1 Problem 1

If the primal LP is unbounded, the dual must be infeasible. We cannot even start.

If the primal LP is infeasible, the dual is either infeasible or unbounded. If it is infeasible, we cannot start; if it is unbounded, because the Primal-Dual scheme can achieve a small difference in polynomial time, one of the followings will happen in a predictable number of iterations:

- DRP is unbounded.
- DRP is bounded, but the step θ is unbounded.

2 Problem 2

There are many optimal solutions to DRP; we consider the variant that update α_s and all reachable vertices α_v from s by edges in E_t. After each iteration, tight edges are still tight, and at least one more slack edge becomes tight. Since there are only $|E|$ edges, the algorithm terminates after at most $|E|$ rounds. In fact, After each iteration, at least one more vertex is reachable from s, so the algorithm terminates after at most $|V|$ rounds.

3 Problem 3

Consider the unweighted maximum matching problem, formulated as follows.

$$\begin{aligned} \text{maximize} \sum_{(u,v)\in E} e_{u,v}&, \\ \sum_{(u,v)\in E} e_{u,v} &\leqslant 1, u \in A, \\ \sum_{(u,v)\in E} e_{u,v} &\leqslant 1, v \in B, \\ e_{u,v} &\geqslant 0, u \in A, v \in B. \end{aligned}$$

Its dual is

$$\begin{aligned} \text{minimize} \sum_{u\in A} \alpha_u + \sum_{v\in B} \alpha_v&, \\ \alpha_u + \alpha_v &\geqslant 1, (u,v) \in E, \\ \alpha_u &\geqslant 0, u \in A, \\ \alpha_v &\geqslant 0, v \in B. \end{aligned}$$

The complementary slackness condition is

$$\begin{aligned} e_{u,v} = 0 &\ \vee\ \alpha_u + \alpha_v = 1, \\ \alpha_u = 0 &\ \vee \sum_{(u,v)\in E} e_{u,v} = 1, \\ \alpha_v = 0 &\ \vee \sum_{(u,v)\in E} e_{u,v} = 1. \end{aligned}$$

The matching $\hat{e}$ produced by the algorithm has no augmenting paths. We construct $\hat{\alpha}$ such that $\hat{e}$ and $\hat{\alpha}$ satisfy the complementary slackness condition. It then follows that $\hat{e}$ is optimal.

We assign α to vertices starting from unmatched vertices, and follow the traverse order when we find alternating paths. That is, we start from 0, and alternate between 0 and 1 as we traverse the graph. α is well defined, or consistent, otherwise we get a augmenting path. For the same reason α is feasible; no edge has two zero-valued ends. By definition α also satisfies the CSC.

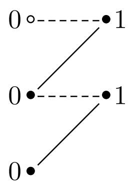

Figure 1. Illustration of the construction of α.

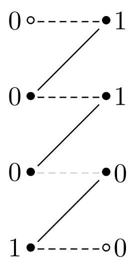

Figure 2. Why α is consistent and feasible. The gray line is impossible.

4 Problem 4

The primal is

$$\begin{aligned}
&\text{minimize} \sum_{(u,v)\in E} b_{u,v}\beta_{u,v}, \\
\alpha_u - \alpha_v + \beta_{u,v} - \gamma_{u,v} &= 0, (u,v)\in E, \\
\alpha_t - \alpha_s &= 1, \\
\alpha_v &\geqslant 0, v\in V, \\
\beta_{u,v} &\geqslant 0, (u,v)\in E, \\
\gamma_{u,v} &\geqslant 0, (u,v)\in E.
\end{aligned}$$

Given J, the RP is

$$\begin{aligned}
&\text{minimize}\, \delta_w + \sum_{(u,v)\in E} \delta_{u,v}, \\
\delta_{u,v} + \alpha_u - \alpha_v + \beta_{u,v} - \gamma_{u,v} &= 0, (u,v)\in E, \\
\delta_w + \alpha_t - \alpha_s &= 1, \\
\alpha_v &\geqslant 0, v\in V, \\
\beta_{u,v} &\geqslant 0, (u,v)\in E, \\
\gamma_{u,v} &\geqslant 0, (u,v)\in E, \\
\delta_{u,v} &\geqslant 0, (u,v)\in E, \\
\delta_w &\geqslant 0, \\
\alpha_s &= 0, \text{ if } \sum_{(s,u)\in E} f_{s,u} - \sum_{(u,s)\in E} f_{u,s} < w, \\
\alpha_t &= 0, \text{ if } \sum_{(t,u)\in E} f_{t,u} - \sum_{(u,t)\in E} f_{u,t} < -w, \\
\alpha_v &= 0, \text{ if } \sum_{(u,v)\in E} f_{u,v} \neq \sum_{(v,u)\in E} f_{v,u}, \\
\beta_{u,v} &= 0, \text{ if } f_{u,v} \neq b_{u,v}, \\
\gamma_{u,v} &= 0, \text{ if } f_{u,v} \neq 0.
\end{aligned}$$

Here $f_{u,v}$s are solution of the dual.

5 Problem 5

We claim that the following algorithm is 3/2-approximate for the zero-endpoint case.

1. Find a minimal spanning tree T of G.
2. Find a perfect matching M_0 of the odd-degree vertices in T. Add edges of M_0 to T. Call it T_0.
3. Find a Euler cycle E_0 of T_0.
4. Chase the E_0 to construct a Hamilton cycle C_0 in G.
5. Remove an edge of C_0 to get a Hamilton path P_0.
6. For each pair of vertices u, v, do the following:
   i. Find a perfect matching $M_{u,v}$ of odd-degree vertices of T, except u and v. Add edges of $M_{u,v}$ to T. Call it $T_{u,v}$.
   ii. Find a Euler path $E_{u,v}$ of $T_{u,v}$. This is feasible because there are only 2 odd-degree vertices, namely u and v.
   iii. Chase $E_{u,v}$ to construct a Hamilton path $P_{u,v}$.
7. Choose the cheapest path $\hat{P}$ from P_0 and $P_{u,v}$.

Figure 3. A 3/2-approximate algorithm for the zero-endpoint case.

The algorithm is evidently polynomial time. To prove our claim, we consider an optimal Hamilton path P^*. We take Figure 4 as an example. Consider the following two cases.

$\boldsymbol{|B|+|D| \leqslant \frac{1}{2}|P^*|}$**.** In this case, M_0 has a cost less than $|B|+|D| \leqslant \frac{1}{2}|P^*|$. Then,

$$|\hat{P}| \leqslant |P_0| \leqslant |C_0| \leqslant |E_0| = |T_0| \leqslant |T| + \frac{1}{2}|P^*| = \frac{3}{2}|P^*|.$$

$\boldsymbol{|B|+|D| \geqslant \frac{1}{2}|P^*|}$**.** In this case, we have $|C| \leqslant |C|+|A|+|E| \leqslant \frac{1}{2}|P^*|$. So M_{v_1,v_4} has a cost less than $|C| \leqslant \frac{1}{2}|P^*|$. Then,

$$|\hat{P}| \leqslant |P_{v_1,v_4}| \leqslant |E_{v_1,v_4}| = |T_{v_1,v_4}| \leqslant |T| + \frac{1}{2}|P^*| = \frac{3}{2}|P^*|.$$

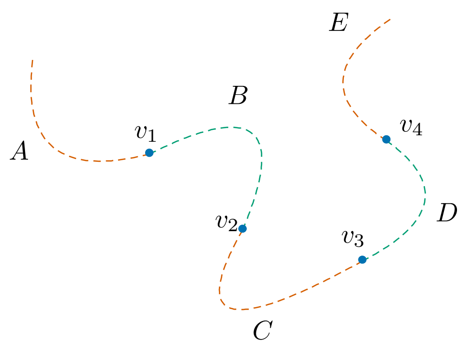

Figure 4. An optimal path to illustrate the proof of Algorithm 3. A to E denote sub paths. v_is denote odd-degree vertices in T.

We claim that the following algorithm is $3/2$-approximate for the one-endpoint case. Assume the staring point is s.

1. Find a minimal spanning tree T of G.
2. Find a perfect matching M_0 of the odd-degree vertices in T and construct C_0. (We omit the details; they are similar to the previous algorithm.)
3. Remove an edge of C_0 to get a Hamilton path P_0 that starts from s.
4. Case spilt on the parity of degree of s.

   odd. For each odd-degree vertex v, find a perfect matching of odd-degree vertices in T—except s and v—and construct P_v. (Again, we omit the details.)

   even. For each odd-degree vertex v, find a perfect matching of odd-degree vertices in T—except v, plus s—and construct P_v. (Again, we omit the details.)
5. Choose the cheapest path $\hat{P}$ from P_0 and P_v.

Figure 5. A $3/2$-approximate algorithm for the one-endpoint case.

The algorithm is evidently polynomial time. We only prove the correctness when the degree of s is odd; the other case is similar. Consider an optimal Hamilton path P^*. We take Figure 6 as an example. Consider the following two cases.

$|B|+|D|\leqslant\frac{1}{2}|P^*|$. In this case, M_0 has a cost less than $|B|+|D|\leqslant\frac{1}{2}|P^*|$. Then,

$$|\hat{P}|\leqslant|P_0|\leqslant|C_0|\leqslant|E_0|=|T_0|\leqslant|T|+\frac{1}{2}|P^*|=\frac{3}{2}|P^*|.$$

$|B|+|D|\geqslant\frac{1}{2}|P^*|$. In this case, we have $|C|\leqslant|C|+|A|\leqslant\frac{1}{2}|P^*|$. So M_{v_1} has a cost less than $|C|\leqslant\frac{1}{2}|P^*|$. Then,

$$|\hat{P}|\leqslant|P_{v_1}|\leqslant|E_{v_1}|=|T_{v_1}|\leqslant|T|+\frac{1}{2}|P^*|=\frac{3}{2}|P^*|.$$

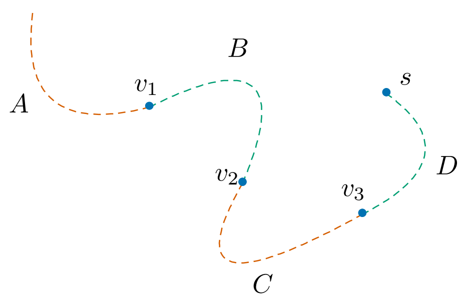

Figure 6. An optimal path to illustrate the proof of Algorithm 5. A to D denote sub paths. s and v_is denote odd-degree vertices in T.

We claim that the following algorithm is $5/3$-approximate for the two-endpoint case. Assume the two endpoints are s and t.

1. Find a minimal spanning tree T of G.
2. Case split on the parity of the degrees of s and t.

   ***t* and *s* odd.**

   i. Find a perfect matching M_1 of odd-degree vertices in T—except s and t—and construct P_1.
   ii. Find a perfect matching M_2 of odd-degree vertices in T and construct C_2.
   iii. Construct a Hamilton path P_2 from C_2. We start from s, follow C_2, skip t, and finally jump to t instead of returning to s.
   iv. Choose the smaller one between P_1 and P_2.

   ***t* and *s* even.** Similar to the previous one.

   ***t* odd, *s* even.**

   i. Find a perfect matching M_1 of odd-degree vertices in T—plus s, except t—and construct P_1.
   ii. Find a perfect matching M_2 of odd-degree vertices in T and construct C_2.
   iii. Construct a Hamilton path P_2 from C_2. We start from s, follow C_2, skip t, and finally jump to t instead of returning to s.
   iv. Choose the smaller one between P_1 and P_2.

   ***s* odd, *t* even.** Similar to the previous case.

Figure 7. A 5/3-approximate algorithm for the two-endpoint case.

The algorithm is evidently polynomial time. We only prove its approximate ratio under condition t and s are odd-degree; arguments for other cases are similar. Consider an optimal Hamilton path P^*. We take Figure 8 as an example. We consider the following two cases.

$\boldsymbol{|M_1| \leqslant \frac{2}{3}|P^*|}$**.** In this case,

$$|\hat{P}| \leqslant |P_1| \leqslant |E_1| = |T_1| \leqslant |T| + \frac{2}{3}|P^*| = \frac{5}{3}|P^*|.$$

$\boldsymbol{|M_1| \geqslant \frac{2}{3}|P^*|}$**.** In this case,

$$|M_2| \leqslant |P^*| - (|B| + |D|) \leqslant |P^*| - |M_1| \leqslant \frac{1}{3}|P^*|.$$

We have $|P_2| \leqslant |C_2| + d_{s,t}$ by the triangular inequality. To bound $d_{s,t}$, we construct a matching of odd-degree vertices except s and t as follows.

1. Remove s and t and the unique path $P_{s,t}$ connecting them in T. Note that the set of odd-degree vertices does not change.
2. Match two old-degree vertices u, v in the same connected component, and remove the unique path $P_{u,v}$ between them. Repeat until nothing can be matched.

Now, this matching $|M|$ satisfies $|M_1| \leqslant |M| \leqslant \sum_{u,v} |P_{u,v}| \leqslant |T| - |P_{s,t}|$ by the minimality of M_1 and triangular inequality. We conclude $d_{s,t} \leqslant |P_{s,t}| \leqslant |T| - \frac{2}{3}|P^*|$ and

$$|P_2| \leqslant |C_2| + d_{s,t} \leqslant |T| + \frac{1}{3}|P^*| + |T| - \frac{2}{3}|P^*| \leqslant \frac{5}{3}|P^*|.$$

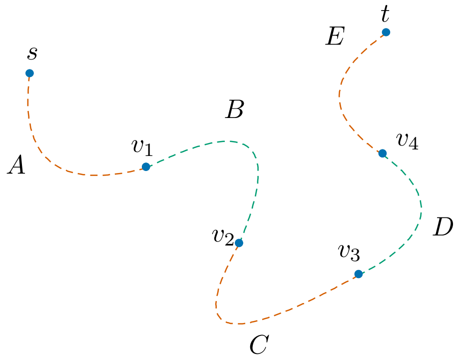

Figure 8. An optimal path to illustrate the proof of Algorithm 7. A to E denote sub paths. v_is denote odd-degree vertices in T.

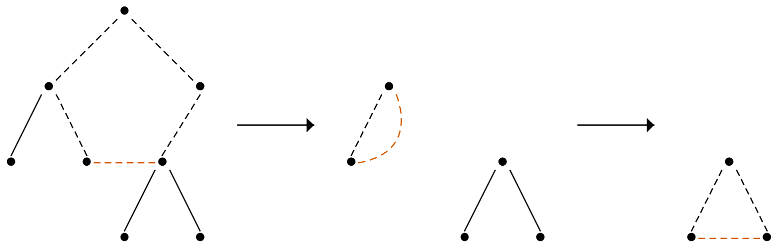

Figure 9. Illustration of the matching argument.

6 Credit

Yuxiao Yang pointed out a neglected case in **Problem 5** (1).