LEMMA 1. *For all $\mathfrak{I} \models \varphi$ where $\mathfrak{I} = (A, \beta)$, there is some finite set A_0 such that for all $A_0 \subset A' \subset A$, if some β' agrees with β on A', then $\mathfrak{I}' \models \varphi$ where $\mathfrak{I}' = (A', \beta')$.*

Here "β and β' agree on A" means $\beta(x) = a \in A \Leftrightarrow \beta'(x) = a \in A$.

Induction on rank. When we see $\neg$ we need to peek inside. The only tricky part is $\neg\exists x\varphi$, which is essentially the same as $\forall x \neg\varphi$. By the induction hypothesis, for every $a \in A$, there is a corresponding A_0 of $\mathfrak{I}\frac{a}{x} \models \neg\varphi$, denoted by A_a. Suppose the free variables of φ are $F = \{v_1, \ldots, v_n\}$, and $\beta(F) = \{a_1, \ldots, a_m\}$, and $a_0 \notin \{a_1, \ldots, a_m\}$ (if there is such an a_0). We contend that $A_0 = \bigcup_{i=0}^{m} A_{a_i}$.

Let $A_0 \subset A' \subset A$ and β' agrees with β on A'. We need to show that $\mathfrak{I}' \models \forall x \neg\varphi$ where $\mathfrak{I}' = (A', \beta')$, that is, $\mathfrak{I}'\frac{a}{x} \models \neg\varphi$ for all $a \in A' \subset A$. If $a \in \beta(F)$ or $a = a_0$ then $A_a \subset A' \subset A$, and $\beta'\frac{a}{x}$ trivially agrees with $\beta\frac{a}{x}$ on A', so we are done. Otherwise $a \notin \beta(F)$. We already know that $\mathfrak{I}'\frac{a_0}{x} \models \neg\varphi$. By a modified version of the Isomorphism Lemma we can show that $\mathfrak{I}'\frac{a}{x} \models \neg\varphi \Leftrightarrow \mathfrak{I}'\frac{a_0}{x} \models \neg\varphi$. The intuition is that a and a_0 are essential the same under the context that neither of them have been "assigned" to some variable.

The theorem implies that the notion of infinity is highly non-trivial. Without additional structures we can not write a consistent sentence which has exactly infinites set as models (this, however, does not need the tedious proof above), and we can not even write a consistent sentence which only has infinite models.