

Simulating the dynamics of the pendulum and cart

1 Kinematics

Consider a cart of mass M , motorized and free to move along the x-axis upon a frictionless track. A pendulum bob of mass m is attached to the cart by a rigid, massless, freely swinging rod of length l . Let's start by writing down the kinematics—that is, the position, velocity, and acceleration—for each of the cart and bob.

For the cart,

$$\begin{aligned}\text{position: } & x\hat{\mathbf{x}} \\ \text{velocity: } & x'\hat{\mathbf{x}} \\ \text{acceleration: } & x''\hat{\mathbf{x}}.\end{aligned}$$

And for the bob,

$$\begin{aligned}\text{position: } & (x + l \sin \theta)\hat{\mathbf{x}} + l \cos \theta \hat{\mathbf{y}} \\ \text{velocity: } & (x' + l\theta' \cos \theta)\hat{\mathbf{x}} - l\theta' \sin \theta \hat{\mathbf{y}} \\ \text{acceleration: } & (x'' + l\theta'' \cos \theta - l\theta'^2 \sin \theta)\hat{\mathbf{x}} - (l\theta'' \sin \theta + l\theta'^2 \cos \theta)\hat{\mathbf{y}}.\end{aligned}$$

2 Forces

We can list the forces acting on each of the cart and bob as well (see Fig. 1).

The cart experiences

$$\begin{aligned}\text{force due to gravity: } & -Mg\hat{\mathbf{y}} \\ \text{rod tension: } & T \sin \theta \hat{\mathbf{x}} + T \cos \theta \hat{\mathbf{y}} \\ \text{normal force on track: } & N\hat{\mathbf{y}} \\ \text{motor force: } & f\hat{\mathbf{x}}.\end{aligned}$$

And the bob experiences

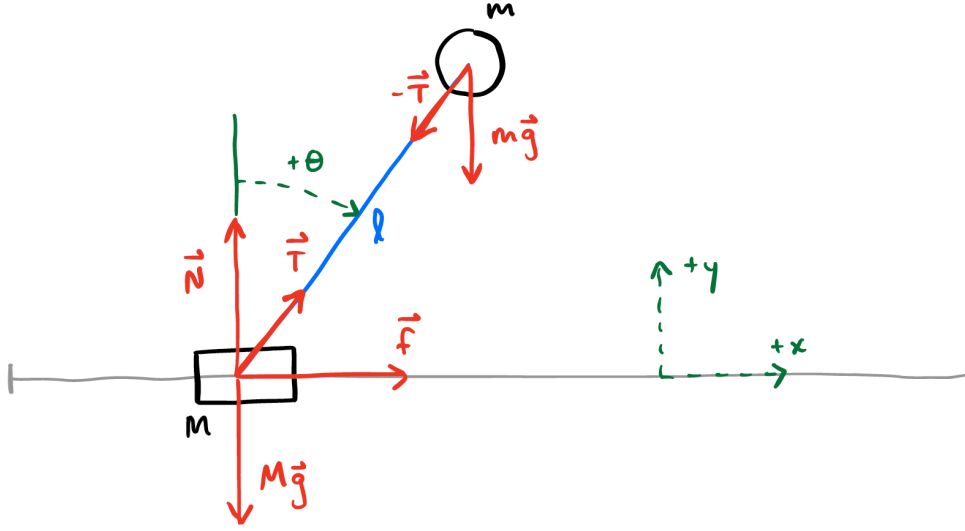


Figure 1: Free body diagram for our setting of the cart and pendulum.

force due to gravity: $-mg\hat{\mathbf{y}}$

rod tension: $-T \sin \theta \hat{\mathbf{x}} - T \cos \theta \hat{\mathbf{y}}$.

3 Equations of motion

Using Newton's law $\mathbf{F}_{\text{net}} = m\mathbf{x}''$ to relate forces with kinematics, we get the two expressions which follow.

For the cart,

$$T \sin \theta \hat{\mathbf{x}} + f \hat{\mathbf{x}} - Mg \hat{\mathbf{y}} + T \cos \theta \hat{\mathbf{y}} + N \hat{\mathbf{y}} = Mx'' \hat{\mathbf{x}}. \quad (1)$$

And for the bob,

$$-T \sin \theta \hat{\mathbf{x}} - mg \hat{\mathbf{y}} - T \cos \theta \hat{\mathbf{y}} = mx'' \hat{\mathbf{x}} + ml\theta'' \cos \theta \hat{\mathbf{x}} - ml\theta'^2 \sin \theta \hat{\mathbf{x}} - ml\theta'' \sin \theta \hat{\mathbf{y}} - ml\theta'^2 \cos \theta \hat{\mathbf{y}}. \quad (2)$$

Separating these two equations into vector components gives four equations.

$$\text{cart x: } T \sin \theta + f = Mx'' \quad (3)$$

$$\text{cart y: } -Mg + T \cos \theta + N = 0 \quad (4)$$

$$\text{bob x: } -T \sin \theta = mx'' + ml\theta'' \cos \theta - ml\theta'^2 \sin \theta \quad (5)$$

$$\text{bob y: } -mg - T \cos \theta = -ml\theta'' \sin \theta - ml\theta'^2 \cos \theta. \quad (6)$$

Adding together equations (3) and (5) to eliminate T gives

$$f = (M + m)x'' + ml\theta'' \cos \theta - ml\theta'^2 \sin \theta. \quad (7)$$

Multiplying (5) through by $\cos \theta$ and multiplying (6) through by $-\sin \theta$ and then adding these together gives

$$mg \sin \theta = mx'' \cos \theta + ml\theta''. \quad (8)$$

Our two equations of motion are (7) and (8). However, these must be rewritten to isolate x'' and θ'' in each equation. This can be done algebraically or using *Mathematica*, yielding

$$x'' = \frac{f + ml\theta'^2 \sin \theta - mg \sin \theta \cos \theta}{M - m - m \cos^2 \theta} \quad (9)$$

$$\theta'' = \frac{-f \cos \theta + (M + m)g \sin \theta - ml\theta'^2 \sin \theta \cos \theta}{l(M + m - m \cos^2 \theta)}. \quad (10)$$

We can turn these two second-order ODEs into four first-order ODEs, yielding a form which will be suitable for numerical simulation via the Runge-Kutta method:

$$x' = v \quad (11)$$

$$\theta' = \omega \quad (12)$$

$$v' = \frac{f + ml\omega^2 \sin \theta - mg \sin \theta \cos \theta}{M + m - m \cos^2 \theta} \quad (13)$$

$$\omega' = \frac{-f \cos \theta + (M + m)g \sin \theta - ml\omega^2 \sin \theta \cos \theta}{l(M + m - m \cos^2 \theta)}. \quad (14)$$

These are the equations of motion we will use for numerical simulation.

4 Runge-Kutta update rule

Now we will work out the update rules for simulating via the Runge-Kutta method. We can represent the system with a 5-dimensional state vector

$$\mathbf{x} = (t, x, \theta, v, \omega) \in \mathbb{R}^5, \quad \text{where, equivalently,} \quad (15)$$

$$x_1 = t$$

$$x_2 = x$$

$$x_3 = \theta$$

$$x_4 = v$$

$$x_5 = \omega.$$

So, our equations of motion become

$$\frac{d}{dt}x_1 = f_1(\mathbf{x}) = 1 \quad (\text{passage of time; } dt = 1)$$

$$\frac{d}{dt}x_2 = f_2(\mathbf{x}) = x_4 \quad (\text{change in cart position; } \frac{dx}{dt} = v)$$

$$\frac{d}{dt}x_3 = f_3(\mathbf{x}) = x_5 \quad (\text{change in pendulum angle; } \frac{d\theta}{dt} = \omega = 1)$$

$$\frac{d}{dt}x_4 = f_4(\mathbf{x}) = \frac{f + mlx_5^2 \sin x_3 - mg \sin x_3 \cos x_3}{M + m - m \cos^2 x_3} \quad (\text{change in cart velocity; } \frac{dv}{dt} = a)$$

$$\frac{d}{dt}x_5 = f_5(\mathbf{x}) = \frac{-f \cos x_3 + (M + m)g \sin x_3 - mlx_5^2 \sin x_3 \cos x_3}{l(M + m - m \cos^2 x_3)} \quad (\text{change in pendulum angular velocity; } \frac{d\omega}{dt} = \alpha).$$

This can be expressed even more generally as

$$\frac{d}{dt}\mathbf{x} = \mathbf{f}(\mathbf{x}) \quad (16)$$

where \mathbf{f} is a loosely defined “vector of functions” $\mathbf{f} = (f_1, f_2, f_3, f_4, f_5)$, each accepting the system state \mathbf{x} as an argument and returning the rate of change of a particular state variable.

The Runge-Kutta method, using a step of size h , numerically approximates four intermediate values of the state as it advances from \mathbf{x}_i to \mathbf{x}_{i+1}

$$\mathbf{k}_1 = \mathbf{f}(\mathbf{x}) \quad (17)$$

$$\mathbf{k}_2 = \mathbf{f}(\mathbf{x} + \frac{h}{2}\mathbf{k}_1) \quad (18)$$

$$\mathbf{k}_3 = \mathbf{f}(\mathbf{x} + \frac{h}{2}\mathbf{k}_2) \quad (19)$$

$$\mathbf{k}_4 = \mathbf{f}(\mathbf{x} + h\mathbf{k}_3). \quad (20)$$

and then returns a weighted average as the result:

$$\mathbf{x}_{i+1} = \mathbf{x}_i + \frac{h}{6}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4). \quad (21)$$

This is the update rule for advancing the simulation by a step h .