CS-1961 Note : Generating Functions, continued

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1 Review

1.1 generating functions, formally

Formally, "generating functions" is the ring of power series $\mathbb{R}[[z]]$, that is, power series $F(z) = \sum_{n\geqslant 0} f_n z^n$[1] equipped with obvious addition and multiplication. From this point of view, other forms of expressions like $\frac{1}{1-z}$ is an *abbreviation* for $1+z+z^2+\cdots$. Luckily, many facts in analysis also hold in this ring. Better, some intuitive equations even hold when a series does not converge at all. We shall operate freely on generating functions (like mathematicians in the 18th century) in later sections without giving proofs.

[1] Or more formally, a sequence $f : \mathbb{N} \to \mathbb{R}$.

1.2 some useful generating functions

Lemma 1

1. *when* $f_n \equiv 1$, $\sum_{n\geqslant 0} f_n z^n = 1 + z + \cdots = \frac{1}{1-z}$.
2. *substitute* z *for* cz, *we get* $\sum_{n\geqslant 0} g_n z^n = 1 + cz + \cdots = \frac{1}{1-cz}$ *when* $g_n = c^n$.
3. *when* $f_k = \begin{pmatrix} n \\ k \end{pmatrix}$, $\sum_{n\geqslant 0} f_n z^n = \begin{pmatrix} n \\ 0 \end{pmatrix} + \cdots + \begin{pmatrix} n \\ n \end{pmatrix} + 0 + \cdots = (1+z)^n$.
   This is binomial coefficients.
4. *when* $f_k = \left(\begin{pmatrix} n \\ k \end{pmatrix}\right)$, $\sum_{n\geqslant 0} f_n z^n = (1 + z + z^2 + \cdots)^n = \frac{1}{(1-z)^n}$.

To prove the last equation, recall the definition of $\left(\begin{pmatrix} n \\ k \end{pmatrix}\right)$: place k identical balls into n distinct boxes. Now look at the right hand side. Every product $z^{k_1} \cdots z^{k_i} = z^{k_1+\cdots+k_i}$ contributes 1 to $f_{k_1+\cdots+k_i}$, which correspond to k_i balls in the ith box. Differentiating both sides gives an alternative proof to $\left(\begin{pmatrix} n \\ k \end{pmatrix}\right) = \begin{pmatrix} n+k-1 \\ k \end{pmatrix}$.

2 Generating function of the partition numbers

Lemma 2 *The generating function of* $P(n)$ *is* $F(z) = \prod_{k\geqslant 1} \frac{1}{(1-z^k)}$.

The right-hand is equal to $(1 + z + z^2 + \cdots)(1 + z^2 + z^4 + \cdots)\cdots$. Similar to the proof of (4), Every product $z^{k_1\cdot 1} \cdots z^{k_i \cdot i} = z^{k_1+\cdots+ik_i}$ contributes 1 to $f_{k_1+\cdots+ik_i}$, which correspond to the partition k_1 1s, k_2 2s, ..., k_i is.

Generating functions can be used to prove the following:

Proposition 3 (Euler) *For every natural number n, the number of partitions of n where every part is odd is equal to the number of these where every part is distinct.*

It is possible to find an explicit bijection, but utilizing generating functions makes the proof simpler. To prove two sequences are identical, we prove their generating functions are equal. We call the sequences d_n, o_n and corresponding generating functions $D(z), O(z)$.

Slightly modify (2) and its proof we can see $D(z) = (1 + z + z^2 + \cdots)(1 + z^3 + z^5 + \cdots)\cdots = \frac{1}{(1-z)(1-z^3)\cdots}$. As for $O(z)$, every number can only be used once in a partition. So higher power monomials in, say, $1 + z^k + z^{2k} + \cdots$ should be discarded because they correspond to having more than one part of number k. Thus $O(z) = (1 + z)(1 + z^2)\cdots$. However, note

$$\begin{aligned}(1+z)(1+z^2)\cdots &= \frac{1-z^2}{1-z}\frac{1-z^4}{1-z^2}\frac{1-z^6}{1-z^3}\cdots \\ &= \frac{1}{(1-z)(1-z^3)\cdots} \\ &= D(z),\end{aligned}$$

because every numerator also appears in the denominators.

3 *Generating function of the Stirling numbers of the second kind*

In this section we derive the generating function of the Stirling numbers of the second kind $S(n, k)$ for a fixed k using techniques last week. Recall the Stirling numbers of the second kind $S(n, k)$ satisfies the recurrence relation (we define $S(n, k) = 0$ if $n < 0$ or $k < 0$)

$$S(n,k) = S(n-1,k-1) + kS(n-1,k) + \mathbb{1}_{n=k=0}.$$

Multiplying and adding both sides gives

$$\begin{aligned}\sum_{n\geqslant 1} S(n,k)z^n &= \sum_{n\geqslant 1} S(n-1,k-1)z^n + \sum_{n\geqslant 1} kS(n-1,k)z^n, \\ S_k(z) - 1 &= zS_{k-1}(z) + kzS_k(z),\ \text{(because } S_0(z) = \sum_{n\geqslant 0} S(n,0)z^n = 1\text{)}, \\ S_k(z) &= \frac{z}{1-kz}S_{k-1}(z).\end{aligned}$$

This then gives $S_k(z) = \frac{z}{1-kz}S_{k-1}(z) = \frac{z}{1-kz}\frac{z}{1-(k-1)z}S_{k-2}(z) = \cdots = \frac{z^k}{\prod_{i=1}^{k}(1-iz)}$.

4 *The Catalan numbers*

The Catalan numbers C_n can be defined as *the number of ways to specify the order to calculate the product of a sequence of variables* $x_0, \ldots, x_n$. In other words, imagine fully parenthesizing the expression until there are no ambiguities. For example, for $n = 3$, we have different orders

$$(x_0 \cdot (x_1 \cdot (x_2 \cdot x_3))), (x_0 \cdot ((x_1 \cdot x_2) \cdot x_3)), ((x_0 \cdot x_1) \cdot (x_2 \cdot x_3)), \ldots.$$

We can even interpret $\cdot$ as `cons _ _` and the Catalan number becomes the number of distinct binary trees. In this section, we use generating functions to calculate C_n.

The Catalan numbers satisfies the recurrence relation $C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}, n \geqslant 2$. This is obvious if we first decide the position of the outermost $\cdot$. Multiplying and adding both sides gives

$$\begin{aligned}
C(z) := \sum_{n\geqslant 2} C_n z^n &= \sum_{n\geqslant 2}\sum_{i=0}^{n-1} C_i C_{n-1-i} z^n \\
&\overset{!}{=} z \sum_{n\geqslant 2} C^2(z) z^{n-1}, \\
C(z) - 1 - z &= zC^2(z) - z, \\
C(z) &= \frac{1 \pm \sqrt{1-4z}}{2z}.
\end{aligned}$$

the second step is "convolution" : $[z^k](F(z)G(z)) = \sum_{i=0}^{k} f_i g_{k-i}$. To determine the sign, remember that $C(z)$ should be a power series and z_0 is meaningful. Since $\sqrt{1-4z}$ expands to $1 + zf(z)$, the sign must be $-$.

To expand $\sqrt{1-4z}$, we use

Theorem 4 (Newton's Binomial Theorem) *Let r be a real number. For all x, y where $0 \leqslant x < y$,*

$$(x+y)^r = \sum_{n\geqslant 0} \binom{r}{n} x^n y^{r-n},$$

where $\binom{r}{n} = \frac{r(r-1)\cdots(r-n+1)}{n!}$.

Proof can be found in calculus textbooks. Using this formula,

$$\begin{aligned}
C(z) &= \frac{1-\sqrt{1-4z}}{2z} \\
&= \frac{1-\sum_{n\geqslant 0}\binom{\frac{1}{2}}{n}(-4z)^n}{2z} \\
&= \frac{\sum_{n\geqslant 1} -\frac{1}{n!}\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n+1\right)(-4z)^n}{2z} \\
&= \sum_{n\geqslant 0}\frac{1}{(n+1)!}2^n\cdot 1\cdot 3\cdot\cdots\cdot(2n-1)z^n \\
&= \sum_{n\geqslant 0}\frac{1}{(n+1)!}2^n\frac{(2n)!}{2^n n!}z^n \\
&= \sum_{n\geqslant 0}\frac{1}{n+1}\binom{n}{2n}z^n,
\end{aligned}$$

$C_n = \frac{1}{n+1}\binom{n}{2n}$.

5 *Exponential generating functions*

Suppose we are given a recurrence relation

$$f_0 = 1, f_{n+1} = (n+1)f_n.$$

Clearly $f_n = n!$. However, what if we try to solve it by means of generating functions? Multiplying and adding both sides gives

$$\begin{aligned}
\sum_{n\geqslant 1} f_n z^n &= \sum_{n\geqslant 1} n f_{n-1} z^n, \\
F(z) - 1 &= z^2 F'(z) + zF(z), \\
z^2 F'(z) + (z-1)F(z) + 1 &= 0.
\end{aligned}$$

It is possible to solve this differential equation, but it is too complex for such a simple task.

Now we introduce a new form of generating functions : for a sequence f_n, we define its *exponential generating function* $\hat{F}(z) = \sum_{n\geqslant 0}\frac{f_n}{n!}z^n$. If we adopt this new form to solve the above recurrence relation, everything is trivial. At first sight the factor $\frac{1}{n!}$ might seem too ad-hoc, but facts below indicate this may be helpful indeed :

1. When $f_n \equiv 1$, $\hat{F}(z) = e^z$. A very compact form.

2. Many combinatoric problems (especially permutations and combinations), the answer is likely to contain factors like $n!$.

A less trivial example : let $f_0 = 1, f_n = nf_{n-1} + 1$.

$$\begin{aligned}\sum_{n\geqslant 1} \frac{f_n z^n}{n!} &= \sum_{n\geqslant 1} \frac{f_{n-1} z^n}{(n-1)!} + \sum_{n\geqslant 1} \frac{1}{n!}, \\ \hat{F}(z) - 1 &= z\hat{F}(z) + e^z - 1, \\ \hat{F}(z) &= \frac{e^z}{1-z}.\end{aligned}$$

Using convolution $[z^k]\hat{F}(z) = \sum_{i=0}^{k} \frac{1}{i!}$, thus $f_n = n![z^n]\hat{F}(z) = n! \sum_{i=0}^{n} \frac{1}{i!}$.

6 The generating function of spanning trees

In this section we calculate the number of spanning trees t_n of a complete graph. t_n satisfies

$$t_0 = 0, t_1 = 1, t_n = \sum_{m>0} \sum_{k_1+\cdots+k_m=n-1, k_i\geqslant 1} \frac{1}{m!} \binom{n-1}{k_1, \ldots, k_m} k_1 \cdots k_m t_{k_1} \cdots t_{k_m},$$

where $\binom{n}{k_1, \ldots, k_m} = \frac{n!}{k_1! k_2! \cdots k_m! (n-k_1-\cdots-k_m)!} = \binom{n}{k_1}\binom{n-k_1}{k_2} \cdots \binom{n-k_1-\cdots-k_{m-1}}{k_m}$.

To understand the equation, fix a vertex v as “root”. First decide how many edges in the spanning tree are incident with v, say m edges. Then partition rest of the vertices to m parts, each part forming a subtree of v. Lastly decide which vertex in each subtree is connected to v.

To solve the recurrence relation, first rewrite the equation as

$$\begin{aligned}t_n &= \sum_{m>0} \frac{1}{m!} \sum_{k_1+\cdots+k_m=n-1, k_i\geqslant 1} \frac{n!}{k_1! k_2! \cdots k_m!} k_1 \cdots k_m t_{k_1} \cdots t_{k_m}, \\ \frac{t_n}{(n-1)!} &= \sum_{m>0} \frac{1}{m!} \sum_{k_1+\cdots+k_m=n-1, k_i\geqslant 1} \frac{t_{k_1}}{(k_1-1)!} \cdots \frac{t_{k_m}}{(k_m-1)!}, n \geqslant 2.\end{aligned}$$

For convenience, let $u_n = nt_n$,

$$\frac{u_n}{n!} = \sum_{m>0} \frac{1}{m!} \sum_{k_1+\cdots+k_m=n-1, k_i\geqslant 1} \frac{u_{k_1}}{k_1!} \cdots \frac{u_{k_m}}{k_m!}, n \geqslant 2$$

In fact, $\sum_{k_1+\cdots+k_m=n-1, k_i\geqslant 0} \frac{u_{k_1}}{k_1!} \cdots \frac{u_{k_m}}{k_m!}$ is the coefficient of z^{n-1} in $\hat{U}^m(z)$. Thus

$$\begin{aligned}\sum_{n\geqslant 2} \frac{u_n}{n!} z^n &= \sum_{n\geqslant 2} z^n \sum_{m>0} [z^{n-1}] \frac{1}{m!} \hat{U}^m(z), \\ \hat{U}(z) - z &= z\left(\sum_{m\geqslant 0} \frac{1}{m!} \hat{U}^m(z)\right) - z, \\ \hat{U}(z) &= ze^{\hat{U}(z)}.\end{aligned}$$

To solve $\hat{U}(z)$, Our approach needs

Theorem 5 (Lagrange Inversion Theorem) *Suppose $z = f(g(z))$ and f is known, we can solve g by*

$$g(z) = a + \sum_{n \geqslant 1} g_n \frac{(z - f(a))^n}{n!}$$

where

$$g_n = \lim_{x \to a} \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} \left(\left(\frac{x - a}{f(x) - f(a)} \right)^n \right).$$

Some requirements are omitted. substitute $z := z, f(x) = \frac{x}{e^x}, a := 0$ and use the theorem we get $\hat{U}(z) = \sum_{n \geqslant 1} \frac{n^{n-2}}{(n-1)!} z^n$, $t_n = n^{n-2}$.