# ISyE/Math/CS 425 – Introduction to Combinatorial Optimization
## Fall 2025

## Assignment 3

Due date: Sunday, October 26 at 11:59pm.

*Instructions and policy*: This assignment contains two sections, one including mandatory exercises, and one with optional exercises, that serve for extra-practice. Handle in only the mandatory exercises. The assignments should be submitted electronically in Canvas. Late submission policy: 20% of total points will be deducted per hour.

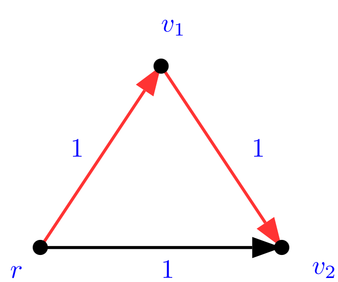
Students are strongly encouraged to work in groups of two on homework assignments. To find a partner you can search for teammates on Piazza. Only one file should be submitted for both group members. In order to submit the assignment for your group please follow these steps in Canvas: Step 1. Click on the "People" tab, then on "Assignment 3", and join one of the available groups for the assignment; Step 2. When also your partner has joined the same group, one of the two can submit the assignment by clicking on the "Assignments" tab, then on the assignment to be submitted, and finally on "Submit assignment". The submission will count for everyone in your group.

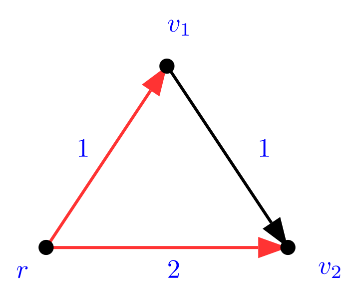
Groups must work independently of each other, may not share answers with each other, and solutions must not be copied from the internet or other sources. In any assignment, students must properly give credit to any outside resources they use (such as books, papers, etc.). If dishonest behavior on an assignment is suspected, all the students in the group will be issued a warning, and the instructor will consider conducting an oral evaluation in addition to the written exams, whose score will be averaged with the scores of the Midterm and Final exams.

## Mandatory exercises

**Exercise 1** ........................................................................... ***10 points***

Show by an example that a spanning directed tree rooted at $r$ can be of minimum cost but not contain least-cost dipaths to all nodes. Also show the converse, that it may contain least-cost dipaths to all nodes, but not be of minimum cost.




**Solution:**

In the first digraph, the spanning directed tree rooted at $r$ (in red) is of minimum cost but does not contain a least-cost dipath to $v_2$ (which would be $P = r, rv_2, v_2$).

In the second digraph, the spanning directed tree rooted at $r$ (in red) contain least-cost dipaths to $v_1$ and $v_2$, but is not of minimum cost (a minimum cost one would be $(V, \{rv_1, v_1v_2\})$).

**Exercise 2** ........................................................................... ***10 points***

Consider the shortest path problem for undirected graphs. Prove that if the costs can be assumed to be nonnegative, then this problem can be solved by reducing it to a shortest path problem in a digraph. When costs are allowed to be negative, what difficulty arises?

**Solution:** Let $G = (V, E)$ be our undirected graph, let $c$ be the nonnegative edge costs and let $r \in V$. We construct an instance $(G', c', r)$ of the shortest dipath problem by replacing each (undirected) edge $e = uv$ of cost $c_e$ with two (directed) arcs $uv$ and $vu$ both of cost $c_e$.

We first prove that for each *simple* path $P$ from $r$ to $v \in V$ in $G$ we can construct a dipath $P'$ from $r$ to $v$ in $G'$ with the same cost. We show this by induction on the number of edges of $P$. If $P$ consists of one edge $rv$, $P'$ will consist of arc $rv$ and the claim is true. Now assume that the claim is true for a simple path with $k$ edges and suppose that $P$ has $k + 1$ edges. Assume that the last edge of $P$ is $wv$ and consider the (simple) subpath of $P$ from $r$ to $w$. By the inductive hypothesis, we know that we can map this simple path to a dipath from $r$ to $w$ in $G'$ with the same cost. By appending to this dipath arc $wv$ our claim is proved.

Viceversa, for each *simple* dipath $P'$ from $r$ to $v$ in $G'$ we can construct a path $P$ from $r$ to $v$ in $G$ with the same cost. The proof is similar to the previous one. We show this by induction on the number of arcs of $P'$. If $P'$ consists of one arc $rv$, $P$ will consist of edge $rv$ and the claim is true. Now assume that the claim is true for a simple dipath with $k$ arcs and suppose that $P'$ has $k + 1$ arcs. Assume that the last arc of $P'$ is $wv$ and consider the (simple) sub-dipath of $P'$ from $r$ to $w$. By the inductive hypothesis, we know that we can map this simple dipath to a path from $r$ to $w$ in $G$ with the same cost. By appending to this path edge $wv$ our claim is proved.

What happens if $P$ or $P'$ are *not* simple? If the edge costs of $G$ are nonnegative, by construction also the arc costs of $G'$ are nonnegative. Consider a path $P$ from $r$ to $v$ in $G$ that is not simple. Then $P$ must contain a cycle of cost 0 and by deleting from $P$ the edges of all such cycles we obtain a simple path $\tilde{P}$ with the same cost of $P$. Then we map $\tilde{P}$ to a directed path $\tilde{P}'$ of $G'$ from $r$ to $v$ with the same cost. Similarly, if $P'$ is a dipath from $r$ to $v$ in $G'$ that is not simple, then $P'$ must contain a directed cycle of cost 0 and by deleting from $P'$ the arcs of all such directed cycles we obtain a simple dipath $\tilde{P}'$ with the same cost of $P'$. Then we map $\tilde{P}'$ to a path $\tilde{P}$ of $G$ from $r$ to $v$ with the same cost.

If $G$ contains a negative cost edge $uv$, the directed cycle in $G'$ given by $uv$ and $vu$ has negative cost, and we have infinitely many non-simple dipaths in $G'$ that cannot be mapped to paths of $G$ of lower or equal cost. This is because all paths of $G$ has nonnegative cost, while we have dipaths in $G'$ whose cost is negative.

**Exercise 3** ........................................................................... ***10 points***

We say that a node $v$ is *incident* to an arc $e$ if $v$ is the tail or the head of $e$. Suppose that we are given a shortest dipath problem on a digraph $G$ such that there is a node $w \neq r$ that is incident with exactly two arcs. Construct a digraph $G'$ on a subset of nodes of $V(G)$, and explain how the solution of the shortest dipath problem on $G'$ yields the solution of the shortest path problem on $G$. *Hint: consider separatly four cases, depending on the direction of the two arcs incident to $w$.*

**Solution:** Let $u$ and $v$ be the nodes adjacent to $w$ in $G$. Depending on the direction of the two arcs incident with $w$ we have four cases.

1. $wu, wv \in E$: Let $G'$ be the digraph obtained from $G$ by deleting $w$. Since no arc enters $w$, no shortest dipath can contain $w$. Therefore there is no dipath in $G$ from $r$ to $w$. For every

$v' \neq w$, the shortest dipath in $G$ from $r$ to $v'$ coincides with the shortest dipath in $G'$ from $r$ to $v'$;

2. $uw, vw \in E$: Let $G'$ be the digraph obtained from $G$ by deleting $w$. In this case no arc leaves $w$, thus the only shortest dipath that contains $w$ is the one from $r$ to $w$, and the next to last node of this dipath is either $u$ or $v$. Therefore, for every $v' \neq w$, the shortest dipath in $G$ from $r$ to $v'$ coincides with the shortest dipath in $G'$ from $r$ to $v'$. Let $P_u$, $P_v$ be the shortest dipaths in $G'$ from $r$ to $u$ and $v$, respectively. The shortest dipath in $G$ from $r$ to $w$ can be obtained by selecting the cheapest among dipaths $P_u, uw, w$ and $P_v, vw, w$ in $G$;

3. $uw, wv \in E$: Let $G'$ be the digraph obtained from $G$ by deleting $w$ and, if $u \neq v$, by adding arc $uv$ with cost $c_{uw} + c_{wv}$. If there was already an arc $uv$ in $G$, we keep only the one of minimum cost.

   Given a dipath $P'$ in $G'$ from $r$ to $v'$, we can construct a dipath in $G$ with the same cost by eventually replacing subpath $u, uv, v$ with $u, uw, w, wv, v$.

   Viceversa, let $P$ be a dipath in $G$ from $r$ to $v'$. We show how to obtain a dipath $P'$ in $G'$ from $r$ to $v'$ with cost at most that of $P$. If $P$ does not contain $w$, then $P' = P$ has the desired property. Otherwise, if $P$ contains $w$, it must contain subpath $u, uw, wv, v$, thus the corresponding dipath $P'$ obtained by replacing such subpath with $u, uv, v$ has cost at most that of $P$.

   The shortest dipath in $G$ from $r$ to $w$ is $P_u, uw$, where $P_u$ is the shortest dipath in $G'$ from $r$ to $u$.

   It follows that an optimal solution of the problem on $G'$ yields an optimal solution of the problem on $G$.

4. $wu, vw \in E$: This case is symmetric to the previous one.

**Exercise 4** ............................................................................ ***10 points***

Let $G = (V, E)$ be a graph with costs $c_e$, for every $e \in E$, and let $r \in V$. Let $T$ be a directed spanning tree of $G$ containing least-cost dipaths from $r$ to all nodes. Suppose that all the costs are increased by 26. Does $T$ still contain least-cost dipaths from $r$ to all nodes?

**Solution:** No. Consider digraph $G = (V, E)$, where $V = \{r, u, v\}$ and $E = \{ru, uv, rv\}$. Define costs $c_{ru} = 1$, $c_{uv} = 1$, and $c_{rv} = 3$. The unique directed spanning tree $T$ of $G$ containing least-cost dipaths from $r$ to all nodes has edge set $E(T) = \{ru, uv\}$.

If all costs are increased by 26 we obtain costs $c'_{ru} = 27$, $c'_{uv} = 27$, and $c'_{rv} = 29$. The unique shortest dipath from $r$ to $v$ is $P = r, rv, v$. However the arc $rv$ is not contained in $E(T)$.

**Exercise 5** ............................................................................ ***10 points***

Prove that if $c$ is integer-valued, $C = 2\max(|c_e| : e \in E) + 1$, and $G$ has no negative-cost dicircuit, then Ford's algorithm terminates after at most $Cn^2$ arc-correction steps.

**Solution:** For every node $v \in V(G)$, the cost of a simple dipath from $r$ to $v$ is an integer number between $-(n-1)\max(|c_e| : e \in E)$ and $(n-1)\max(|c_e| : e \in E)$. Therefore the value of $y_v$ can decrease at most

$$\begin{aligned}
2(n-1)\max(|c_e| : e \in E) + 1 &\leq 2n \max(|c_e| : e \in E) + n \\
&\leq (2\max(|c_e| : e \in E) + 1)n = Cn
\end{aligned}$$

times during the execution of the algorithm. Considering every node of $G$, the total number of arc-correction steps is bounded by $Cn^2$.