CS-1961 Note : Posets, continued

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1 From König to Dilworth

In this section we show how to derive

Theorem 1 (Dilworth) *Any finite poset can be partition to n chains but no less where n is the size of a maximum antichain.*

from the following

Theorem 2 (König) *In a bipartite graph, the size of a maximum matching equals the size of a minimum vertex cover.*

Proof. Suppose we are given a finite poset $(P, \leqslant)$. We construct the following poset :

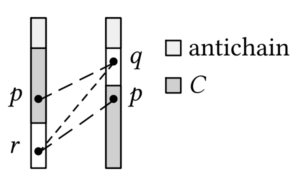
- vertices V : two copies of P, P_1 and P_2.
- edges E : $(p \in P_1, q \in P_2)$ exists iff $p < q$ (not $\leqslant$).

First, we inspect an arbitrary matching $M \subset E$ in that graph. We can recover a partition of P to chains $\{C_1, \ldots, C_k\}$ from M : we put p, q into the same chain if $(p, q) \in M$. This is reasonable since $(p, q) \in M \subset E$ implies p and q are comparable. Further,

$$|P| = |C_1| + \cdots + |C_k| = |m_1 + 1| + \cdots + |m_2 + 1| = |M| + k \tag{1}$$

where m_i is the number of corresponding edges in C_i. This is obvious with the observation that C_i along with edges in M is literally a "chain" because M is a matching, where a vertex can only be incident with only one edge.

Then, we inspect an arbitrary vertex cover $C \subset V$. We can assume no p appears in both copies of P in C because if so, we can remove one of them. After removing it is still a legitimate cover since if not, we must have some $p < q$ and $r < p$ where $q, r \in V - C$ (abbreviation for "two copies of p are both in $V - C$"). However, this means C does not cover the edge (q, r), a contradiction.



We can recover an antichain from this vertex cover : those $p \in P$ such that $p \in V - C$ forms an antichain, since $p < q$ implies $(p, q) \in E$, and either

p or q is in C. From our assumption

$$l + |C| = |P| \tag{2}$$

where l is the size of the antichain.

Now

$$\begin{aligned} \max l &\leqslant \min k \quad \text{(easily shown)} \\ &\leqslant |P| - \max |M| \quad (1) \\ &= |P| - \min |C| \quad (1) \\ &\leqslant \max l \quad (2). \end{aligned}$$

□

2 *Sperner theorem with application*

In this section we prove and show an interesting consequence of

Theorem 3 (Sperner) *The size of a maximum antichain of* $(\mathcal{P}([n]), \subset)$ *is* $\binom{n}{\lceil \frac{n}{2} \rceil}$.

Proof. Suppose $\{A_1, \ldots, A_k\}$ is a maximum antichain of $(\mathcal{P}([n]), \subset)$. For each A_i, we calculate the number of maximal chains C_i that contains A_i. In fact, a maximal chain is simply $\emptyset \subset \{n\} \subset \{n, m\} \subset \cdots \subset [n]$ where each set is one element larger than the previous. So it is clear now $C_i = |A_i|!(n - |A_i|)$. Further, we find that C_is are mutually exclusive because they are chains. Now

$$\begin{aligned} 1 &\geqslant \frac{\sum_{i=1}^{k} |C_i|}{|\{\text{all maximal chains}\}|} \\ &= \frac{\sum_{i=1}^{k} |A_i|!(n - |A_i|)}{n!} \\ &= \sum_{i=1}^{k} \frac{1}{\binom{n}{|A_i|}} \\ &\geqslant \frac{k}{\binom{n}{\lceil \frac{n}{2} \rceil}}. \end{aligned}$$

□

An example of such an antichain is the set of all subsets of size $\binom{n}{\lceil \frac{n}{2} \rceil}$.

An interesting corollary :

Theorem 4 (Erdös) *For any n reals $x_1, \ldots, x_n$ where $|x_i| \geqslant 1$ and an interval $(s, s+2)$, among the 2^n sums $\sum_{i=1}^n \pm x_i$ there are at most $\binom{n}{\lceil \frac{n}{2} \rceil}$ of them in the interval.*

Proof. Without loss of generality, assume $x_i \geqslant 1$. There is a one-to-one correspondence between subsets $S \subset [n]$ and sums by mapping S to a sum where only the coefficients of $x_i, i \in S$ are 1. Let f be this map. We note that if $S_1 \subset S_2$ then $|f(S_1) - f(S_2)| \geqslant 2$, that is, at least one of $f(S_1), f(S_2)$ does not falls in the interval. In other words, the sums that do lie in the interval have corresponding subsets that form an antichain. As we already know from 3, it is at most $\binom{n}{\lceil \frac{n}{2} \rceil}$. □

3 Mobiös inversion

In some ways Mobiös inversion is a generalisation of the inclusion-exclusion principle. Recall the generalised inclusion-exclusion principle : for some sets $\mathcal{A} = \{B_1, B_2, \ldots, B_n\}$,

$$N_=(\mathcal{I}) = \sum_{J:\mathcal{I} \subset J \subset \mathcal{A}} (-1)^{|J|-|\mathcal{I}|} N_\geqslant(J)$$

where

$$N_=(\mathcal{I}) = |\{x | x \in \cap \mathcal{I}, x \notin \cup(\mathcal{A} - \mathcal{I})\}|, N_\geqslant(J) = |\{x | x \in \cap J\}|.$$

We have given an instinctive proof for this, but there is an even more instinctive perspective. What we are trying to do is to express $N_=$s using $N_\geqslant$s, but the other direction is rather trivial :

$$N_\geqslant(J) = \sum_{J \subset \mathcal{I} \subset \mathcal{A}} N_=(\mathcal{I}).$$

In fact, this means $N_\geqslant(J)$ is a linear combination of $N_=(\mathcal{I})$s. For example,

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} N_=(\emptyset) \\ N_=(\{B_1\}) \\ N_=(\{B_2\}) \\ N_=(\{B_3\}) \\ N_=(\{B_1, B_2\}) \\ N_=(\{B_1, B_3\}) \\ N_=(\{B_2, B_3\}) \\ N_=(\{B_1, B_2, B_3\}) \end{pmatrix} = \begin{pmatrix} N_\geqslant(\emptyset) \\ N_\geqslant(\{B_1\}) \\ N_\geqslant(\{B_2\}) \\ N_\geqslant(\{B_3\}) \\ N_\geqslant(\{B_1, B_2\}) \\ N_\geqslant(\{B_1, B_3\}) \\ N_\geqslant(\{B_2, B_3\}) \\ N_\geqslant(\{B_1, B_2, B_3\}) \end{pmatrix}.$$

Now our task is just to compute the inversion of the matrix. Although this does not give us an explicit formula, it does guide our way to Mobiös inversion.

We generalise in two directions :

1. Replace $(\mathcal{P}(\mathcal{A}), \subset)$ with arbitrary locally finite posets $(P, \leqslant)$. Here "locally finite" means forall $x \leqslant y$, there are only finite zs such that $x \leqslant z \leqslant y$.
2. Replace $N_=, N_\geqslant$ with arbitrary function $F, G : P \to \mathbb{R}$ where $G(x) = \sum_{z \leqslant x} F(z)$ and the sum is finite. A sufficient condition is P has a bottom element $\perp$.

To achieve our goal we need to generalise the notion of matrix since the poset can be infinite. We define the incidence algebra on $(P, \leqslant)$:

- elements : $\mathcal{F} = \{f \in P \times P \to \mathbb{R} | \forall x \nleqslant y, f(x, y) = 0\}$.
- $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F} : (f + g)(x, y) = f(x, y) + g(x, y)$.
- $\cdot : \mathbb{R} \times \mathcal{F} \to \mathcal{F} : (c \cdot f)(x, y) = cf(x, y)$.
- $* : \mathcal{F} + \mathcal{F} \to \mathcal{F} : (f * g)(x, y) = \sum_{x \leqslant z \leqslant y} f(x, z)g(z, y)$ (this is reasonable because P is locally finite).

Here we focus on $*$. As in the world of matrices, $*$ is associative (though not important here); there exists a unit $\delta(x, y) = 1$ iff $x = y$; not all elements in $\mathcal{F}$ has an inverse, but just like upper triangular matrices, those $f(y, y) \neq 0, \forall y \in P$ do have inverses :

$$f^{-1}(x, y) = \begin{cases} \frac{1}{f(y,y)}, & x = y \\ -\frac{1}{f(y,y)} \sum_{x \leqslant z < y} g(x, z) f(z, y), & x < y \end{cases}.$$

Luckily, the function we are interested in

$$\zeta(x, y) = \begin{cases} 1, & x \leqslant y \\ 0, & \text{otherwise} \end{cases}$$

satisfies this condition, and its inverse is

$$\mu(x, x) = 1, \mu(x, y) = -\sum_{x \leqslant z < y} \mu(x, z), \forall x < y.$$

Look again at the matrix example. Now we can prove our main theorem :

Theorem 5 (Mobiös inversion) $F(x) = \sum_{y \leqslant x} G(y)\mu(y, x)$.

Proof.

$$\begin{aligned} \sum_{y \leqslant x} G(y)\mu(y, x) &= \sum_{y \leqslant x} \sum_{z \leqslant y} F(z)\mu(y, x) \\ &= \sum_{y \leqslant x} \mu(y, x) \sum_{z \in P} \zeta(z, y) F(z) \\ &= \sum_{z \in P} \sum_{y \leqslant x} \mu(y, x)\zeta(z, y) F(z) \\ &= \sum_{z \in P} \left(\sum_{z \leqslant y \leqslant x} \zeta(z, y)\mu(y, x) \right) F(z) \\ &= \sum_{z \in P} \delta(z, x) F(z) \\ &= F(x). \end{aligned}$$

□

The inclusion-exclusion formula is a special case of it, as you can check.