -w_1w_n \\ -w_2w_1 & \ddots & \ddots & \ddots & -w_2w_n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -w_{n-1}w_1 & \ddots & \ddots & \ddots & -w_{n-1}w_n \\ -w_nw_1 & -w_nw_2 & \dots & -w_nw_{n-1} & 1 - w_nw_n \end{bmatrix}$$

$$A\mathbf{w} = \begin{bmatrix} (w_1 - w_1^3) - (w_1w_2^2) - \dots - (w_1w_{n-1}^2) - (w_1w_n^2) \\ \vdots \\ \vdots \\ \vdots \\ -w_nw_1^2 - (w_nw_2^2) - \dots - (w_nw_{n-1}^2) - (w_n - w_n^3) \end{bmatrix}$$

$$A\mathbf{w} = \begin{bmatrix} w_1(1 - (w_1^2 + \dots w_n^2)) \\ \vdots \\ \vdots \\ \vdots \\ w_n(1 - (w_1^2 + \dots w_n^2)) \end{bmatrix}$$

$$A\mathbf{w} = \begin{bmatrix} w_1(1 - 3^2) \\ \vdots \\ \vdots \\ \vdots \\ w_n(1 - 3^2) \end{bmatrix} = -8\mathbf{w}$$

Thus by ***Definition 5.1***, $\mathbf{w}$ is an eigenvector of $A$ and its corresponding eigenvalue is -8.

(b) From (a) we have

$$A\mathbf{v} = \begin{bmatrix} (v_1 - w_1(v_1w_1)) - w_1(v_2w_2) - \cdots - w_1(w_{n-1}v_{n-1}) - w_1(w_nv_n) \\ \vdots \\ \vdots \\ \vdots \\ w_n(v_1w_1) - w_n(v_2w_2) - \cdots - w_n(w_{n-1}v_{n-1}) + (v_n - w_n(w_nv_n)) \end{bmatrix}$$

$$A\mathbf{v} = \begin{bmatrix} v_1 - 0 \\ \vdots \\ \vdots \\ \vdots \\ v_n - 0 \end{bmatrix} = 1\mathbf{v} \qquad \textit{group the terms which are the dot product two orthogonal vectors}$$

Thus by ***Definition 5.1***, $\mathbf{v}$ is an eigenvector of $A$ and its corresponding eigenvalue is 1.

(c) i. From (a), we know that $A$ is a self-joint matrix, because for every pair of non-diagonal entries of $A$ where one entry is of (row $i^{th}$, column $j^{th}$) and one entry is of (row $j^{th}$, column $i^{th}$), both of them have the value $-w_iw_j$. Thus $A = A^*$ and $A$ is a self-joint matrix by ***Definition 7.1***.

ii. Also, we prove that the singular values of $A$ are precisely the absolute values of the eigenvalues of $A$, by reusing the proof from ***Problem 2, Module 13*** as follows *(precisely recite the proof from the solution)*
By ***Theorem 7.36***, $\text{eig}(A^2) = \{\lambda^2 : \lambda \in \text{eig(A)}\}$ . Thus the singular values of $A$ are the nonnegative square roots of $\lambda^2$ for each eigenvalue $\lambda$ of $A$, which is exactly $|\lambda|$ for each eigenvalue $\lambda$ of $A$.

iii. From (b), we know that every vector which is orthogonal to $\mathbf{w}$ is an eigenvector of $A$. Together with $\mathbf{w}$, the set of all orthorgonal vectors to $\mathbf{w}$ will span $A$. Thus the basis would have $n$ vectors including $\mathbf{w}$. Thus there are $n-1$ linearly independent vectors orthogonal to $\mathbf{w}$ and the eigenvalue of each of them is 1.

From (i), (ii), (iii) and along with ***Theorem 7.46***, we have $Z = \text{diag}(1, ..., 1, 8) \in \mathbb{R}^{n \times n}$.

(d) i. From (c), we know that $A$ is a self joint matrix, and by ***Definition 7.1***, $A$ is a normal matrix.

ii. From ***Theorem 7.35***, we know that every normal matrix is similar to a diagonal matrix.

(i), (i) imply that $A$ is a diagonalizable matrix by ***Definitin 6.1***.

(e) i. From (c), 0 is not an eigenvalue of $A$.

ii. From ***Eigenvalue Characterization of Invertibility Theorem, Module 10***, $A$ is invertible if and only if 0 is not an eigenvalue of $A$.

(i), (ii) imply that $A$ is invertible.